Medical Image Processing for Interventional Applications

Eight Point Algorithm

Online Course – Unit 32

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Topics

The Eight Point Algorithm

- read point correspondences $\{(\tilde{\boldsymbol{p}}_k^{\mathrm{i}}, \tilde{\boldsymbol{q}}_k^{\mathrm{i}}), k = 1, \dots, N\}$
- rewrite the epipolar constraints in terms of linear equations: $\boldsymbol{Me} = 0$
- compute the SVD of $\boldsymbol{M} = \boldsymbol{U}\Sigma\boldsymbol{V}^{\mathsf{T}}$
- $\sigma_9 > \varepsilon$ ($\boldsymbol{M}$ has full rank)
  - `false`:
    - nullspace of $\boldsymbol{M}$ defines the components of the essential matrix $\boldsymbol{E}$
    - compute the SVD of $\boldsymbol{E} = \boldsymbol{U}'\Sigma'\boldsymbol{V}'^{\mathsf{T}}$
    - $\sigma_3 > \varepsilon$ ($\boldsymbol{E}$ has rank 3)
      - `false`:
        - $\sigma_1 - \sigma_2 > \varepsilon$
          - `false`: return $\boldsymbol{E} = \boldsymbol{U}' \begin{pmatrix} \frac{\sigma_1+\sigma_2}{2} & 0 & 0 \\ 0 & \frac{\sigma_1+\sigma_2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{V}'^{\mathsf{T}}$
          - `true`: stop (essential matrix has different singular values σ_1, σ_2)
      - `true`:
        - stop (essential matrix violates rank criterion)
        - $\varnothing$
  - `true`:
    - stop (epipolar constraint not fulfilled)
    - $\varnothing$

Figure 1: Eight point algorithm for $\boldsymbol{E}$

Input Data

For now we ...

... have two images of a patient with one camera,

... use point features,

... assume the correspondence problem to be already solved:

$$\{(\tilde{\boldsymbol{p}}^{\mathrm{i}}_k, \tilde{\boldsymbol{q}}^{\mathrm{i}}_k), k = 1, \ldots, N\},$$

i. e., point $\tilde{\boldsymbol{p}}^{\mathrm{i}}_k$ in image 1 corresponds to point $\tilde{\boldsymbol{q}}^{\mathrm{i}}_k$ in image 2 [the tilde indicates homogeneous coordinates],

... have the points given as normalized homogeneous image coordinates, i. e., the third component is set to 1,

... use perspective projection (pinhole camera).

Intrinsic Camera Parameters

- Intrinsic parameters (summarized by the matrix $\boldsymbol{K} \in \mathbb{R}^{3\times3}$) are known.
- Intrinsic parameters do not change when camera moves.
- Origin of image coordinate system does not coincide with the intersection of optical axis and image plane in general.
- Axes of the camera's CCD-Chip are not orthogonal.
- Pixels are non-quadratic.

