Theory of Mine (Exhaustible resources)

Lecture Notes for the course Natural Resource Economics (AEC-608)

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1 Introduction

Exhaustible resources, such as fossil fuels and minerals, are finite and deplete with extraction, posing unique challenges in economic management. Their finite nature necessitates strategies that balance current extraction with future availability, impacting economic policies, environmental sustainability, and intergenerational equity.

How do exhaustible resources differ from other resources?

- **Limited stock:** The total quantity available is finite; once extracted and used, it cannot be replenished within a relevant timeframe.
- **Non-producibility:** These resources cannot be recreated or produced by human or natural means at a meaningful rate.

Because of this, there is an **opportunity cost** to current consumption: extracting one unit today means one less unit is available for the future. This opportunity cost must be explicitly considered when deciding the optimum extraction path for a non-renewable resource.

2 Basic Concepts in Exhaustible Resource Extraction

Exhaustible resources are natural assets available in limited quantities, unable to be renewed within a human timeframe, leading to eventual depletion. Examples include fossil fuels (oil, coal, natural gas) and minerals (copper, gold) *[Exhaustible Resources]*. The economics of these resources involves managing supply, demand, and allocation over time to maximize societal benefits.

- **Extraction Cost**: The expenses incurred in extracting the resource, including labor, equipment, and operational costs. These costs may be constant or vary with the quantity extracted or the remaining stock, influencing extraction timing and intensity.
- **Rent or Opportunity Cost**: The scarcity rent, or the value of leaving the resource in the ground for future extraction, reflecting the trade-off between immediate benefits and future value. This rent, often called resource rent, equals the shadow value of the natural resource.

3 Myopic Extraction and Opportunity Cost

Traditionally, economic theory suggests that the optimal use of a resource occurs where Marginal Cost (MC) equals Marginal Return (Price), i.e., Price = MC.

However, for exhaustible resources:

- Applying this rule ignores the opportunity cost of depleting a finite resource.
- This can lead to *myopic extraction*: using up all the resource in the current period, without regard for future scarcity.
- The correct rule for exhaustible resources is:

$$\text{Price} = \text{MC} + \text{Opportunity Cost}$$

- **Myopic extraction** occurs at the point Y^{ii} (where only current costs are considered).
- **Optimum extraction** occurs at Y^{i} (where both current and future values are balanced).

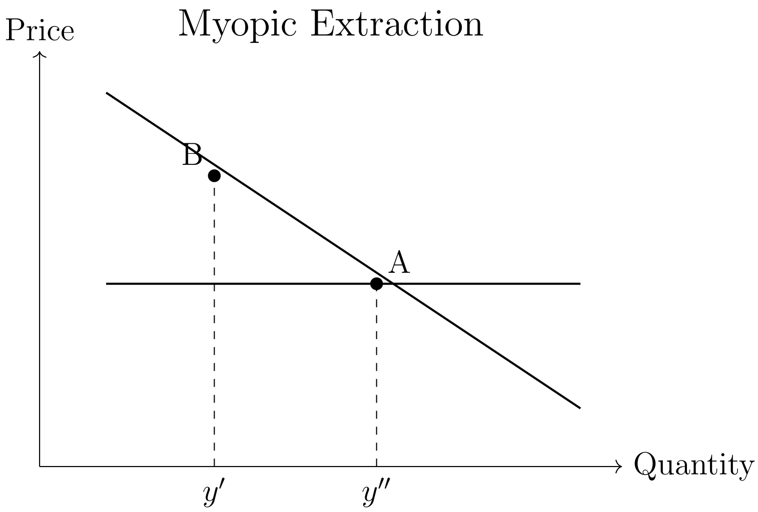

Figure 1: Myopic Extraction: The intersection of the demand curve and marginal cost line determines the myopic extraction quantity y''. Point A is the equilibrium under myopic behavior, while B shows a lower extraction at a higher price.

Implication: The optimal extraction rate for a non-renewable resource is lower than that for a renewable or reproducible resource. This principle was first formalized by C.L. Gray in 1914 for an individual mine owner.

4 Concept of Resource Rent, Opportunity Cost, User Cost, and Royalty

- **Resource Rent:** The difference between the market price and the marginal cost of extraction. It represents the surplus or economic profit available due to the resource's scarcity.

- **Opportunity Cost (User Cost):** The value forgone by extracting and consuming a resource unit today rather than preserving it for future use, when it might be more valuable.
- **Royalty:** The net social benefit derived from the resource, calculated as the total social benefit minus the cost of extraction. It can also be interpreted as the net value of the resource left in the ground.

