Medical Image Processing for Interventional Applications

Factorization for Orthographic Projections

Online Course – Unit 39

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Factorization Methods

Preliminaries:

- Orthogonal projection model
- Number of frames $N_\text{F} \geq 3$
- Each world point $\tilde{\boldsymbol{p}}_j^w$ is visible in **all** frames.
- The world points are **not** all coplanar.
- $(x_{ij}, y_{ij})^\text{T} \in \mathbb{R}^2$ is the j-th image point in the i-th frame.

Factorization Methods

Idea:

- Put all image points together in one matrix $\boldsymbol{M}$,
- then **factorize** $\boldsymbol{M}$ into a product of two matrices, a projection-matrix $\boldsymbol{R}$, and a matrix $\boldsymbol{S}$ (world-points):

$$\boldsymbol{M} = \boldsymbol{R}\boldsymbol{S}.$$

In general we have:

- $\boldsymbol{R}$ is a $3N_\mathrm{F} \times 4$ matrix containing all projection matrices,
- $\boldsymbol{S}$ is a $4 \times N_\mathrm{p}$ matrix containing all world points (N_p = number of all points).

In the case of orthogonal projections, the homogeneous form is not necessary, thus:

- $\boldsymbol{R}$ is $2N_\mathrm{F} \times 3$,
- $\boldsymbol{S}$ is $3 \times N_\mathrm{p}$.

Measurement Matrix

Form the so-called ***measurement matrix*** $\boldsymbol{M}$ of size $2N_{\text{F}} \times N_{\text{p}}$ from the image points:

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{pmatrix},$$

where

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1N_{\text{p}}} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N_{\text{F}}1} & x_{N_{\text{F}}2} & \dots & x_{N_{\text{F}}N_{\text{p}}} \end{pmatrix}, \qquad \boldsymbol{Y} = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1N_{\text{p}}} \\ \vdots & \vdots & \ddots & \vdots \\ y_{N_{\text{F}}1} & y_{N_{\text{F}}2} & \dots & y_{N_{\text{F}}N_{\text{p}}} \end{pmatrix}.$$

Registered Measurement Matrix

For factorization we need the **registered** measurement matrix $\widehat{\boldsymbol{M}}$ containing all 2-D points $(x_{ij}, y_{ij})^{\mathrm{T}}$ shifted so that their mean is 0, i. e.,

$$\widehat{\boldsymbol{M}} = \begin{pmatrix} \widehat{\boldsymbol{X}} \\ \widehat{\boldsymbol{Y}} \end{pmatrix},$$

where the entries of $\widehat{\boldsymbol{X}}$, $\widehat{\boldsymbol{Y}}$ are:

$$\widehat{x}_{ij} = x_{ij} - \bar{x}_i\,, \qquad \widehat{y}_{ij} = y_{ij} - \bar{y}_i\,,$$

with

$$\bar{x}_i = \frac{1}{N_\mathrm{p}} \sum_{j=1}^{N_\mathrm{p}} x_{ij}\,, \qquad \bar{y}_i = \frac{1}{N_\mathrm{p}} \sum_{j=1}^{N_\mathrm{p}} y_{ij}\,.$$

Representation of 2-D Image Points

Now consider the following representation of image points:

$$x_{ij} = \boldsymbol{u}_i^{\mathrm{T}} \left(\tilde{\boldsymbol{p}}_j^w - \boldsymbol{t}_i \right), \quad y_{ij} = \boldsymbol{v}_i^{\mathrm{T}} \left(\tilde{\boldsymbol{p}}_j^w - \boldsymbol{t}_i \right),$$

where

- $\boldsymbol{u}_i$, $\boldsymbol{v}_i$ are unit vectors of image reference frame i (3-D vectors),
- $\boldsymbol{t}_i$ is the translation vector from world-origin to frame origin,
- $\tilde{\boldsymbol{p}}_j^w$ is a 3-D world point,
- the world coordinate system is object-centered:

$$\frac{1}{N_{\mathrm{p}}} \sum_{j=1}^{N_{\mathrm{p}}} \tilde{\boldsymbol{p}}_j^w = 0.$$

