Topic: A Reduced-Bias least square estimation of the Extreme Value Index using dependent uniformly distributed weight.

The report is organised as follows. The reduced-bias weighted least square estimator and its asymptotic properties with their proofs are presented in this section.

We propose a regression model given as

$$Z_j = \gamma + b_{n,k} C_j + \varepsilon_j, \quad j \in \{1, 2, \cdots, k\} \tag{1}$$

where $b_{n,k} = b\left(\frac{k}{n}\right)$ is the slope of Z_j, $C_j = \left(\frac{j}{k+1}\right)^{-\rho}$ is the covariate (an independent variable that can influence the outcome of a given statistical trial, but which is not of direct interest), γ is the Intercept in 1, $\varepsilon_i \sim N\left(0, \gamma^2\right)$ and ρ is a second-order parameter which would be estimated externally.

The proposed estimator that uses weighted least squares on the $Z_j's$ in (1), to estimate γ and $b_{n,k}$, the coefficient of the model from the loss function of the weighted least square is given by

$$L_k\left(\gamma, b_{n,k}; W\right) = \sum_{j=1}^{k} W_j \left(Z_j - \gamma - b_{n,k} C_j\right)^2$$

for $j \in \{1, 2, \ldots, k\}$. The weight function is defined as

$$W_j = \left(1 - \theta_j^{\alpha(k)} \frac{j}{k+1}\right), 1 \leq j \leq k,$$

where $\theta_j \overset{i.i.d}{\sim} U(0,1)$.Thus, $W_j \in (0,1)$ and decreases linearly with respect to j. We minimise (1) with respect to γ and $b_{n,k}$ to obtain the following Weighted Least Squares (WLS) estimators:

Differentiating the loss function partially in respect to γ

$$\frac{\partial L_k}{\partial \gamma} = -2\sum_{j=1}^{k} W_j \left(Z_j - \gamma - b_{n,k}C_j\right), \text{ but } \frac{\partial L_k}{\partial \gamma} = 0$$

$$0 = \sum_{j=1}^{k} W_j Z_j - \gamma \sum_{j=1}^{k} W_j - b_{n,k} \sum_{j=1}^{k} W_j C_j$$

$$\gamma \sum_{j=1}^{k} W_j = \sum_{j=1}^{k} W_j Z_j - b_{n,k} \sum_{j=1}^{k} W_j C_j, \text{ dividing through by } \sum_{j=1}^{k} W_j$$

$$\gamma = \sum_{j=1}^{k} \left(\frac{W_j}{\sum_{i=1}^{k} W_j} \right) Z_j - b_{n,k} \sum_{j=1}^{k} \left(\frac{W_j}{\sum_{j=1}^{k} W_j} \right) C_j, \text{ but } \overline{W_j} = \frac{W_j}{\sum_{j=1}^{k} W_j}$$

$$\gamma = \sum_{j=1}^{k} \overline{W_j} Z_j - b_{n,k} \sum_{j=1}^{k} \overline{W_j} C_j \tag{2}$$

Differentiating partially in respect to the $b_{n,k}$

$$\frac{\partial L_k}{\partial b_{n,k}} = -2C_j \sum_{j=1}^{k} W_j \left(Z_j - \gamma - b_{n,k}C_j\right), \text{but } \frac{\partial L_k}{\partial b_{n,k}} = 0$$

$$0 = \sum_{j=1}^{k} C_j W_j Z_j - \gamma \sum_{j=1}^{k} W_j C_j - b_{n,k} \sum_{j=1}^{k} W_j C_j^2$$

$$b_{n,k} \sum_{j=1}^{k} W_j C_j^2 = \sum_{j=1}^{k} C_j W_i Z_j - \gamma \sum_{j=1}^{k} W_j C_j, \text{ dividing through by } \sum_{j=1}^{k} W_j$$

$$b_{n,k} C_j^2 = \sum_{j=1}^{k} C_j \left(\frac{W_j}{\sum_{j=1}^{k} W_j} \right)_j - \gamma \sum_{j=1}^{k} \left(\frac{W_j}{\sum_{j=1}^{k} W_j} \right) C_j$$

$$b_{n,k} C_j^2 = \sum_{j=1}^{k} C_j \overline{W_j} Z_j - \gamma \sum_{j=1}^{k} \overline{W_j} C_j \tag{3}$$

From (2) $\gamma = \sum_{j=1}^{k=} \overline{W_j} Z_j - b_{n,k} \sum_{j=1}^{k} \overline{W_j} C_j$, putting Eqn. (2) into Eqn. (3), we have

$$
\begin{aligned}
b_{n,k}C_j^2 &= \sum_{j=1}^{k} C_j\overline{W_j}Z_j - \left(\sum_{j=1}^{k}\overline{W_j}Z_j - b_{n,k}\sum_{j=1}^{k}\overline{W_j}C_j\right)\sum_{j=1}^{k}\overline{W_j}C_j \\
b_{n,k}C_j^2 &= \sum_{j=1}^{k} C_j\overline{W_j}Z_j - \sum_{j=1}^{k}\overline{W_j}Z_j\sum_{j=1}^{k}\overline{W_j}C_j + b_{n,k}\left(\sum_{j=1}^{k}\overline{W_j}C_j\right)^2 \\
b_{n,k}C_j^2 - b_{n,k}\left(\sum_{j=1}^{k}\overline{W_j}C_j\right)^2 &= \sum_{j=1}^{k} C_j\overline{W_j}Z_j - \sum_{j=1}^{k}\overline{W_j}Z_j\sum_{j=1}^{k}\overline{W_j}C_j \\
b_{n,k}\left[C_j^2(1) - \left(\sum_{j=1}^{k}\overline{W_j}C_j\right)^2\right] &= \sum_{j=1}^{k}\overline{W_j}\left(C_j - \sum_{j=1}^{k}\overline{W_j}C_j\right)Z_j \\
b_{n,k} &= \frac{\sum_{j=1}^{k}\overline{W_j}\left(C_j - \sum_{j=1}^{k}\overline{W_j}C_j\right)Z_j}{C_j^2(1) - \left(\sum_{j=1}^{k}\overline{W_j}C_j\right)^2}
\end{aligned}
$$

But W_j is normalised and sum up to 1 , i.e., $0 \leq W_j \leq 1$ and $\sum_{j=1}^{k}\overline{W}_j = 1$, it follows that

$$
\hat{b}_{n,k} = \frac{\sum_{j=1}^{k}\overline{W_j}\left(C_j - \sum_{j=1}^{k}\overline{W_j}C_j\right)Z_j}{\sum_{j=1}^{k}\overline{W_j}C_j^2 - \left(\sum_{j=1}^{k}\overline{W_j}C_j\right)^2} \tag{4}
$$

Properties of the proposed estimator

We proposed that the weighted least square estimator is asymptotically unbiased, consistent and normal. To show that the estimator is said to be unbiased, We have to show that, the bias(γ) = 0 and bias is defined as $E[\gamma] - \gamma$

