LESSON-6

COMPLEX NUMBERS

1. INTRODUCTION

Let us take the quadratic equation $x^2 - 2x + 10 = 0$. The formal solution of this equation is $\frac{2 \pm \sqrt{4-40}}{2}$ i.e., $1 \pm 3\sqrt{-1}$, which is not meaningful in the set of real numbers.

It is therefore, the symbol *i*, is thought of to possess the following properties:

(i) It combines with itself and with real numbers satisfying the laws of algebra.

(ii) Whenever we come across –1 we may substitute i^2.

So the roots of the equation discussed earlier may be taken as 1 + 3*i* , 1 – 3*i*.

It is taken that 1 is real part and 3(or – 3) is the imaginary part of this complex number 1 + 3*i* or 1 – 3i respectively.

Example: **If $x = -5 + 2\sqrt{-4}$ find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.**

Solution: $x = -5 + 4i \ (i = \sqrt{-1})$

$x + 5 = 4i$

Squaring, $x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now $x^4 + 9x^3 + 35x^2 - x + 4 = (x^2 + 10x + 41)\ (x^2 - x + 4) - 160$ and $x^2 + 10x + 41 = 0$

Hence given expression = 0 − 160 = − 160

2. COMPLEX NUMBERS

A complex number, represented by an expression of the form *x* + *iy* (*x*, *y* are real), is taken to be an ordered pair (*x*, *y*) of two real numbers, combined to form a complex number and an algebra is defined on the set of such numbers, represented by an ordered pair (*x*, *y*) to satisfy the following:

(addition) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

(subtraction) $(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$

(multiplication) $(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

(division) $(x_1, y_1) \div (x_2, y_2) = \left(\dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \dfrac{x_2y_1 + x_1y_2}{x_2^2 + y_2^2}\right)$

The representation of a complex number in the form (x, y) has a uniqueness property; and for a complex number it is not possible to have two different ordered pairs form of representation.

Example: **Find the sum and product of the two complex numbers $Z_1 = 2 + 3i$ and $Z_2 = -1 + 5i$**

Solution: $Z_1 + Z_2 = 2 + 3i + (-1 + 5i) = 2 - 1 + 8i = 1 + 8\mathrm{i}$

$$Z_1Z_2 = (2 + 3i)(-1 + 5i) = -2 + 15i^2 - 3i + 10i = -17 + 7\mathrm{i} \quad (i^2 = -1)$$

Based on the above discussion we are listing a few points:

1. If $z = a + ib$, then real part of $z = \text{Re}(z) = a$ and Imaginary part of $z = Im(z) = b$.
2. If Re $(z) = 0$, the complex number is purely imaginary.
3. If Im $(z) = 0$, the complex number is real.
4. The complex number $0 = 0 + 0i$ is both purely imaginary and real.
5. Two complex numbers are equal if and only if their real parts and imaginary parts are separately equal i.e. $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.

Example: **Express $\dfrac{1}{(1 - \cos\theta + i\sin\theta)}$ in the form $a + ib$.**

Solution:
$$\frac{1}{(1 - \cos\theta + i\sin\theta)} = \frac{(1 - \cos\theta) - i\sin\theta}{(1 - \cos\theta + i\sin\theta)(1 - \cos\theta - i\sin\theta)}$$

$$= \{(1 - \cos\theta) - i\sin\theta\}/\{(1 - \cos\theta)^2 + \sin^2\theta\} = \frac{(1 - \cos\theta) - i\sin\theta}{2 - 2\cos\theta}$$

$$= \frac{1 - \cos\theta}{2(1 - \cos\theta)} - \frac{i \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{1}{2} - i \cdot \cot\frac{\theta}{2}$$

3. REPRESENTATION OF A COMPLEX NUMBER

1. GEOMETRICAL REPRESENTATION

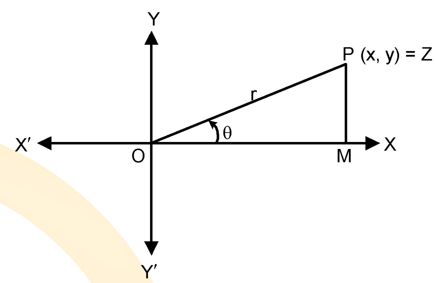
It is known, from coordinate geometry, that the ordered pair (x, y) represents a point in the Cartesian plane.