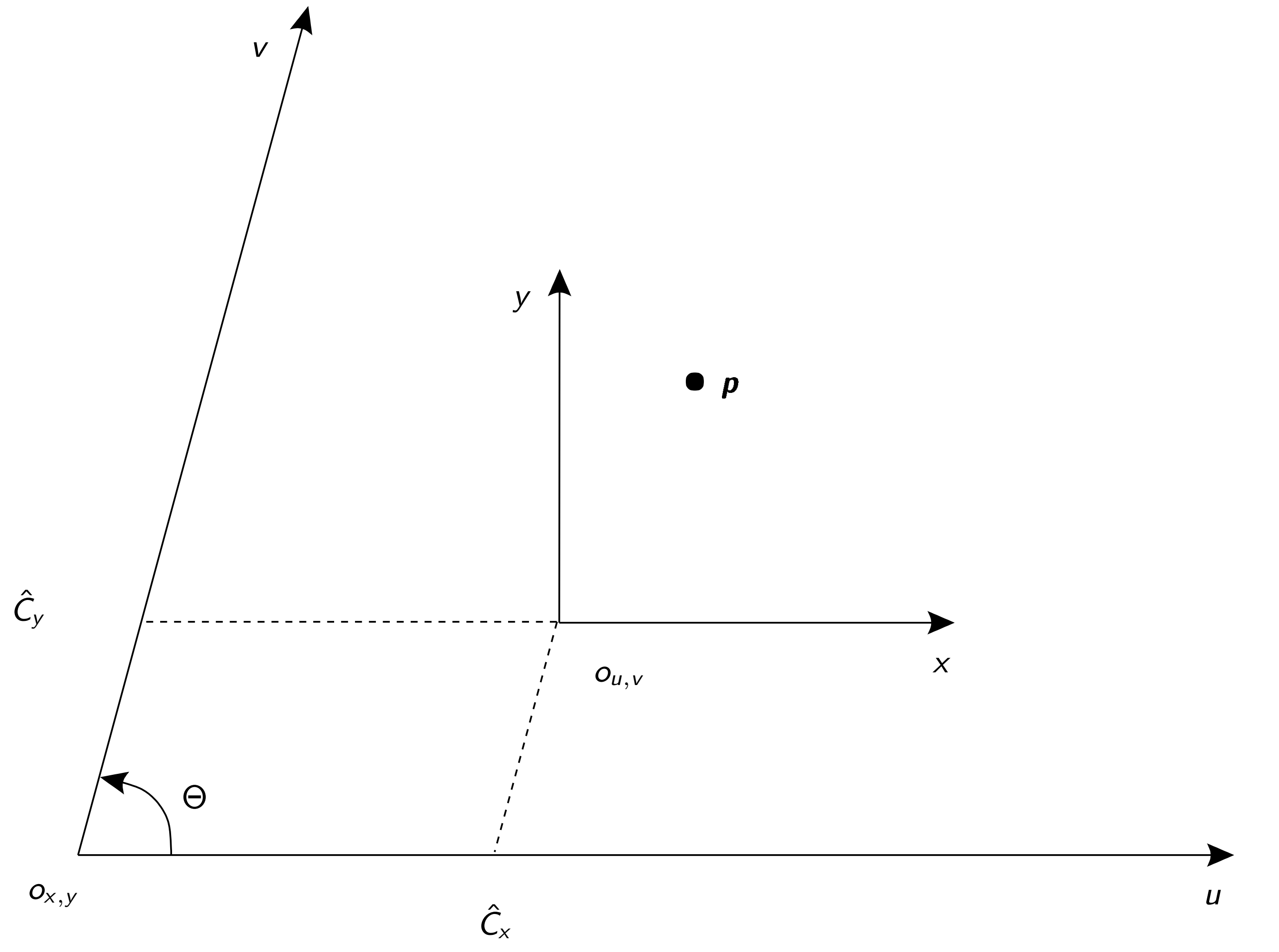

Figure 2: Pixel coordinate system

Coordinate System

(x, y) - coordinate system:

- ideal coordinate system used so far (image coordinate system with origin o^i)

(u, v) - coordinate system:

- real system, in which pixels are addressed, (pixel coordinate system, origin o^p)
- Θ: angle between axes, skew $s = -k_x \tan\Theta$
- k_x, k_y: units of u and v axis, with respect to units in x/y system