Now, having $\gamma = \sum_{j=1}^{k}\overline{W_j}Z_j - \hat{b}_{n,k}\sum_{j=1}^{k}\overline{W_j}C_j$

$$E[\gamma] = \sum_{j=1}^{k} \overline{W_j} E\left(Z_j\right) - \hat{b}_{n,k} \sum_{j=1}^{k} \overline{W_j} C_j, \text{ but } E(Z_j) = \gamma + \hat{b}_{n,k} C_j$$
$$E[\gamma] = \sum_{j=1}^{k} \overline{W_j} \left(\gamma + \hat{b}_{n,k} C_j\right) - \hat{b}_{n,k} \sum_{j=1}^{k} \overline{W_j} C_j$$
$$E[\gamma] = \sum_{j=1}^{k} \overline{W_j} \gamma + \hat{b}_{n,k} \sum_{j=1}^{k} \overline{W_j} C_j - \hat{b}_{n,k} \sum_{j=1}^{k} \overline{W_j} C_j$$
$$E[\gamma] = \gamma \sum_{j=1}^{k} \overline{W_j}, \text{ but } \sum_{j=1}^{k} \overline{W_j} = 1$$

$$E[\gamma] = \gamma \tag{5}$$

Now,

$$\begin{aligned} bias(\gamma) &= E[Y] - \gamma \\ &= \gamma - \gamma \\ &= 0 \end{aligned}$$

since γ is unbiased, the MSE is the same as variance of the estimator. We can see from Eqn. (2) and Eqn. (4) that we can also rewrite γ as

$$\gamma = \sum_{j=1}^{k} \overline{W_j} Z_j - \hat{b}_{n,k} \sum_{j=1}^{k} \overline{W_j} C_j, \text{ from Eqn. (4), it follows that}$$

$$\gamma = \sum_{j=1}^{k} \overline{W_j} Z_j - \left(\frac{\sum_{j=1}^{k} \overline{W_j} \left(C_j - \sum_{j=1}^{k} \overline{W_j} C_j \right) Z_j}{\sum_{j=1}^{k} \overline{W_j} C_j^2 - \left(\sum_{j=1}^{k} \overline{W_j} C_j \right)^2} \right) \sum_{j=1}^{k} \overline{W_j} C_j$$

$$\gamma = \sum_{j=1}^{k} \overline{W_j} \left(1 - \frac{\left(C_j - \sum_{j=1}^{k} \overline{W_j} C_j \right) \sum_{j=1}^{k} \overline{W_j} C_j}{\sum_{j=1}^{k} \overline{W_j} C_j^2 - \left(\sum_{j=1}^{k} \overline{W_j} C_j \right)^2} \right) Z_j$$

$$\gamma = \sum_{j=1}^{k} \overline{W_j} \left(1 + \frac{\left(\sum_{j=1}^{k} \overline{W_j} C_j - C_j \right) \sum_{j=1}^{k} \overline{W_j} C_j}{\sum_{j=1}^{k} \overline{W_j} C_j^2 - \left(\sum_{j=1}^{k} \overline{W_j} C_j \right)^2} \right) Z_j$$

$$\gamma = \sum_{j=1}^{k} \overline{W_j} \left(1 + \frac{(S_1 - C_j) S_1}{S_2} \right) Z_j$$

$$\gamma = \sum_{j=1}^{k} \overline{W_j} \left(1 + \frac{\left(S_1^2 - S_1 C_j \right)}{S_2} \right) Z_j$$

where $S_1(\rho, \theta) = \sum_{j=1}^{k} \overline{W_j} C_j$, and $S_2(\rho, \theta) = \sum_{j=1}^{k} \overline{W_j} C_j^2 - \left(\sum_{j=1}^{k} \overline{W_j} C_j \right)^2$

For the variance, We have

$$\gamma = \sum_{j=1}^{k} \overline{W}_j \left(1 + \frac{s_1}{s_2} (s_1 - c_j) \right) Z_j$$

$$\mathrm{Var}(\gamma) = \frac{\phi^2(k)}{k^2} \sum_{j=1}^{k} \left(1 + \frac{s_1}{s_2} (s_1 - c_j) \right)^2 \mathrm{Var}\left(W_j Z_j \right) \text{ where}$$

$$
\begin{aligned}
&= \gamma^2 \sum_{j=1}^{k} \overline{W_j}^2 \left(1 + \frac{2}{S_2}\left(S_1^2 - S_1 C_j\right) + \frac{\left(S_1^2 - S_1 C_j\right)^2}{S_2^2}\right) \\
&= \gamma^2 \left(\sum_{j=1}^{k} \overline{W_j}^2 + \sum_{j=1}^{k} \overline{W_j}^2 \frac{2S_1}{s_2}\left(S_1 - C_j\right) + \sum_{j=1}^{k} \overline{W_j}^2 \frac{S_1^2}{S_2^2}\left(S_1 - C_j\right)^2\right) \\
&= \gamma^2 \left(\sum_{j=1}^{k} \overline{W_j}^2 + \frac{2S_1}{s_2} \sum_{j=1}^{k} \overline{W_j}^2 \left(S_1 - C_j\right) + \frac{S_1^2}{S_2^2} \sum_{j=1}^{k} \overline{W_j}^2 \left(S_1 - C_j\right)^2\right) \\
&= \gamma^2 \left(\sum_{j=1}^{k} \overline{W_j}^2 + \frac{2S_1}{s_2} \sum_{j=1}^{k} \overline{W_j}^2 \left(S_1 - C_j\right) + \frac{S_1^2}{S_2^2} \sum_{j=1}^{k} \overline{W_j}^2 \left(S_1 - C_j\right)^2\right) \\
&= \gamma^2 \left(\sum_{j=1}^{k} \overline{W_j}^2 + \frac{2S_1 \dot{S}}{S_2} + \frac{S_1^2 \ddot{S}}{S_2^2}\right)
\end{aligned}
$$

where $\dot{S}(\rho,\theta) = \sum_{j=1}^{k} \overline{W_j}^2 \left(S_1 - S_j\right)$, $\ddot{S}(\rho,\theta) = \sum_{j=1}^{k} \overline{W_j}^2 \left(S_1 - C_j\right)^2$ and simplifying $\sum_{j=1}^{k} \overline{W}_j^2$, We begin by simplifying

$$
W_j = \left(1 - \theta_j^{\alpha(k)} \frac{j}{k+1}\right) \tag{6}
$$

$$
\begin{aligned}
\sum_{j=1}^{k} W_j &= \sum_{j=1}^{k} \left(1 - \theta_j^{\alpha(k)} \frac{j}{k+1}\right) \\
&= k - k \sum_{j=1}^{k} \theta_j^{\alpha(k)} \frac{j}{k(k+1)} \\
&= k \left[\int_1^0 \left(1 - \mathbb{E}\left(\theta_j^{\alpha(k)}\right) u\right) du\right] \\
&= k \left[\int_1^0 \left(1 - \left(\int_0^1 \theta_j^{\alpha(k)} d\theta\right) u\right) du\right] \\
&= k \left[\int_1^0 \left(1 - \left.\frac{\theta_j^{\alpha(k)+1}}{\alpha(k)+1}\right|_0^1 u\right) du\right] \\
&= k \int_1^0 \left(1 - \frac{u}{\alpha(k)+1}\right) du
\end{aligned}
$$

$$= k\left[u - \frac{u^2}{2\alpha(k)+2}\right]_0^1$$

$$= k\left(1 - \frac{1}{2\alpha(k)+2}\right)$$

$$= k\left(\frac{2\alpha(k)+2-1}{2\alpha(k)+2}\right)$$

$$\sum_{j=1}^{k} W_j = \frac{k}{2}\left(\frac{2\alpha(k)+1}{\alpha(k)+1}\right) \tag{7}$$

Therefore from Eqn. 6 and Eqn. 7

$$\overline{W_j} \approxeq \frac{2}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\left(1 - \theta_j^{\alpha(k)}\frac{j}{k+1}\right) \tag{8}$$