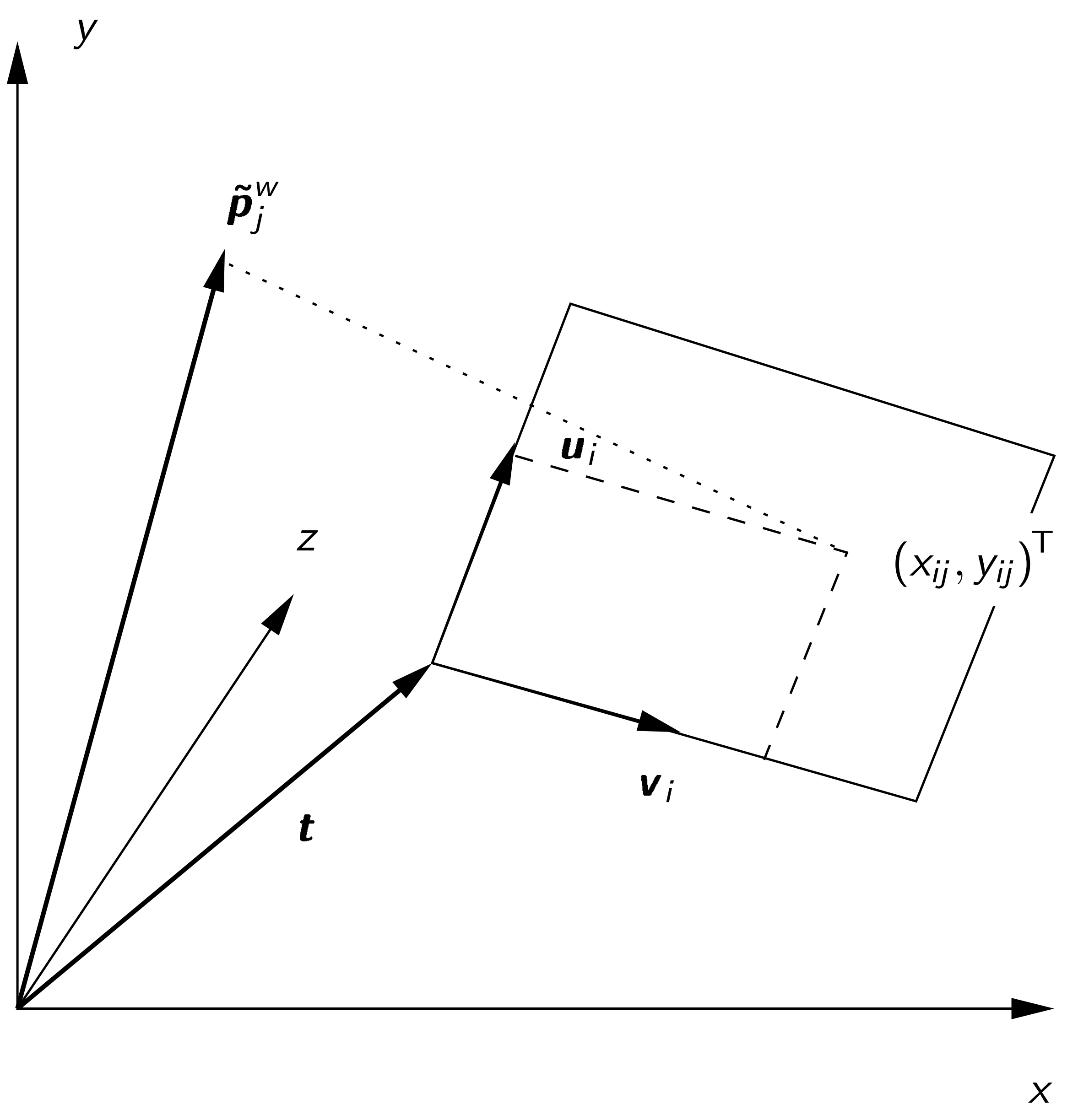

Figure 1: Image planes in 3-D

Representation of 2-D Image Points

Thus we get:

$$
\begin{aligned}
\widehat{x}_{ij} = x_{ij} - \bar{x}_i &= \boldsymbol{u}_i^\mathsf{T}(\tilde{\boldsymbol{p}}_j^w - \boldsymbol{t}_i) - \frac{1}{N_\mathrm{p}} \sum_{m=1}^{N_\mathrm{p}} \left(\boldsymbol{u}_i^\mathsf{T}(\tilde{\boldsymbol{p}}_m^w - \boldsymbol{t}_i)\right) \\
&= \boldsymbol{u}_i^\mathsf{T}\tilde{\boldsymbol{p}}_j^w - \boldsymbol{u}_i^\mathsf{T}\boldsymbol{t}_i - \boldsymbol{u}_i^\mathsf{T}\left(\left(\frac{1}{N_\mathrm{p}} \sum_{m=1}^{N_\mathrm{p}} \tilde{\boldsymbol{p}}_m^w\right) - \boldsymbol{t}_i\right) \\
&= \boldsymbol{u}_i^\mathsf{T}\tilde{\boldsymbol{p}}_j^w \,.
\end{aligned}
$$

