LESSON-7

QUADRATIC EQUATIONS

1. QUADRATIC EQUATION

An equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$, a, b, c are real numbers, is called a quadratic equation.

If α is a root of this equation, then α satisfies this equation and hence $a\alpha^2 + b\alpha + c = 0$.

The quantity $D = b^2 - 4ac$ is called the discriminant of quadratic equation

$$ax^2 + bx + c = 0 \quad (a \neq 0). \qquad \ldots(1)$$

The roots of the quadratic equation, generally denoted by α and β are

$$\alpha = \frac{-b+\sqrt{D}}{2a} \quad \text{and} \quad \beta = \frac{-b-\sqrt{D}}{2a}.$$

1. NATURE OF THE ROOTS

1. Suppose $a, b, c \in R$ and $a \neq 0$. Then the following hold good:
   (a) The equation (1) has real and distinct roots if and only if $D > 0$.
   (b) The equation (1) has real and equal roots if and only if $D = 0$.
   (c) The equation (1) has complex roots with non-zero imaginary parts if and only if $D < 0$.
   (d) $p + iq$ $(p, q \in R, q \neq 0)$ is a root of (1) if and only if $p - iq$ is a root of (1).
2. If $a, b, c \in Q$ and D is a perfect square of a rational number, then (1) has rational roots.
3. If $a, b, c \in Q$ and $p + \sqrt{q}$ $(p, q \in Q)$ is an irrational root of (1) then $p - \sqrt{q}$ is also a root of (1).
4. If (1) is satisfied by more than two complex numbers, then (1) becomes an identity, that is $a = b = c = 0$.

Example: **Let $a > 0$, $b > 0$ and $c > 0$. Then prove that both the roots of the equation $ax^2 + bx + c = 0$ have negative real parts.**

Solution: We have $D = b^2 - 4ac$. If $D \geq 0$, then the roots of the equation are given by

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

As $D = b^2 - 4ac < b^2$ (∵ $a > 0$, $c > 0$), it follows that the roots of the quadratic equation are negative. In case $D < 0$, then the roots of the equation are given by

$$x = \frac{-b \pm i\sqrt{-D}}{2a}$$

which have negative real parts.

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2. RELATION BETWEEN ROOTS AND COEFFICIENTS AND SYMMETRIC FUNCTIONS OF ROOTS

Let α and β be the roots of the equation $ax^2 + bx + c = 0$ then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Location of real roots on the number line

1. When D 0

   (a) if both roots are positive, then $\alpha + \beta = \frac{-b}{a} > 0$ and $\alpha\beta = \frac{c}{a} > 0$

   (b) if both roots are negative, then $\alpha + \beta = \frac{-b}{a} < 0$ and $\alpha\beta = \frac{c}{a} > 0$

2. When $D > 0$

   (a) if one root say α is positive and the other root β is negative, then $\alpha\beta = \frac{c}{a} < 0$

3. When D 0

   (a) if both roots are greater than k, then $\alpha - k > 0$ and $\beta - k > 0$

   $\therefore\ \alpha - k + \beta - k > 0$ and $(\alpha - k)(\beta - k) > 0$

   (b) if both roots are less than k, then $\alpha - k < 0$ and $\beta - k < 0$

   $\alpha + \beta - 2k < 0$ and $(\alpha - k)(\beta - k) > 0$

   (c) If both roots lie between k and l, $k < l$ then $\alpha - k > 0$, $\beta - k > 0$, $\alpha - l < 0$, $\beta - l < 0$

   Therefore $\alpha - k + \beta - k > 0$, $\alpha - l + \beta - l < 0$ and $(\alpha - k)(\beta - k) > 0$ and $(\alpha - l)(\beta - l) > 0$

   (d) If $\alpha < k$ and $\beta > k$, then $\alpha - k < 0$ and $\beta - k > 0$

$\therefore\ (\alpha-k)(\beta-k)<0$

(e) If $\alpha < k$ and $\beta > l$, $k < l$

then $\alpha-k<0,\ \beta-k>0,\ \alpha-l<0,\ \beta-l>0$

Symmetric functions of the roots are algebraic expressions, which are symmetric in α and β.

For examples: $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\dfrac{3abc-b^3}{a^3}$

Example: **If α, β are the roots of the equation $6x^2-6x+1=0$, then prove that**

$$\frac{1}{2}(a+b\alpha+c\alpha^2+d\alpha^3)+\frac{1}{2}(a+b\beta+c\beta^2+d\beta^3)=\frac{a}{1}+\frac{b}{2}+\frac{c}{3}+\frac{d}{4}$$

Solution:

$\alpha+\beta=1$ and $\alpha\beta=\dfrac{1}{6}$

Now $\dfrac{1}{2}(a+b\alpha+c\alpha^2+d\alpha^3)+\dfrac{1}{2}(a+b\beta+c\beta^2+d\beta^3)$

$$=\frac{1}{2}\left[(a+a)+b(\alpha+\beta)+c(\alpha^2+\beta^2)+d(\alpha^3+\beta^3)\right]$$

$$=\frac{1}{2}\left[2a+b+c\left\{(\alpha+\beta)^2-2\alpha\beta\right\}+d\left\{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\right\}\right]$$

$$=\frac{1}{2}\left[2a+b+c\left\{1-\frac{1}{3}\right\}+d\left\{1-\frac{1}{2}\right\}\right]$$

$$=a+\frac{b}{2}+\frac{c}{3}+\frac{d}{4}$$