# CSC418 (Computer Graphics) Assignment 1

Maxwell Huang-Hobbs

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## Part A: Questions

1. a. **Find the tangent vector to the wiggly-ellipse as a function of the time.**

$$\begin{aligned}
v_x(t) &= 4\cos(2\pi t) + \frac{1}{16}\cos(32\pi t)\\
v_y(t) &= 2\sin(2\pi t) + \frac{1}{16}\sin(32\pi t)\\
\text{with } u &= \frac{dv}{dt}\\
u_x(t) &= -8\pi\sin(2\pi t) - 2\pi sin(32\pi t)\\
u_y(t) &= 4\pi\cos(2\pi t) + 2\pi cos(32\pi t)
\end{aligned} \tag{1}$$

Since the derivative of a parametric curve is the same as the derivative of its components, $u$ is the tangent to $v$ at time $t$.

b. **Find the normal vector**

in 2d space, the normal vector to a line has slope equal to $m_n = \frac{-1}{m_v}$ $u = \frac{dv}{dt}$, so $n$, the normal vector to $v$ at any point in time has

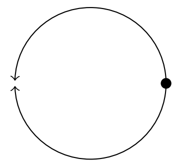
$$n_x(t) = u_y(t) = 4\pi\cos(2\pi t) + 2\pi cos(32\pi t)$$
$$n_x(t) = -u_x(t) = 8\pi\sin(2\pi t) + 2\pi sin(32\pi t)$$

c. **Is the curve symmetric around the X-axis? Y-axis?**

$v$ can be thought of as the sum of two ellipses with different periods,

$$\begin{aligned}
a_x(t) &= 4\cos(2\pi t)\\
a_y(t) &= 2\sin(2\pi t)\\
b_x(t) &= \frac{1}{16}\cos(32\pi t)\\
b_y(t) &= \frac{1}{16}\sin(32\pi t)
\end{aligned} \tag{2}$$

An ellipse is symmetric about both the x and the y axies, so we know that $a_x$ is symmetric about the x and y axies. If we trace the value of $a$ over the ranges $t = (0..0.5)$ and $t = (0..-0.5)$, we see that $v$ is symmetrical about the x axis, because the following relationships hold between a at $t$ and $-t$.



$$\begin{aligned} a_x(t) &= a_t(0.5 - t) \\ a_y(t) &= -a_x(0.5 - t) \end{aligned} \tag{3}$$

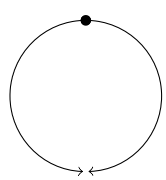
Figure 1: paths of corresponding points traced from left to right of the ellipse

Since $v(t) = a(t) + b(t)$, x will only be symmetric about a given axis if the same property holds for $b$ over the same ranges.

$$\begin{aligned} b_x(t) &= b_x(-t) && \text{(assert that } b_x \text{ symmetric in the x axis)} \\ \cos(32\pi t) &= \frac{1}{16}\cos(32\pi(-t)) \\ \cos(32\pi t) &= \cos(-32\pi t) \\ \cos(t) &= \cos(-t) && \text{(known true)} \end{aligned}$$

$$\begin{aligned} b_y(t) &= -b_y(-t) && \text{(assert that } b_y \text{ symmetric in the y axis)} \\ \frac{1}{16}\sin(32\pi t) &= -\frac{1}{16}\sin(32\pi(-t)) \\ \sin(32\pi t) &= -\sin(-32\pi t) \\ \sin(t) &= -\sin(-t) && \text{(known true)} \end{aligned}$$

so $v$ is symmetric about the x axis



$$\begin{aligned} a_x(t) &= -a_x(0.5 - t), t \in [0.25, 0.75] \\ a_y(t) &= a_y(0.5 - t), t \in [0.25, 0.75] \end{aligned} \tag{4}$$

Figure 2: paths of equal points traced from top to bottom of the ellipse

$$\begin{aligned} b_x(t) &= -b_x(0.5 - t), & t &\in (0, 25, 0.75) && \text{(assert that } b_y \text{ symmetric about the y axis)} \\ \frac{1}{16}\cos(32\pi t) &= \frac{1}{16}\cos(32\pi(0.5 - t)), & t &\in [0.25, 0.75] \\ \cos(2\pi t) &= \cos(\pi - 2\pi t), & t &\in [0.25, 0.75] \\ \cos(t) &= \cos(\pi - t), & t &\in [\frac{\pi}{2}, \frac{3}{2}\pi] \\ \cos(\frac{\pi}{2} + t) &= \cos(\frac{\pi}{2} - t), & t &\in [0, \pi] \\ 1 &= 0 & &\text{(for } t = \pi) && \text{(contradiction)} \end{aligned}$$

