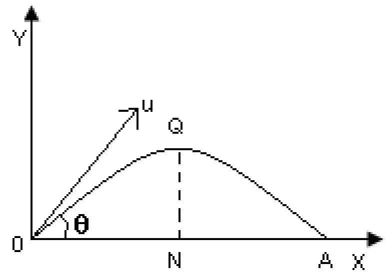
PROJECTILE MOTION

Projectile motion:- The two dimensional motion of a particle projected obliquely into the air is called projectile motion and the particle projected is called projectile.



A particle projected from the point O with an initial velocity u at angle Q with the horizontal. On the horizontal surface through 0.

Here, point O is called point of projection

θ is called angle of projection

0A is called the horizontal range = R

QN = is called the maximum height (H)

Time of Flight:- The total time taken by the particle in deserving the path 0QA is called the time of flight (T)

Trajectory:- Both followed by projectile motion is called trajectory. In projectile motion a constant (and hence constant acceleration) acts on a particle at an angle θ (≠0 or 180^o) with the direction of its initial velocity (≠ 0). The path followed by the particle is a parabola and motion of the particle is contrasted in a plane. A particle is thrown obliquely near the earth's surface and it moves is a parabolic path provided the particle remains done the surface and the air resistance is negligible.

In any problem of projectile motion, we usually follow the three steps given below:-

Step-1:- Select two mutually perpendicular direction x and y.

Step-2:- Write down the proper value of with sigh.

Step-3:- Select among the six listed equations which are required in the problem.

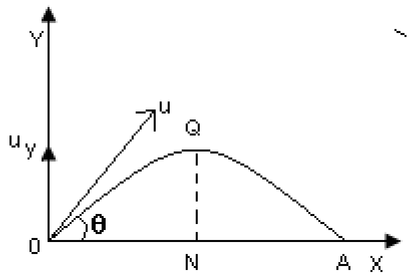
(i) $V_x = u_x + a_x t$ (ii) $S_x = u_x t + \frac{1}{2} a_x t^2$ (iii) $V_x^2 = u_x^2 + 2a_x S_x$

(iv) $V_y = u_y + a_y t$ (v) $S_y = u_y t + \frac{1}{2} a_y t^2$ (vi) $V_y^2 = u_y^2 + 2a_y S_y$

Time of flight:- Here x and y axes are in the direction shown. x-axies is along horizontal direction and y-axies is vertically upward.

$\therefore u_x = u\cos\theta$

$u_y = u\sin\theta \uparrow$



$a_x = 0$

$a_y = g \downarrow \quad = -g$

At point A,

$S_y = 0$

so,

$S_y = u_y t + \frac{1}{2} a_y t^2$

$0 = u \sin\theta - \frac{1}{2} g t^2$

$t(\sin\theta - \frac{1}{2} g t) = 0$

$t = 0, \frac{2u\sin\theta}{g}$

t = T (time of flight)

$\therefore T = 0, \quad \frac{2u\sin\theta}{g}$

Here, Both t = 0, $t = \frac{2u\sin\theta}{g}$ correspond to the situation where $S_y = 0$, The time t = 0 corresponds to point O and $t = \frac{2u\sin\theta}{g}$ corresponds to point A.

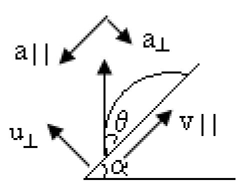
Thus, Time of flight of projectile is

T $= \frac{2u\sin\theta}{g}$

In general, $T = \frac{2u_\perp}{a_\perp}$

where $u_\perp$ and $a_\perp$ are respective velocity and acceleration perpendicular to the surface

Here $u_\perp = v_0 \sin\theta$ and $a_\perp = g \cos\alpha$



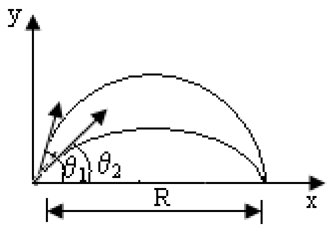
Thus, $T = \dfrac{2u_{\perp}}{a_{\perp}} = \dfrac{2v_0 \sin\theta}{g\cos\alpha}$

Horizontal Range (R):- Distance OA is the range R. This is equal to the displacement of particle along x-axies in time t =T.

Thus,

$$S_x = u_x t + \frac{1}{2} g_x t^2 \qquad \left[T = \frac{2u\sin\theta}{g} \right.$$

$$R = u\cos\theta x \frac{2u\sin\theta}{g}$$

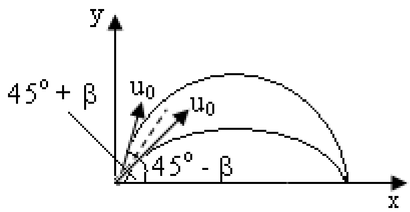


$$R = \frac{2u\sin\theta\cos\theta}{g} = \frac{u^2\sin\theta}{g}$$

- Furthermore, same ranges also occur for angles symmetrically located about the angle to 45^o,

  i.e. $\theta_1 = 45^o + \beta$

  and $\theta_2 = 45^o - \beta$



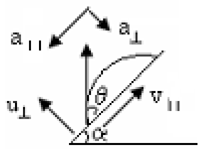
- For complementary angles of projection if T_1 and T_2 are the respective times of flight then,

  $T_1T_2 = \dfrac{T_1}{T_2}$

Here, two points are important regarding regarding the range of a projectile.

