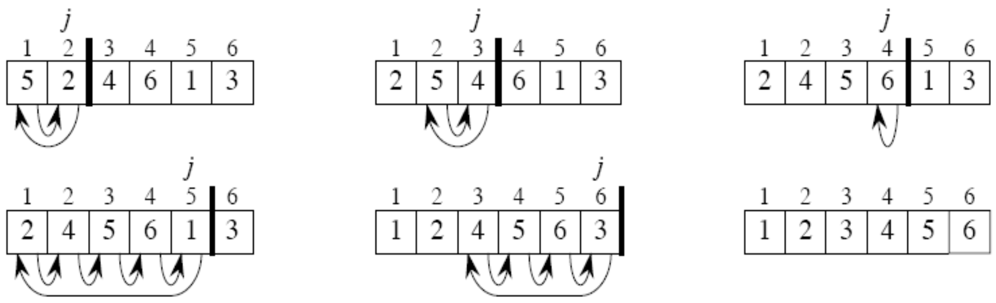
Insertion Sort and its Time complexity Analysis

- A good algorithm for sorting a small number of elements.
- It works the way you might sort a hand of playing cards:
  - Start with an empty left hand and the cards face down on the table.
  - Then remove one card at a time from the table, and insert it into the correct position in the left hand.
  - To find the correct position for a card, compare it with each of the cards already in the hand, from right to left.
  - At all times, the cards held in the left hand are sorted, and these cards were originally the top cards of the pile on the table.

Insertion Sort Example



Insertion Sort Algorithm

```
INSERTION-SORT(A)                                            cost    times
for $j \leftarrow 2$ to $n$                                  $c_1$   $n$
    do $key \leftarrow A[j]$                                 $c_2$   $n-1$
        ▷ Insert $A[j]$ into the sorted sequence $A[1..j-1]$.  0     $n-1$
        $i \leftarrow j-1$                                   $c_4$   $n-1$
        while $i > 0$ and $A[i] > key$                       $c_5$   $\sum_{j=2}^{n} t_j$
            do $A[i+1] \leftarrow A[i]$                      $c_6$   $\sum_{j=2}^{n}(t_j-1)$
                $i \leftarrow i-1$                           $c_7$   $\sum_{j=2}^{n}(t_j-1)$
        $A[i+1] \leftarrow key$                              $c_8$   $n-1$
```

How do we analyze an algorithm's running time?

- ***Input size:*** Depends on the problem being studied.
  - Usually, the number of items in the input. Like the size *n* of the array being sorted.
  - But could be something else. If multiplying two integers, could be the total number of bits in the two integers.
  - Could be described by more than one number. For example, graph algorithm running times are usually expressed in terms of the number of vertices and the number of edges in the input graph.
- ***Running time:*** On a particular input, it is the number of primitive operations (steps) executed.
  - Want to define steps to be machine-independent.
  - Figure that each line of pseudo code requires a constant amount of time.
  - One line may take a different amount of time than another, but each execution of line *i* takes the same amount of time *ci* .
  - This is assuming that the line consists only of primitive operations.
    - If the line is a subroutine call, then the actual call takes constant time, but the execution of the subroutine being called might not.
    - If the line specifies operations other than primitive ones, then it might take
    - more than constant time.

Analysis of Insertion Sort

- Assume that the *i* th line takes time *ci* , which is a constant. (Since the third line is a comment, it takes no time.)
- For *j* = 2, 3, . . . , *n*, let *tj* be the number of times that the **while** loop test is executed for that value of *j* .
- Note that when a **for** or **while** loop exits in the usual way-due to the test in the loop header-the test is executed one time more than the loop body.

Running Time of Insertion Sort

The running time of the algorithm is

$$\sum_{\text{all statements}} (\text{cost of statement}) \cdot (\text{number of times statement is executed}) .$$

Let $T(n)$ = running time of INSERTION-SORT.

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) .$$

The running time depends on the values of t_j. These vary according to the input.

Best case: The array is already sorted.

- Always find that $A[i] \le key$ upon the first time the **while** loop test is run (when $i = j - 1$).
- All t_j are 1.
- Running time is

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\ &= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) \ . \end{aligned}$$

- Can express $T(n)$ as $an + b$ for constants a and b (that depend on the statement costs c_i) $\Rightarrow T(n)$ is a *linear function* of n.

Worst case: The array is in reverse sorted order.

- Always find that $A[i] > key$ in while loop test.
- Have to compare *key* with all elements to the left of the jth position $\Rightarrow$ compare with $j - 1$ elements.
- Since the while loop exits because i reaches 0, there's one additional test after the $j - 1$ tests $\Rightarrow t_j = j$.
- $\sum_{j=2}^{n} t_j = \sum_{j=2}^{n} j$ and $\sum_{j=2}^{n} (t_j - 1) = \sum_{j=2}^{n} (j - 1)$.
- $\sum_{j=1}^{n} j$ is known as an ***arithmetic series***, and equation (A.1) shows that it equals $\frac{n(n+1)}{2}$.
- Since $\sum_{j=2}^{n} j = \left(\sum_{j=1}^{n} j\right) - 1$, it equals $\frac{n(n+1)}{2} - 1$.
- Letting $k = j - 1$, we see that $\sum_{j=2}^{n} (j-1) = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$.
- Running time is

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) \\ &\quad + c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1) \\ &= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)n \\ &\quad - (c_2 + c_4 + c_5 + c_8) \ . \end{aligned}$$

- Can express $T(n)$ as $an^2 + bn + c$ for constants a, b, c (that again depend on statement costs) $\Rightarrow T(n)$ is a *quadratic function* of n.