Introduction to Fractional 2^k Designs

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Introduction

- Remember from our prior lessons that one of the advantages of 2^k designs is that they are very efficient in terms of the number of runs required to estimate the effects of k factors.
  - We group the levels of the factors into two levels, $-$ and $+$, and then run the experiment at all combinations of these levels.

- However, even for a reasonably small value of k, the number of required runs can become prohibitively large.
  - For example, if $k = 6$, then the number of runs required is $2^6 = 64$.

- What do we mean by this exactly and why is it a problem? Let's dive deeper.

Introduction

- Suppose we have a 2^6 design. What this means is that we have 6 main effects, but we also have all of the two, three, four, five, and six way interactions.

- The formula for the fully crossed design is so large, I'm not going to include it here. But as you can imagine, we have six main effects, 15 two-way interactions, 20 three-way interactions, 15 four-way interactions, 6 five-way interactions, and one six-way interaction.

Introduction

- To estimate all of these effects, we have to have 64 runs (or observations).

- Recall from regression, if we have n observations, at most, we can estimate $p = n$ parameters. The number of parameters is equal to the number of main effects plus the number of interactions and the overall mean.

- So in this case, we have 64 observations and 64 parameters. This means that we are estimating one parameter per observation.
  - This is not a good design. We are not estimating the parameters well, and we are not estimating the interactions well. This is a problem because we are not getting enough information from our design to make good conclusions about the effects of our factors on the response variable.

Introduction

- Additionally, it is highly unlikely that any interaction terms beyond the two-way interactions are important practically or statistically.
  - So, even if we did have more than one replicate per parameter, we probably don't care too much about the three-way, four-way, five-way, and six-way interactions.

- If this is the case, then we can save a lot of time (and potentially money) by not running the full 2^k design.
  - Instead, we can run a fractional factorial design, which is a subset of the full factorial design.

- In such a design, we expect that information on the main effects and low-order interactions can be obtained from this smaller number of runs (observations).

Introduction

- These **fractional factorial designs** are very useful in practice, especially when the number of factors is large and the number of runs is limited.
- We can think of fractional factorial designs as a type of variable importance analysis.
  - We are trying to determine which factors are important and which factors are not important as efficiently as possible.
- Let's dive in! What would a **One-Half Fraction** look like?

One-Half Fractional Factorial Design

- Let's suppose we are working with a 2^3 design.
  - This means we have three factors, A, B, and C, each at two levels, $-$ and $+$.
- Check out our table on the next slide. Note, the I column is the identity column and denotes the overall, grand mean of the response variable.

One-Half Fractional Factorial Design

Table 1: Full Factorial Design for a 2^3 Experiment

Treatment	I	A	B	C	AB	AC	BC	ABC
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
c	+	-	-	+	+	-	-	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	-	+	-	-	-
ac	+	+	-	+	-	+	-	-
bc	+	-	+	+	-	-	+	-
(1)	+	-	-	-	+	+	+	-

One-Half Fractional Factorial Design

- Suppose that instead of estimating all of the effects, we only want to estimate the main effects (a, b, and c) as well as the three-way interaction abc as our one-half (i.e., $2^3/2 = 2^{3-1} = 4$) fraction.
- Here, we are selecting only those treatments that have a $+$ in the ABC column.
- Thus, in this case, we call ABC the **generator** of this particular fraction.

One-Half Fractional Factorial Design

- Additionally, the identity column is always $+$, so we can call $I = ABC$ the **defining relation** for our design.
- In general, the defining relation for a fractional factorial will always be the set of all columns that are equal to the identity column, I.
- The treatment combinations of our now 2^{3-1} design yield three degrees of freedom to help us estimate the main effects.

