

Portal constraint links two bodies together (the body and it's portal clone) via two portals defined on a further two objects.

These portals are defined by a local anchor and direction on the 'portal' bodies  $\vec{a}_i$  and  $\vec{d}_i$  to portal bodies  $A$  and  $B$  together with a scaling  $\eta$  from portal 1 to 2. A body enter in the direction of  $\vec{d}_1$  and is emitted in the direction of  $\vec{d}_2$ . It is assumed that the mass/inertia of body 2 are  $\eta \times$  mass/inertia of body 1 so that the bodies can be treat equally in terms of the constraint.

We then have:

$$\begin{aligned} \vec{n}_1 &= R(\theta_A) \vec{d}_1, \quad \vec{n}_2 = R(\theta_B) \vec{d}_2 \text{ as the world-space portal directions} \\ \vec{p}_1 &= R(\theta_A) \vec{a}_1, \quad \vec{p}_2 = R(\theta_B) \vec{a}_2 \text{ as the world-space relative portal positions} \\ \alpha_1 &= \theta_A + \arg(\vec{d}_1), \quad \alpha_2 = \theta_B + \arg(\vec{d}_2) \text{ as the world-space portal angles} \end{aligned}$$

Furthermore, defining:

$$\vec{s}_1 = \vec{x}_1 - \vec{p}_1 - \vec{x}_A, \quad \vec{s}_2 = \vec{x}_2 - \vec{p}_2 - \vec{x}_B$$

as the vectors from portal position in world space to object positions, we can define the constraint by:

$$C(\vec{x}) = \begin{bmatrix} \eta \vec{s}_1 \cdot \vec{n}_1 + \vec{s}_2 \cdot \vec{n}_2 \\ \eta \vec{s}_1 \times \vec{n}_1 + \vec{s}_2 \times \vec{n}_2 \\ (\theta_1 - \alpha_1) - (\theta_2 - \alpha_2) \end{bmatrix}$$

Computing the time derivatives of the quantities above:

$$\begin{aligned} \frac{d}{dt} \vec{n}_1 &= \omega_A \times \vec{n}_1, \quad \frac{d}{dt} \vec{n}_2 = \omega_B \times \vec{n}_2 \\ \frac{d}{dt} \vec{p}_1 &= \omega_A \times \vec{p}_1, \quad \frac{d}{dt} \vec{p}_2 = \omega_B \times \vec{p}_2 \\ \frac{d}{dt} \vec{s}_1 &= \vec{v}_1 - (\omega_A \times \vec{p}_1) - \vec{v}_A, \quad \frac{d}{dt} \vec{s}_2 = \vec{v}_2 - (\omega_B \times \vec{p}_2) - \vec{v}_B \\ \frac{d}{dt} \alpha_1 &= \omega_A, \quad \frac{d}{dt} \alpha_2 = \omega_B \end{aligned}$$

We can find the velocity constraint:

$$V(\vec{v}) = \begin{bmatrix} \eta \left[ \left( \frac{d}{dt} \vec{s}_1 \right) \cdot \vec{n}_1 + \vec{s}_1 \cdot \left( \frac{d}{dt} \vec{n}_1 \right) \right] + \left[ \left( \frac{d}{dt} \vec{s}_2 \right) \cdot \vec{n}_2 + \vec{s}_2 \cdot \left( \frac{d}{dt} \vec{n}_2 \right) \right] \\ \eta \left[ \left( \frac{d}{dt} \vec{s}_1 \right) \times \vec{n}_1 + \vec{s}_1 \times \left( \frac{d}{dt} \vec{n}_1 \right) \right] + \left[ \left( \frac{d}{dt} \vec{s}_2 \right) \times \vec{n}_2 + \vec{s}_2 \times \left( \frac{d}{dt} \vec{n}_2 \right) \right] \\ \frac{d}{dt} (\theta_1 - \alpha_1) - \frac{d}{dt} (\theta_2 - \alpha_2) \end{bmatrix}$$

Expanding the inner parts of the equations:

$$\begin{aligned} \left( \frac{d}{dt} \vec{s}_1 \right) \cdot \vec{n}_1 &= (\vec{v}_1 - (\omega_A \times \vec{p}_1) - \vec{v}_A) \cdot \vec{n}_1 \\ \vec{s}_1 \cdot \left( \frac{d}{dt} \vec{n}_1 \right) &= \vec{s}_1 \cdot (\omega_A \times \vec{n}_1) \\ &+ \text{similar results.} \end{aligned}$$

