Medical Image Processing for Interventional Applications

Properties of the SVD

Online Course – Unit 3

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Properties of the SVD: Rank and Norm

The singular value decomposition shows many extremely useful properties that are listed here without proof:

- ***rank*** of matrix $\boldsymbol{A}$:

$$\operatorname{rank}(\boldsymbol{A}) = \#\{\sigma_i > 0\},$$

- ***numerical*** ε***-rank*** of matrix $\boldsymbol{A}$:

$$\operatorname{rank}_\varepsilon(\boldsymbol{A}) = \#\{\sigma_i > \varepsilon\},$$

- the ***Frobenius norm*** of the matrix $\boldsymbol{A} = (a_{i,j})_{i,j}$ is given by

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}^2} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2}.$$

Properties of the SVD: Eigenvectors

The singular value decomposition shows many extremely useful properties that are listed here without proof:

- decomposition into rank 1 – matrices:

$$\boldsymbol{A} = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{T}}, \quad r = \mathrm{rank}(\boldsymbol{A}),$$

- $\boldsymbol{A}\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ and $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_i = \sigma_i \boldsymbol{v}_i$,
- the column vectors of $\boldsymbol{U}$ are the eigenvectors of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$:

$$\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_i = \sigma_i^2 \boldsymbol{u}_i,$$

- the column vectors of $\boldsymbol{V}$ are the eigenvectors of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$:

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{v}_i = \sigma_i^2 \boldsymbol{v}_i.$$

Properties of the SVD

- The SVD yields orthonormal bases for the kernel (null-space) and the range of a matrix $\boldsymbol{A}$:
  - The ***kernel*** of $\boldsymbol{A}$ is spanned by the column vectors $\boldsymbol{v}_i$ of $\boldsymbol{V}$, where the corresponding singular values fulfill $\sigma_i = 0$.
  - The ***range*** of $\boldsymbol{A}$ is spanned by the column vectors $\boldsymbol{u}_i$ of $\boldsymbol{U}$, where σ_i are the corresponding non–zero singular values.
- For the 2-norm of matrix $\boldsymbol{A}$ we get:

$$\|\boldsymbol{A}\|_2^2 = \max_{\|\boldsymbol{x}\|_2=1} \boldsymbol{x}^\mathsf{T}\boldsymbol{A}^\mathsf{T}\boldsymbol{A}\boldsymbol{x} = \sigma_1^2,$$

  and if $\boldsymbol{A}$ is regular, we even have:

$$\|\boldsymbol{A}^{-1}\|_2^2 = \frac{1}{\sigma_p^2}.$$

Example

$$\boldsymbol{A} = \begin{pmatrix} 11 & 10 & 14 \\ 12 & 11 & -13 \\ 14 & 13 & -66 \end{pmatrix} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathsf{T}},$$

where

$$\boldsymbol{U} = \begin{pmatrix} 0.1285 & 0.8375 & 0.5311 \\ -0.2396 & 0.5459 & -0.8028 \\ -0.9623 & -0.0241 & 0.2708 \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} 71.3967 & 0 & 0 \\ 0 & 21.7831 & 0 \\ 0 & 0 & 0.0006 \end{pmatrix},$$

$$\boldsymbol{V} = \begin{pmatrix} -0.2092 & 0.7082 & -0.6743 \\ -0.1941 & 0.6458 & 0.7384 \\ 0.9584 & 0.2854 & 0.0024 \end{pmatrix}.$$

Example

- Obviously, matrix $\boldsymbol{A}$ has a rank deficiency if we select $\varepsilon = 10^{-3}$.
- The kernel of $\boldsymbol{A}$ is given by:

$$\ker(\boldsymbol{A}) = \left\{ \lambda \cdot \begin{pmatrix} -0.6743 \\ 0.7384 \\ 0.0024 \end{pmatrix} ; \lambda \in \mathbb{R} \right\}.$$

- The range (or image) of $\boldsymbol{A}$ is:

$$\operatorname{im}(\boldsymbol{A}) = \left\{ \lambda \cdot \begin{pmatrix} 0.1285 \\ -0.2396 \\ -0.9623 \end{pmatrix} + \mu \cdot \begin{pmatrix} 0.8375 \\ 0.5459 \\ -0.0241 \end{pmatrix} ; \lambda, \mu \in \mathbb{R} \right\}.$$

Ill-conditioned Matrix

Definition

A matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is called ***ill-conditioned*** if for a given linear system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

minor changes in $\boldsymbol{b} \in \mathbb{R}^m$ cause major changes in $\boldsymbol{x} \in \mathbb{R}^n$.

