Medical Image Processing for Interventional Applications

Super-Resolution: ML Estimation

Online Course – Unit 22

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Topics

Problem Statement

Given: Set of low-resolution frames given as continuous functions (irradiance light fields)

$$y^{(1)}(\boldsymbol{u}), \ldots, y^{(K)}(\boldsymbol{u}),$$

where $\boldsymbol{u} \in \mathbb{R}^2$ (pixel grid)

We want to reconstruct a high-resolution image $x(\boldsymbol{u})$ that generated these frames according to:

$$y^{(k)}(\boldsymbol{u}) = \mathcal{W}^{(k)}\left\{x(\boldsymbol{u})\right\}, \qquad \text{for all } k = 1, \ldots, K,$$

where $\mathcal{W}^{(k)}\{\cdot\}$ is the (frame-wise) image formation model that:

- models characteristics of the camera optics,
- models spatial sampling on the sensor array.

$\rightarrow$ We investigate different approaches to model and solve this inverse problem.

Image Formation Model

Mathematical description of the image formation process:

Given an ideal image $x(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{R}^2$, as continuous function, we can model the formation of a low-resolution image $y^{(k)}(\boldsymbol{u})$:

$$y^{(k)}(\boldsymbol{u}) = \mathcal{D}\left\{\mathcal{M}^{(k)}\{x(\boldsymbol{u})\} * h^{(k)}(\boldsymbol{u})\right\} + \varepsilon(\boldsymbol{u}).$$

$\mathcal{D}\{\cdot\}$ and $\mathcal{M}^{(k)}\{\cdot\}$: sampling and motion operators
$h^{(k)}(\boldsymbol{u})$: space invariant point spread function (PSF)
$\varepsilon(\boldsymbol{u})$: additive noise

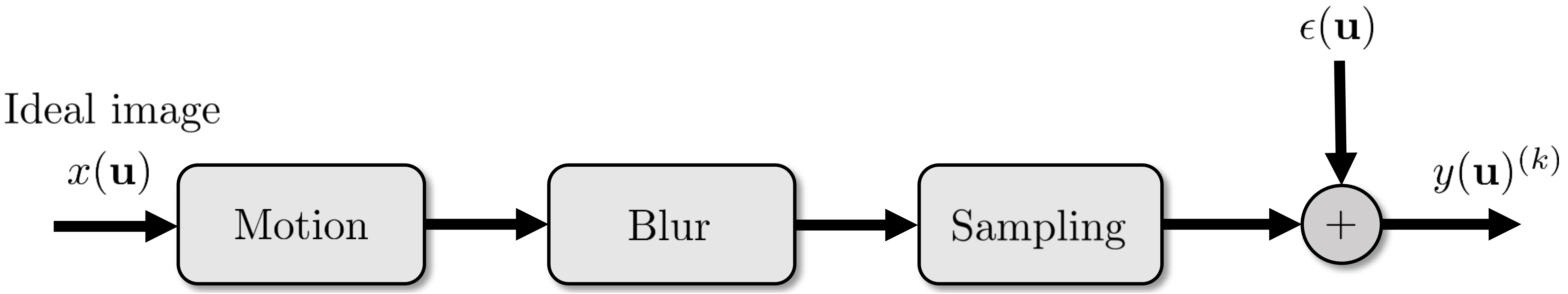

Figure 1: Steps of the image formation from an ideal image to a low-resolution output

Image Formation Model

Discretization of the continuous model:

- We need to discretize the image formation model to employ it for digital super-resolution algorithms:

$$\boldsymbol{y}^{(k)} = \boldsymbol{W}^{(k)}\boldsymbol{x} + \boldsymbol{\varepsilon}^{(k)}.$$

- Image formation modeled by matrix/vector operations:

$\boldsymbol{x}$: high-resolution image $\boldsymbol{x} \in \mathbb{R}^N$,
$\boldsymbol{y}^{(k)}$: k-th low-resolution frame $\boldsymbol{y}^{(k)} \in \mathbb{R}^M$ where $M < N$,
$\boldsymbol{W}^{(k)}$: system matrix of k-th frame to model motion, PSF and downsampling.

