Medical Image Processing for Interventional Applications

Epipolar Constraint and Essential Matrix

Online Course – Unit 31

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Topics

Epipolar Constraint and Essential Matrix

In the last unit we have introduced the ***epipolar constraint***:

$$(\tilde{\boldsymbol{q}}^{\mathrm{i}})^{\mathsf{T}} \cdot \boldsymbol{E} \cdot \tilde{\boldsymbol{p}}^{\mathrm{i}} = 0,$$

for normalized coordinates ($f = 1$) and with the essential matrix $\boldsymbol{E}$.

Now we want to discuss some of its properties.

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- If we inspect the matrix
$$\boldsymbol{E}^{\mathsf{T}}\boldsymbol{E} = (\boldsymbol{R}[\boldsymbol{t}]_{\times})^{\mathsf{T}}\boldsymbol{R}[\boldsymbol{t}]_{\times} = [\boldsymbol{t}]_{\times}{}^{\mathsf{T}}[\boldsymbol{t}]_{\times},$$
we recognize that $\boldsymbol{E}^{\mathsf{T}}\boldsymbol{E}$ is independent from $\boldsymbol{R}$.

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- Two nonzero singular values are identical.
- It is $\boldsymbol{l}_2 \boldsymbol{E} \tilde{\boldsymbol{p}}^i = 0$ and $(\tilde{\boldsymbol{q}}^i)^T \boldsymbol{E} \boldsymbol{l}_1{}^T = 0$

System of Linear Equations

For all points $(\tilde{\boldsymbol{q}}_k^{\mathrm{i}}, \tilde{\boldsymbol{p}}_k^{\mathrm{i}})$, $k = 1, 2, \ldots, N$, we have

$$\left(\tilde{\boldsymbol{q}}_k^{\mathrm{i}}\right)^{\mathrm{T}} \cdot \boldsymbol{E} \cdot \tilde{\boldsymbol{p}}_k^{\mathrm{i}} = 0.$$

These equations are **linear** in the unknowns, i. e., the components of $\boldsymbol{E}$:

$$\left(\tilde{q}_{k,1}^{\mathrm{i}}, \tilde{q}_{k,2}^{\mathrm{i}}, \tilde{q}_{k,3}^{\mathrm{i}}\right) \begin{pmatrix} e_{1,1} & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2} & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix} \begin{pmatrix} \tilde{p}_{k,1}^{\mathrm{i}} \\ \tilde{p}_{k,2}^{\mathrm{i}} \\ \tilde{p}_{k,3}^{\mathrm{i}} \end{pmatrix} = 0.$$

For the k-th equation we get:

$$\left(\tilde{q}_{k,1}^{\mathrm{i}} e_{1,1} + \tilde{q}_{k,2}^{\mathrm{i}} e_{2,1} + \tilde{q}_{k,3}^{\mathrm{i}} e_{3,1}\right) \tilde{p}_{k,1}^{\mathrm{i}} + \left(\tilde{q}_{k,1}^{\mathrm{i}} e_{1,2} + \tilde{q}_{k,2}^{\mathrm{i}} e_{2,2} + \tilde{q}_{k,3}^{\mathrm{i}} e_{3,2}\right) \tilde{p}_{k,2}^{\mathrm{i}} + \left(\tilde{q}_{k,1}^{\mathrm{i}} e_{1,3} + \tilde{q}_{k,2}^{\mathrm{i}} e_{2,3} + \tilde{q}_{k,3}^{\mathrm{i}} e_{3,3}\right) \tilde{p}_{k,3}^{\mathrm{i}} = 0.$$

Measurement Matrix

$$
\boldsymbol{M}\boldsymbol{e} = \begin{pmatrix}
\tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,1} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,2} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,3} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,2}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,1} & \cdots & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{1,3}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{1,3} \\
\tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,1} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,2} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,3} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,2}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,1} & \cdots & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{2,3}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{2,3} \\
\vdots & & \ddots & & & \vdots \\
\tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,1} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,2} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,1}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,3} & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,2}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,1} & \cdots & \tilde{\boldsymbol{q}}^{\mathrm{i}}_{N,3}\tilde{\boldsymbol{p}}^{\mathrm{i}}_{N,3}
\end{pmatrix}
\begin{pmatrix} e_{1,1} \\ e_{1,2} \\ e_{1,3} \\ e_{2,1} \\ e_{2,2} \\ e_{2,3} \\ e_{3,1} \\ e_{3,2} \\ e_{3,3} \end{pmatrix}
$$

The Seaman's Algorithm

read point correspondences $\{(\tilde{\boldsymbol{p}}_i^{\mathrm{i}}, \tilde{\boldsymbol{q}}_i^{\mathrm{i}}), i = 1, \dots, N\}$
rewrite the epipolar constraints in terms of linear equations: $\boldsymbol{Me} = 0$
compute the SVD of $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}$
return last column of $\boldsymbol{V}$

Figure 1: Seaman's algorithm for $\boldsymbol{E}$

Topics

Take Home Messages

- We can compute the essential matrix by solving a linear system of equations.
- One possibility to implement this is by using Seaman's algorithm.

Further Readings

Epipolar geometry is nicely introduced in:

Emanuele Trucco and Alessandro Verri. *Introductory Techniques for 3-D Computer Vision*. Upper Saddle River, NJ, USA: Prentice Hall, 1998

All the math regarding epipolar geometry can be found in:

Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685

Magnetic navigation:

Michelle P. Armacost et al. “Accurate and Reproducible Target Navigation with the Stereotaxis Niobe® Magnetic Navigation System”. In: *Journal of Cardiovascular Electrophysiology* 18 (Jan. 2007), S26–S31. DOI: 10.1111/j.1540-8167.2007.00708.x