Key question: When should the resource be extracted to maximize the present value of royalty (or resource rent)? This requires balancing immediate returns from extraction against the potential future value if extraction is delayed.

Reference: See George Santoprieto, "Alternative methods for estimating resource rent and depletion cost".

5 Resource Rent: An Illustration

A mineral deposit in situ is an asset, much like any other asset owned by an individual or firm. The value of such an asset can be decomposed into three components:

1. **Product Flow:** The output or product generated by utilizing the asset. For minerals still in the ground, this is zero until extraction occurs.
2. **Depreciation:** The decline in asset value over time due to physical or qualitative deterioration. For most minerals, this is negligible as their quality does not degrade significantly while in situ.
3. **Appreciation:** The rate at which the value of the resource increases over time, often due to rising scarcity or prices.

Since the mineral is in the ground, it does not produce anything (1 is zero), and does not depreciate (2 is zero). What is left is the net value of the marginal unit of the resource held in the ground:

Net Value of Marginal Unit = Value of Marginal Unit$-$Marginal Cost of Extraction = $P - MC$

This net value is known as the **asset price**, **resource rent**, or **royalty**.

6 Gray's Model: Historical Context and Key Ideas

Lewis Cecil Gray's 1914 paper, "Rent Under the Assumption of Exhaustibility", was a pioneering contribution to the economics of exhaustible resources, focusing on a single mine owner. Gray modified traditional rent theory to account for the finite nature of resources, analyzing how exhaustion affects economic decisions.

Assumptions:

- Perfect competition and constant price expectations: The expected selling price of the resource remains unchanged over time.
- Homogeneous quality: The quality of the resource does not vary across the deposit or over time.
- Identical cost curves across periods: The cost of extraction per unit remains the same in each period.
- Known stock: The owner is fully aware of the total quantity of resource available.

Key insights:

- **Rent under Exhaustibility**: Gray redefined rent to include the scarcity value due to finite stocks, adjusting classical land rent theories to reflect resource depletion.
- **Effect of Exhaustion**: He examined how the finite nature of resources influences the intensity of utilization, affecting extraction rates.
- **Influence of Interest Rate**: Gray highlighted the role of interest rates in determining whether to extract now or later, linking present and future values.

Decision rules:

1. Extraction should occur where Price = MC + Rent, accounting for both extraction costs and the opportunity cost of depletion.
2. The present value of resource rent must be equalized across all periods, or equivalently, the undiscounted rent should grow at the rate of interest:

$$P_t - MC = (P_{t+1} - MC)e^{-rt}$$
$$P_t = MC + (P_{t+1} - MC)e^{-rt}$$
$$P_t = MC + (P_0 - MC)e^{rt}$$

3. **Stock Condition:** The sum of all units extracted over all periods must not exceed the total available stock.
4. **Terminal Condition:** For the last unit extracted, the average cost of extraction is minimized. This ensures that extraction ceases when it is no longer profitable.

7 Discount Rate and Sustainability

The discount rate (interest rate) reflects the preference for current versus future income. A higher discount rate increases the attractiveness of immediate extraction, as future benefits are valued less.

Illustration: The present value of receiving Rupees 3,000 after 3 years:

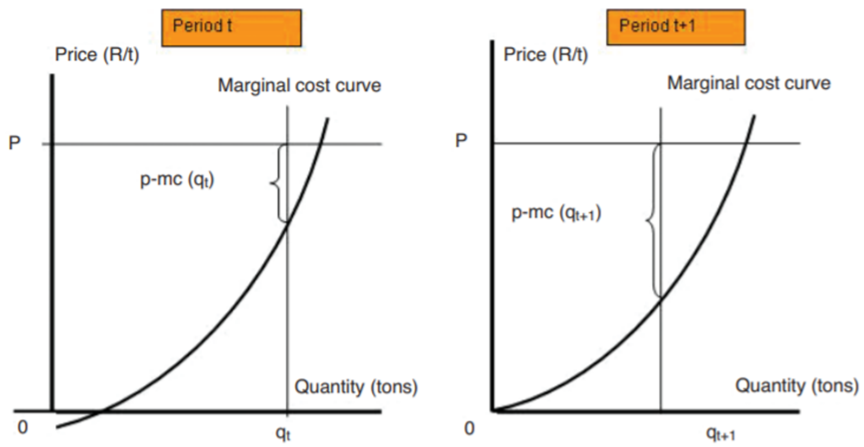

Figure 2: Gray Model: price = MC + rent, where rent grows at the market rate of interest.

- At 10% interest: Present value = Rupees 2,253
- At 20% interest: Present value = Rupees 1,736

Thus, a higher discount rate leads to increased current extraction and reduced conservation for the future.