(a) Range Along the Inclined Plane

The range of the projectile along the inclined plane is given by



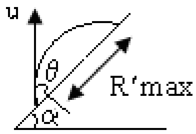
$$R' = V_{||}\, T - \frac{1}{2} a_{||} T^2$$

Since $T = \dfrac{2u_{\perp}}{a_{\perp}} = \dfrac{2v_0\sin\theta}{g\cos\alpha}$

$$R' = \frac{2v_0^2}{g}\frac{\sin\theta\cos(\theta+\alpha)}{\cos^2\alpha}$$

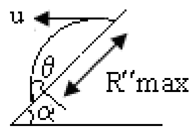
Important Points

- The minimum rage occurs when



$$\theta = \frac{\pi}{4} - \frac{\alpha}{2}$$

- The minimum range along the inclined plane when the projectile is through downwards is given by



$$R'_{max} = \frac{v_0^2}{g(1+\sin\alpha)}$$

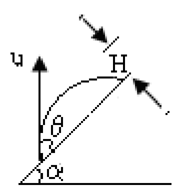
- The maximum range along the inclined plane when the projectile is through downwards is given by

$$R'_{max} = \frac{v_0^2}{g(1-\sin\alpha)}$$

Maximum Height of the Projectile

When a projectile attains its maximum height its vertical velocity becomes zero.
I.e. $V_v = 0$
The minimum height of the projectile is given by

$$H = \frac{v_0^2\sin^2\theta}{2g}$$



In general, $H = \frac{v_\perp^2}{2a_\perp}$

If a projectile up an inclined plane, as shown in fig. the maximum height attained si given by

$$H = \frac{(v_0\sin\theta)^2}{2g\cos\alpha} = \frac{v_0^2\sin^2\theta}{2g\cos\alpha}$$

Important Points

- The maximum height H and the rang R related to each other as,
  $R = 4H\cot\theta$
- For complementary angles of projection, i.e.,

$\theta_1 = 45^0 + \alpha$ and $\theta_2 = 45^0 - \alpha$

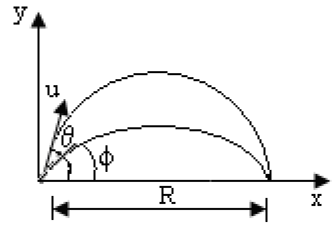
the ratio of the maximum heights attained is given by

$$\frac{H_2}{H_1} = \tan^2\left(45^0 - \alpha\right)$$

- The angle of projection at which the range and the maximum height attained by a projectile are is $\tan^{-1}4$.
- If the range of projectile is η times the maximum height of the projectiles, then the angle of projectile is given by

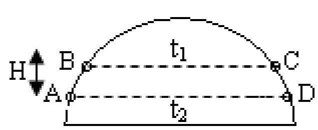
$$\theta = \tan^{-1}\left(\frac{4}{\eta}\right)$$

- The angle of elevation ϕ of the highest point of the projectile and the angle projection θ are related to each other as



$$\tan\phi = \frac{1}{2}\tan\theta$$

- In the fig. B and C are at the same level, the difference between these two position is t_1; A and D are also at the same level, the time difference between these two position is t_2.

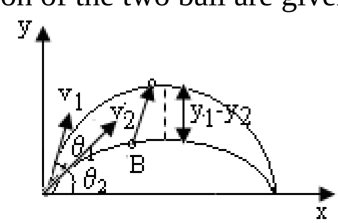


Then, $t_2^2 - t_1^2 = \frac{BH}{g}$

- **Motion of a Projection as Observed from Another Projectile**

  Suppose two balls A and B are projected simultaneously from the origin, with initial velocities v1 and v2 at angle θ1 and θ2, respectively with the horizontal.

