# Medical Image Processing for Interventional Applications

## Factorization for Perspective Projections

Online Course – Unit 40

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Perspective Factorization

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# Perspective Factorization

**Historical remarks:**

- Projective factorization method introduced by Sturm and Triggs (1996)
- Algorithm very similar to orthographic factorization

# Perspective Factorization

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**Preliminaries:**

- Perspective projection model
- Projection matrix for the $i$-th frame is denoted by $\boldsymbol{P}_i \in \mathbb{R}^{3\times 4}$.
- Number of frames $N_\mathrm{F} \geq 3$
- $j$-th world point $\tilde{\boldsymbol{p}}_j^w \in \mathbb{R}^4$ is represented in homogeneous coordinates and is visible in **all** frames.
- World points are **not** all coplanar.
- $\tilde{\boldsymbol{q}}_{i,j}^{\boldsymbol{i}} = (x_{i,j}, y_{i,j}, 1)^\mathrm{T} \in \mathbb{R}^3$ is the homogeneous vector associated with the $j$-th image point in the $i$-th frame.
- $\lambda_{i,j}$ is the scaling factor (**projective depth**) of the $j$-th image point in the $i$-th frame.

# Perspective Factorization

**Factorization using homogeneous coordinates:**

The world point is projected to the image point by perspective projection:

$$\lambda_{i,j}\tilde{\boldsymbol{q}}^{i}_{i,j} = \boldsymbol{P}_i\tilde{\boldsymbol{p}}^{w}_{j}.$$

In components:

$$\lambda_{i,j}\begin{pmatrix} x_{i,j} \\ y_{i,j} \\ 1 \end{pmatrix} = \begin{pmatrix} p_{i,1,1} & p_{i,1,2} & p_{i,1,3} & p_{i,1,4} \\ p_{i,2,1} & p_{i,2,2} & p_{i,2,3} & p_{i,2,4} \\ p_{i,3,1} & p_{i,3,2} & p_{i,3,3} & p_{i,3,4} \end{pmatrix}\begin{pmatrix} x_j \\ y_j \\ z_j \\ 1 \end{pmatrix}$$

# Perspective Factorization

Considering all points simultaneously, matrix notation gives us the following factorization of the measurement matrix $\boldsymbol{M}$:

$$\begin{pmatrix} \lambda_{1,1}\tilde{\boldsymbol{q}}^i_{1,1} & \lambda_{1,2}\tilde{\boldsymbol{q}}^i_{1,2} & \cdots & \lambda_{1,N_\text{p}}\tilde{\boldsymbol{q}}^i_{1,N_\text{p}} \\ \lambda_{2,1}\tilde{\boldsymbol{q}}^i_{2,1} & \lambda_{2,2}\tilde{\boldsymbol{q}}^i_{2,2} & \cdots & \lambda_{2,N_\text{p}}\tilde{\boldsymbol{q}}^i_{2,N_\text{p}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N_\text{F},1}\tilde{\boldsymbol{q}}^i_{N_\text{F},1} & \lambda_{N_\text{F},2}\tilde{\boldsymbol{q}}^i_{N_\text{F},2} & \cdots & \lambda_{N_\text{F},N_\text{p}}\tilde{\boldsymbol{q}}^i_{N_\text{F},N_\text{p}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \vdots \\ \boldsymbol{P}_{N_\text{F}} \end{pmatrix} \left(\tilde{\boldsymbol{p}}^w_1, \tilde{\boldsymbol{p}}^w_2, \dots, \tilde{\boldsymbol{p}}^w_{N_\text{p}}\right) .$$

# Perspective Factorization

**Assumption:** Projective depth $\lambda_{i,j}$ is known.

- In the perspective case the measurement matrix has rank 4, since it is a product of two matrices where the first factor has 4 columns and the second one 4 rows.
- Rank criterion can be enforced by SVD: all but the first four singular values are set to zero.

# Perspective Factorization

- Use the SVD of the rank enforced measurement matrix

$$\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}.$$

- Up to a $4 \times 4$ projective transform we get the following motion and scene parameters:
  - projection matrices result from:

$$\begin{pmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \vdots \\ \boldsymbol{P}_{N_\mathrm{F}} \end{pmatrix} = \boldsymbol{U}\boldsymbol{\Sigma},$$

  - 3-D scene points are:

$$\left(\tilde{\boldsymbol{p}}_1^w, \tilde{\boldsymbol{p}}_2^w, \ldots, \tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w\right) = \boldsymbol{V}^{\mathrm{T}}.$$

# Perspective Factorization

**Estimating projective depths:**

Apply the following iterative scheme:

1. Initialize $\lambda_{i,j} := 1$.
2. Normalize projective depths by scaling $\boldsymbol{M}$ such that column and row vectors have norm 1.
3. Enforce the rank criterion for the measurement matrix.
4. Use SVD to estimate projection matrices and 3-D structure.
5. Project estimated points anew into each frame and update all $\lambda_{i,j}$.
6. If the projective depths change significantly, go back to step 2.

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# Take Home Messages

- In many ways, perspective factorization is similar to orthogonal factorization, but here we additionally need to estimate the projective depths.
- The projective depths are estimated iteratively along with the factorization algorithm.

## Further Readings


- Carlo Tomasi and Takeo Kanade. "Shape and Motion from Image Streams Under Orthography: A Factorization Method". In: *International Journal of Computer Vision* 9.2 (Nov. 1992), pp. 137–154. DOI: 10.1007/BF00129684
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- Mei Han and Takeo Kanade. "A Perspective Factorization Method for Euclidean Reconstruction with Uncalibrated Cameras". In: *The Journal of Visualization and Computer Animation* 13.4 (2002), pp. 211–223. DOI: 10.1002/vis.290
- Peter Sturm and Bill Triggs. "A Factorization Based Algorithm for Multi-Image Projective Structure and Motion". In: *Computer Vision — ECCV '96: 4th European Conference on Computer Vision Cambridge, UK, April 15–18, 1996 Proceedings Volume II*. ed. by Bernard Buxton and Roberto Cipolla. Vol. 1065. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, 1996, pp. 709–720. DOI: 10.1007/3-540-61123-1_183