8 Hotelling's Model: Formalization and Extensions

Harold Hotelling's 1931 paper, "The Economics of Exhaustible Resources", formalized the theory of exhaustible resources, extending Gray's ideas to competitive markets. Hotelling's rule posits that in a competitive market, the net price (price minus marginal extraction cost) rises at the interest rate, ensuring optimal resource allocation over time. The model can also be extended to case of monopoly.

8.1 Assumptions

Hotelling's model assumes:

- A perfectly competitive industry (can be relaxed).
- Homogeneous resource with known stock.
- Constant unit extraction cost c.
- No fixed costs or stock effects.
- Rational expectations.

8.2 Hotelling's Rule

The rule states that the net price $p(t) - c$ grows at the interest rate r, expressed as:

$$p(t) - c = (p(0) - c)e^{rt}$$

For zero extraction cost ($c = 0$), the price itself grows at the interest rate: $p(t) = p(0)e^{rt}$.

8.3 Derivation for Competitive Markets

Consider a resource owner maximizing the present value of profits:

$$\max \int_0^\infty e^{-rt}[p(t)q(t) - c(q(t))]\, dt$$

subject to the resource constraint:

$$\dot{S}(t) = -q(t), \quad S(0) = S_0, \quad S(t) \geq 0$$

where $S(t)$ is the resource stock, $q(t)$ is the extraction rate, $p(t)$ is the price, and $c(q(t))$ is the cost function.

Using the current-value Hamiltonian:

$$\mathcal{H} = p(t)q(t) - c(q(t)) + \mu(t)(-q(t))$$

the first-order conditions are:

1. $\frac{\partial \mathcal{H}}{\partial q} = p(t) - c'(q(t)) - \mu(t) = 0 \Rightarrow p(t) - c'(q(t)) = \mu(t)$
2. $\dot{\mu}(t) = r\mu(t) - \frac{\partial \mathcal{H}}{\partial S}$. Since $\mathcal{H}$ does not depend on S, $\frac{\partial \mathcal{H}}{\partial S} = 0$, so $\dot{\mu}(t) = r\mu(t)$.

From condition 2, $\mu(t) = \mu(0)e^{rt}$. Substituting into condition 1:

$$p(t) - c'(q(t)) = \mu(0)e^{rt}$$

If the marginal cost $c'(q(t)) = c$ (constant), then:

$$p(t) - c = \mu(0)e^{rt}$$

Differentiating with respect to time:

$$\dot{p}(t) = \mu(0)re^{rt} = r(p(t) - c)$$

Thus:

$$\frac{\dot{p}(t)}{p(t) - c} = r$$

This is Hotelling's rule, indicating that the net price grows at the interest rate.

8.4 Monopoly Case

For a monopolist, the objective is to maximize:

$$\max \int_0^{\infty} e^{-rt}[R(q(t)) - c(q(t))]\, dt$$

where $R(q(t)) = p(q(t))q(t)$ is the revenue function. The Hamiltonian becomes:

$$\mathcal{H} = R(q(t)) - c(q(t)) + \mu(t)(-q(t))$$

First-order conditions yield:

$$MR(q(t)) - c'(q(t)) = \mu(t), \quad \dot{\mu}(t) = r\mu(t)$$

Thus, $\mu(t) = \mu(0)e^{rt}$, and:

$$MR(q(t)) - c'(q(t)) = \mu(0)e^{rt}$$

If $c'(q(t)) = c$, then $MR(q(t)) - c$ grows at the interest rate, leading to slower initial extraction compared to competitive markets due to the monopolist's incentive to restrict output.

9 Application of Hotelling Rule to an Individual Firm

Assuming a constant price P and marginal cost $MC > 0$, the present value of rent across two periods must be equal:

$$\left(\frac{1}{1+r}\right)^t (P - MC(Q_t)) = \left(\frac{1}{1+r}\right)^{t+1} (P - MC(Q_{t+1}))$$

This leads to the expression:

$$r = \frac{(P - MC(Q_{t+1})) - (P - MC(Q_t))}{P - MC(Q_t)}$$

This is another form of Hotelling's r-percent rule, ensuring that the resource rent grows at the rate of interest. Other conditions—such as the stock constraint and terminal condition—remain as in the Gray model.

10 Concept of Backstop Technology

In planning the extraction of a non-renewable resource, the mine owner must consider the potential emergence of **backstop technology**—an alternative resource or technology that becomes economically viable when the price of the non-renewable resource exceeds a certain threshold.

