LESSON-5

MATHEMATICAL INDUCTION

1. INTRODUCTION

In mathematics there are some results or statements that are formulated in terms of *n*, where $n \in N$. To prove such statements we use a well suited method, based on the specific technique, which in known as **principle of mathematical induction**.

2. FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

The statement $P(n)$ is true for all $n \in N$ if

(i) $P(1)$ is true.

(ii) $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

The above statement can be generalized as $P(n)$ is true for all $n \in N$ and $n \geq k$ if

(i) $P(k)$ is true.

(ii) $P(m)$ is true $(m > k)$ $\Rightarrow$ $P(m+1)$ is true.

Example: **Prove by the principle of mathematical induction that for all $n \in N$:**

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Solution: Let $P(n)$ be the statement given by

$$P(n) : \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Step I: We have, $P(1) : \frac{1}{1.2} = \frac{1}{1+1}$

Since, $\frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$

So, $P(1)$ is true.

Step II: Let $P(m)$ be true, then $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......+\frac{1}{m(m+1)}=\frac{m}{m+1}$ (i)

We shall now show that $P(m+1)$ is true. If $P(m)$ is true.

For this we have to show that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......+\frac{1}{m(m+1)}+\frac{1}{(m+1)(m+1+1)}=\frac{m+1}{(m+1)+1}$$

Now, $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......\frac{1}{m(m+1)}+\frac{1}{(m+1)\,(m+1+1)}$

$$=\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......\frac{1}{m(m+1)}\right]+\frac{1}{(m+1)\,(m+1+1)}$$

$$=\frac{m}{m+1}+\frac{1}{(m+1)((m+1)+1)}$$ [using (i)]

$$=\frac{1}{(m+1)}\left\{\frac{m}{1}+\frac{1}{m+2}\right\}=\frac{1}{(m+1)}\frac{(m^2+2m+1)}{(m+2)}=\frac{(m+1)^2}{(m+1)(m+2)}=\frac{m+1}{m+2}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.

Example: **For every positive integer *n*, prove that 7^n-3^n is divisible by 4.**

Solution: We have $P(n): 7^n-3^n$ is divisible by 4

We note that $P(1): 7^1-3^1=4$, which is divisible by 4. Thus $P(n)$ is true for $n = 1$

Let $P(k)$ be true for some natural number k.

i.e., $P(k): 7^k-3^k$ is divisible by 4.

We get $7^k-3^k = 4d$, where $d \in N$ …(i)

Now, we wish to prove that $P(k+1)$ is true whenever $P(k)$ is true.

i.e., we have show $7^{k+1}-3^{k+1}=4m$

Now, $7^{(k+1)}-3^{(k+1)}=7^{(k+1)}-7.3^k+7\ 3^k-3^{(k+1)}$

$$=7(7^k-3^k)+(7-3)3^k=7(4d)+(7-3)3^k$$

$= 7(4d) + 4.3^k$ [using (i)]

$= 4(7d + 3^k) = 4m$

From the last line, we see that $7^{(k+1)} - 3^{(k+1)}$ is divisible by 4.

Thus, $P(k+1)$ is true when $P(k)$ is true.

Therefore, by principle of mathematical induction the statement is true for every positive integer *n*.

Example: **Prove the rule of exponents $(ab)^n = a^n b^n$ by using principle of mathematical induction for every natural number.**

Solution: Let $P(n)$ be the given statement i.e, $P(n): (ab)^n = a^n b^n$

We note that $P(n)$ is true for *n* = 1 since $(ab)^1 = a^1 b^1$

Let $P(k)$ be true, i.e., $(ab)^k = a^k b^k$ …(i)

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true.

Now, we have $(ab)^{k+1} = (ab)^k (ab)$

$= (a^k b^k)(ab)$ [by (i)]

$= (a^k . a^1)(b^k . b^1) = a^{k+1} . b^{k+1}$

Therefore, $P(k+1)$ is also true whenever $P(k)$ is true.

Hence, by principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example: **Using mathematical induction, show that**

$$\cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos(2^{n-1}\theta) = \frac{\sin 2^n\theta}{2^n \sin\theta}, \quad \forall\ n \in N$$

Solution: Let $P(n): \cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos(2^{n-1}\theta) = \frac{\sin 2^n\theta}{2^n \sin\theta}$

Step I: For *n* = 1

L.H.S. of (i) = $\cos\theta$ and R.H.S. of (i) = $\frac{\sin 2\theta}{2\sin\theta} = \cos\theta$

Therefore, *P*(1) is true.

Step II: Assume it is true for *n* = *k*, then

$$p(k): \cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos(2^{k-1}\theta) = \frac{\sin 2^k\theta}{2^k \sin\theta} \quad \ldots(i)$$

Step III: For $n = k + 1$

$$P(k+1): \cos\theta\cos 2\theta\cos 4\theta ...... \cos\left(2^{k-1}\theta\right)\cos\left(2^k\theta\right) = \frac{\sin 2^{k+1}\theta}{2^{k+1}\sin\theta}$$

L.H.S. = $\cos\theta\cos 2\theta\cos 4\theta ...... \cos\left(2^{k-1}\theta\right)\cos\left(2^k\theta\right)$

$= \frac{\sin\left(2^k\theta\right)}{2^k\sin\theta}.\cos\left(2^k\theta\right) = \frac{2\sin\left(2^k\theta\right)\cos\left(2^k\theta\right)}{2^{k+1}\sin\theta}$ [using (i)]

$= \frac{\sin\left(2.2^k\theta\right)}{2^{k+1}\sin\theta} = \frac{\sin\left(2^{k+1}\theta\right)}{2^{k+1}\sin\theta}$ = R.H.S.

This shows that the $P(k+1)$ is true if $P(k)$ is true.

Hence by the principle of mathematical induction, the result is true for all $n \in N$.

Example: **Prove that $2^n > n$ for all positive integers *n*.**

Solution: Let $P(n): 2^n > n$

When $n = 1$, $2^1 > 1$

Hence $P(1)$ is true.

Assume that $P(k)$ is true for any positive integers k i.e.,

$2^k > k$ ...(i)

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true.

Multiplying both sides of (i) by 2, we get

$2.2^k > 2k$

i.e., $2^{k+1} > 2k = k + k > k + 1$

$\therefore$ $P(k+1)$ is true when $P(k)$ is true.

Hence, by principle of mathematical induction, $P(n)$ is true for every positive integer n.