From Eqn. (6) and Eqn. (7), we have

$$\sum_{j=1}^{k} \overline{W}_j^2 = \frac{W_j^2}{\sum_{j=1}^{k} W_j^2} = \sum_{j=1}^{k} \frac{4}{k^2} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \left(1 - \theta_j^{\alpha(k)} \frac{j}{k+1} \right)^2$$

$$= \frac{4}{k^2} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \sum_{j=1}^{k} \left(1 - 2\theta_j^{\alpha(k)} \frac{j}{k+1} + \theta_j^{2\alpha(k)} \left(\frac{j}{k+1} \right) \right)$$

$$= \frac{4}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \sum_{j=1}^{k} \frac{1}{k} \left(1 - 2\theta_j^{\alpha(k)} \frac{j}{k+1} + \theta_j^{2\alpha(k)} \left(\frac{j}{k+1} \right)^2 \right)$$

$$= \frac{4}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \frac{1}{k} \left(k - 2\sum_{j=1}^{k} \theta_j^{\alpha(k)} \frac{j}{k+1} + \sum_{j=1}^{k} \theta_j^{2\alpha(k)} \left(\frac{j}{k+1} \right)^2 \right)$$

$$= \frac{4}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \left(1 - 2\sum_{j=1}^{k} \mathbb{E}\left(\theta_j^{\alpha(k)} \right) \frac{j}{k+1} + \sum_{j=1}^{k} \mathbb{E}\left(\theta_j^{2\alpha(k)} \right) \left(\frac{j}{k+1} \right)^2 \right) \frac{1}{k}$$

$$= \frac{4}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \left(1 - 2\left(\int_0^1 \theta_j^{\alpha(k)} d\theta \right) \sum_{j=1}^{k} \frac{j}{k+1} + \left(\int_0^1 \theta_j^{2\alpha(k)} d\theta \right) \sum_{j=1}^{k} \left(\frac{j}{k+1} \right)^2 \right) \frac{1}{k}$$

$$= \frac{4}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \left(1 - \frac{2}{\alpha(k)+1} \sum_{j=1}^{k} \left(\frac{j}{k+1} \right) + \frac{1}{2\alpha(k)+1} \sum_{j=1}^{k} \left(\frac{j}{k+1} \right)^2 \right) \frac{1}{k}$$

$$= \frac{4}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \int_0^1 \left(1 - \frac{2u}{\alpha(k)+1} + \frac{u^2}{2\alpha(k)+1} \right) du$$

$$= \frac{4}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right)^2 \left(u - \frac{u^2}{(\alpha(k)+1)} + \frac{u^3}{3(2\alpha(k)+1)} \right)_0^1$$

$$= \frac{4}{k} \lim_{k\to\infty} \left(\frac{\frac{\alpha(k)}{k} + \frac{1}{k}}{\frac{2\alpha(k)}{k} + \frac{1}{k}} \right)^2 \left(1 - \frac{\frac{1}{k}}{(\frac{\alpha(k)}{k} + \frac{1}{k})} + \frac{\frac{1}{k}}{3(\frac{2\alpha(k)}{k} + \frac{1}{k})} \right)$$

$$= \frac{4}{k} \lim_{k\to\infty} \left(\frac{\frac{\alpha(k)}{k} + \frac{1}{k}}{\frac{2\alpha(k)}{k} + \frac{1}{k}} \right)^2$$

$$= \frac{4}{k} \left(\frac{\Delta}{2\Delta} \right)^2$$

$$= \frac{1}{k}$$

Hence, the MSE of the weighted least squares estimator is given by

$$0 < MSE\left(\hat{\gamma}\right) = \gamma^2 \left(\frac{1}{k} + \frac{2S_1\dot{S}}{S_2} + \frac{S_1^2\ddot{S}}{S_2^2} \right) + O(1/k)$$

Note that, the following basic properties are required to study the asymptotic behavior of the reduced-bias weighted least squares estimator.

$$W_j = \left(1 - \theta_j^{\alpha(k)} \frac{j}{k+1}\right), j \in \{1, 2, \ldots, k\},$$

where $\theta_j \overset{i.i.d}{\sim} U(0,1)$. Thus, $W_j \in (0,1)$ and decreases linearly with respect to j. The exponent $\alpha(k) \geq 0$ is chosen such that

$$\Delta := \lim_{k\to\infty} \frac{\alpha(k)}{k} < \infty \tag{9}$$

In this study, we consider $0 < \Delta \leq 1$.

Thus, we would define $\alpha(k)$ such that $0 < \alpha(k) \leq k$. Note that, W_j is random through θ_j, and when the exponent is $0, W_j$ is deterministic. In particular, when $\alpha(k) = 0$, we obtain the weight function $h_j = 1 - j/(k+1)$. Nevertheless, we can approximate the weight h_j as a limit of the current result by allowing Δ to approach 0.

Note that, since the weight function depends on the θ_j 's, the estimators in Eqn.(4) and Eqn. (2) also depend on the θ_j 's through the weight function. Therefore, we would find the MSE by conditioning on the θ_j 's. From Eqn. (4) and Eqn. (2), the MSE for $\hat{\gamma}$ is obtained as

where

$$S_1(\theta) = \sum_{j=1}^{k} \overline{W}_j C_j \tag{10}$$

$$S_1(\theta) = \sum_{j=1}^{k} \overline{W}_j C_j \tag{11}$$

$$\dot{S}(\theta) = \sum_{j=1}^{k} \overline{W}_j^2 \left(S_1(\theta) - C_j\right) \tag{12}$$

$$\ddot{S}(\theta) = \sum_{j=1}^{k} \overline{W}_j^2 \left(S_1(\theta) - C_j\right)^2 \tag{13}$$

Asymptotic Properties of the Proposed estimators

Unbiasedness, consistency and normality are desirable properties of a good estimator. In this section, we investigate desirable properties of the proposed estimator possesses.

We shall summarise the asymptotic behaviour of the statistics used to build the MSE of the proposed estimator in Lemma 1. These properties will be required in the proof of the asymptotic consistency and sampling distribution of the proposed estimator. Henceforth, anytime we use the term $a.s$ it is with respect to the law of the i.i.d sequence $\theta_1, \theta_2, \theta_3, \ldots, \theta_k$.

Lemma 1. Assume that ρ is estimated by a consistent estimator $\hat{\rho}$ and Eqn. (9) holds, then as $k \to \infty$ and $k/n \to 0$;

i. $S_1(\rho, \theta) = \displaystyle\sum_{j=1}^{k} \overline{W}_j C_j \xrightarrow{a.s} 1/(1-\rho).$

ii. $S_2(\rho, \theta) = \displaystyle\sum_{j=1}^{k} \overline{W}_j C_j^2 - \left(\sum_{j=1}^{k} \overline{W}_j C_j\right)^2 \xrightarrow{a.s} \rho^2/(1-2\rho)(1-\rho)^2.$

iii. $\dot{S}(\rho, \theta) = \displaystyle\sum_{j=1}^{k} \overline{W}_j^2 \left(S_1 - C_j\right) \xrightarrow{a.s} 0.$

iv. $\ddot{S}(\rho, \theta) = \displaystyle\sum_{j=1}^{k} \overline{W}_j^2 \left(S_1 - C_j\right)^2 \xrightarrow{a.s} 0.$

Proof of Lemma 1.