One-Half Fractional Factorial Design

- So how do we estimate the main effect A in this design? Looking in the A column, we have the sequence $+$, -, -, $+$ which are associated with treatments a, b, c, and abc, respectively.
- Thus, we can estimate the main effect A as:

$$[A] = \frac{1}{2}(a - b - c + abc)$$

One-Half Fractional Factorial Design

- Similarly, we can estimate the main effect B and C as:

$$[B] = \frac{1}{2}(-a + b - c + abc)$$

$$[C] = \frac{1}{2}(-a - b + c + abc)$$

- Note, we use the $[A]$, $[B]$, $[C]$ notation to indicate the linear combinations (i.e., the formulas derived from the $+/$- table) associated with the main effects.

One-Half Fractional Factorial Design

- We can also define the two-way interactions as:

$$[AB] = \frac{1}{2}(-a - b + c + abc)$$

$$[AC] = \frac{1}{2}(-a + b - c + abc)$$

$$[BC] = \frac{1}{2}(a - b - c + abc)$$

One-Half Fractional Factorial Design

- Note that, interestingly: $[A] = [BC]$, $[B] = [AC]$, and $[C] = [AB]$.
  - What does this mean??
- This means that the main effects are **aliased** with the two-way interactions.
  - In other words, we cannot differentiate between the main effects and the two-way interactions.
- In fact, when we estimate A, B, and C, we are really estimating $A + BC$, $B + AC$, and $C + AB$, respectively.

One-Half Fractional Factorial Design

- The alias structure can be determined by using the defining relation, $I = ABC$.
- Multiplying any column by the defining relation will yield the alias of that column.
  - For example, to determine the alias of A, we can multiply the A column by the defining relation, $I = ABC$:

$$A \cdot I = A \cdot ABC = A^2 \cdot BC$$

- If we square any column, the $+$ remain $+$ and the - become $+$. This implies that the square of any column is equal to the identity column, I.

One-Half Fractional Factorial Design

- Thus: $A^2 \cdot BC = I \cdot BC = BC$.
  - This means that A is aliased with BC.

- Similarly, we can find the aliases of B and C as:

$$B \cdot I = B \cdot ABC = B^2 \cdot AC = I \cdot AC = AC$$

$$C \cdot I = C \cdot ABC = C^2 \cdot AB = I \cdot AB = AB$$

One-Half Fractional Factorial Design

- In this last example, we chose $I = +ABC$ as our defining relation (we call this the **principal fraction**).
  - What if we chose the other one-half fraction? That is, the treatment combinations in the $+$/- table associated with the -'s of the ABC column?

- Here, we our defining relation would be $I = -ABC$ and our treatment combinations would be:

$$[A]' = \frac{1}{2}(ab + ac - bc - (1)) \to A - BC$$

$$[B]' = \frac{1}{2}(ab - ac + bc - (1)) \to B - AC$$

$$[C]' = \frac{1}{2}(-ab + ac + bc - (1)) \to C - AB$$

One-Half Fractional Factorial Design

- In this case, we can see that the main effects are aliased with the two-way interactions, but they are different than the previous example.
  - In this case, we have A aliased with $-BC$, B aliased with $-AC$, and C aliased with $-AB$.
- This means that we are estimating $A - BC$, $B - AC$, and $C - AB$.
- So how can we de-alias our main effect estimates?

One-Half Fractional Factorial Design

- The answer is to run a second experiment with the opposite sign of the defining relation.
  - For example, if we run the first experiment with $I = +ABC$, then we can run a second experiment with $I = -ABC$. When we run a second design with the intention of de-aliasing the first design, we call the full experiment a **fold-over design**.

- If we average the effects $[A]$ and $[A]'$, this implies:

$$\frac{1}{2}([A] + [A]') = \frac{1}{2}(A + BC + A - BC) = \frac{1}{2}(2A) = A$$

- And:

$$\frac{1}{2}([A] - [A]') = \frac{1}{2}(A + BC - A + BC) = \frac{1}{2}(2BC) = BC$$

One-Half Fractional Factorial Design

- Thus, for all three pairs of linear combinations, we would obtain the following:

i	$0.5([i]+[i]')$	$0.5([i]-[i]')$
A	A	BC
B	B	AC
C	C	AB

One-Half Fractional Factorial Design: Design Resolution

- The resolution of a fractional factorial design is a measure of how well the design can separate the main effects from the two-way interactions.
  - The resolution is defined as the minimum number of factors that are aliased with each other.