It is now seen that the ordered pair (x, y) taken as Z represents a complex number.

2. ARGAND DIAGRAM

The graphical representation of a complex number $Z = (x, y)$ by a point $P(x, y)$ is called representation in the Argand's Diagram also called Gaussian plane. In this representation, all complex numbers like (2, 0), (3, 0), (–1, 0), (α, 0) with imaginary part 0 will be represented by points on the x-axis. Since the real number α is represented as a complex number (α, 0), all real numbers will get marked on the x-axis. For this reason, the x-axis is called the real axis. Similarly all purely imaginary numbers (with real part 0) like (0, 1), (0, 2), (0, – 3), (0,β) will be marked on the y-axis. Hence the y-axis is also called the imaginary axis in this context. The Cartesian plane (two dimensional plane) is also called the complex plane.

3. POLAR REPRESENTATION



Let $P(x, y)$ be any point on the complex plane representing the complex number $z = (x, y)$, with $X'OX$ and $Y'OY$ as the axes of coordinates.

Let $OP = r$ and $\angle XOP = \theta$ (measured in anticlockwise).

Then from ΔOMP, we find that $x = OM = r\cos\theta$ and $y = MP = r\sin\theta$

Thus $\quad z = (x, y) = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

where $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$ by eulers formula

Thus $\quad z = r(\cos\theta + i\sin\theta)$ can be written as

$\boldsymbol{z = re^{i\theta}}$

This form of representation of Z is called the **trigonometric form** or the **polar form** or the **modulus amplitude form**.

When z is written in the form $r(\cos\theta + i\sin\theta)$, r is called the modulus of z and is written as $|z|$; $|z| = r = \sqrt{x^2+y^2}$, a non-negative number. $|z| = 0$ for the only number (0, 0).

Example: **Represent the given complex numbers in polar form:**

(i) $(1+i\sqrt{3})^2 / 4i(1-i\sqrt{3})$ (ii) $\sin\alpha - i\cos\alpha$ (α acute) (iii) $1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

Solution: (i) $i(1-i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

$$\therefore \frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})} = \frac{(1+i\sqrt{3})^2}{4(\sqrt{3}+i)} = \frac{-2+2i\sqrt{3}}{4(\sqrt{3}+i)} = \frac{(-1+i\sqrt{3})(\sqrt{3}-i)}{2(\sqrt{3}+i)(\sqrt{3}-i)}$$

$$= \frac{-\sqrt{3}+\sqrt{3}+4i}{2(3+1)} = \frac{i}{2}$$

and $\dfrac{i}{2} = \dfrac{1}{2}\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$. Hence $\dfrac{(1+i\sqrt{3})^2}{4i\,(1-i\sqrt{3})} = \dfrac{1}{2}\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right) = \dfrac{1}{2}\,e^{i\pi/2}$

(ii) Real part > 0; Imaginary part < 0

argument of sin α − i cos α is in the nature of a negative acute angle.

$$\therefore \sin\alpha - i\cos\alpha = \cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right) = e^{i\left(\alpha - \frac{\pi}{2}\right)}$$

(iii) $$1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = 2\cos^2\frac{\pi}{6} + i\cdot 2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$$

$$= 2\cos\frac{\pi}{6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6}\,e^{i\pi/6}$$

4. CONJUGATE OF A COMPLEX NUMBER

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$, where a and b are real numbers, $i = \sqrt{-1}$ and $b \neq 0$ are said to be complex conjugate of each other. (Here the complex conjugate is obtained by just changing the sign of i).