i. From Eqn. (6) and Eqn. (10), we have

$$S_1(\rho, \theta) = \sum_{j=1}^{k} \bar{W}_j C_j$$

$$= \sum_{j=1}^{k} \frac{2}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \left(1 - \theta_j^{\alpha(k)} \frac{j}{k+1} \right) \left(\frac{j}{k+1} \right)^{-\rho}$$

$$= \frac{2}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \sum_{j=1}^{k} \left(1 - \theta_j^{\alpha(k)} \frac{j}{k+1} \right) \left(\frac{j}{k+1} \right)^{-\rho}$$

$$= \frac{2}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \left(k - \sum_{j=1}^{k} \theta_j^{\alpha(k)} \frac{j}{k+1} \right) \left(\frac{j}{k+1} \right)^{-\rho}$$

$$= \frac{2}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \left(k - \sum_{j=1}^{k} \mathbb{E}\left(\theta_j^{\alpha(k)} \right) \frac{j}{k+1} \right) \left(\frac{j}{k+1} \right)^{-\rho}$$

$$= \frac{2}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \left(k - \sum_{j=1}^{k} \left(\int_0^1 \theta_j^{\alpha(k)} d\theta \right) \frac{j}{k+1} \right) \left(\frac{j}{k+1} \right)^{-\rho}$$

$$= \frac{2}{k} \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \left(k - \sum_{j=1}^{k} \frac{1}{\alpha(k)+1} \frac{j}{k+1} \right) \left(\frac{j}{k+1} \right)^{-\rho}$$

$$= 2 \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \int_0^1 \left(1 - \frac{u}{\alpha(\alpha)+1} \right) u^{-1} du + O(1)$$

$$= 2 \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \int_0^1 \left(u^{-\rho} - \frac{u^{1-\rho}}{\alpha(k)+1} \right) du + O(1)$$

$$= 2 \left(\frac{\alpha(k)+1}{2\alpha(k)+1} \right) \left[\frac{u^{1-\rho}}{1-\rho} - \frac{u^{2-\rho}}{(2-\rho)(\alpha(k)+1)} \right]_0^1 + O(1)$$

$$= \lim_{k\to\infty} 2 \left(\frac{\frac{\alpha(k)}{k} + \frac{1}{k}}{\frac{2\alpha(k)}{k} + \frac{1}{k}} \right) \left(\frac{1}{1-\rho} - \frac{\frac{1}{k}}{(2-\rho)(\frac{\alpha(k)}{k} + \frac{1}{k})} \right) + O(1)$$

$$S_1(\rho, \theta) = 2 \left(\frac{\Delta}{2\Delta} \right) \left(\frac{1}{1-\rho} \right) = \frac{1}{1-\rho} \tag{14}$$

where $\Delta = \lim_{k\to\infty} \alpha(k)/k$ and hence, as $k \to \infty$, we have $S_1(\theta) \xrightarrow{\text{a.s}} 1/(1-\rho)$

ii. $S_2(\rho, \theta) = \sum_{j=1}^{K} \overline{W}_j C_j^2 - \left(\sum_{j=1}^{k} \overline{W}_j C_j \right)^2$, from Eqn. (14), it follows that

$$S_2(\rho, \theta) = \sum_{j=1}^{K} \overline{W}_j C_j^2 - \left(\frac{1}{1-\rho} \right)^2 \tag{15}$$

Simplifying $\sum_{j=1}^{K} \overline{W}_j C_j^2$

$$\sum_{j=1}^{k} \bar{W}_j C_j^2 = \frac{2}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\sum_{j=1}^{k}\left(1-\theta_1^{\alpha(k)}\frac{j}{k+1}\right)\left(\frac{j}{k+1}\right)^{-2\rho}$$

$$= \frac{2}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\left(k-\sum_{j=1}^{k}\theta_j^{\alpha(k)}\frac{j}{k+1}\right)\left(\frac{j}{k+1}\right)^{-2\rho}$$

$$= \frac{2}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\left(k-\sum_{j=1}^{k}\left(\theta_j^{\alpha(k)}\right)_{\frac{j}{k+1}}\right)\left(\frac{j}{k+1}\right)^{-2\rho}$$

$$= \frac{2}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\left(k-\sum_{j=1}^{k}\left(\int_0^1 \theta_j^{\alpha(k)}d\theta\right)\frac{j}{k+1}\right)\left(\frac{j}{k+1}\right)^{-2\rho}$$

$$= \frac{2}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\left(k-\sum_{j=1}^{k}\frac{1}{\alpha(k)+1}\frac{j}{k+1}\right)\left(\frac{j}{k+1}\right)^{-2\rho}$$

$$= 2\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\int_0^1 \cdot\left(1-\frac{u}{\alpha(k)+1}\right)u^{-2\rho}du$$

$$= 2\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\int_0^1\left(u^{-2\rho}-\frac{u^{1-2\rho}}{\alpha(k)+1}\right)du$$

$$= 2\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)\left[\frac{u^{1-2p}}{1-2p}-\frac{u^{2-2p}}{(2-2\rho)(\alpha(k)+1)}\right]_0^1$$

$$= 2\left(\frac{2(k)+1}{2\alpha(k)+1}\right)\left(\frac{1}{1-2p}-\frac{1}{(2-2\rho)(\alpha(k)+1)}\right)$$

$$= \lim_{k\to\infty} 2\left(\frac{\alpha(k)}{\frac{k}{2\alpha(k)}+\frac{1}{k}}\right)\left(\frac{1}{k}\right)-\frac{\frac{1}{k}}{1-2p}(2-2\rho)\left(\frac{\alpha(k)}{k}+\frac{1}{k}\right)\bigg)$$

$$= \lim_{k\to\infty} 2\left(\frac{\frac{\alpha(k)}{k}+\frac{1}{k}}{\frac{2\alpha(k)}{k}+\frac{1}{k}}\right)\left(\frac{1}{1-2\rho}-\frac{\frac{1}{k}}{(2-2\rho)(\frac{\alpha(k)}{k}+\frac{1}{k})}\right)+O(1)$$

$$= 2\left(\frac{\Delta}{2\Delta}\right)\left(\frac{1}{1-2p}\right)=\frac{1}{1-2\rho}$$

Now

$$S_2(p,\theta)=\frac{1}{(1-2\rho)}-\frac{1}{(1-\rho)^2},\ \text{from Eqn. (15)}$$

$$=\frac{1-2\rho+\rho^2-1+2\rho}{(1-2\rho)(1-\rho)^2}=\frac{\rho^2}{(1-2\rho)(1-\rho)^2}$$

where $\Delta = \lim_{k\to\infty}\alpha(k)/k$ and hence, as $k\to\infty$, we have $S_2(\rho,\theta)\xrightarrow{\text{a.s}}\rho^2/(1-2\rho)(1-\rho)^2$

iii. $\dot{S}(\rho,\theta) = \sum_{j=1}^{K} \bar{W}_j^2 \left(S_1 - C_j\right)$

$$= \frac{4}{k^2}\sum_{j=1}^{k}\left(1-\theta_j^{\alpha(k)}\frac{j}{k+1}\right)^2\left(S_1(\rho,\theta)-\left(\frac{j}{k+1}\right)^{-\rho}\right)$$

$$= \frac{4}{k^2}\sum_{j=1}^{k}\left(1-2\theta_j^{\alpha(k)}\frac{j}{k+1}+\theta_j^{2\alpha(k)}\left(\frac{j}{k+1}\right)^2\right)\left(\frac{1}{1-\rho}-\left(\frac{j}{k+1}\right)^{-\rho}\right), \text{ from Eqn. (14)}$$

$$= \frac{4}{k^2}\left(k-2\sum_{j=1}^{k}\theta_j^{\alpha(k)}\frac{j}{k+1}+\sum_{j=1}^{k}\theta_j^{2\alpha(k)}\left(\frac{j}{k+1}\right)^2\right)\left(\frac{1}{1-\rho}-\left(\frac{j}{k+1}\right)^{-\rho}\right)$$

$$= \frac{4}{k^2}\left(k-2\sum_{j=1}^{k}\mathbb{E}\left(\theta_j^{\alpha(k)}\right)\frac{j}{k+1}+\sum_{j=1}^{k}\mathbb{E}\left(\theta_j^{2\alpha(k)}\right)\left(\frac{j}{k+1}\right)^2\right)\left(\frac{1}{1-\rho}-\left(\frac{j}{k+1}\right)^{-\rho}\right)$$

