Finite Element Analysis of Electromagnetic Waves in Anisotropic Media

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Introduction to Finite Element Method

- **What is FEM?**
  - A numerical technique for solving differential equations in engineering and mathematical models.
- **Applications:**
  - Structural analysis
  - Heat transfer
  - Fluid dynamics
  - Electromagnetics
- **Key Idea:** Break down a complex domain into smaller, simpler parts (elements), and solve locally.

Key Concepts in FEM

- **Meshing:** Divide the domain into finite elements.
- **Shape Functions:** Local functions that approximate the solution over each element.
- **Stiffness Matrix:** Matrix representation of the system based on element contributions.
- **Boundary Conditions:** Ensure physical accuracy of the model.

Advantages and Limitations of FEM

- **Advantages:**
  - Handles complex geometries and boundary conditions.
  - Provides local accuracy through adaptive meshing techniques thus capturing local effects.
  - Easily incorporates non-linear material properties thus suitable for anisotropic media.
  - Allows for efficient parallel processing and distributed computing.
- **Limitations:**
  - Computationally intensive, particularly for large-scale models.
  - Solution accuracy depends on mesh quality and element size.
  - Requires significant memory for storing large stiffness matrices.

Steps in FEM

1. **Discretization**
   - Divide the domain into small elements (meshing).
   - Types of elements: Triangular, Quadrilateral, etc.
2. **Interpolation**
   - Approximate the solution over each element using shape functions.
3. **Formulation**
   - Set up governing equations (e.g., weak form of the differential equation).
   - Construct a global system from local contributions.

Step 1: Governing Differential Equation

The problem is defined by a partial differential equation (PDE):

$$\mathcal{L}(u) = f \quad \text{in} \quad \Omega$$

where:

- $\mathcal{L}$ is the differential operator (e.g., Laplace, Helmholtz).
- f is the source term.
- Ω is the domain.

Example: Poisson equation in 1D

$$-u'' = f, \Omega = (0, 1)$$

Step 2: Weak Formulation of the PDE

Convert the PDE into its weak form:

$$\int_\Omega v\mathcal{L}(u)\, d\Omega = \int_\Omega vf\; d\Omega$$

Apply integration by parts to reduce the order of derivatives:

$$\int_\Omega \nabla v \cdot \mathbf{k}\nabla u\, d\Omega = \int_\Omega vf\; d\Omega$$

where v is a test function.

Step 3: Discretization (Meshing)

The domain Ω is divided into smaller, simpler elements:

- Triangular or quadrilateral elements in 2D.
- Tetrahedral or hexahedral elements in 3D.

Each element contains nodes where the solution is approximated. The number of nodes is denoted as N.

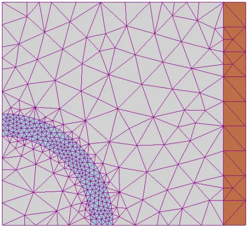

Figure: Example of triangular mesh in 2D

Step 4: Approximation of the Solution

Approximate the solution $u(\mathbf{x})$ using shape functions:

$$u(\mathbf{x}) \approx \sum_{i=0}^{N} u_i N_i(\mathbf{x})$$

where:

- u_i are the nodal values.
- $N_i(\mathbf{x})$ are shape functions (e.g., polynomials).

The test function v is also approximated using the same shape functions.

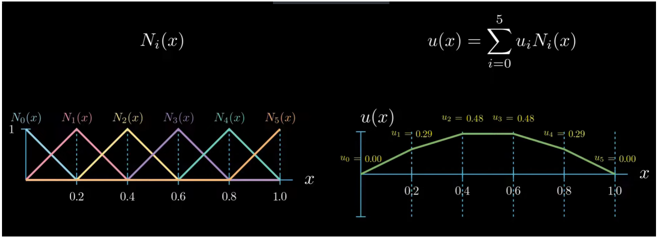

Figure: a univariate function approximated by simple shape functions

Step 5: Assembly of the Elemental System

Substitute the shape functions into the weak form. For each element, we get a system of equations:

$$\sum_e \int_{\Omega_e} \nabla N_j \cdot \mathbf{k} \nabla N_i \, d\Omega_e = \sum_e \int_{\Omega_e} N_j f \, d\Omega_e$$

The global system is:

$$\mathbf{KU} = \mathbf{F}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{U}$ is the vector of nodal values, and $\mathbf{F}$ is the load vector.

