Abstract Integration, Chapter 1

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1. **Exercise.** Does there exists an infinite σ-algebra which has only countably many members?
   Solution. No. Let X be a measurable set with an infinite σ-algebra $\mathfrak{M}$. Since $\mathfrak{M}$ is infinite, there exists a nonempty set $E \in \mathfrak{M}$ which is properly contained in X. By letting the measurable subsets of E and E^c be the intersections of X with E and E^c, E and E^c are therefore measurable spaces. Since $\mathfrak{M}$ is infinite, at least one of E and E^c must be infinite.
   A rooted binary tree will be inductively built in the following way. First, define the root as X. Second, given a vertex which is a measurable subset E of X (which can be X of course), if it contains a proper measurable subset E', define the two successors to be E' and $E - E'$. Since at least one of E' and $E - E'$ is infinite, the tree is infinite.
   Pick one path consisting of subsets $E_n \supset E_{n+1}$, then the sets $F_i = E_i - E_{i+1}$ form an infinite collection of disjoint nonempty measurable subsets of X. By the definition of σ-algebra, $\mathfrak{M}$ needs to contain every union of such sets, which forms a bijection with the set of subsets of $\mathbb{N}$ (that is, $\{\{0\}, \{0,1\}, \{0,1,2\}, \cdots\}$), which is uncountable. Hence $\mathfrak{M}$ is not countable.

2. **Exercise.** Prove an analogue of Theorem 1.8 for n functions.
   Solution. The theorem to be proved is:

   Let $u_1, \cdots, u_n$ be real measurable functions on a measurable space X, let Φ be a continuous mapping of $\mathbb{R}^n$ into a topological space Y, and define

   $$h(x) = \Phi(u_1(x), \cdots, u_n(x))$$

   for $x \in X$. then h is measurable.

   Put $f_n(x) = (u_1(x), \cdots, u_n(x))$, then f maps X into $\mathbb{R}^n$. Since $h(x) = \Phi(f(x))$ and the composition of a continuous function (Φ) and a measurable function (f) is measurable, it's enough to prove that f is measurable.
   if R is any open n-cell in the space, with sides parallel to the axes, then R is the cartesian product of two segments $I_1, I_2, \cdots, I_n$ and

   $$f^{-1}(R) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap \cdots \cap u_n^{-1}(I_n)$$

   which is measurable, by the assumption of u_i. Every open set V in the space is a countable union of such rectangle R_i, and since

   $$f^{-1}(V) = f^{-1}(\bigcup_{n=1}^{\infty} R_i) = \bigcup_{n=1}^{\infty} f^{-1}(R_i),$$

   $f^{-1}(V)$ is measurable, which shows the measurability of f.

3. **Exercise.** Prove that if f is a real function on a measurable space X such that $\{x | f(x) \geq r\}$ is measurable for every rational r, then f is measurable.
   Solution. This exercise is a weakened version of 1.12(c). Hence it suffices to prove

$f^{-1}((\alpha,\infty]) \in \mathfrak{M}$ for every real α.
Let Ω be the collection of all $E \subset [-\infty,\infty]$ such that $f^{-1}(E) \in \mathfrak{M}$. Then Ω is a σ-algebra. Pick $\alpha \in \mathbb{R}$, and choose a sequence of rational numbers $\{\alpha_n\}$ such that $\alpha_n > \alpha$ and $\lim_{n\to\infty} \alpha_n = \alpha$. This is possible since $\mathbb{Q}$ is dense in $\mathbb{R}$. Observe that

$$(\alpha,\infty] = \bigcup_{n=1}^{\infty} [\alpha_n,\infty]$$

and $[\alpha_n,\infty] \in \Omega$, $(\alpha,\infty]$ is therefore a member of Ω. That is, $f^{-1}((\alpha,\infty]) \in \mathfrak{M}$.

4. **Exercise.** Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty,\infty]$, and prove the following assertions:

   (a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$

   (b) $\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$
   provided that none of the sum is of the form $\infty - \infty$. Show by an example that strict inequality can hold here.