$$= \frac{4}{k^2}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)^2\left(k-2\sum_{j=1}^{k}\left(\int_0^1\theta_j^{\alpha(k)}d\theta\right)\frac{j}{k+1}+\sum_{j=1}^{k}\left(\int_0^1\theta_j^{2\alpha(k)}d\theta\right)\left(\frac{j}{k+1}\right)^2\right)$$
$$\left(\frac{1}{1-\rho}-\left(\frac{j}{k+1}\right)^{-\rho}\right)$$

$$= \frac{4}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)^2\int_0^1\left(\frac{1}{1-\rho}-\frac{2u}{(1-p)(\alpha(k)+1)}+\frac{u^2}{(1-\rho)(2\alpha(k)+1)}-\right.$$
$$\left. u^{-\rho}+\frac{2u^{1-\rho}}{\alpha(k)+1}-\frac{u^{2-\rho}}{2\alpha(k)+1}\right)du$$

$$= \frac{4}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1)}\right)^2\left[\frac{u}{1-\rho}-\frac{u^2}{(1-\rho)(\alpha(k)+1)}+\frac{u^3}{3(1-\rho)(2\alpha(k)+1)}-\frac{u^{1-\rho}}{1-\rho}+\right.$$
$$\left.\frac{2u^{2-\rho}}{(\alpha(k)+1)(2-\rho)}-\frac{u^{3-\rho}}{(3-\rho)(x(k+1))}\right]_0^1$$

$$= \frac{4}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)^2\left(\frac{1}{1-\rho}-\frac{1}{(1-\rho)(\alpha(k)+1)}+\frac{1}{3(1-\rho)(2\alpha(k)+1)}-\frac{1}{1-\rho}+\right.$$
$$\left.\frac{2}{(\alpha(k)+1)(2-\rho)}-\frac{1}{(3-\rho)(2\alpha(k)+)}\right)$$

$$= \lim_{k\to\infty}\frac{4}{k}\left(\frac{\frac{\alpha(k)}{k}+\frac{1}{k}}{\frac{2\alpha(k)}{k}+\frac{1}{k}}\right)^2\left(-\frac{\frac{1}{k}}{(1-\rho)(\frac{\alpha(k)}{k}+1)}+\frac{\frac{1}{k}}{3(1-\rho)(2\frac{\alpha(k)}{k}+1)}+\frac{\frac{2}{k}}{(\frac{\alpha(k)}{k}+\frac{1}{k})(2-\rho)}\right.$$
$$\left.-\frac{\frac{1}{k}}{(3-\rho)(2\frac{\alpha(k)}{k}+\frac{1}{k})}\right)$$

$$= 0$$

Therefore, as $k \to \infty,$ we have $\dot{S}(\rho, \theta) \xrightarrow{a.s} 0$.

iv.

$$\begin{aligned}
\ddot{S}(\rho,\theta) &= \sum_{j=1}^{k} \bar{W}_j^2 \left(S_1(\rho,\theta) - C_j\right)^2 \\
&= \frac{4}{k^2}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)^2 \sum_{j=1}^{k}\left(1-\theta_j^{\alpha(k)}\frac{j}{k+1}\right)^2 \left(S_1^2(\rho,\theta) - 2S_1(\rho,\theta)C_j + C_j^2\right) \\
&= \frac{4}{k^2}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)^2 \sum_{j=1}^{k}\left(1-2\theta_j^{\alpha(k)}\frac{j}{k+1} + \theta_j^{2\alpha(k)}\left(\frac{j}{k+1}\right)^2\right) \\
&\left(\frac{1}{(1-\rho)^2} - \frac{2}{(1-\rho)}\left(\frac{j}{k+1}\right)^{-\rho} + \left(\frac{j}{k+1}\right)^{-2\rho}\right) \\
&= \frac{4}{k}\left(\frac{\alpha(k)+1}{2\alpha(k)+1}\right)^2 \int_0^1 \left(1-2\mathbb{E}\left(\theta_j^{\alpha(k)}\right)u + \mathbb{E}\left(\theta_j^{2\alpha(k)}\right)u^2\right)\left(\frac{1}{(1-\rho)^2} - \frac{2u^{-\rho}}{(1-\rho)} + u^{-2\rho}\right)du \\
&+ O(1) \\
&= \lim_{k\to\infty}\frac{4}{k}\left(\frac{\alpha(k)+1}{2\alpha((k)+1}\right)^2 \int_0^1 \left(1 - \frac{2u}{\alpha(k)+1} + \frac{u^2}{2\alpha(k)+1}\right)\left(\frac{1}{(1-\rho)^2} - \frac{2u^{-\rho}}{(1-\rho)} + u^{-2\rho}\right)du \\
&+ O(1) \\
&= 0
\end{aligned}$$

Therefore, as $k \to \infty,$ we have $\ddot{S}(\rho, \theta) \xrightarrow{a.s} 0$ which completes the proof of Lemma 1.

A desirable property of an estimator is consistency. If more observations are included, we hope to obtain a lot of information about the unknown parameter. From Lemma 1, the MSE of $\hat{\gamma}$ approaches 0 , as $k \to \infty$. This implies that the estimator $\hat{\gamma}$ is "MSE-Consistent".

We conclude this section by stating and proving the sampling distribution of the weighted least squares estimator, $\hat{\gamma}$.

Lemma 2. Let $C_j = \left(\frac{j}{k+1}\right)^{-\rho}, \overline{W}_j = \frac{W_j}{\sum_{j=1}^{k} W_j}, j \in \{1, 2, \ldots, k\}$ and $\rho < 0$. Then as $k \to \infty$,

$$\mathfrak{B}_k(\theta, C_j) \to O\left(1/k^{1+\tau+\rho}\right)$$

where

$$\mathfrak{B}_k(\theta, C_j) = \frac{\sum_{j=1}^{k} \overline{W_j} \left(C_j - \sum_{j=1}^{k} \overline{W_j} C_j \right)}{\sum_{j=1}^{k} \overline{W_j} C_j^2 - \left(\sum_{j=1}^{k} \overline{W_j} C_j \right)^2} \tag{16}$$

from 4, $0 < \tau \leq 0.1$.

Proof of Lemma 2. We notice that

$$-\frac{k^{\rho}}{|k^{1+\tau+\rho} + n^{\rho}|} \leq \mathfrak{B}_k(\theta, C_j) \leq \frac{k^{\rho} C_j}{S_2(\rho, \theta)\, |k^{1+\tau+\rho} + n^{\rho}|}$$

Therefore, we have $\mathfrak{B}_k(\theta, C_j) \to O\left(1/k^{1+\tau}\right)$, $0 < \tau \leq 0.1$, as $k \to \infty$, which completes the proof of Lemma 2.

Lemma 3. Let $Z_1, Z_2, Z_3, \ldots$ be independent random variables, $a_i \in \mathbb{R}, i \geq 1$ and for every $p > 0$ $\mathbb{E}\left|Z_i^P\right| < \infty$. We write

$$\begin{aligned} Y &= a_1 Z_1 + a_2 Z_2 + \ldots + a_m Z_m \\ &= \sum_{j=1}^{m} a_i Z_i \end{aligned}$$

then the moment generating function (m.g.f) of Y is described as

$$M_Y(t) = \prod_{i=1}^{m} M_{Z_i}(a_i t)$$

where M_{Z_1} is the m.g.f of Z_i.