- For example, a design with resolution III means that the main effects are aliased with two-way interactions, but not with each other.
  - The 2^{3-1} design we've been working with is of resolution III.

- A resolution IV design means that the main effects are not aliased with each other *or* with any two-way interactions. Two-way interactions may be aliased with each other.
  - A 2^{4-1} design with $I = ABCD$ is a resolution IV design.

One-Half Fractional Factorial Design: Design Resolution

- Finally, a design with resolution V means that the main effects are not aliased with each other or with any two-way interactions, and two-way interactions are not aliased with each other. But two-way interactions may be aliased with three-way interactions.
  - A 2^{5-1} design with $I = ABCDE$ is a resolution V design.

Analysis of a One-Half Fractional Factorial Design

- Okay so you may be asking yourself, “Self, this is all well and good but how do I use this in a practical sense?”
- Well, let’s take a look at an example!

Analysis of a One-Half Fractional Factorial Design

- Suppose we are designing a new adventure video game about a heroic dog named Buster. We want to assess how different game design choices affect **player engagement time** (measured as the amount of time, in minutes, players actively engage with a 20-minute game demo.).

- Below represents our design table for our factors of interest:

Factor	Description	Low Level (−)	High Level (+)
A	Game Difficulty	Easy	Hard
B	Visual Style	Cartoon-style	Realistic
C	Narrative Depth	Lighthearted	Emotional / Deep

Analysis of a One-Half Fractional Factorial Design

- Since it can be time consuming and expensive to create full video games with all combinations of these factors, we will use a 2^3 one-half fractional factorial design to run this experiment. The number of runs will be 4 with two replications each (8 total observations).

- As before, let's use $I = ABC$ as our defining relation. This means we will run the following treatment combinations:

Run	A	B	C = AB	Description
1	$-$	$-$	$+$	Easy, Cartoon-style, Emotional
2	$-$	$+$	$-$	Easy, Realistic, Lighthearted
3	$+$	$-$	$-$	Hard, Cartoon-style, Lighthearted
4	$+$	$+$	$+$	Hard, Realistic, Emotional

Analysis of a One-Half Fractional Factorial Design

- Note, we could automatically generate this table using the `FrF2` function within the library of the same name:

```
library(FrF2)
design <- FrF2(4, #Number of total runs
               3, #Number of factors
               generators = "AB", #Since C=AB when I=ABC
               randomize=F)
print(design)
```

```
   A  B  C
1 -1 -1  1
2  1 -1 -1
3 -1  1 -1
4  1  1  1
class=design, type= FrF2.generators
```

Analysis of a One-Half Fractional Factorial Design

- Remember, because of the defining relation, $I = ABC$, the following aliasing occurs:
  - A = BC
  - B = AC
  - C = AB

- To reiterate, this means that our main effects are **confounded** with the two-factor interactions.

Analysis of a One-Half Fractional Factorial Design

- Alright, so now that the overall design is set up, let's look at some data (contained in the `Hero Buster Game.xlsx` file):

```r
library(tidyverse)
library(readxl)
## Read in Data ##
buster <- read_excel("Hero Buster Game.xlsx")
buster |>
  glimpse()
```

```
Rows: 8
Columns: 5
$ Difficulty <chr> "Easy", "Easy", "Hard", "Hard", "Easy", "Easy", "Hard", "Ha~
$ Style      <chr> "Cartoon", "Cartoon", "Cartoon", "Cartoon", "Realistic", "R~
$ Narrative  <chr> "Emotional", "Emotional", "Lighthearted", "Lighthearted", "~
$ Replicate  <dbl> 1, 2, 1, 2, 1, 2, 1, 2
$ Engagement <dbl> 15.1, 12.2, 14.5, 14.9, 14.1, 13.3, 18.8, 16.4
```

