Medical Image Processing for Interventional Applications

Guided Filter

Online Course – Unit 16

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Topics

Guided Filter

Summary

Take Home Messages

Further Readings

Guided Filtering

- Proposed at the ECCV 2010 (later published as He, Sun and Tang, 2013)
- Applications:
  - Non-approximative edge-preserving denoising
  - HDR compression
  - Multi-modal image upsampling
  - ...
- Complexity: $\mathcal{O}(N)$
- **Basic idea:** guide the filtering process by a dedicated image $i(\boldsymbol{x})$.

Guided Filtering: Cost Function

Assumption: Output as a linear transform of the guidance $i(\boldsymbol{x})$, i.e.:

$$f(\boldsymbol{x}') = a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}}, \forall \boldsymbol{x}' \in \omega_{\boldsymbol{x}}$$

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Gradient relationship induced by this model:

$$\nabla f = a \nabla i$$

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Cost function:

$$\mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \left(\left(a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}} - g(\boldsymbol{x}') \right)^2 + \epsilon a_{\boldsymbol{x}}^2 \right)$$

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Cost function:

Output

$$\mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \left(\left(\boxed{a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}}} - g(\boldsymbol{x}') \right)^2 + \epsilon a_{\boldsymbol{x}}^2 \right)$$

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$$\nabla f = a \nabla i$$

Cost function:

$$\mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \Big(\big(\underbrace{\boxed{a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}}}}_{\text{Output}} - \underbrace{\boxed{g(\boldsymbol{x}')}}_{\text{Input}} \big)^2 + \epsilon a_{\boldsymbol{x}}^2 \Big)$$

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Gradient relationship induced by this model:

$$\nabla f = a \nabla i$$

Cost function:

Output Input

$$\mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \left(\left(\boxed{a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}}} - \boxed{g(\boldsymbol{x}')} \right)^2 + \boxed{\epsilon} a_{\boldsymbol{x}}^2 \right)$$

Regularization

Guided Filtering: Partial Derivatives of the Cost Function

Cost function:

$$\mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \left((a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}} - g(\boldsymbol{x}'))^2 + \epsilon a_{\boldsymbol{x}}^2 \right)$$

Guided Filtering: Partial Derivatives of the Cost Function

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Partial derivatives:

$$\frac{\partial}{\partial a_{\boldsymbol{x}}} \mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = 2 \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \left((a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}} - g(\boldsymbol{x}')) i(\boldsymbol{x}') + \epsilon a_{\boldsymbol{x}} \right) \overset{!}{=} 0$$

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$$\frac{\partial}{\partial a_{\boldsymbol{x}}} \mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = 2 \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \left((a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}} - g(\boldsymbol{x}')) i(\boldsymbol{x}') + \epsilon a_{\boldsymbol{x}} \right) \overset{!}{=} 0$$

$$\frac{\partial}{\partial b_{\boldsymbol{x}}} \mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = 2 \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} (a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}} - g(\boldsymbol{x}'))$$

$$= 2 a_{\boldsymbol{x}} \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}') + 2 b_{\boldsymbol{x}} \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} 1 - 2 \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} g(\boldsymbol{x}') \overset{!}{=} 0$$

Guided Filtering: Partial Derivatives of the Cost Function

Cost function:

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$$\frac{\partial}{\partial b_{\boldsymbol{x}}} \mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = 2 \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \left(a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}} - g(\boldsymbol{x}')\right)$$

$$= 2 a_{\boldsymbol{x}} \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}') + 2 b_{\boldsymbol{x}} \boxed{\sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} 1} - 2 \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} g(\boldsymbol{x}') \stackrel{!}{=} 0$$

$$\boxed{\sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} 1 = |\omega_{\boldsymbol{x}}|}$$

Relation between Parameter b_x and Mean Filtering

Deriving $b_{\boldsymbol{x}}$:

$$\frac{1}{2}\frac{\partial}{\partial b_{\boldsymbol{x}}}\mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = a_{\boldsymbol{x}}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} i(\boldsymbol{x}') + b_{\boldsymbol{x}}|\omega_{\boldsymbol{x}}| - \sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} g(\boldsymbol{x}') \stackrel{!}{=} 0$$

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$$b_{\boldsymbol{x}} = \frac{1}{|\omega_{\boldsymbol{x}}|} \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} g(\boldsymbol{x}') - a_{\boldsymbol{x}}\left(\frac{1}{|\omega_{\boldsymbol{x}}|} \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}')\right)$$

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mean/box filter

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mean/box filter

Probabilistic formulation: Mean filtering yields the expectation value $\mathrm{E}_{\omega_{\boldsymbol{x}}}$ if we interprete $g(\boldsymbol{x}), i(\boldsymbol{x})$ as uniformly distributed random variables.