  This instantaneous position of the two ball are given by



Ball A: $x_1 = (v_1\cos\theta_1)t$ $\qquad$ $x_2 = (v_1\sin\theta_1)t - \frac{1}{2}gt^2$

Ball B: $x_2 = (v_2\cos\theta_2)t$ $\qquad$ $y_2 = (v_2\sin\theta_1)t - \frac{1}{2}gt^2$

The position of the ball A with respect to ball B is given by

$x = x_1 - x_2 = (v_1\cos\theta_1 - v_2\cos\theta_2)t$
$y = y_1 - y_2 = (v_1\sin\theta_1 - v_2\sin\theta_2)t$

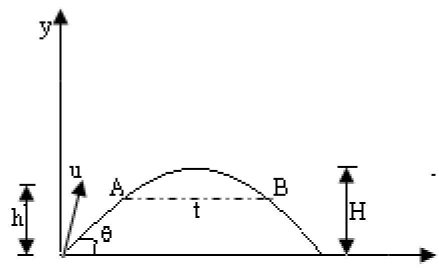
Now $\frac{y}{x} = \left(\frac{v_1 \sin\theta_1 - v_2 \sin\theta_2}{v_1 \cos\theta_1 - v_2 \cos\theta_2}\right) = \text{constant}$

Thus, motion of a projectile relative to another projectile is a straight line.

Example:- The figure shows two position A and B at the same height h above the ground. If the maximum height of the projectile is H, then determine the time t elapsed between the position A and B in terms of H.

Ans:- Let T be the time of flight. We can now write

$$T_2 - t_1 = \frac{8h}{g}$$



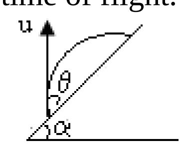
Since $T = \frac{2v_0 \sin\theta}{g}$ or $T^2 = \frac{4v_0^2 \sin^2\theta}{g^2} = \frac{8H}{g}$

Thus, $t^2 = T^2 - \frac{8h}{g} = \frac{8}{g}(H - h)$

or $t = \sqrt{\frac{8}{g}(H - h)}$

Example:- At what angle should a ball be projected up an inclined plane with a velocity v_0 so that it may hit the incline normally. The angle of the inclined plane with the horizontal is α

Ans:- The ball will hit the incline normally if its parallel component of velocity reduces to zero during the time of flight.



The time of flight is given by $T = \frac{2v_0 \sin\theta}{g\cos\alpha}$ (1)

Applying the equation of kinematics parallel to the incline, we get

$0 = v0 \cos\theta - (g\sin\alpha)T$

or $T = \frac{2v_0 \sin\theta}{g \sin\alpha}$ (2)

Solving equation (1) and (2), we get

$2 \tan\theta \tan\alpha = 1$

or $\theta = \tan^{-1}\left[\frac{1}{2}\cot\alpha\right]$

Range is maximum

R_{max} is maximum when sin2θ = I

$\sin 2\theta = \sin 90^o$

$2\theta = \sin 90^o$

$\theta = 45^o$

$$R_{max} = \frac{u^2}{g}$$

(i) For given value of u range at θ and (90-θ) are equal although times of flight and maximum heights may be different.

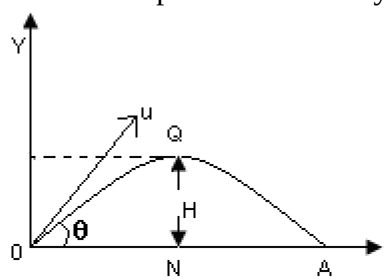
$$R_{90-\theta} = \frac{u^2 \sin 2(90-\theta)}{g} = \frac{u^2 \sin(180-2\theta)}{g}$$

$$= \frac{u^2 \sin\theta}{g} = R\theta$$

$$R_{30} = R_{60} \quad , R_{20} = R_{70}$$

Maximum height:-

At maximum height Q, the vertical component of velocity becomes zero.



$V_y = 0$

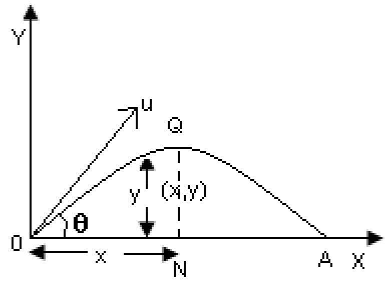
$V_y^2 = u_y^2 + 2a_y S_y$

$0 = (u\sin\theta)^2 - 2gH$

$$\therefore H = \frac{u^2 \sin^2\alpha}{2g}$$

Equation of trajectory of projectile :-

Let a body is projected up with an critical velocity u is a direction making an angle Q with horizontal. Let 0 be the point of projection of the body.



0X and 0Y be horizontal under constant acceleration of acting vertically downwards. So, the horizontal velocity u_x remains uncharged throughout the motion (proved the resistance of air is negligible), but velocity u_y continuously charges.

Let at any constant particle is at position (m_y)

The displacement of the body along the horizontal direction after a time t is

x = u cosαt

$$\therefore\ t = \frac{x}{u\cos\theta}$$

$$y = u\sin\theta t - \frac{1}{2}gt^2$$

$$y = \cancel{u}\sin\theta x\frac{x}{\cancel{\cos\theta}} - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$ [Putting ‘t’ is given]

$$y = \tan\theta.x - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$

This equation is quadratic in x and liner in y. Therefore it represents a parabola. Hence the path of the projectile is parabolic.