Analysis of a One-Half Fractional Factorial Design

- Let's add the effect columns to our data set.
  - We will use the formulas we derived earlier to calculate the main effects.

```
buster <- buster |>
  mutate(Effect = rep(c("c","b","a","abc"),
                      each=2)
         )
buster |>
  glimpse()
```

```
Rows: 8
Columns: 6
$ Difficulty <chr> "Easy", "Easy", "Hard", "Hard", "Easy", "Easy", "Hard", "Ha~
$ Style      <chr> "Cartoon", "Cartoon", "Cartoon", "Cartoon", "Realistic", "R~
$ Narrative  <chr> "Emotional", "Emotional", "Lighthearted", "Lighthearted", "~
$ Replicate  <dbl> 1, 2, 1, 2, 1, 2, 1, 2
$ Engagement <dbl> 15.1, 12.2, 14.5, 14.9, 14.1, 13.3, 18.8, 16.4
$ Effect     <chr> "c", "c", "b", "b", "a", "a", "abc", "abc"
```

Analysis of a One-Half Fractional Factorial Design

- Now, let's calculate some summary statistics, just as we have done before!

```
library(rstatix)
## Game Difficulty ##
buster |>
  group_by(Difficulty) |>
  get_summary_stats(Engagement,type="mean_sd") |>
  select(-variable)
```

```
# A tibble: 2 x 4
  Difficulty      n  mean    sd
  <chr>       <dbl> <dbl> <dbl>
1 Easy            4  13.7  1.23
2 Hard            4  16.2  1.95
```

Analysis of a One-Half Fractional Factorial Design

```r
## Visual Style ##
buster |>
  group_by(Style) |>
  get_summary_stats(Engagement,type="mean_sd") |>
  select(-variable)
```

```
# A tibble: 2 x 4
  Style           n  mean    sd
  <chr>       <dbl> <dbl> <dbl>
1 Cartoon         4  14.2  1.34
2 Realistic       4  15.6  2.48
```

Analysis of a One-Half Fractional Factorial Design

```
## Narrative Depth ##
buster |>
  group_by(Narrative) |>
  get_summary_stats(Engagement,type="mean_sd") |>
  select(-variable)
```

```
# A tibble: 2 x 4
  Narrative         n  mean    sd
  <chr>         <dbl> <dbl> <dbl>
1 Emotional         4  15.6 2.75
2 Lighthearted      4  14.2 0.683
```

Analysis of a One-Half Fractional Factorial Design

- So from what we see here, we can see that:
  1. The Easy game difficulty has a slightly lesser mean engagement time than the Hard game difficulty, but it may not necessarily be a meaningful difference.
  2. For the Visual Style, we see that while they Realistic style games have a slightly greater mean engagement style.
  3. Finally, we observe the Emotional style games have a greater mean engagement time than the lighthearted style games.
- Are these meaningful from a statistical or practical perspective? Let's take a look!

Analysis of a One-Half Fractional Factorial Design

```
## Fit Model ##
mod <- aov(Engagement ~ Difficulty + Style + Narrative,
           data = buster)
```

Analysis of a One-Half Fractional Factorial Design

- Checking assumptions:

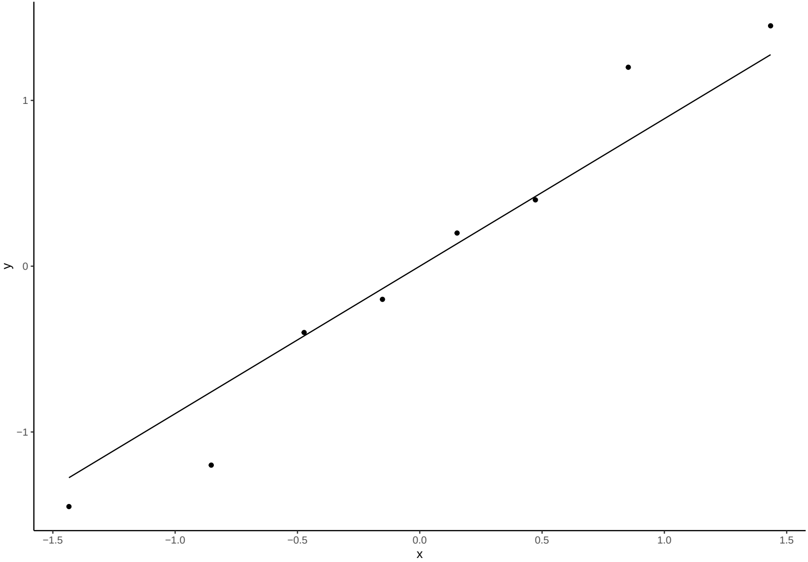
```r
library(broom)
## Shaprio-Wilk Test of Normality ##
mod |>
  resid() |>
  shapiro.test() |>
  tidy()
```

```
# A tibble: 1 x 3
  statistic p.value method
      <dbl>   <dbl> <chr>
1     0.957   0.782 Shapiro-Wilk normality test
```

Analysis of a One-Half Fractional Factorial Design

```
buster |>
  ggplot(aes(sample=resid(mod))) +
  geom_qq() +
  geom_qq_line() +
  theme_classic()
```

Analysis of a One-Half Fractional Factorial Design



Analysis of a One-Half Fractional Factorial Design

- Normality looks good! What about constant variance?

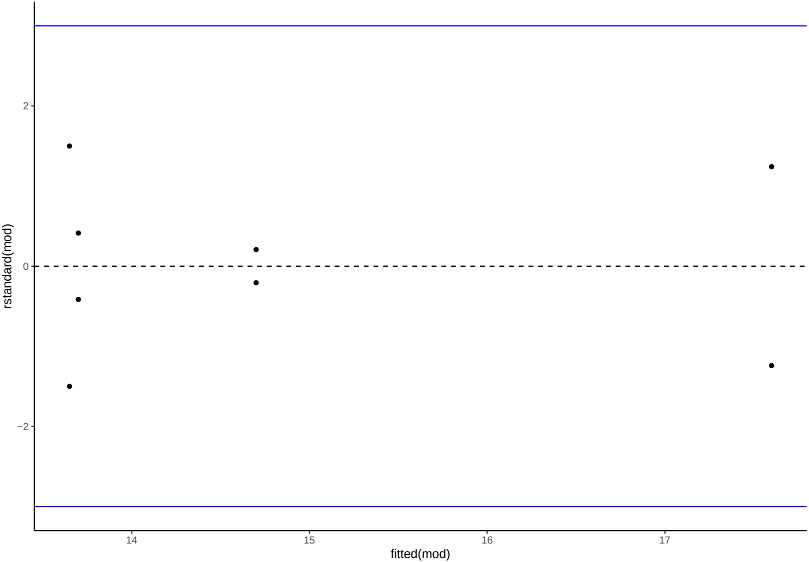
```
## B-P Test ##
library(lmtest)
mod |>
  bptest() |>
  tidy()
```

```
# A tibble: 1 x 4
  statistic p.value parameter method
      <dbl>   <dbl>     <dbl> <chr>
1         8  0.0460         3 studentized Breusch-Pagan tes
```

Analysis of a One-Half Fractional Factorial Design

```r
buster |>
  ggplot(aes(x=fitted(mod),y=rstandard(mod))) +
  geom_point() +
  geom_hline(yintercept=0,linetype="dashed") +
  geom_hline(yintercept=3,color='blue') +
  geom_hline(yintercept=-3,color='blue') +
  theme_classic()
```

Analysis of a One-Half Fractional Factorial Design



Analysis of a One-Half Fractional Factorial Design

- The Breush-Pagan test tells us that the model residuals may not have equality of variance.
- We can somewhat see this in the residual plot as well.
- However, given our small sample size, we can proceed but should be cautious about our conclusions.
  - Replications of the whole experiment may be needed to confirm our results.