Step 6: Imposing Boundary Conditions

Apply boundary conditions to the system:

- **Dirichlet boundary conditions:** Known values of u at the boundary.
- **Neumann boundary conditions:** Known flux $\mathbf{k}\nabla u \cdot \hat{n}$ at the boundary.

Modify the global stiffness matrix and load vector accordingly.

Step 7: Solve the System of Equations

Solve the algebraic system:

$$\mathbf{KU} = \mathbf{F}$$

- Use **direct solvers** (e.g., Gaussian elimination) for small systems.
- Use **iterative solvers** (e.g., Conjugate Gradient) for large systems.

Step 8: Post-Processing

After solving for $\mathbf{U}$, the solution can be:

- **Interpolated** within elements using the shape functions.
- Compute **derived quantities** such as fluxes and gradients.

Example: In electromagnetics, compute the electric or magnetic field from the solution u.

Simulation with FEniCS

FEniCS is a popular open-source computing platform for solving partial differential equations (PDEs) with the finite element method (FEM). FEniCS enables users to quickly translate scientific models into efficient finite element code.
We are using its high level Python interface.
Basic workflow:

- Define the mesh and function spaces.
- Formulate the problem (weak form).
- Solve using FEM.

FEniCS Website

Maxwell's equations in the differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{1}{\mu_0}\frac{\partial \mathbf{E}}{\partial t}$$

- $\mathbf{E}$: Electric field vector, representing the force per unit charge.
- $\mathbf{B}$: Magnetic field vector, related to the magnetic influence on moving charges and currents.
- $\mathbf{H}$: Magnetic field intensity vector, related to the magnetic field $\mathbf{B}$ by the relation $\mathbf{B} = \mu_0 \mathbf{H}$ in free space.
- ρ: Electric charge density, representing the amount of charge per unit volume.
- $\mathbf{J}$: Electric current density vector, representing the current per unit area.
- ε_0: Permittivity of free space, $\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F/m}$ (farads per meter).
- μ_0: Permeability of free space, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}$ (henries per meter).

Deriving Electromagnetic Waves from Maxwell's Equations

Step 1: Set Free Space Conditions

- No free charges ($\rho = 0$)
- No free currents ($\mathbf{J} = 0$)

Step 2: Take the Curl of Faraday's Law

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Using Ampere's Law:

$$\nabla \times \mathbf{B} = \frac{1}{\mu_0}\frac{\partial \mathbf{E}}{\partial t}$$

Step 3: Apply Vector Identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

In free space, $\nabla \cdot \mathbf{E} = 0$, so we get:

$$-\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Result: Wave Equation for E

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similarly, we can derive the wave equation for **B**:

$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Conclusion: The electric and magnetic fields propagate as waves with velocity $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, where c is the speed of light.

Anisotropic Media

- Anisotropic media are materials whose properties differ based on direction.
- Example: Crystals, certain composites.
- They affect how electromagnetic waves propagate, reflecting different velocities in different directions.

Constitutive relations in anisotropic media:

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \text{(Electric displacement)}$$
$$\mathbf{B} = \mu \mathbf{H} \quad \text{(Magnetic flux density)}$$

where the permittivity ε and permeability μ are tensors:

$$\varepsilon_{ij}, \mu_{ij}$$

Challenges

- **1. Find the Modified Wave Equation in Anisotropic Medium**
  - Identify the governing equations that describe wave propagation in an anisotropic medium.
  - Consider the material properties that influence the wave behavior, such as permittivity and permeability.
- **2. Derive its Weak Formulation**
  - Apply the variational principle to convert the modified wave equation into its weak form.
  - Choose suitable test functions and integrate the equation over the domain.
  - Incorporate boundary conditions to ensure a well-posed problem.
- **3. Identify Suitable Mesh Schema**
  - Analyze the geometry of the problem to determine an appropriate mesh type .
  - Ensure mesh quality to maintain accuracy and stability in the numerical solution.
- **4. Code Implementation**