Therefore v not symmetric about the y axis.

d. **What is the formula to compute this curve's perimeter?**

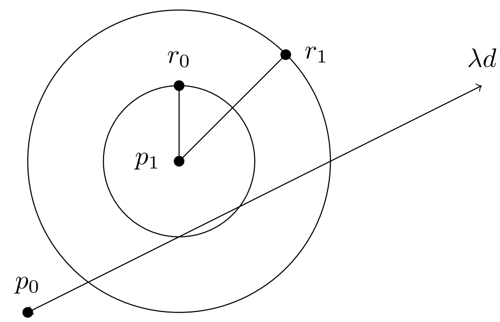
The curve intersects itself at $t = 1$, so the perimeter is the sum of the changes in position from $t = 0$ to $t = 1$

$$\int_0^1 (u_x(t)^2(+u_y(t)^2)^2 dt$$

e. **How can one piecewise approximate this perimeter?**
walk along the curve in $n$ discrete time steps, summing the resulting differences

$$\sum_{n=1}^{N}([v_x(n/N) - v_x((n-1)/N]^2 + [v_y(n/N) - v_y((n-1)/N]^2)$$

2. Given the following doughnut:



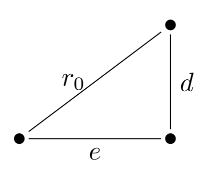

a. **What is the area of the doughnut?**
$2\pi {r_1}^2 - 2\pi {r_0}^2$

b. **How many intersections between the line and the boundary of the doughnut can you have?**
you can have between 0 & 4 intersections between a line and a doughnut's surface (no intersections, tangent to the outside circle, intersecting the outside circle twice but not the inner circle, tangent to the inner circle, or intersecting both circles twice)

c. **Describe an algorithm to determine the number of intersections between the surface of a doughnut and a line**
project vector $\overrightarrow{p_0, p_1}$ onto $d$ by taking the cross product $\overrightarrow{p_0 p_1} \times d$. This is the point on line $p_0 + \lambda d$ closest to $p_1$. Then, switch on the distance

$$\text{numCollissions}(d) = \begin{cases} 4 & d < r_0 \\ 3 & d = r_0 \\ 2 & r_0 < d < r_2 \\ 1 & d = r_2 \\ 0 & r_2 < d \end{cases}$$

d. **Describe an algorithm to determine the locations of intersections between the surface of a doughnut and a line**
as in part (c.), project vector $\overrightarrow{p_0, p_1}$ onto $d$ to get the point on $d$ closest to $p_1$. Call this point $m$. let $d = \|\overrightarrow{m, p_0}\|$ Construct the following right triangle and solve for the length of e




$$\begin{aligned} e^2 + d^2 &= {r_0}^2 \\ e &= \sqrt{{r_0}^2 - d^2} \end{aligned} \tag{5}$$

repeat this process substituting $r_1$ for $r_0$.

e. **If the line and donut are both transformed by a non-uniform scale $(S_x, S_y)$ around the origin, how do the number of intersections and their locations change?**

There is no change in the number of intersections. Their locations are changed by scaling by a factor of $(S_x, S_y)$

f. **How do the number of intersection(s) and their location(s) change if this transformation is only applied to the donut?**

Scaling the doughnut by the nonuniform factor $S_x, S_y$ is the same as scaling the line by $(\frac{1}{S_x}, \frac{1}{S_y})$ in the reference frame of the doughnut. We can therefore maintain the assumption of a uniform circle and recompute the number / location of intersections using the algorithms from parts (c.) and (d.) with the original circle and the new line $p(\lambda) = \begin{bmatrix} \frac{1}{S_x} p_{0,x} \\ \frac{1}{S_y} p_{0,y} \end{bmatrix} + \lambda \begin{bmatrix} \frac{1}{S_x} d_x \\ \frac{1}{S_y} d_y \end{bmatrix}$

3. **Prove / disprove the commutativity of the following transforms:**

a. **Translation and Uniform Scale**)