Analysis of a One-Half Fractional Factorial Design

- Now let's look at the overall results!

```
mod |>
  tidy()
```

```
# A tibble: 4 x 6
  term          df sumsq meansq statistic p.value
  <chr>      <dbl> <dbl>  <dbl>     <dbl>   <dbl>
1 Difficulty     1 12.3   12.3       6.55  0.0627
2 Style          1  4.35   4.35      2.33  0.202
3 Narrative      1  4.06   4.06      2.17  0.215
4 Residuals      4  7.48   1.87     NA    NA
```

Analysis of a One-Half Fractional Factorial Design

- Nothing is statistically significant!!
  - This is not surprising given our small sample size. We'll calculate partial η^2 in a bit.
- Let's now tabulate our main effects. First, let's add the effect name as a column in our dataframe:

Analysis of a One-Half Fractional Factorial Design

```
buster <- buster |>
  mutate(Effect = rep(c("c","a","b","abc"),
                      each=2))
```

Analysis of a One-Half Fractional Factorial Design

```r
## Calculating the Main Effects ##
library(tidyr)
buster |>
  group_by(Effect) |>
  summarise(Engagement = mean(Engagement)) |>
  pivot_wider(names_from = Effect,
              values_from = Engagement) |>
  mutate(A = 0.5*(a-b-c+abc),
         B = 0.5*(-a+b-c+abc),
         C = 0.5*(-a-b+c+abc),
         .keep='unused' #keeps only created columns
  )
```

```
# A tibble: 1 x 3
      A     B     C
  <dbl> <dbl> <dbl>
1  2.48  1.48  1.43
```

Analysis of a One-Half Fractional Factorial Design

- Cool! So what does this mean?
  - Exactly the same interpretation as our interpretation of the means!!

1. As difficulty goes from the - level of Easy to the + level of hard, we expect an increase of 2.48 minutes.
2. As visual style goes from the - level of Cartoon to the + level of Realistic, we expect an increase of 1.48 minutes.
3. As narrative depth goes from the - level of Lighthearted to the + level of Emotional, we expect an increase of 1.43 minutes.

Analysis of a One-Half Fractional Factorial Design

- Remember, we aren't able to distinguish between the main effects and the two-way interactions.
  - So, we need to be careful about our conclusions.

- If we were to run a second experiment with the opposite sign of the defining relation $(I = -ABC)$, we could de-alias our main effects and two-way interactions.
  - This would allow us to estimate the main effects and two-way interactions separately.

Analysis of a One-Half Fractional Factorial Design

- So lastly, let's calculate the partial η^2 for our main effects.
  - This will help us determine how much of the variance in the response variable is explained by each of the main effects.

```
mod |>
  partial_eta_squared()
```

```
Difficulty      Style  Narrative
 0.6207486  0.3676207  0.3517376
```

Analysis of a One-Half Fractional Factorial Design

- This is telling us that the main effects of Difficulty, Style, and Narrative are accounting for 62.07%, 36.76%, and 35.17% of the variance in the response variable, respectively.
  - This is a pretty good amount of variance explained for a small sample size.
- All of these effects would be considered large.
  - This is a good example of ensuring that we don't solely rely on p-values to determine the importance of our factors.

Final Thoughts

- So, in summary, we have learned about one-half fractional factorial designs and how to analyze them.
  - We have also learned about the alias structure of these designs and how to de-alias the main effects and two-way interactions using fold-over designs.
- We also learned that we have to be careful about our conclusions and not to solely rely on p-values as the measure of variable importance.
- If we wanted to go a step further, we could take a one-quarter fractional factorial design, which would allow us to estimate the main effects and two-way interactions separately.
  - This would be a more efficient design, but it would require more runs.