- **Threshold Price:** The price at which the cost of producing the alternative (with a typically falling average cost curve) is less than the current market price of the exhaustible resource.
- Once this threshold is reached, the alternative replaces the exhaustible resource, rendering further extraction unprofitable.

Example: Consider the case of solar energy versus coal:

- Previously, solar energy cost Rupees 17 per unit, while coal-based electricity cost Rupees 1 per unit.
- Technological advances have reduced solar costs to about Rupees 4 per unit, while coal-based electricity costs have risen to around Rupees 4.5 per unit.
- At this point, solar energy becomes a viable backstop technology, potentially replacing coal as the preferred energy source.

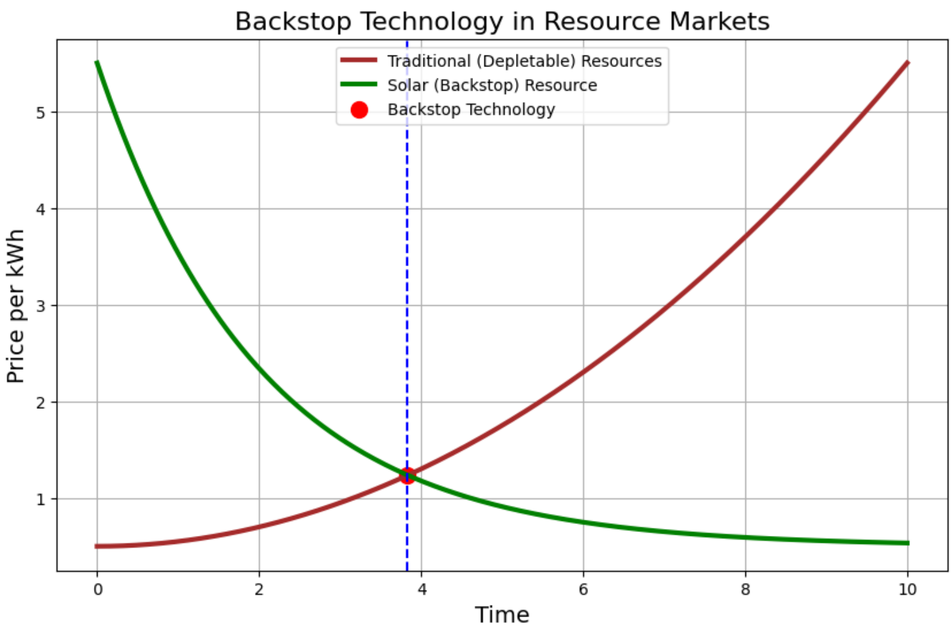

Figure 3: Backstop Technology and Threshold Price: When the price of the exhaustible resource rises to the level of the backstop technology, the alternative becomes viable and extraction of the exhaustible resource ceases.

11 Two-Period Model

The two-period model simplifies the analysis by considering two time periods ($t = 1, 2$), with a total resource stock S, and extractions q_1 and q_2 such that $q_1 + q_2 \leq S$. Assume zero extraction cost and an inverse demand function $p(q)$.

$$\pi = max_{q_1,q_2}\Big[p_1\, q_1 - C(q_1)\Big] \;+\; \frac{1}{1+r}\Big[p_2\, q_2 - C(q_2)\Big]$$

Resource constraint:

$$q_1 + q_2 = S$$

Lagrangian:

$$\mathcal{L} = p_1\, q_1 - C(q_1) + \frac{1}{1+r}\big(p_2\, q_2 - C(q_2)\big) + \lambda\,(S - q_1 - q_2)$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial q_1} : \; p_1 - C'(q_1) - \lambda = 0, \qquad \frac{\partial \mathcal{L}}{\partial q_2} : \; \frac{1}{1+r}\big(p_2 - C'(q_2)\big) - \lambda = 0$$

Eliminate λ to get the Euler (Hotelling) condition:

$$p_1 - C'(q_1) = \frac{1}{1+r}\big(p_2 - C'(q_2)\big) \quad\Longleftrightarrow\quad (1+r)\big[p_1 - C'(q_1)\big] = p_2 - C'(q_2)$$

Constant marginal cost c :

$$p_1 - c = \frac{1}{1+r}(p_2 - c) \quad\Longrightarrow\quad p_2 - c = (1+r)\,(p_1 - c)$$

Zero-cost special case ($c = 0$):

$$p_2 = (1+r)\,p_1$$

11.1 Competitive Markets- Zero cost case

We consider a price-taking firm that extracts an exhaustible resource over two periods. Let q_1, q_2 be the quantities extracted in period 1 and 2, and S the fixed stock.