Anatomy of the System Matrix

Definition of the matrix: The system matrix models the mapping from $\mathbf{x}$ to $\mathbf{y}^{(k)}$:

$$W_{mn}^{(k)} = h(\mathbf{v}_n - \mathbf{u}'_m)\,,$$

where

$h(\mathbf{u})$:	camera PSF as space and time invariant kernel,
$\mathbf{v}_n$:	coordinates of n-th pixel in $\mathbf{x}$,
$\mathbf{u}'_m$:	coordinates of m-th pixel in $\mathbf{y}$ warped to $\mathbf{x}$.

The elements are normalized according to:

$$\sum_n W_{mn}^{(k)} = 1\,.$$

Example: Isotropic Gaussian kernel of width σ_{PSF}

$$h(\mathbf{u}) = \exp\left(-\frac{||\mathbf{u}||_2^2}{2\sigma_{\text{PSF}}^2}\right)$$

Anatomy of the System Matrix

Properties and practical considerations:

- The system matrix $\mathbf{W}^{(k)}$ consists of:
  - N columns, where N denotes the number of high-resolution pixels,
  - M rows, where M is the number of low-resolution pixels.

  $\longrightarrow$ This is infeasible to store for larger instances (e. g., $N = 1024^2$ and $M = 512^2$).

- For a practical computation, we approximate $\mathbf{W}^{(k)}$ as **sparse matrix** by assuming a narrow kernel $h(\mathbf{u})$:

$$W_{mn}^{(k)} := 0 \quad \text{if } \left|\left|\mathbf{v}_n - \mathbf{u}'_m\right|\right|_2 > d_{max},$$

  e. g., $d_{max} = 3\sigma$ for isotropic Gaussian PSF.

Topics

Super-Resolution as an Inverse Problem

Maximum Likelihood Estimation

- Bayesian Formulation
- Maximum Likelihood Estimation
- Numerical Optimization

Summary

- Take Home Messages
- Further Readings

Bayesian Formulation of Multi-Frame Super-Resolution

Definitions and nomenclature:

Let us assign probability distributions to the quantities of the image formation model:

- We model a high-resolution image with a prior distribution $\mathbf{x} \sim p(\mathbf{x})$.
- Similarly, we model a low-resolution image as random variable $\mathbf{y}^{(k)} \sim p(\mathbf{y}^{(k)})$.

According to Bayes rule we obtain the posterior distribution:

$$p(\mathbf{x} \,|\, \mathbf{y}) = p(\mathbf{x} \,|\, \mathbf{y}^{(1)} \ldots \mathbf{y}^{(K)}) = \frac{p(\mathbf{x}) \cdot p(\mathbf{y}^{(1)} \ldots \mathbf{y}^{(K)} \,|\, \mathbf{x})}{p(\mathbf{y}^{(1)} \ldots \mathbf{y}^{(K)})} = \frac{p(\mathbf{x}) \cdot p(\mathbf{y} \,|\, \mathbf{x})}{p(\mathbf{y})},$$

under the assumption of independent and identically distributed (i.i.d.) observations $\boldsymbol{y}$.

Maximum Likelihood Estimation

Derivation of the log-likelihood:

- For maximum likelihood (ML) estimation, $\boldsymbol{x}$ is assumed to be uniformly distributed (no prior available).
- The negative log-likelihood under this assumption is given by:

$$L(\mathbf{x}, \mathbf{y}^{(1)} \dots \mathbf{y}^{(K)}) = -\log p(\mathbf{y}^{(1)} \dots \mathbf{y}^{(K)} \,|\, \mathbf{x}).$$

  $p(\mathbf{y}^{(1)} \dots \mathbf{y}^{(K)} \,|\, \mathbf{x})$ is referred to as the Bayesian observation model.
- Reconstruct $\mathbf{x}$ that explains $\mathbf{y}$ best:

$$\hat{\mathbf{x}}_{ML} = \arg\max_{\mathbf{x}} p(\boldsymbol{y} \,|\, \boldsymbol{x}) = \arg\min_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}^{(1)} \dots \mathbf{y}^{(K)}).$$