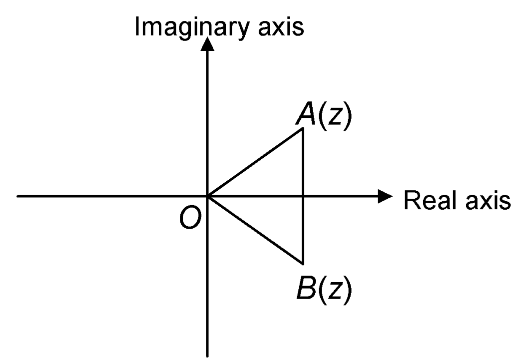
Note that, sum $= (a + ib) + (a - ib) = 2a$ which is real

and product $= (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2b^2$

$= a^2 - (-1)\,b^2 = a^2 + b^2$ which is real.

1. PROPERTIES OF CONJUGATE

- $\overline{(\bar{z})} = z$
- $z = \bar{z} \Leftrightarrow z$ is real
- $z = -\bar{z} \Leftrightarrow z$ is purely imaginary
- $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$
- $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$
- $\overline{z^n} = (\bar{z})^n$
- If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$



5. MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = x + iy$ is a real number given by $|z| = \sqrt{x^2 + y^2}$. It is always non-negative and $|z| = 0$ only for $z = 0$ i.e. origin of Argand plane. Geometrically it represents the distance of the point complex number from its origin.

1. PROPERTIES OF MODULUS

- $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1||z_2|$

  In general $|z_1 z_2 z_3 \ldots\ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots\ldots |z_n|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$
- $|z^n| = |z|^n$
- Unimodular : i.e., unit modulus

  If z is unimodular then $|z| = 1$. A unimodular complex number can always be expressed as $\cos\theta + i\sin\theta$, $\theta \in \boldsymbol{R}$.

Example: **If $|z - 2 + i| \leq 2$ then find the greatest and least value of $|z|$.**

Solution: Given that

$$|z - 2 + i| \leq 2 \qquad \ldots(i)$$

$$|z - 2 + i| \geq ||z| - |2 - i||$$

$$\therefore \quad |z - 2 + i| \geq ||z| - \sqrt{5}| \qquad \ldots(ii)$$

From (i) and (ii)

$$||z| - \sqrt{5}| \leq |z - 2 + i| \leq 2$$

$$\therefore \quad ||z| - \sqrt{5}| \leq 2$$

$$\Rightarrow \quad -2 \leq |z| - \sqrt{5} \leq 2 \Rightarrow \sqrt{5} - 2 \leq |z| \leq \sqrt{5} + 2$$

Hence greatest value of $|z|$ is $\sqrt{5} + 2$ and least value of $|z|$ is $\sqrt{5} - 2$.

Example: **If Z_1 and Z_2 be two complex numbers such that $\left|\frac{Z_1 - 2Z_2}{2 - Z_1\bar{Z}_2}\right| = 1$ and $|Z_2| \neq 1$. What is the value of $|Z_1|$?**

Solution: $|Z_1 - 2Z_2| = |2 - Z_1\bar{Z}_2|$

$\therefore |Z_1 - 2Z_2|^2 = |2 - Z_1\overline{Z}_2|^2$

$\therefore (Z_1 - 2Z_2)(\overline{Z}_1 - 2\overline{Z}_2) = (2 - Z_1\overline{Z}_2)(2 - \overline{Z}_1 Z_2)$

$\therefore Z_1\overline{Z}_1 - 2\overline{Z}_1 Z_2 - 2Z_1\overline{Z}_2 + 4Z_2\overline{Z}_2 = 4 - 2Z_1\overline{Z}_2 - 2\overline{Z}_1 Z_2 + Z_1\overline{Z}_1 Z_2\overline{Z}_2$

$\therefore Z_1\overline{Z}_1 + 4Z_2\overline{Z}_2 - 4 - Z_1\overline{Z}_1 Z_2\overline{Z}_2 = 0$

$|Z_1|^2 + 4|Z_2|^2 - |Z_1|^2 |Z_2|^2 - 4 = 0$ i.e. $\left(|Z_1|^2 - 4\right)\left(|Z_2|^2 - 1\right) = 0$

Since $|Z_2| \neq 1$ it is that $|Z_1|^2 = 4$ i.e. $|Z_1| = 2$