We can define:

$$\vec{u}_1 = \vec{v}_1 - (\omega_A \times \vec{p}_1) - \vec{v}_A, \quad \vec{u}_2 = \vec{v}_2 - (\omega_B \times \vec{p}_2) - \vec{v}_B$$

And express the velocity constraint more succintly by:

$$V(\vec{v}) = \begin{bmatrix} \eta (\vec{u}_1 \cdot \vec{n}_1 + \omega_A (\vec{s}_1 \times \vec{n}_1)) + (\vec{u}_2 \cdot \vec{n}_2 + \omega_B (\vec{s}_2 \times \vec{n}_2)) \\ \eta (\vec{u}_1 \times \vec{n}_1 + \omega_A (\vec{s}_1 \cdot \vec{n}_1)) + (\vec{u}_2 \times \vec{n}_2 + \omega_B (\vec{s}_2 \cdot \vec{n}_2)) \\ (\omega_1 - \omega_A) - (\omega_2 - \omega_B) \end{bmatrix}$$

Noting the following results of partial differentiations:

$$\begin{aligned}\frac{\partial}{\partial \vec{u}} \vec{u} \cdot \vec{v} &= \frac{\partial}{\partial \vec{u}} \vec{v}^\top \vec{u} = \vec{v}^\top \\ \frac{\partial}{\partial \vec{u}} \vec{u} \times \vec{v} &= -\frac{\partial}{\partial \vec{u}} [\vec{v}]_\times^\top \vec{u} = -[\vec{v}]_\times^\top\end{aligned}$$

We find the jacobian of our portal constraint as:

$$\mathbf{J} = \begin{bmatrix} \begin{bmatrix} \eta \vec{n}_1^\top \\ -\eta [\vec{n}_1]_\times^\top \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \eta \vec{n}_2^\top \\ -\eta [\vec{n}_2]_\times^\top \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \end{bmatrix}$$

And our effective mass matrix:

$$\begin{aligned}\mathbf{K} &= \frac{1}{m_1} \begin{bmatrix} \eta \vec{n}_1^\top \\ -\eta [\vec{n}_1]_\times^\top \\ 0 \end{bmatrix} \begin{bmatrix} \eta \vec{n}_1^\top \\ -\eta [\vec{n}_1]_\times^\top \\ 0 \end{bmatrix}^\top + \frac{1}{i_1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^\top \\ &+ \frac{1}{m_2} \begin{bmatrix} \vec{n}_2^\top \\ -[\vec{n}_2]_\times^\top \\ 0 \end{bmatrix} \begin{bmatrix} \vec{n}_2^\top \\ -[\vec{n}_2]_\times^\top \\ 0 \end{bmatrix}^\top + \frac{1}{i_2} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}^\top\end{aligned}$$

Noting the lovely facts:

$$\begin{aligned}\vec{u}^\top (\vec{u}^\top)^\top &= \|\vec{u}\| = [\vec{u}]_\times^\top ([\vec{u}]_\times^\top)^\top, \quad \|\vec{n}_i\| = 1 \\ \vec{u}^\top ([\vec{u}]_\times^\top)^\top &= \vec{u} \cdot [\vec{u}]_\times = \vec{u} \times \vec{u} = 0\end{aligned}$$

We show:

$$\begin{bmatrix} \eta \vec{n}_1^\top \\ -\eta [\vec{n}_1]_\times^\top \\ 0 \end{bmatrix} \begin{bmatrix} \eta \vec{n}_1^\top \\ -\eta [\vec{n}_1]_\times^\top \\ 0 \end{bmatrix}^\top = \begin{bmatrix} \eta^2 & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^\top = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And so we arrive at a very simple effective mass matrix:

$$\mathbf{K} = \begin{bmatrix} \left( \frac{\eta^2}{m_1} + \frac{1}{m_2} \right) \mathbf{E}_2 & 0 \\ 0 & \frac{1}{i_1} + \frac{1}{i_2} \end{bmatrix}$$