Proof of Lemma 3. Keep in mind that $Z_1, Z_2, Z_3, \ldots$ be independent random variables and the moment generating function of Y is described as

$$\begin{aligned} M_Y(t) &= \mathbb{E}\left(e^{Yt}\right) \\ &= \mathbb{E}\left(e^{t(a_1 Z_1 + a_2 Z_2 + \ldots + a_m Z_m)}\right) \\ &= \mathbb{E}\left(e^{a_1 Z_1 t} e^{a_2 Z_2 t} \ldots e^{a_m Z_m t}\right) \\ &= \mathbb{E}\left(e^{a_1 Z_1 t}\right) \mathbb{E}\left(e^{a_2 Z_2 t}\right) \ldots \mathbb{E}\left(e^{a_m Z_m t}\right) \\ &= \prod_{i=1}^{m} M_{Z_1}(a_i t). \end{aligned}$$

Lemma 4. Let $Z_1, Z_2, Z_3, \ldots$ be independent random variables from an exponential distribution with mean $\mu_i < \infty$, for all i. Then for any $\epsilon > 0$

$$\lim_{k \to \infty} \mathbb{P}\left(\sqrt{k} b_{n,k} \geq \epsilon\right) = 0$$

where $b_{n,k}$ is defined by Eqn. (4) .

Proof of Lemma 4. The proof requires the use of large deviation principles (LDP). From Eqn.(4) and Eqn.(16), $\hat{b}_{n,k}(\theta,\hat{\rho}) = \sum_{j=1}^{k} \mathfrak{B}(\theta, C_j) Z_j$. Given $\{\hat{\rho} = \rho\}$, Z_j is exponentially distributed with mean

$$\mu_j(\theta,\rho) = \mathbb{E}[Z_j|\hat{\rho} = \rho, \theta] = \gamma + \hat{b}_{n,k}(\hat{\rho},\theta)C_j(\hat{\rho}).$$

Hence, we have

$$\begin{aligned}
M_{Z_k,\theta,\rho}(t) &= \mathbb{E}[e^{Z_k t}|\theta, \hat{\rho} = \rho] \\
&= \int_0^\infty \frac{e^{z_k t} e^{-\dfrac{z_k}{\mu_k(\theta,\rho)}}}{\mu_k(\theta,\rho)} dz_k \\
&= \frac{1}{\mu_k(\theta,\rho)} \int_0^\infty e^{-z_k\left(\dfrac{1}{\mu_k(\theta,\rho)} - t\right)} dz_k \\
&= \frac{1}{\mu_k(\theta,\rho)} \int_0^\infty e^{-z_k\left(\dfrac{1}{\mu_k(\theta,\rho)} - t\right)} dz_k \\
&= \frac{1}{\mu_k(\theta,\rho)} \left(\frac{\mu_k(\theta,\rho)}{1-\mu_k(\theta,\rho)t}\right)
\end{aligned}$$

$$M_{Z_k,\theta,\rho}(t) = \frac{1}{1-\mu_k(\theta,\rho)t} \tag{17}$$

Using Lemma 2, and Eqn. (17), the moment generating function of $b_{n,k}$ is

$$\begin{aligned}
&= \left\{ \prod_{j=1}^{k} \int_0^\infty e^{z_k t} e^{-\dfrac{z_k}{\mu_k(\theta,,)\mathfrak{B}\,(\theta,C_j)}} dz_k \right\} \\
&= \left\{ \prod_{j=1}^{k} \int_0^\infty \rho^{-z_k\left(\dfrac{1}{\mu_k(\theta,\rho)\mathfrak{B}(\theta,C_j)} - t\right)} dz_k \right\} \\
&= \left\{ \prod_{j=1}^{k} \frac{1}{\mu_k(\theta,\rho)\mathfrak{B}\,(\theta,C_j)} \left(\frac{\mu_k(\theta,\rho)\mathfrak{B}\,(\theta,C_j)}{1-\mu_k(\theta,\rho)\mathfrak{B}\,(\theta,C_j)\,t}\right) \right\}
\end{aligned}$$

$$M_{\hat{b}_{n,k}(\rho)}(t) = \left\{ \prod_{j=1}^{k} \left(\frac{1}{1 - \mu_j(\theta,\rho)\mathfrak{B}(\theta, C_j)t} \right) \right\}.$$

Consequently,

$$M_{\hat{b}_{n,k}(\rho)}(kt) = \left\{ \prod_{j=1}^{k} \left(\frac{1}{1 - k\mu_j(\theta,\rho)\mathfrak{B}(\theta, C_j)t} \right) \right\}.$$

It should be noted that, from Lemma 2, we have $k\mu_k\mathfrak{B}_k(\theta, C_j) \to O\left(1/k^{1+\tau}\right)$ as $k \to \infty$ and hence, $\exists\ \delta > 0$ such that $-\delta \leq k\mu_k\mathfrak{B}_k(\theta, C_j) \leq \delta$. This implies that

$$\begin{aligned}
&\prod_{j=1}^{k} \left(\frac{1}{1+\delta} \right) \leq M_{b_{n,k}}(kt) \leq \prod_{j=1}^{k} \left(\frac{1}{1-\delta} \right) \\
&\left(\frac{1}{1+\delta} \right)^k \leq M_{b_{n,k}}(kt) \leq \left(\frac{1}{1-\delta} \right)^k \\
&\left(1 - \delta + \delta^2 - \delta^3 + \ldots\right)^k \leq M_{b_{n,k}}(kt) \leq \left(1 + \delta + \delta^2 + \delta^3 + \ldots\right)^k \\
&k \log\left(1 - \delta + \delta^2 - \delta^3 + \ldots\right) \leq \log M_{b_{n,k}}(kt) \leq k \log\left(1 + \delta + \delta^2 + \delta^3 + \ldots\right) \\
&\log\left(1 - \delta + \delta^2 - \delta^3 + \ldots\right) \leq \lim_{k\to\infty} \frac{1}{k} \log M_{b_{n,k}}(kt) \leq \log\left(1 + \delta + \delta^2 + \delta^3 + \ldots\right)
\end{aligned}$$

Now letting $\delta \longrightarrow 0$ and applying the Squeeze Theorem, we get

$$\lim_{k\to\infty} \frac{1}{k} \log M_{b_{n,k}}(kt) = 0.$$

Hence, the logarithmic moment generating function converges to 0 with speed k and by the Gärtner Ellis Theorem, $b_{n,k}$ follows a Large Deviation Principle (LDP) with speed k and a rate function $I(x)$ defined as

$$\begin{aligned}
I(x) &= \sup_{\eta\in\mathbb{R}}\{\eta x - 0\} = \sup_{\eta\in\mathbb{R}}\{\eta x\} \\
&= \begin{cases} 0 \text{ if } x = 0 \\ \infty \text{ if } x \neq 0 \end{cases}
\end{aligned}$$

where $x \in [0, \infty)$ and $\eta > 0$. If $b_{n,k}$ obeys LDP with a rate function $I(x)$, then $\exists \epsilon > 0$ such that

$$\begin{aligned}
\mathbb{P}\left(\sqrt{k} b_{k,n} > \epsilon\right) &= \mathbb{P}\left(b_{n,k} > \frac{\epsilon}{\sqrt{k}} \middle| (\hat{\theta}, \hat{\rho}) = (\theta, \rho)\right) \mathbb{P}\left[(\hat{\theta}, \hat{\rho}) = (\theta, \rho)\right] \\
&= \mathbb{P}\left(b_{n,k} > \mathit{\Gamma}_k\right) \mathbb{P}\left[(\hat{\theta}, \hat{\rho}) = (\theta, \rho)\right] \leq e^{-kI(x)+o(k)},
\end{aligned}$$

where $\Gamma_k = \epsilon/k$. The typical behaviour of the rate function is when $x \neq 0$, therefore

$$\lim_{k\to\infty} \mathbb{P}\left(\sqrt{k}b_{n,k} > \epsilon\right) \leq 0.$$

Thus, $\sqrt{k}b_{n,k} \to_p 0$ as $k \to \infty$ and the proof is now complete.