Translation and uniform scale do not commute

Take some point $p=(0,3), a translation $(0,1)$ and a scale by $(2,2)$. If the translation is applied first, the result is

$$(0,3) \stackrel{translate}{\rightarrow} (0,4) \stackrel{scale}{\rightarrow} (0,8)$$

However, if the scale is applied first, the result is

$$(0,3) \stackrel{scale}{\rightarrow} (0,6) \stackrel{translate}{\rightarrow} (0,7)$$

b. **Translation and Nonuniform Scale**

Translation and nonuniform scale do not commute

Take some point $p=(0,3), a translation $(0,1)$ and a scale by $(1,2)$. If the translation is applied first, the result is

$$(0,3) \stackrel{translate}{\rightarrow} (0,4) \stackrel{scale}{\rightarrow} (0,8)$$

However, if the scale is applied first, the result is

$$(0,3) \stackrel{scale}{\rightarrow} (0,6) \stackrel{translate}{\rightarrow} (0,7)$$

c. **Scaling and Rotation**

Scaling and rotation do not commute.

Take some point $p=(0, 3), a rotation by $90\,\deg$ and a scale by $(1,2)$. If the rotation is applied first, the result is

$$(0,3) \stackrel{rotation}{\rightarrow} (3,0) \stackrel{scale}{\rightarrow} (3,0)$$

However, if the scale is applied first, the result is

$$(0,3) \stackrel{scale}{\rightarrow} (0,6) \stackrel{rotation}{\rightarrow} (6,0)$$

d. **Scaling and Scaling, not sharing the same fixed points**

Scaling and scaling always commute, so scaling will always commute with scaling even if they have different fixed points. take the following two scale operations

$$\begin{bmatrix} S_{2,1} & 0 & .. & 0 \\ 0 & S_{2,2} & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & S_{2,n} \end{bmatrix} \left( \begin{bmatrix} S_{1,1} & 0 & .. & 0 \\ 0 & S_{1,2} & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & S_{2,n} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ .. \\ P_n \end{bmatrix} \right)$$

The result of this product is $\begin{bmatrix} S_{1,1} * S_{2,1} * P_1 \\ S_{1,2} * S_{2,2} * P_1 \\ .. \\ S_{1,n} * S_{2,n} * P_n \end{bmatrix}$

The $i$th entry in the resulting vector is $S_{1,i} * S_{2,1}P_i$. If you were to invert the order of the scale operations, the result would be $S_{2,i} * S_{1,1}P_i = S_{1,i} * S_{2,1}P_i$. Since multiplication commutes, scaling therefore also commutes.

e. **Translation and Shearing**

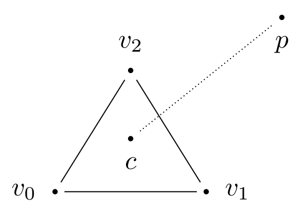
Translation and shear do not commute. Take the translation and shear matrices

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, S = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and the point } P = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(SP) = T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$S(TP) = S \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

4. Take some triangle $< v_1, v_2, v_3 >$ and a point $p$




a. **Write a procedure for determining if a point p is inside / outside the triangle.**

i. to begin, find some point $c$ inside the triangle by averaging the locations of $v_1$, $v_2$, and $v_3$

ii. let $p', v'_{1-3}$ be $p, v_{1-3}$ translated by $-c$. (translate the system s.t. c is at the origin)

iii. rewrite each $\overrightarrow{v_0v_1}$, $\overrightarrow{v_1v_2}$, $\overrightarrow{v_2v_0}$ as a function of the form $\overrightarrow{v_av_b} \to f(\lambda) = v_a + \lambda(v_b - v_a)$

iv. find the values of $t_1$ and $t_2$ s.t. $f(t_1) = t_2 * d$

for each $f(t_1) = \bar{b} + t_1 * \overrightarrow{l}$

$$\begin{aligned} f(t_1) &= \bar{b} + t_2 * \overrightarrow{d} \\ \bar{b} + t_1 * \overrightarrow{l} &= \bar{b} + t_2 * \overrightarrow{d} \\ \overrightarrow{l} t_2 - \overrightarrow{d} t_1 &= \bar{b} \end{aligned} \tag{6}$$