Computation of Registered Image Points

With $\widehat{x}_{ij} = \boldsymbol{u}_i^\mathrm{T} \tilde{\boldsymbol{p}}_j^w$ and $\widehat{y}_{ij} = \boldsymbol{v}_i^\mathrm{T} \tilde{\boldsymbol{p}}_j^w$ the registered measurement matrix looks as follows:

$$\widehat{\boldsymbol{M}} = \begin{pmatrix} \boldsymbol{u}_1^\mathrm{T} \tilde{\boldsymbol{p}}_1^w & \boldsymbol{u}_1^\mathrm{T} \tilde{\boldsymbol{p}}_2^w & \dots & \boldsymbol{u}_1^\mathrm{T} \tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{u}_{N_\mathrm{F}}^\mathrm{T} \tilde{\boldsymbol{p}}_1^w & \boldsymbol{u}_{N_\mathrm{F}}^\mathrm{T} \tilde{\boldsymbol{p}}_2^w & \dots & \boldsymbol{u}_{N_\mathrm{F}}^\mathrm{T} \tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \\ & & & \\ \boldsymbol{v}_1^\mathrm{T} \tilde{\boldsymbol{p}}_1^w & \boldsymbol{v}_1^\mathrm{T} \tilde{\boldsymbol{p}}_2^w & \dots & \boldsymbol{v}_1^\mathrm{T} \tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{v}_{N_\mathrm{F}}^\mathrm{T} \tilde{\boldsymbol{p}}_1^w & \boldsymbol{v}_{N_\mathrm{F}}^\mathrm{T} \tilde{\boldsymbol{p}}_2^w & \dots & \boldsymbol{v}_{N_\mathrm{F}}^\mathrm{T} \tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_1^\mathrm{T} \\ \vdots \\ \boldsymbol{u}_{N_\mathrm{F}}^\mathrm{T} \\ \boldsymbol{v}_1^\mathrm{T} \\ \vdots \\ \boldsymbol{v}_{N_\mathrm{F}}^\mathrm{T} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{p}}_1^w & \tilde{\boldsymbol{p}}_2^w & \dots & \tilde{\boldsymbol{p}}_{N_\mathrm{p}}^w \end{pmatrix}.$$

Notes

- $\widehat{\boldsymbol{M}}$ can be factorized into:
  - a $2N_\mathrm{F} \times 3$ matrix $\boldsymbol{R}$ containing camera movement,
  - and a $3 \times N_\mathrm{p}$ matrix $\boldsymbol{S}$ containing 3-D points.
- $\widehat{\boldsymbol{M}}$ is always of rank 3, since
  - $\boldsymbol{u}_i$, $\boldsymbol{v}_i$, $\tilde{\boldsymbol{p}}_j^w$ are 3-vectors,
  - and the world points are **not** all coplanar.
- Factorization can be done using the SVD.
- The factorization is not unique.

Rank theorem: $\widehat{\boldsymbol{M}}$ has rank 3.

Factorization of the Measurement Matrix

If the factorization is $\widehat{\boldsymbol{M}} = \boldsymbol{R}\boldsymbol{S}$, then

$$\widehat{\boldsymbol{M}} = (\boldsymbol{R}\boldsymbol{Q})(\boldsymbol{Q}^{-1}\boldsymbol{S})$$

is also a valid factorization. The matrix $\boldsymbol{Q}$ is an invertible 3×3 matrix.

The following constraints are useful:

- $\boldsymbol{u}_i$, $\boldsymbol{v}_i$ are orthogonal,
- $|\boldsymbol{u}_i| = |\boldsymbol{v}_i| = 1$.