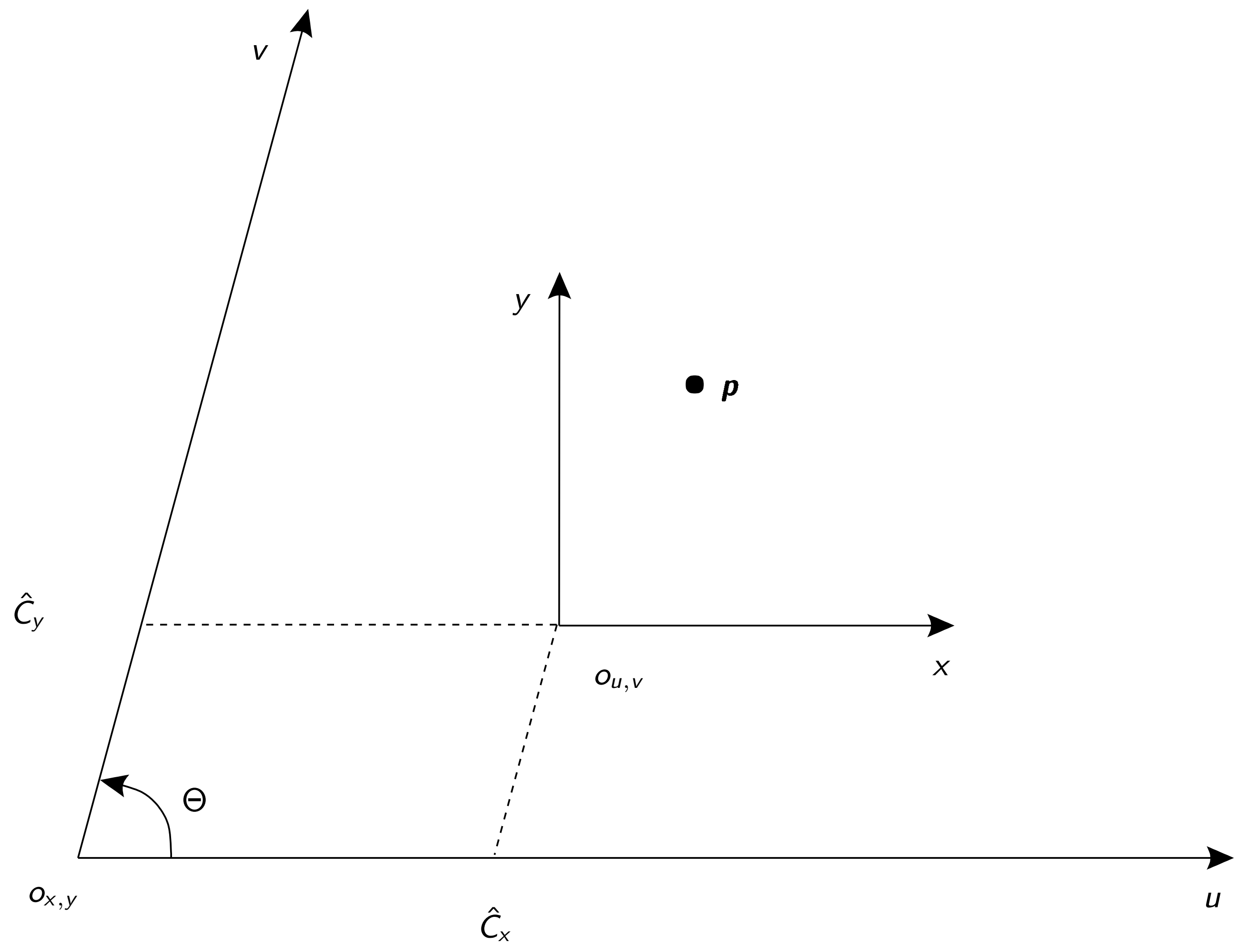

Figure 3: Pixel coordinate system

Fundamental Matrix

If pixel coordinates are used instead of ideal image coordinates

$$\tilde{\boldsymbol{p}}^{\mathrm{p}} = \boldsymbol{K}\tilde{\boldsymbol{p}}^{\mathrm{i}},$$

we substitute

$$\tilde{\boldsymbol{p}}^{\mathrm{i}} = \boldsymbol{K}^{-1}\tilde{\boldsymbol{p}}^{\mathrm{p}}, \qquad \tilde{\boldsymbol{q}}^{\mathrm{i}} = \boldsymbol{K}^{-1}\tilde{\boldsymbol{q}}^{\mathrm{p}},$$

and get:

$$(\tilde{\boldsymbol{q}}^{\mathrm{p}})^{\mathsf{T}} \cdot \underbrace{\left(\left(\boldsymbol{K}^{-1}\right)^{\mathsf{T}} \cdot \boldsymbol{E} \cdot \boldsymbol{K}^{-1}\right)}_{\boldsymbol{F}} \cdot \tilde{\boldsymbol{p}}^{\mathrm{p}} = 0.$$

F is called the ***fundamental matrix***.