This yields a system of equations with two unknowns($t_1$ and $t_2$) and two equations (the x and y dimensions). The $t$ values can therefore be expressed in terms of known constants $\overrightarrow{l}$, $\overrightarrow{d}$, and $\bar{b}$ [1].

$$\begin{aligned} t_1 &= \frac{b_y - l_y \frac{b_x}{l_x}}{d_y - l_x \frac{-d_x}{l_x}} \\ t_2 &= \frac{b_x}{l_x} - \frac{-d_x}{l_x} \frac{b_y - l_y \frac{b_x}{l_x}}{d_y - l_x \frac{-d_x}{l_x}} \end{aligned} \tag{7}$$

[1] full explanation at end of writeup

if t_1 and t_2 are both $\in [0, 1)$ for any $f$, then $\overrightarrow{c, p}$ intersects with one of the edges of the triangle, and $p$ is outside the triangle.

b. **Write a procedure for determining if a point q is on the edge of a triangle**
use the algorithm in part (a.)

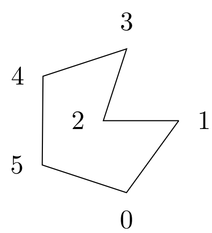
c. **How can one triangulate a quadrilateral such that it is the union of two triangles**
pick two opposite sides of the quadrilateral and draw a line between them. Construct two triangles, each with one of the remaining points in the quadrilateral and the line.

d. **Give a procedure that can triangulate any n-sided convex polygon**
given the points of an N sided convex polygon as an array `points`

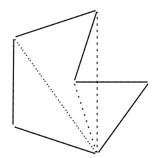
```
for n in [1..N-2]:
    triangle(points[0], points[N], points[N+1])
```

This works because for any convex polygon, a straight line can be drawn between any two nonconsecutive points without crossing the boundary of the polygon.

e. **the procedure will not work in general for concave polygons provide a counterexample**
This procedure will fail on some concave polygons




Input Polygon



Failed Tessellation

f. **How can one use the point in/out/on a triangle procedure (or the idea behind it) to perform a point in/out/on a convex polygon test**
Use the same procedure used in (a.) and (b.) (finding an intersection between an interior point on the polygon and a side of the polygon), but test against each side of the polygon instead of just against the 3 sides of the triangle.

## Math at the End

$$
\begin{array}{c}
\begin{array}{cc} t_2 & t_1 \end{array} \\
\left[\begin{array}{cc|c} l_x & -d_x & b_x \\ l_y & -d_y & b_y \end{array}\right]
\end{array}
$$

$$
\begin{aligned}
r_1 &= r_1/l_x \\
&\left[\begin{array}{cc|c} 1 & -\frac{-d_x}{l_x} & \frac{b_x}{l_x} \\ l_y & -d_y & b_y \end{array}\right] \\
r_2 &= r_2 - l_y r_1 \\
&\left[\begin{array}{cc|c} 1 & -\frac{-d_x}{l_x} & \frac{b_x}{l_x} \\ 0 & -d_y - l_y \frac{-d_x}{l_x} & b_y - l_y \frac{b_x}{l_x} \end{array}\right] \\
r2 &= r2/(d_y - l_x \frac{-d_x}{l_x}) \\
&\left[\begin{array}{cc|c} 1 & -\frac{-d_x}{l_x} & \frac{b_x}{l_x} \\ 0 & 1 & \frac{b_y - l_y \frac{b_x}{l_x}}{d_y - l_x \frac{-d_x}{l_x}} \end{array}\right] \\
r_1 &= r_1 + -\frac{-d_x}{l_x} r_2 \\
&\left[\begin{array}{cc|c} 1 & 1 & \frac{b_x}{l_x} - \frac{-d_x}{l_x} \frac{b_y - l_y \frac{b_x}{l_x}}{d_y - l_x \frac{-d_x}{l_x}} \\ 0 & 1 & \frac{b_y - l_y \frac{b_x}{l_x}}{d_y - l_x \frac{-d_x}{l_x}} \end{array}\right] \\
t_1 &= \frac{b_y - l_y \frac{b_x}{l_x}}{d_y - l_x \frac{-d_x}{l_x}} \\
t_2 &= \frac{b_x}{l_x} - \frac{-d_x}{l_x} \frac{b_y - l_y \frac{b_x}{l_x}}{d_y - l_x \frac{-d_x}{l_x}}
\end{aligned}
\tag{8}
$$