Math Foundations of ML, Fall 2018

Homework #1

Due Friday August 31, at the beginning of class

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Sign up for Piazza if you have not already done it, https://piazza.com/gatech/fall2018/ececsisyebmed8843880388438813

2. Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

3. Suppose that $f(t)$ is a second-order spline that defined by the overlap of 5 B-splines:

$$f(t) = \sum_{k=0}^{4} \alpha_k b_2(t-k),$$

where $b_2(t)$ is defined as on page 11 of the notes,

$$b_2(t) = \begin{cases} (t+3/2)^2/2 & -3/2 \le t \le -1/2 \\ -t^2 + 3/4 & -1/2 \le t \le 1/2 \\ (t-3/2)^2/2 & 1/2 \le t \le 3/2 \\ 0 & |t| \ge 3/2 \end{cases}$$

(a) Write a MATLAB[1] function

```
ft = piecepoly2(t, alpha);
```

which takes $\boldsymbol{\alpha} = \{\alpha_0, \ldots, \alpha_4\}$ and returns samples of $f(t)$ at the locations specified in the vector `t`. Turn in a plot of $f(t)$ for $\boldsymbol{\alpha} = \{3, 2, -1, 4, -1\}$. Sample t densely enough so that your plot looks like a smooth function.

(b) Suppose I tell you that

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = -4, \quad f(3) = -5, \quad f(4) = -2.$$

What are the corresponding α_k? (Hint: you will have to construct a system of equations then solve it.)

(c) To generalize this, suppose that $f(t)$ is now a superposition of N B-splines:

$$f(t) = \sum_{n=0}^{N-1} \alpha_n b_2(t-n).$$

[1]The code syntax in the homeworks will be for MATLAB, but please feel free to use Python instead if you are more comfortable.

Describe how to construct the $N \times N$ matrix that maps the coefficients $\boldsymbol{\alpha}$ to the N samples $f(0), \ldots, f(N-1)$. That is, find $\boldsymbol{A}$ such that

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix} = \begin{bmatrix} & & \\ & \boldsymbol{A} & \\ & & \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

(d) To take this even further, suppose that

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n b_2(t-n),$$

so $f(t)$ is described by the (possibly infinite) sequence of numbers $\{\alpha_n\}_{n \in \mathbb{Z}}$. Show that there is a convolution operator that maps the sequence $\{\alpha_n\}$ to the sequence $\{f(n)\}$. That is, find a sequence of numbers $\{h_n\}_{n \in \mathbb{Z}}$ such that

$$f(n) = \sum_{\ell=-\infty}^{\infty} h_\ell \, \alpha_{n-\ell}.$$

4. In the file `nonuniform_samples`[2] there are a set of samples locations t_m and sample values y_m, $m = 1, \ldots, 9$.

(a) Find a ninth-order polynomial that passes through these points. Plot your answer with the samples overlaid.

(b) Find $f(t)$ such that

$$f(t) = \sum_{k=0}^{9} \alpha_k \, b_2(t-k)$$

and $f(t_m) = y_m$ for $m = 0, \ldots, 9$. Plot your answer with the samples overlaid.

5. (a) Let $f(t) = |t|^p$ for $p \geq 1$. Prove that

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}.$$

Show that this extends to the more general result that

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b), \quad \text{for all } \ 0 \leq \lambda \leq 1.$$

That is, prove that $f(t)$ is convex. (Hint: mean value theorem.)

(b) Let $\mathcal{S}$ be the set of all (infinite length) sequences

$$\mathcal{S}_p = \left\{ \{x_n\}_{n=1}^{\infty} \ : \ \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty \right\}.$$

Show that $\mathcal{S}$ is indeed a linear vector space. (You will probably find the assertion in part (a) handy.)

[2] There are two versions: a .mat file if you are using MATLAB (which contains vectors `t` and `y`) and a text file if you are using Python. For the latter, the first ten numbers are the t_m and the second ten are the y_m.