Properties of Fundamental Matrix

- $\boldsymbol{F}$ has rank 2.
- $\boldsymbol{F}$ encodes intrinsic and extrinsic parameters.
- $\boldsymbol{F}$ maps a point $\tilde{\boldsymbol{p}}^{\mathrm{p}}$ to its epipolar line $\boldsymbol{l}$ in pixel coordinates by $\boldsymbol{l}^{\mathsf{T}} = \boldsymbol{F} \cdot \tilde{\boldsymbol{p}}^{\mathrm{p}}$:

$$\boldsymbol{l}_2{}^{\mathsf{T}} = \boldsymbol{F}\tilde{\boldsymbol{p}}^{\mathrm{p}}, \qquad \boldsymbol{l}_1{}^{\mathsf{T}} = \boldsymbol{F}^{\mathsf{T}}\tilde{\boldsymbol{q}}^{\mathrm{p}}.$$

- All epipolar lines intersect in the epipole (left epipole $\tilde{\boldsymbol{e}}_l^p$, right epipole $\tilde{\boldsymbol{e}}_r^p$):
  - computation of the left null space:

$$(\tilde{\boldsymbol{e}}_r^p)^{\mathsf{T}} \boldsymbol{l}_1{}^{\mathsf{T}} = (\tilde{\boldsymbol{e}}_r^p)^{\mathsf{T}} \boldsymbol{F}\tilde{\boldsymbol{p}}^{\mathrm{p}} = 0$$

    implies $(\tilde{\boldsymbol{e}}_r^p)^{\mathsf{T}} \boldsymbol{F} = 0$,
  - computation of the right null space:

$$\boldsymbol{l}_2 \tilde{\boldsymbol{e}}_r^p = \left(\tilde{\boldsymbol{e}}_r^p \boldsymbol{F}\right)^{\mathsf{T}} \tilde{\boldsymbol{q}}^{\mathrm{p}} = 0$$

    implies $\boldsymbol{F}^{\mathsf{T}} \tilde{\boldsymbol{e}}_l^p = 0$.

Eight Point Algorithm for $\boldsymbol{F}$

Computation of $\boldsymbol{F}$:

- We get N equations of the form $\left(\tilde{\boldsymbol{q}}_i^{\mathrm{p}}\right)^{\mathsf{T}} \cdot \boldsymbol{F} \cdot \tilde{\boldsymbol{p}}_i^{\mathrm{p}} = 0$.
- This system of equations is **linear** in the components of $\boldsymbol{F} = [f_{ij}]_{i,j \in \{1,2,3\}}$:

$$\boldsymbol{M} \cdot \boldsymbol{f} = 0, \quad \boldsymbol{f} = \begin{pmatrix} f_{11} \\ f_{12} \\ \vdots \\ f_{33} \end{pmatrix}, \quad \boldsymbol{M} \in \mathbb{R}^{N \times 9},$$

  where $\operatorname{rank}(\boldsymbol{M}) = 8$.
- Solve this system using singular value decomposition.
- Make sure that $\operatorname{rank}(\boldsymbol{F}) = 2$.

Eight Point Algorithm

Starting point:

- Over-determined system of equations $\boldsymbol{M} \cdot \boldsymbol{f} = 0$
- $\boldsymbol{f}$ lies in the null space of $\boldsymbol{M}$. The null space is non-trivial, since $\boldsymbol{M} \in \mathbb{R}^{N \times 9}$ and $\operatorname{rank}(\boldsymbol{M}) = 8$.