Maximize the discounted sum of revenues:

$$\max_{q_1,q_2} \pi \;=\; p(q_1)\, q_1 \;+\; \delta\, p(q_2)\, q_2, \qquad q_1 + q_2 = S, \quad \delta = \frac{1}{1+r}\,.$$

Use the constraint $q_2 = S - q_1$ to rewrite

$$\pi(q_1) = p(q_1)\, q_1 + \delta\, p\big(S - q_1\big)\big(S - q_1\big).$$

Differentiate with respect to q_1 and set to zero:

$$\frac{d\pi}{dq_1} = p'(q_1)\, q_1 + p(q_1) - \delta\big[p'(q_2)\,(S - q_1) + p(q_2)\big] = 0,$$
$$\Longrightarrow \quad MR(q_1) = \delta\, MR(q_2), \quad MR(q) \equiv p(q) + p'(q)\, q.$$

Compute the second derivative:

$$\frac{d^2\pi}{dq_1^2} = p''(q_1)\, q_1 + 2\, p'(q_1) + \delta\big[p''(q_2)\,(S - q_1) + 2\, p'(q_2)\big].$$

Under standard assumptions $p' < 0$ and $p'' \leq 0$, this is negative, so the solution is a maximum.

In a competitive market $MR(q) = p(q)$. Hence the FOC reduces to

$$p(q_1) = \delta\, p(q_2) \quad \Longrightarrow \quad p(q_2) = (1 + r)\, p(q_1),$$

i.e. prices grow at the interest rate r, which is Hotelling's rule.

Let $p_t = a - b\, q_t$. Then the Euler-equation becomes

$$a - b\, q_1 = \delta\big[a - b\,(S - q_1)\big], \quad q_1 + q_2 = S.$$

Solve for q_1:

$$a - bq_1 = \delta a - \delta bS + \delta bq_1 \implies q_1\big(b(1+\delta)\big) = a(1-\delta) + \delta bS,$$

$$\boxed{q_1 = \frac{a(1-\delta) + \delta\, b\, S}{b\,(1+\delta)}, \qquad q_2 = S - q_1.}$$

The two prices are then

$$p_1 = a - b\, q_1, \qquad p_2 = a - b\, q_2 = (1 + r)\, p_1,$$

confirming that the price path grows at rate r.

11.2 Monopolist

The monopolist maximizes:

$$\pi = p(q_1)q_1 + \delta p(q_2)q_2, \quad q_1 + q_2 = S$$

Substitute $q_2 = S - q_1$:

$$\pi = p(q_1)q_1 + \delta p(S - q_1)(S - q_1)$$

Differentiate:

$$\frac{d\pi}{dq_1} = p'(q_1)q_1 + p(q_1) + \delta[p'(q_2)(-q_2) + p(q_2)(-1)] = 0$$

This yields:

$$MR(q_1) = \delta MR(q_2)$$

where $MR(q) = p'(q)q + p(q)$. The monopolist equates marginal revenue across periods, adjusted for discounting, leading to slower initial extraction.

12 N-Period Model

The n-period model generalizes the two-period framework to multiple periods, maximizing:

$$\sum_{t=0}^{T} \delta^t [p(q_t)q_t - c(q_t)]$$

subject to:

$$\sum_{t=0}^{T} q_t \leq S$$

This is solved using dynamic programming or optimal control, extending Hotelling's rule to ensure the net price grows at the interest rate over multiple periods. The solution involves determining the extraction path $\{q_t\}$ that satisfies the resource constraint and maximizes present value, often requiring numerical methods for complex demand or cost functions.

13 Conclusion

The economics of exhaustible resources is critical for sustainable resource management and policy design. The main difference between the theory of resource use and exhaustible resource extraction is due to the "resource rent" which is included to account for the scarcity of the resources. Gray's early work highlighted the importance of scarcity rent and interest rates for a single mine owner, laying the groundwork for modern theories. Hotelling's model provided a rigorous framework for competitive and monopoly markets, with Hotelling's rule serving as a cornerstone for optimal extraction strategies.

14 Summary Table

Table 1: Comparison of Models and Decision Rules

Model	**Focus**	**Key Decision Rule**	**Derivation Method**
Gray's Model	Single mine owner, rent	Balance rent, interest rate influence	Qualitative, 1914 paper
Hotelling's Model	Competitive, monopoly markets	Net price rises at interest rate	Hamiltonian, optimal control
Two-Period	Simplified trade-off	$p_1 = \delta p_2$ or $MR_1 = \delta MR_2$	Lagrange, calculus
N-Period	Long-term planning	Dynamic optimization	Dynamic programming