   (c) $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$ if $a_n \leq b_n$ holds for all n.

   Solution.

   (a) Put $A_k = \sup\{-a_k, -a_{k+1}, \cdots\}$. In this case, $-a_j \leq A_k$ for all $j \geq k$. If $A < A_k$ then there exist some $j \geq k$ such that $-a_j > A$. This is the definition of supremum.
   Now consider a_j. $a_j \geq -A_k$ for all $j \geq k$. If $-A > -A_k$ then there exist some $j \geq k$ such that $a_j < -A$. Hence, $A'_k = \inf\{a_k, a_{k+1}, \cdots\} = -A_k$.
   Following the same way, we obtain

   $$\inf\{A_k\} = -\sup\{-A_k\} = -\sup A'_k,$$

   which is equivalent to the assertion (a) according to the definition of upper/lower limit.

   (b) Put $A_k = \sup\{a_k, a_{k+1}, \cdots\}$, $B_k = \sup\{b_k, b_{k+1}, \cdots\}$. Then $a_j + b_j \leq A_k + B_k$ for all $j \geq k$. Hence $C_k = \sup\{a_k + b_k, a_{k+1} + b_{k+1}, \cdots\} \leq A_k + B_k$. Observe that all $\{A_k\}$, $\{B_k\}$ and $\{C_k\}$ are monotonically decreasing, the desired inequality can be obtained by dealing with the limit k:

   $$\lim_{k\to\infty} C_k = \inf C_k \leq \lim_{k\to\infty}(A_k + B_k) = \lim_{k\to\infty} A_k + \lim_{k\to\infty} B_k = \inf A_k + \inf B_k$$

   Again, this is equivalent to (b).
   For strict inequality, consider $a_n = (-1)^n$, $b_n = (-1)^{n-1}$. In this case, the left side is equal to 0 while the right side is equal to 2.

   (c) For $A_k = \inf\{a_k, a_{k+1}, \cdots\}$ and $B_k = \inf\{b_k, b_{k+1}, \cdots\}$. Since $a_k \leq b_k$, $A_k \leq B_k$ holds for all $k > 0$. Hence $\sup A_k \leq \sup B_k$, which proves (c).

5. **Exercise.**

   (a) Suppose $f : X \to [-\infty,\infty]$ and $g : X \to [-\infty,\infty]$ are measurable. Prove that the sets

   $$\{x|f(x) < g(x)\}, \quad \{x|f(x) = g(x)\}$$

   are measurable.

   (b) Prove the set pf points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

   Solution.

(a) The key to this question is to rewrite the two sets. Of course, infinity should be considered.
Define $F_+ = \{x|f(x) = \infty\}$, $F_- = \{x|f(x) = -\infty\}$. Respectively, define G_+ and G_- for $g(x)$. These four sets are all measurable. For example, F_+ can be rewritten as $\bigcap\limits_{n=1}^{\infty} \{x|f(x) \geq n\}$.
Define $X' = \{x| -\infty < f(x) < \infty, -\infty < g(x) < \infty\}$ and $h(x) = f(x) - g(x)$ on X', which is measurable. Then the first set can be written as

$$h^{-1}([-\infty, 0)) \cup (Y_+ - Z_+) \cup (Z_- - Y_-)$$

which means, if f and g are finite, then take the set where $h(x) < 0$; if not, take the set where f is infinite but g is not, or the set where g is negatively infinite but f is not. According to the definition of σ-algebra, all these 3 sets are measurable, so is the first set of (a).
The second set can be written as

$$(X' - h^{-1}([-\infty, 0) \cup (0, \infty])) \cup (Y_+ \cap Z_+) \cup (Y_- \cup Z_-)$$

which is also measurable. This notation means, the set where $h = 0$ or $f = g = \infty$ or $f = g = -\infty$.