Lemma 5 Suppose that $Z_1, Z_2, ...$ are independent random variables such that $\mathbb{E}(Z_k) = \hat{\mu}_k$ and $Var(Z_k) = \sigma_k^2 < \infty$, then there exists $\delta > 0$ such that

$$\lim_{k\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{j=1}^{k} \mathbb{E}\left(|Z_j - \hat{\mu}_j|^{2+\delta}\right) = 0,$$

where $z_k = \mathbb{E}_\theta[\tilde{W}_k(\theta_k)]T_k$.

Proof of Lemma 5 We observe that $z_1, z_2, z_3, ...$ are independent but not identical distributed random variables.

Let $f(z_k) = \dfrac{1}{\mu_k} e^{-\dfrac{z_k}{\mu_k}}$ be the probability density function of z_k given $(\theta, \hat{\rho})$ and observe that we

$$\begin{aligned}
\mathbb{E}\left(|z_k - \mu_k|^{2+\delta}\middle|\theta, \hat{\rho} = \rho\right) &= \int_0^\infty |z_k - \mu_k|^{2+\delta} f(z_k)\, dz_k \\
&= -\int_0^{\mu_k} (z_k - \mu_k)^{2+\delta} f(z_k)\, dz_k + \int_{\mu_k}^\infty (Z_k - \mu_k)^{2+\delta} f(z_k)\, dz_k \\
&= -\int_0^{\mu_k} (z_k - \mu_k)^{2+6} \frac{1}{\mu_k} e^{-\dfrac{z_k}{\mu_k}} dz_k + \int_{\mu_k}^\infty (z_k - \mu_k)^{2+\delta} \frac{1}{\mu_k} e^{-\dfrac{z_k}{\mu_k}} dz_k \\
&= \int_0^{-\mu_k} \xi^{2+\delta} \frac{1}{\mu_k} e^{-\dfrac{(\xi + \mu_k)}{\mu_k}} d\xi + \int_0^\infty \xi^{2+\delta} \frac{1}{\mu_k} e^{-\dfrac{(\xi + \mu_k)}{\mu_k}} d\xi \\
&= \frac{e^{-1}}{\mu_k} \int_0^{-\mu_k} \xi^{2+\delta} e^{-\dfrac{u}{\mu_k}} d\xi + \frac{e^{-1}}{\mu_k} \int_0^\infty \xi^{2+\delta} e^{-\dfrac{\xi}{\mu_k}} d\xi. \\
&= \frac{e^{-1}}{\mu_k} \left\{ \int_0^{-\mu_k} \xi^{2+\delta} e^{-\dfrac{\xi}{\mu_k}} d\xi + \int_0^\infty \xi^{2+\delta} e^{-\dfrac{\xi}{\mu_k}} d\xi \right\} \\
&= \frac{e^{-1}}{\mu_k} \left\{ \int_0^{-\mu_k} \xi^{2+\delta} \left(1 - \frac{\xi}{\mu_k}\right) d\xi + \frac{(2+\delta)!}{\left(\frac{1}{\mu_k}\right)^{3+\delta}} \right\} \{1 + O(1)\} \\
&= \frac{e^{-1}}{\mu_k} \left\{ \int_0^{-\mu_k} \xi^{2+\delta} d\xi - \int_0^{-\mu_k} \frac{\xi^{3+\delta}}{\mu_k} d\xi + (\mu_k)^{3+\delta} (2+\delta)! \right\} \{1 + O(1)\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-1}}{\mu_k}\left\{\left.\frac{\xi^{3+\delta}}{3+\delta}\right|_0^{-\mu_k} - \left.\frac{\xi^{4+\delta}}{(4+\delta)\mu_k}\right|_0^{-\mu_k} + (\mu_k)^{3+\delta}(2+\delta)!\right\}\{1+O(1)\} \\
&= \frac{e^{-1}}{\mu_k}\left\{\frac{(-\mu_k)^{3+\delta}}{(3+\delta)} - \frac{(-\mu_k)^{4+\delta}}{\mu_k(4+\delta)} + (\mu_k)^{3+\delta}(2+\delta)!\right\}\{1+O(1)\} \\
&= \frac{e^{-1}}{\mu_k}\left\{\frac{(-\mu_k)^{3+\delta}}{3+\delta} - \frac{(-\mu_k)(-\mu_k)^{3+\delta}}{\mu_k(4+\delta)} + (\mu_k)^{3+\delta}(2+\delta)!\right\}\{1+O(1)\} \\
&= \frac{e^{-1}}{\mu_k}\left\{\frac{(-\mu_k)^{3+\delta}}{3+\delta} + \frac{(-\mu_k)^{3+\delta}}{4+\delta} + (\mu_k)^{3+\delta}(2+\delta)!\right\}\{1+O(1)\} \\
&= \frac{e^{-1}}{\mu_k}\left\{(-\mu_k)^{3+\delta}\left(\frac{4+\delta+3+\delta}{(3+\delta)(4+\delta)}\right) + (\mu_k)^{3+\delta}(2+\delta)!\right\}\{1+O(1)\} \\
&= \frac{e^{-1}}{\mu_k}\left\{\frac{(-\mu_k)^{3+\delta}(7+2\delta)}{(3+\delta)(4+6)} + (\mu_k)^{3+\delta}(2+\delta)!\right\}\{1+O(1)\} \\
&= \frac{e^{-1}}{\mu_k}\left\{\frac{(-\mu_k)^{3+\delta}(7+2\delta)}{(3+\delta)(4+\delta)} + \mu_k^{3+\delta}(2+\delta)!\right\}\{1+O(1)\} \\
&= \frac{e^{-1}\mu_k}{\mu_k}\left\{(-1)^{3+\delta}\frac{(\mu_k)^{2+\delta}(7+2\delta)}{(3+\delta)(4+\delta)} + \mu_k^{2+\delta}(2+\delta)!\right\}\{1+O(1)\} \\
&= \mu_k^{2+\delta}\eta(\delta)\Big\{1+O(1)\Big\}
\end{aligned}
$$

where $\eta(\delta) = \frac{1}{e}\Big\{\frac{(-1)^{3+\delta}(7+2\delta)}{(3+\delta)(4+\delta)} + (2+\delta)!\Big\} < \infty$ for $0 < \delta < \infty$. As a result, we have

$$
\mathbb{E}\left\{|Z_k-\mu_k|^{2+\delta}\right\} = \mathbb{E}\left\{\mathbb{E}\left\{|Z_k-\mu_k|^{2+\delta}\big|(\theta,\hat{\rho})\right\}\right\} \le \eta(\delta)\{1+O(1)\}\mu_k^{2+\delta}.
$$

$$
\begin{aligned}
&= \eta(\delta) \sum_{j=1}^{k} \mu_k^{2+\delta} \{1 + O(1)\} \\
&= \eta(s) \sum_{j=1}^{k} \left(\gamma e^{D\left(\frac{j}{n+1}\right)^{-\rho}} \right)^{2+\delta} \{1 + O(1)\} \\
&= \eta(\delta)\gamma^{2+\delta} \sum_{j=1}^{k} \left(e^{(2+\delta)D\left(\frac{j}{n+1}\right)^{-\rho}} \right) \{1 + O(1)\} \\
&= \eta(\delta)\gamma^{2+\delta} \sum_{j=1}^{k} \sum_{r=0}^{\infty} \frac{\left((2+\delta)D \left(\frac{j}{n+1}\right)^{-\rho} \right)^r \{1 + O(1)\}}{r!} \\
&= \eta(\delta)\gamma^{2+\delta} \sum_{1=1}^{k} \left(1 + (2+\delta)D \left(\frac{j}{m+n} \right)^{-\rho} \right) \{1 + o(1)\} \\
&= \eta(\delta)\gamma^{2+\delta} \left(k + (2+\delta)D \sum_{k=1}^{k} \left(\frac{j}{n+1} \right)^{-\rho} \right) \{1 + o(1)\} \qquad \text{since } 1 \leq j \leq k \text{ and } p < 0 \text{ we have,} \\
&= \eta(k)\gamma^{2+\delta}(k + (2+\delta)Dk)\{1 + O(1)\} \\
&= \eta(\delta)k\gamma^{2+\delta}(1 + (2+\delta)D)\{1 + O(1)\}
\end{aligned}
$$