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$$b_{\boldsymbol{x}} = \boxed{\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}}} g(\boldsymbol{x}') - a_{\boldsymbol{x}}\left(\boxed{\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}}} i(\boldsymbol{x}')\right)$$

mean/box filter

Probabilistic formulation: Mean filtering yields the expectation value $\mathrm{E}_{\omega_{\boldsymbol{x}}}$ if we interprete $g(\boldsymbol{x}), i(\boldsymbol{x})$ as uniformly distributed random variables.

$$\Rightarrow \quad b_{\boldsymbol{x}} = E_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})] - a_{\boldsymbol{x}} E_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})]$$

Relation between Parameter $a_{\boldsymbol{x}}$ and Mean Filtering

Deriving $a_{\boldsymbol{x}}$:

$$\frac{1}{2}\frac{\partial}{\partial a_{\boldsymbol{x}}}\mathcal{J}(a_{\boldsymbol{x}}, b_{\boldsymbol{x}}) = \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} \left((a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}} - g(\boldsymbol{x}')) i(\boldsymbol{x}') + \epsilon a_{\boldsymbol{x}} \right) \overset{!}{=} 0$$

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$$b_{\boldsymbol{x}} = E_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})] - a_{\boldsymbol{x}} E_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})]$$

Some calculations later …

$$a_{\boldsymbol{x}} \left(\sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}')i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})] \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}') + \epsilon \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} 1 \right) = \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} g(\boldsymbol{x}')i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})] \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}')$$

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$$\boxed{\cdot \frac{1}{|\omega_{\boldsymbol{x}}|}}$$

$$\boxed{a_{\boldsymbol{x}} \left(\sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}') i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})] \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}') + \epsilon \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} 1 \right)} = \boxed{\sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} g(\boldsymbol{x}') i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})] \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} i(\boldsymbol{x}')}$$

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Some calculations later …

$$\boxed{\cdot\frac{1}{|\omega_{\boldsymbol{x}}|}}$$

$$\boxed{a_{\boldsymbol{x}}\left(\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} i(\boldsymbol{x}')i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})]\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} i(\boldsymbol{x}') + \epsilon\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} 1\right)} = \boxed{\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} g(\boldsymbol{x}')i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})]\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} i(\boldsymbol{x}')}$$

$$\Rightarrow \quad a_{\boldsymbol{x}}\left(\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} i(\boldsymbol{x}')i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})]\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} i(\boldsymbol{x}') + \epsilon\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} 1\right)$$

$$= \frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} g(\boldsymbol{x}')i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})]\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}} i(\boldsymbol{x}')$$

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$$\frac{1}{2}\frac{\partial}{\partial a_{\boldsymbol{x}}}\mathcal{J}(a_{\boldsymbol{x}},b_{\boldsymbol{x}}) = \sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}}\left((a_{\boldsymbol{x}} i(\boldsymbol{x}')+b_{\boldsymbol{x}}-g(\boldsymbol{x}'))i(\boldsymbol{x}')+\epsilon a_{\boldsymbol{x}}\right) \overset{!}{=} 0$$

Some calculations later …

mean/box filter

$$\Rightarrow \quad a_{\boldsymbol{x}}\left(\boxed{\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}}} i(\boldsymbol{x}')i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})]\boxed{\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}}} i(\boldsymbol{x}') + \epsilon\cancel{\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}}} 1\right)$$

$$= \boxed{\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}}} g(\boldsymbol{x}')i(\boldsymbol{x}') - E_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})]\boxed{\frac{1}{|\omega_{\boldsymbol{x}}|}\sum_{\boldsymbol{x}'\in\omega_{\boldsymbol{x}}}} i(\boldsymbol{x}')$$