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1. SVD of $\boldsymbol{M} = \boldsymbol{U} \Sigma \boldsymbol{V}^\mathsf{T}$:
   - $\sigma_9 \approx 0 \quad \Rightarrow \quad \boldsymbol{f} = \lambda \cdot \boldsymbol{v}_9, \quad$ and since $\|\boldsymbol{F}\|_F = \|\boldsymbol{f}\|_2 = 1 \quad \Rightarrow \quad \boldsymbol{f} = \boldsymbol{v}_9$
   - If $\sigma_9 > \varepsilon \rightarrow$ error

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   - If $\sigma_9 > \varepsilon \rightarrow$ error
2. Enforce $\text{rank}(\boldsymbol{F}) = 2$ using SVD of $\boldsymbol{F} = U_{\boldsymbol{F}}\Sigma_{\boldsymbol{F}}V_{\boldsymbol{F}}^{\mathsf{T}}$:
   - For the fundamental matrix it is: $\sigma_1 \geq \sigma_2 > 0$, $\sigma_3 = 0$.
   - If $\sigma_3 > \varepsilon \rightarrow$ error
   - Set $\sigma_3 = 0$, and compute $\boldsymbol{F}$ using Σ'_F anew:

$$\boldsymbol{F} = U_{\boldsymbol{F}} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_{\boldsymbol{F}}^{\mathsf{T}}.$$

Numerical Instabilities

- Image coordinates are usually defined with respect to the top left corner of the image.
- Thus coordinates vary from 0 to a few hundred.
- The third (homogeneous) coordinate is usually set to 1.

Numerical Instabilities

Normalize the coordinates $\tilde{\boldsymbol{p}}_i^{\mathrm{p}} = \left(\boldsymbol{p}_{1,i}, \boldsymbol{p}_{2,i}, 1\right)^{\mathsf{T}}$ and $\tilde{\boldsymbol{q}}_i^{\mathrm{p}} = \left(\boldsymbol{q}_{1,i}, \boldsymbol{q}_{2,i}, 1\right)^{\mathsf{T}}$ such that the entries of $\boldsymbol{M}$ are of comparable size:

- Translate the origin of the image coordinate system to the centroid of the feature points, that is:
  - to $\left(\frac{1}{N}\sum_i^N \boldsymbol{p}_{1,i}, \frac{1}{N}\sum_i^N \boldsymbol{p}_{2,i}, 1\right)^{\mathsf{T}}$ for the left side,
  - and $\left(\frac{1}{N}\sum_i^N \boldsymbol{q}_{1,i}, \frac{1}{N}\sum_i^N \boldsymbol{q}_{2,i}, 1\right)^{\mathsf{T}}$ for the right image.
- Scale the feature points such that the mean homogeneous point looks like $\frac{1}{\sqrt{2}}(1, 1, 1)^{\mathsf{T}}$, i. e., the mean norm of a 2-D point is $\sqrt{2}$.

This is called ***balancing***.

Topics

Take Home Messages

- We have gone over the eight point algorithm using the fundamental matrix.
- The fundamental matrix maps a point to its epipolar line.
- The epipolar constraint is linear in components of $\boldsymbol{E}$ and $\boldsymbol{F}$.
- Balancing can be used to make an estimation of an essential matrix numerically robust.

Further Readings

Epipolar geometry is nicely introduced in:

Emanuele Trucco and Alessandro Verri. *Introductory Techniques for 3-D Computer Vision*. Upper Saddle River, NJ, USA: Prentice Hall, 1998

All the math regarding epipolar geometry can be found in:

Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685

Magnetic navigation:

Michelle P. Armacost et al. “Accurate and Reproducible Target Navigation with the Stereotaxis Niobe® Magnetic Navigation System”. In: *Journal of Cardiovascular Electrophysiology* 18 (Jan. 2007), S26–S31. DOI: 10.1111/j.1540-8167.2007.00708.x