Maximum Likelihood Estimation

Definition of the observation model:

Let $\varepsilon \sim N(0, \sigma^2 \mathbf{I})$ be spatially uncorrelated, additive Gaussian noise:

$$p(\mathbf{y}^{(k)} \,|\, \mathbf{x}) = \left(\frac{1}{2\pi\sigma}\right)^{\frac{M}{2}} \exp\left(-\frac{\left|\left|\mathbf{y}^{(k)} - \mathbf{W}^{(k)}\mathbf{x}\right|\right|_2^2}{2\sigma^2}\right).$$

Using the observation model $p(\mathbf{y}^{(k)} \,|\, \mathbf{x})$, ML estimation is equivalent to the energy minimization:

$$\hat{\mathbf{x}}_{ML} = \arg\min_{\mathbf{x}} \sum_{k=1}^{K} \left|\left|\mathbf{y}^{(k)} - \mathbf{W}^{(k)}\mathbf{x}\right|\right|_2^2 = \arg\min_{\mathbf{x}} ||\mathbf{y} - \mathbf{W}\mathbf{x}||_2^2.$$

Numerical Optimization

Optimization of the log-likelihood:

- **Closed-form solution**: Solve for $\hat{\mathbf{x}}_{ML}$ using the pseudoinverse $\mathbf{W}^{+}$:

$$\hat{\mathbf{x}}_{ML} = \mathbf{W}^{+}\mathbf{y}.$$

  For a large system $\mathbf{W}$, it is not feasible to compute $\mathbf{W}^{+}$ directly.

- **Iterative numerical optimization** to determine $\hat{\mathbf{x}}_{ML}$ from an initial guess $\boldsymbol{x}^0$:
  - Gradient descent iterations: $\boldsymbol{x}^{t+1} = \boldsymbol{x}^t + \alpha^t \cdot \boldsymbol{p}^t$
  - Calculation of the search direction $\boldsymbol{p}^t$ according to steepest descent:

$$\boldsymbol{p}^t = \nabla_{\boldsymbol{x}} ||\mathbf{y} - \mathbf{W}\mathbf{x}||_2^2 = -2\boldsymbol{W}^\top(\mathbf{y} - \mathbf{W}\mathbf{x})$$

    $\longrightarrow$ Different strategies available to compute $\boldsymbol{p}^t$
  - Calculation of α^t by line search or use of constant step size ($\alpha^t = \alpha$)

Topics

Take Home Messages

- The multiframe super-resolution problem can be stated as an inverse problem and yields a system matrix after discretization of the image formation model.
- The system matrix is normally quite large, so that sparsity assumptions are made.
- One possibility to solve the inverse problem is maximum likelihood estimation where high- and low-resolution images are regarded as probability distributions.

Further Readings

Theory of image super-resolution (books and review articles):

- Hayit Greenspan. "Super-Resolution in Medical Imaging". In: *The Computer Journal* 52.1 (Feb. 2008), pp. 43–63. DOI: 10.1093/comjnl/bxm075
- Peyman Milanfar, ed. *Super-Resolution Imaging*. Digital Imaging and Computer Vision. CRC Press, 2011
- Sina Farsiu et al. "Advances and Challenges in Super-Resolution". In: *International Journal of Imaging Systems and Technology* 14.2 (Aug. 2004), pp. 47–57. DOI: 10.1002/ima.20007
- Sung Cheol Park, Min Kyu Park, and Moon Gi Kang. "Super-Resolution Image Reconstruction: A Technical Overview". In: *IEEE Signal Processing Magazine* 20.3 (May 2003), pp. 21–36. DOI: 10.1109/MSP.2003.1203207

ML/MAP super-resolution:

- Lyndsey C. Pickup. "Machine Learning in Multi-frame Image Super-resolution". PhD Thesis. Robotics Research Group, University of Oxford, 2007
- Michael Elad and Arie Feuer. "Restoration of a Single Superresolution Image from Several Blurred, Noisy, and Undersampled Measured Images". In: *IEEE Transactions on Image Processing* 6.12 (Dec. 1997), pp. 1646–1658. DOI: 10.1109/83.650118