Relation between Parameter $a_{\boldsymbol{x}}$ and Mean Filtering

Therefore we found:

$$a_{\boldsymbol{x}}\left(\mathrm{E}_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})i(\boldsymbol{x})] - \mathrm{E}_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})]\mathrm{E}_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})] + \epsilon\right) = \mathrm{E}_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})i(\boldsymbol{x})] - \mathrm{E}_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})]\mathrm{E}_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})].$$

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Using the computational formulas for variance and covariance:

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2, \qquad \text{and} \qquad \mathrm{Cov}[X,Y] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y],$$

we finally obtain:

$$a_{\boldsymbol{x}} = \frac{\mathrm{Cov}_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x}), i(\boldsymbol{x})]}{\mathrm{Var}_{\omega_{\boldsymbol{x}}}[i(\boldsymbol{x})] + \epsilon}.$$

Special Cases

- Guided filtering, linear model: $f(\boldsymbol{x}') = a_{\boldsymbol{x}} i(\boldsymbol{x}') + b_{\boldsymbol{x}}, \forall \boldsymbol{x}' \in \omega_{\boldsymbol{x}}$
- Lets consider the case that $i(\boldsymbol{x}) = g(\boldsymbol{x})$:

$$a_{\boldsymbol{x}} = \frac{\mathrm{Var}_{\omega_{\boldsymbol{x}}}\big(g(\boldsymbol{x})\big)}{\mathrm{Var}_{\omega_{\boldsymbol{x}}}\big(g(\boldsymbol{x})\big) + \epsilon}, \qquad b_{\boldsymbol{x}} = \mathrm{E}_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})] - a_{\boldsymbol{x}} \mathrm{E}_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})].$$

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- Trivial case $\epsilon = 0$:
  - **Flat patch**: if $g(\boldsymbol{x})$ is constant across $\omega_{\boldsymbol{x}}$: $a_{\boldsymbol{x}} = 0, b_{\boldsymbol{x}} = \mathrm{E}_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})]$
  - **High variance**: if $g(\boldsymbol{x})$ changes across $\omega_{\boldsymbol{x}}$: $a_{\boldsymbol{x}} = 1, b_{\boldsymbol{x}} = 0$

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- The criterion of **flat patch** or **high variance** is given by ϵ.

Guided Filtering

- So far, only **one** local window was considered.
  - → Apply the model to **all** local windows.
  - → Pixel $\boldsymbol{x}$ is involved in all local windows $\omega_{\boldsymbol{x}'}$ that contain $\boldsymbol{x}$.
  - → Average all possible local coefficients $a_{\boldsymbol{x}'}, b_{\boldsymbol{x}'}$.

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- Final filter output for all local windows:

$$f_{\mathrm{GF}}(\boldsymbol{x}) = \frac{1}{\left|\omega_{\boldsymbol{x}'}\right|} \sum_{\boldsymbol{x}' :\, \boldsymbol{x} \in \omega_{\boldsymbol{x}'}} \left(a_{\boldsymbol{x}'} i(\boldsymbol{x}) + b_{\boldsymbol{x}'}\right) = \mathrm{E}_{\omega_{\boldsymbol{x}'}}[a_{\boldsymbol{x}}] i(\boldsymbol{x}) + \mathrm{E}_{\omega_{\boldsymbol{x}'}}[b_{\boldsymbol{x}}]$$

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➔ **Guided filtering can be expressed with mean filters only!**

Guided Filtering

- Mean or box filtering is the backbone of guided filtering:

$$\mathrm{E}_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})] = \frac{1}{|\omega_{\boldsymbol{x}}|} \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} g(\boldsymbol{x}') = \{g * \mathcal{M}_r\}(\boldsymbol{x})$$

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- Integral images (Viola and Jones, 2001):

$$\vartheta(x, y) = \sum_{\substack{x' \leq x \\ y' \leq y}} g(x', y')$$

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$$\vartheta(x, y) = \sum_{\substack{x' \leq x \\ y' \leq y}} g(x', y')$$