Tomasi's Factorization Algorithm

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3. Set all σ_k for $k \geq 4$ to zero, since $\operatorname{rank}\left(\widehat{\boldsymbol{M}}\right) = 3$.
4. Let $\boldsymbol{U}'$ be the $2N_{\mathrm{F}} \times 3$ submatrix of $\boldsymbol{U}$, and $\boldsymbol{V}'$ the $3 \times N_{\mathrm{p}}$ submatrix of $\boldsymbol{V}$ corresponding to σ_1, σ_2, and σ_3. Let $\boldsymbol{\Sigma}' = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$, then compute:
$$\hat{\boldsymbol{R}} = \boldsymbol{U}'\boldsymbol{\Sigma}'^{\frac{1}{2}}, \qquad \hat{\boldsymbol{S}} = \boldsymbol{\Sigma}'^{\frac{1}{2}}\boldsymbol{V}'^{\mathsf{T}}.$$

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$$\hat{\boldsymbol{R}} = \boldsymbol{U}'\boldsymbol{\Sigma}'^{\frac{1}{2}}, \qquad \hat{\boldsymbol{S}} = \boldsymbol{\Sigma}'^{\frac{1}{2}}\boldsymbol{V}'^{\mathsf{T}}.$$
5. Solve the following (nonlinear) equations for $\boldsymbol{Q}$:
$$\hat{\boldsymbol{u}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{u}}_i = 1,$$
$$\hat{\boldsymbol{v}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{v}}_i = 1,$$
$$\hat{\boldsymbol{u}}_i^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}}\hat{\boldsymbol{v}}_i = 0.$$

Tomasi's Factorization Algorithm

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$$\widehat{\boldsymbol{M}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\mathsf{T}.$$
3. Set all σ_k for $k \geq 4$ to zero, since $\operatorname{rank}\left(\widehat{\boldsymbol{M}}\right) = 3$.
4. Let $\boldsymbol{U}'$ be the $2N_\mathrm{F} \times 3$ submatrix of $\boldsymbol{U}$, and $\boldsymbol{V}'$ the $3 \times N_\mathrm{p}$ submatrix of $\boldsymbol{V}$ corresponding to σ_1, σ_2, and σ_3.
   Let $\boldsymbol{\Sigma}' = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$, then compute:
$$\hat{\boldsymbol{R}} = \boldsymbol{U}'\boldsymbol{\Sigma}'^{\frac{1}{2}}, \qquad \hat{\boldsymbol{S}} = \boldsymbol{\Sigma}'^{\frac{1}{2}}\boldsymbol{V}'^\mathsf{T}.$$
5. Solve the following (nonlinear) equations for $\boldsymbol{Q}$:
$$\begin{aligned} \hat{\boldsymbol{u}}_i^\mathsf{T}\boldsymbol{Q}\boldsymbol{Q}^\mathsf{T}\hat{\boldsymbol{u}}_i &= 1, \\ \hat{\boldsymbol{v}}_i^\mathsf{T}\boldsymbol{Q}\boldsymbol{Q}^\mathsf{T}\hat{\boldsymbol{v}}_i &= 1, \\ \hat{\boldsymbol{u}}_i^\mathsf{T}\boldsymbol{Q}\boldsymbol{Q}^\mathsf{T}\hat{\boldsymbol{v}}_i &= 0. \end{aligned}$$
6. Compute the output:
$$\boldsymbol{R} = \hat{\boldsymbol{R}}\boldsymbol{Q}, \qquad \boldsymbol{S} = \boldsymbol{Q}^{-1}\hat{\boldsymbol{S}}.$$

Remarks

- Nonlinear optimization for $\boldsymbol{Q}$ is not very pleasant.
- Elegant “democratic” method: All points are treated equally.
- It is mathematically simple and stable.
- The algorithm yields only the **rotation** of the world points.
- It is used in industry.
- Translation parallel to the image plane is proportional to the translation of the image centroid between two frames.
- The translational component along the optical axis cannot be computed because of the orthogonal projection model.
- Adding new frames is easy and gives a more stable reconstruction.
- *Problem:* **All** 3-D points must be visible in **all** frames.
- Check the assumption that the camera gives an orthogonal image.

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- Preliminaries
- Registered Measurement Matrix
- Factorization of the Measurement Matrix

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- Take Home Messages
- Further Readings

Take Home Messages

- If we put all image points from several ultrasound acquisitions into a single measurement matrix for 3-D reconstruction, we can perform a factorization of this matrix.
- One of the factorized matrices contains the projective information, the other contains the world points.
- We need to register a given measurement matrix towards the centroid center of the image points.
- Tomasi's algorithm can be used to compute a factorization in case of orthogonal projections.

Further Readings

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