We define

$$
V_k = Z_k - \mu_k, 1 \leq j \leq k
$$

$$
\begin{aligned}
S_n &= \sqrt{\operatorname{Var}\left(\sum_{j=1}^{k} V_k \right)} \\
&= \sqrt{\operatorname{Var}\left(\sum_{j=1}^{k} (Z_k - \mu_k) \right)} \\
&= \sqrt{\sum_{j=1}^{k} \operatorname{Var}(Z_k)} \\
&= \gamma\sqrt{k}
\end{aligned}
$$

This ends te proof.

Now

$$
\begin{aligned}
\lim_{k\to\infty}\frac{1}{S_n^{2+\delta}}\sum_{j=1}^{k}\mathbb{E}\left(|Z_k-M_k|^{2+\delta}\right) &= \lim_{k\to\infty}\frac{\eta(\delta)k(\gamma+O(1/k^{\tau})+O(1))^{2+\delta}(1+(2+\delta)D)\{1+O(1)\}}{(\gamma\sqrt{k})^{2+\delta}} \\
&= \lim_{k\to\infty}\frac{\eta(\delta)k\gamma^{2+\delta}(1+(2+\delta)D)\{(1+0(1)\}}{kk^{\delta}\gamma^{2+\delta}} \\
&= \lim_{k\to\infty}\frac{\eta(\delta)(1+(2+\delta)D)\{1+0(1)\}}{\sqrt{k^{\delta}}} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma &= \sum_{j=1}^{k}\overline{W}_j\left(1+\frac{s_1}{s_2}(s_1-c_j)\right)Z_j \\
\operatorname{Var}(\gamma) &= \frac{\phi^2(k)}{k^2}\sum_{j=1}^{k}\left(1+\frac{s_1}{s_2}(s_1-c_j)\right)^2\operatorname{Var}(W_jZ_j)
\end{aligned}
$$

Simplifying $\operatorname{Var}(W_j, Z_j)$, $-0(-1)\leq F(x)\leq 0(1)$, it follows

$$
\left(1-0(1)\frac{j}{k+1}\right)\leq W_j\leq\left(1+0(1)\frac{j}{k+1}\right)
$$

$$
\begin{aligned}
Y_k &= \overline{W}_kZ_k \\
\sum_{1=1}^{k}Y_k &= \frac{\phi(k)}{k}\sum_{j=1}^{k}\left(Z_k+o(\theta)^{\Delta k}\right) \quad \theta\sim U(0,1) \\
\sum_{j=1}^{k}Y_j &= \frac{\phi(k)}{k}\sum_{1=1}^{k}Z_j+o(\theta)^{\Delta k} \\
\hat{\gamma} &\stackrel{d}{=} \phi(k)\hat{\gamma}^H
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{k}(\hat{\gamma}-\gamma) &\stackrel{d}{=} \sqrt{k}\left(\phi(k\hat{\gamma}^H-\gamma\right)-\phi(k)\sqrt{k}\gamma+\phi(k)\sqrt{k}\gamma \\
\sqrt{k}(\hat{\gamma}-\gamma) &\stackrel{d}{=} \sqrt{k}\phi(k)\hat{\gamma}^H-\sqrt{k}\gamma-\phi(k)\sqrt{k}\gamma+\phi(k)\sqrt{k}\gamma \\
\sqrt{k}(\hat{\gamma}-\gamma) &\stackrel{d}{=} \sqrt{k}\phi(k)\left(\hat{\gamma}^H-\gamma\right)+\sqrt{k}\gamma(\phi(k)-1) \\
\sqrt{k}(\hat{\gamma}-\gamma) &\stackrel{d}{=} \sqrt{k}\phi(k)\left(\hat{\gamma}^H-\gamma\right)+o(1/k)
\end{aligned}
$$

$\left(1-o(1)\frac{j}{k+1}\right)^2\mathbb{E}\left(Z_j^2\right)-\left(1+o(1)\frac{j}{k+1}\right)^2(\mathbb{E}(Z_j))^2\leq\operatorname{Var}(W_jZ_j)\leq\left(1+o(1)\frac{j}{k+1}\right)^2\mathbb{E}(Z_j^2)-\left(1-o(1)\frac{j}{k+1}\right)^2(\mathbb{E}(Z_j))^2$

$$
\begin{aligned}
\mathbb{E}\left(Z_j^2\right) - \left(\mathbb{E}\left(Z_j\right)\right)^2 \leq \operatorname{Var}\left(W_j Z_j\right) &\leq \mathbb{E}\left(Z_j^2\right) - \left(E\left(Z_j\right)\right)^2 \\
\operatorname{Var}\left(Z_j\right) \leq \operatorname{Var}\left(W_j Z_j\right) &\leq \operatorname{Var}\left(Z_j\right) \\
\operatorname{Var}\left(W_j Z_j\right) &\overset{a \cdot s}{\sim} \operatorname{Var}\left(Z_j\right)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\operatorname{Var}(\gamma) &= \frac{2\phi^2(k)}{k^2} \sum_{j=1}^{k} \left(1 + \frac{s_1}{s_2}\left(s_1 - c_j\right)\right)^2 \gamma^2 \\
&= \gamma^2 \frac{2\phi^2(k)}{k} \int_0^1 \left(1 + 2\frac{s_1}{s_2}\left(s_1 - u^{-\rho}\right) + \frac{s_1^2}{s_2^2}\left(s_1^2 - 2s_1 u^{-\rho} + u^{-2\rho}\right)\right) du \\
&= \frac{2\gamma^2}{k}\phi^2(k)\left(1 + \frac{2s_1}{s_2}\left(s_1 - \frac{1}{1-\rho}\right) + \frac{s_1^2}{s_2^2}\left(s_1^2 - \frac{2}{(1-\rho)^2} + \frac{1}{1-2\rho}\right)\right) \\
&= \frac{2\gamma^2}{k}\phi^2(k)\left(1 - \frac{s_1^2}{s_2^2}\left(s_2\right)\right) \\
&= \frac{2\gamma^2}{k}\frac{\phi^2(k)(\rho^2 + 2\rho - 1)}{\rho^2} \\
&= \frac{\gamma^2}{k}\frac{(\rho^2 + 2\rho - 1)}{\rho^2}, \quad \text{at large } k.
\end{aligned}
$$

```
data(soa)

# Hill estimator
H <- Hill(soa$size, plot=FALSE)
# Moment estimator
M <- Moment(soa$size)
# Generalised Hill estimator
gH <- genHill(soa$size, gamma=H$gamma)

# Plot estimates
plot(H$k[1:5000], M$gamma[1:5000], xlab="k", ylab=expression(gamma),
 type="l", ylim=c(0.2,0.5))
lines(H$k[1:5000], gH$gamma[1:5000], lty=2)
legend("topright", c("Moment", "Generalised Hill"), lty=1:2)
```