- Mean filtering using integral images:

$$\{g * \mathcal{M}_r\}(x, y) = \big(\vartheta(x + r, y + r) - \vartheta(x + r, y - r - 1)\big) - \big(\vartheta(x - r - 1, y + r) - \vartheta(x - r - 1, y - r - 1)\big)$$

Guided Filtering

- Mean or box filtering is the backbone of guided filtering:

$$\mathrm{E}_{\omega_{\boldsymbol{x}}}[g(\boldsymbol{x})] = \frac{1}{|\omega_{\boldsymbol{x}}|} \sum_{\boldsymbol{x}' \in \omega_{\boldsymbol{x}}} g(\boldsymbol{x}') = \{g * \mathcal{M}_r\}(\boldsymbol{x})$$

- Integral images (Viola and Jones, 2001):

$$\vartheta(x, y) = \sum_{\substack{x' \leq x \\ y' \leq y}} g(x', y')$$

- Mean filtering using integral images:

$$\{g * \mathcal{M}_r\}(x, y) = \overbrace{\boxed{\big(\vartheta(x + r, y + r) - \vartheta(x + r, y - r - 1)\big) - \big(\vartheta(x - r - 1, y + r) - \vartheta(x - r - 1, y - r - 1)\big)}}^{\mathcal{O}(1)}$$

Example

1	32	20	3	31	16
8	15	16	21	1	8
30	9	26	13	18	16
26	22	18	8	30	19
29	24	1	21	19	3
17	12	11	24	29	2

Figure 1: Image data

1	33	53	56	87	103
9	56	92	116	148	172
39	95	157	194	244	284
65	143	223	268	348	407
94	196	277	343	442	504
111	225	317	407	535	599

Figure 2: Integral image representation

Example

$\mathcal{M}_1$

1	32	20	3	31	16
8	15	16	21	1	8
30	9	26	13	18	16
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29	24	1	21	19	3
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Figure 2: Integral image representation

$$26 + 13 + 18 + 18 + 8 + 30 + 1 + 21 + 19 = 154$$

Example

$\vartheta(x-r-1, y-r-1)$

$\mathcal{M}_1$

1	32	20	3	31	16
8	15	16	21	1	8
30	9	26	13	18	16
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Figure 2: Integral image representation

$\vartheta(x+r, y+r)$

$$26 + 13 + 18 + 18 + 8 + 30 + 1 + 21 + 19 = 154$$

Example

$\mathcal{M}_1$

1	32	20	3	31	16
8	15	16	21	1	8
30	9	26	13	18	16
26	22	18	8	30	19
29	24	1	21	19	3
17	12	11	24	29	2

Figure 1: Image data

$\vartheta(x-r-1, y-r-1)$

$\vartheta(x+r, y-r-1)$

$\vartheta(x-r-1, y+r)$

$\vartheta(x+r, y+r)$

1	33	53	56	87	103
9	56	92	116	148	172
39	95	157	194	244	284
65	143	223	268	348	407
94	196	277	343	442	504
111	225	317	407	535	599

Figure 2: Integral image representation

$$26 + 13 + 18 + 18 + 8 + 30 + 1 + 21 + 19 = 154$$

$$(442 - 148) - (196 - 56) = 154$$

Topics

Guided Filter

Summary

Take Home Messages

Further Readings

Take Home Messages

- As the name expresses, a guided filter makes use of a guidance image to model, e.g., a smooth but edge-preserving filter.
- From the derivation of a guided filter by using a model that is linear in a neighborhood $\omega_{\boldsymbol{x}}$ of $\boldsymbol{x}$, we found a relationship of the filter with mean filtering of both guidance and target image.
- Mean or box filtering can be computed efficiently using integral images.

Further Readings

- Kaiming He, Jian Sun, and Xiaoou Tang. "Guided Image Filtering". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 35.6 (June 2013), pp. 1397–1409. DOI: 10.1109/TPAMI.2012.213
- Paul Viola and Michael Jones. "Rapid Object Detection Using a Boosted Cascade of Simple Features". In: *Proceedings of the 2001 IEEE Computer Society Conference on Computer Vision and Pattern Recognition. CVPR 2001*. Vol. 1. IEEE, Dec. 2001, pp. I-511–I-518. DOI: 10.1109/CVPR.2001.990517