Medical Image Processing for Interventional Applications

Random Walker – Algorithm

Online Course – Unit 41 Andreas Maier, Stefan Steidl, Frank Schebesch Pattern Recognition Lab (CS 5)













Topics

Random Walks for Image Segmentation

Algorithm

Dirichlet Integral

Decomposition

Solution

Summary

Take Home Messages

Further Readings







K-way image segmentation

- User-defined seeds
- Indicating regions of the image belonging to K objects

Random walk

- Labeling an **unseeded** pixel by resolving the question: What is the probability of a random walker starting at this pixel that it first reaches seed point *k*?
- Selecting the label of the most probable seed destination for each pixel
- Biasing the random walker to avoid crossing sharp intensity gradients







Image as discrete object

- Graph with a fixed number of vertices and edges
- Each node represents one pixel in the image.
- Edges connect neighboring pixels: e.g., 4-connectivity (2-D), 6-connectivity (3-D), 8-connectivity (2-D).
- A real-valued weight is assigned to each edge representing the likelihood that a random walker will cross this edge.
 - → Weight of zero: the random walker may not move along that edge.
- Purely combinatorial operators:
 - No discretization
 - No discretization errors or ambiguities







Edge weights for adjacent pixels *i* and *j*

Gaussian weighting function:

$$w_{ij} = \exp\left(-\beta(g_i - g_j)^2\right)$$

where

- g_i : image intensity at pixel i
- β : only free parameter!
- Useful operation: prior normalization of the square gradients:

$$\forall e_{ij} \in E : (g_i - g_j)^2 \in [0, 1]$$

• Modification to handle color or general vector-valued data: $(g_i - g_j)^2 \longrightarrow ||\boldsymbol{g}_i - \boldsymbol{g}_j||^2$







Four mathematically equivalent ways (Grady, 2006)

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- 3. "Assign the pixel to the label for which its seeds have the largest effective conductance (i. e., smallest effective resistance) with the pixel."







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- 4. "If a 2-tree is drawn randomly from the graph (with probability given by the product of weights in the 2-tree), assign the pixel to the label for which the pixel is most likely to remain connected to."







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Combinatorial Laplacian matrix L

$$L_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -w_{ij} & \text{if } v_i \text{ and } v_j \text{ are adjacent nodes,} \\ 0 & \text{otherwise,} \end{cases}$$

where

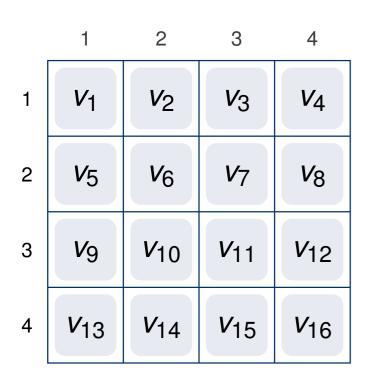
- L_{ii} is indexed by vertices v_i and v_i ,
- $d_i = \sum w(e_{ij})$ for all edges e_{ij} incident on node v_i .







Example: Pixels of a 4×4 *image*

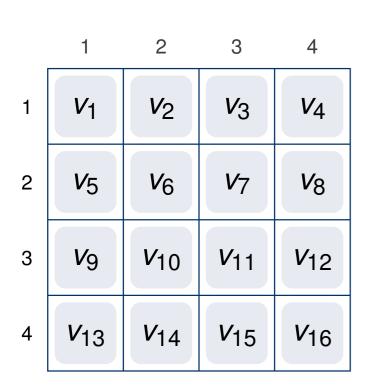








Example: Pixels of a 4×4 *image* and the according *combinatorial Laplacian matrix L*









Combinatorial formulation of the Dirichlet integral

$$D(x) = \frac{1}{2}x^{T}Lx = \frac{1}{2}\sum_{e_{ij}\in E}w_{ij}(x_{i}-x_{j})^{2}$$

Partitioning the vertices into two sets:

- marked/seed nodes V_M ,
- unseeded nodes V_U ,

such that $V_M \cup V_U = V$ and $V_M \cap V_U = \emptyset$.

Without loss of generality: The nodes in L and x are ordered, i. e., seed nodes are first, unseeded nodes are second.







Decomposition

$$D[\mathbf{x}_U] = \frac{1}{2} (\mathbf{x}_M^\mathsf{T} \ \mathbf{x}_U^\mathsf{T}) \begin{bmatrix} \mathbf{L}_M \ \mathbf{B} \end{bmatrix} (\mathbf{x}_M) \\ \mathbf{B}^\mathsf{T} \ \mathbf{L}_U \end{bmatrix} (\mathbf{x}_M) = \frac{1}{2} (\mathbf{x}_M^\mathsf{T} \mathbf{L}_M \mathbf{x}_M + 2\mathbf{x}_U^\mathsf{T} \mathbf{B}^\mathsf{T} \mathbf{x}_M + \mathbf{x}_U^\mathsf{T} \mathbf{L}_U \mathbf{x}_U)$$

L is positive semi-definite, i. e., the only critical points of D[x] will be minima.







Differentiating w. r. t. x_U and finding the critical points:

$$\boldsymbol{L}_{U}\boldsymbol{x}_{U}=-\boldsymbol{B}^{\mathsf{T}}\boldsymbol{x}_{M}$$

- System of linear equations with $|V_U|$ unknowns
- Equation will be non-singular
 - if the graph is connected, or
 - if every connected component contains a seed.







Solution to the combinatorial Dirichlet problem for label s

- x_i^s : probability (potential) assumed at node v_i for label s
- Set of labels: $\forall v_j \in V_M$: $Q(v_j) = s, s \in \mathbb{Z}, 0 < s \le K$
- $V_M \times 1$ vector m^s :

$$m_j^s = \begin{cases} 1 & \text{if } Q(v_j) = s, \\ 0 & \text{if } Q(v_j) \neq s \end{cases}$$







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Solution for all labels:

$$\boldsymbol{L}_{U}\boldsymbol{X} = -\boldsymbol{B}^{\mathsf{T}}\boldsymbol{M}$$

where X, M are matrices with K columns taken by each x^s and m^s , respectively.







Note:

• At any node the probabilities x_i^s will sum to unity:

$$\forall v_i \in V : \sum_{s} x_i^s = 1.$$

• Hence, only K-1 sparse linear systems must be solved.







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Take Home Messages

- For the segmentation using a random walker, we describe pixel transitions from one pixel to a neighboring pixel in form of graph edges.
- In the algorithm the combinatorial Dirichlet problem has to be solved to find a segmentation result.







Further Readings

These slides are based on the following publication:

L. Grady. "Random Walks for Image Segmentation". In: IEEE Transactions on Pattern Analysis and Machine Intelligence 28.11 (Nov. 2006), pp. 1768–1783. DOI: 10.1109/TPAMI.2006.233

His implementations in Matlab can be downloaded here:

- Graph Analysis Toolbox
- Random Walker