LESSON-11

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BINOMIAL THEOREM

1. BINOMIAL EXPRESSION

Any algebraic expression consisting of only two terms is known as binomial expression. The terms may consist of variables x, y etc. or constants or their mixed combinations. For example: $2x + 3y$, $4xy + 5$ etc.

2. BINOMIAL THEOREM FOR POSITIVE INDEX

Binomial theorem gives a formula for the expansion of a binomial expression raised to any positive integral power.

In general for a positive integer n

$$(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \ldots.. + {}^nC_n\, x^0 y^n$$, where $${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

for $r = 0, 1, 2, \ldots\ldots., n$ is called binomial coefficient.

Example: **Expand** $\left(\boldsymbol{x - \frac{1}{x}}\right)^{\mathbf{6}}$

Solution:

$$\left(x - \frac{1}{x}\right)^6 = {}^6C_0 x^6 + {}^6C_1 x^5\left(\frac{-1}{x}\right) + {}^6C_2 x^4\left(\frac{-1}{x}\right)^2 + {}^6C_3 x^3\left(\frac{-1}{x}\right)^3 + {}^6C_4 x^2\left(\frac{-1}{x}\right)^4$$

$$+ {}^6C_5 x\left(\frac{-1}{x}\right)^5 + {}^6C_6 x^0\left(\frac{-1}{x}\right)^6$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

3. GENERAL TERM IN THE BINOMIAL EXPANSION

The general term in the expansion of $(x + y)^n$ is $(r + 1)^{th}$ term, given by $t_{r+1} = {}^nC_r\, x^{n-r}\, y^r$ where $r = 0, 1, 2 \ldots\ldots\ldots n$.

- Every term in the expansion is of nth degree in variables x and y.
- The total number of terms in the expansion is $n + 1$.
- Binomial expansion can also be expressed as $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$.

Example: **Find the 11[th] term in the expansion of** $\left(3x - \dfrac{1}{x\sqrt{3}}\right)^{20}$.

Solution: The general term $= t_{r+1} = (-1)^r \; {}^{20}C_r \; (3x)^{20-r} \left(\dfrac{1}{x\sqrt{3}}\right)^r$

For the 11th term, we must take $r = 10$

$$\therefore t_{11} = t_{10+1} = (-1)^{10} \; {}^{20}C_{10} \; (3x)^{20-10} \left(\frac{1}{x\sqrt{3}}\right)^{10}$$

$$= {}^{20}C_{10} \; 3^{10} \; x^{10} \; \frac{1}{x^{10} \; (\sqrt{3})^{10}} = {}^{20}C_{10} \; (\sqrt{3})^{10} = {}^{20}C_{10} \; 3^5$$

Example: **The 2nd, 3rd and 4th terms of $(x + y)^n$ are 240, 720 and 1080 respectively. Find x, y and n.**

Solution:

$$t_2 = {}^nC_1 \; x^{n-1} \; y = 240$$

$$t_3 = {}^nC_2 \; x^{n-2} \; y^2 = 720$$

$$t_4 = {}^nC_3 \; x^{n-3} \; y^3 = 1080$$

$$\frac{{}^nC_2 \; x^{n-2} \; y^2}{{}^nC_1 \; x^{n-1} \; y} = \frac{720}{240}; \text{ i.e., } \frac{n-1}{2}\frac{y}{x} = 3 \qquad \ldots \text{(i)}$$

$$\frac{{}^nC_3 \; x^{n-3} \; y^3}{{}^nC_2 \; x^{n-2} \; y^2} = \frac{1080}{720}; \text{ i.e., } \frac{n-2}{3}\frac{y}{x} = \frac{3}{2} \qquad \ldots \text{(ii)}$$

Dividing equation (i) by equation (ii) we have

$$\frac{3(n-1)}{2(n-2)} = 2 \quad \Rightarrow n = 5$$

Using $n = 5$ in (i) we have $\dfrac{y}{x} = \dfrac{3}{2}$

Substituting in t_2 we have ${}^5C_1 \; x^4 \; \dfrac{3}{2} x = 240$

$\therefore$ $x^5 = 32$ $\Rightarrow$ $x = 2$ and $y = 3$

$\therefore$ $x = 2$, $y = 3$ and $n = 5$.

Example: **Find the term independent of x in** $\left(\dfrac{3}{2}x^2 - \dfrac{1}{3x}\right)^9$

Solution: The general term $= {}^9C_r \left(\frac{3x^2}{2}\right)^{9-r} \left(\frac{-1}{3x}\right)^r$

$$= (-1)^r \; {}^9C_r \; \frac{3^{9-2r}}{2^{9-r}} \cdot x^{18-3r}$$

The term independent of *x*, (or the constant term) corresponds to x^{18-3r} being

x^0 or $18 - 3r = 0 \Rightarrow r = 6$

$\therefore$ the term independent of *x* is the 7th term and its value is

$$(-1)^6 \; {}^9C_6 \frac{3^{9-12}}{2^{9-6}} = {}^9C_3 \frac{3^{-3}}{2^3} = \frac{9.8.7}{3.2.1} \cdot \frac{1}{(6)^3} = \frac{7}{18}$$

4. MIDDLE TERMS OF THE EXPANSION

In the binomial expansion of $(x + y)^n$

1. WHEN *n* IS ODD

There are $(n + 1)$ i.e. even terms in the expansion and hence two middle terms are given by

$$t_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}} \, x^{\frac{n+1}{2}} \, y^{\frac{n-1}{2}} \quad \text{for} \quad r = \frac{n-1}{2}$$

and

$$t_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}} \, x^{\frac{n-1}{2}} \, y^{\frac{n+1}{2}} \quad \text{for} \quad r = \frac{n+1}{2}$$

2. WHEN *n* IS EVEN

There are odd terms in the expansion and hence only one middle term is given by

$$t_{\frac{n}{2}+1} = {}^nC_{n/2} \, x^{n/2} \, y^{n/2} \quad \text{for} \quad r = \frac{n}{2}$$

Example: **Find the middle term in the expression of $(1 - 2x + x^2)^n$.**

Solution: $(1 - 2x + x^2)^n = [(1 - x)^2]^n = (1 - x)^{2n}$

Here $2n$ is even integer, therefore, $\left(\frac{2n}{2}+1\right)^{\text{th}}$ i.e. $(n + 1)^{\text{th}}$ term will be the middle term.

Now $(n + 1)^{\text{th}}$ term in $(1 - x)^{2n} = {}^{2n}C_n \, (1)^{2n-n} \, (-x)^n$

$$= {}^{2n}C_n (-x)^n = \frac{(2n)!}{n!\,n!}(-x)^n.$$