Definition

The ***condition number*** of a regular matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with respect to a matrix norm $\|.\|$ is defined by

$$\kappa(\boldsymbol{A}) = \left\|\boldsymbol{A}^{-1}\right\| \cdot \|\boldsymbol{A}\| \, .$$

If $\boldsymbol{A}$ is singular, $\kappa(\boldsymbol{A}) = +\infty$.

Ill-conditioned Matrix: Remarks

- A matrix with $\kappa(\boldsymbol{A})$ close to 1 is called ***well-conditioned***.
- A matrix with $\kappa(\boldsymbol{A})$ significantly greater than 1 is said to be ***ill-conditioned***.
- The condition number is a measure of the stability or sensitivity of a matrix.
- Using the 2-norm, the condition number of a regular matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ can be computed by SVD:

$$\kappa(\boldsymbol{A}) = \frac{\sigma_1}{\sigma_n},$$

  where σ_1 is the largest, and σ_n is the smallest singular value.
- The SVD allows for the exact computation of the condition number, but this is computationally expensive.

Ill-conditioned Matrix

Example

Consider the previous matrix

$$\boldsymbol{A} = \begin{pmatrix} 11 & 10 & 14 \\ 12 & 11 & -13 \\ 14 & 13 & -66 \end{pmatrix},$$

where we have $\det(\boldsymbol{A}) = 1$. The singular value decomposition of $\boldsymbol{A}$ results in the singular values:

$$\sigma_1 = 71.3967, \sigma_2 = 21.7831, \text{ and } \sigma_3 = 0.0006.$$

Thus the condition number is $\kappa(\boldsymbol{A}) = 118994.5 \gg 1$.

Exercise: Show that a variation in $\boldsymbol{b}$ by 0.1% implies a change in $\boldsymbol{x}$ by 240%.

Take Home Messages

- We learned about important properties of the SVD, like
  - analytical and numerical rank definition,
  - Frobenius norm and 2-norm,
  - the relation between $\boldsymbol{U}$, $\boldsymbol{V}$ and the eigenvectors of $\boldsymbol{A}\boldsymbol{A}^\mathsf{T}$, $\boldsymbol{A}^\mathsf{T}\boldsymbol{A}$,
  - the relation between the kernel/range of $\boldsymbol{A}$ and the columns of $\boldsymbol{V}$, $\boldsymbol{U}$.
- For every matrix a condition number can be computed. Ill-conditioned matrices are numerically rather instable.

Further Readings

Read the original:

Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. 3rd ed. Johns Hopkins Studies in the Mathematical Sciences. Baltimore: The Johns Hopkins University Press, Oct. 1996

A very detailed and easy to follow introduction of the SVD can be found in:

Carlo Tomasi's class notes, chapter 3 (a **must-read**).

The theory is described in an easy to read format in:

Lloyd N. Trefethen and David Bau III. *Numerical Linear Algebra*. Philadelphia: SIAM, 1997

For details about the numerical computation of SVD see:

William H. Press et al. *Numerical Recipes – The Art of Scientific Computing*. 3rd ed. Cambridge University Press, 2007. Get at `http://numerical.recipes/` (August 2016).

A good reference for properties of matrices is the following script:

Kaare Brandt Petersen and Michael Syskind Pedersen. *The Matrix Cookbook*. Online. Technical University of Denmark, Nov. 2012. URL: `http://www2.imm.dtu.dk/pubdb/p.php?3274`