(b) Let f_n and E be the sequence of real functions and set mentioned in (b). If E can be shown as a countable union/intersection of measurable sets, then it's measurable.
Since f_n converges, Cauthy criteria can be used here. For every $n \geq 1$, there is an integer $m \geq 1$ such that $|f_i(x) - f_j(x)| < \frac{1}{n}$ holds for all $x \in E$ where $i, j \geq m$. Hence E can be rewritten in the following form

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{i,j \geq m} \{x||f_i(x) - f_j(x)| < \frac{1}{n}\} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{i,j \geq m} (f_i - f_j)^{-1}(-\frac{1}{n}, \frac{1}{n})$$

Since $(f_i - f_j)$ is measurable, the set $(f_i - f_j)^{-1}(-\frac{1}{n}, \frac{1}{n})$ is measurable. Hence E has been shown as a countable union and intersection of measurable sets, and is measurable.

6. **Exercise.** Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable (finite or countable), and define $\mu(E) = 0$ in the first case and $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measure functions and their integrals.
Solution. The question includes 3 parts.

(a) $\mathfrak{M}$ a σ-algebra in X.
  i. Since $X \in \mathfrak{M}$ is uncountable, $X^c = \emptyset$ is at most countable, $\emptyset \in \mathfrak{M}$.
  ii. If $A \in \mathfrak{M}$, then either $A = (A^c)^c$ or A^c is at most countable; showing $A^c \in \mathfrak{M}$.
  iii. Suppose $A_i \in \mathfrak{M}$ for $i \in \mathbb{N}$ and $A = \bigcup A_i$ in this section. The fact that $A \in \mathfrak{M}$ shall be proved here.
    A. If A_i is at most countable for all i, then A is at most countable while the complement $A^c = \bigcap A_i^c$ is uncountable. Hence $A \in \mathfrak{M}$.
    B. If A is uncountable, there is at least one uncountable set. Let A_1 be such set. Notice the fact that A_1^c is at most countable and $(\bigcup A_i)^c = \bigcap A_i^c \subset A_1^c$, the set $(\bigcup A_i)^c$ is therefore at most countable. Hence $\bigcup A_i \in \mathfrak{M}$.

(b) μ is a measure on $\mathfrak{M}$.
  i. Since μ takes values 0 and 1, $\mu(A) \in [0, \infty]$ for all $A \in \mathfrak{M}$.
  ii. Suppose A_i for $i \in \mathbb{N}$ are disjoint measurable sets. $A = \bigcup A_i$ The fact that μ is countable addictive shall be proved here.

A. If $\mu(A) = 0$ then all of A_i is countable, so

$$\sum \mu(A_i) = \mu(A) = 0$$

B. If $\mu(A) = 1$, then at least one of A_i is uncountable. Let A_1 be such set. Notice the fact that since all A_i are disjoint, A_1^c is countable and $A_i \subset A_1^c$ for $j > 1$, $\mu(A_i) = 0$ for $j > 1$. Hence

$$\mu(A) = \mu(A_1) = \sum \mu(A_i) = 1$$

(c) Characterization of measurable functions and their integrals.
Let $f : X \to \mathbb{R}$ be the corresponding functions. For every $r \in \mathbb{R}$, either $f^{-1}(r)$ or $f^{-1}(\mathbb{R} - r)$ is countable. This describes the corresponding real functions. Let $A \in \mathbb{R}$ denote the set of points such that $f^{-1}(r)$ is not at most countable, then $\int_X f d\mu = \sum_{r \in A} r$.

7. **Exercise.** Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \cdots$,, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that this conclusion does not follow if the condition $f_1 \in L^1(\mu)$ is omitted.
Solution. Take $g = f_1$ in the theorem 1.34, Lebesgue's Dominated Convergence Theorem, the equation is concluded.
For showing $f_1 \in L^1(\mu)$ is a necessary condition for the conclusion, it suffices to show a counterexample. Take $X = \mathbb{R}$ and $f_n(x) = \chi_{[n,\infty)}$. In this case,

$$\int_X f d\mu = 0$$

while

$$\lim_{n\to\infty} \int_X f_n d\mu = \infty.$$

8. **Exercise.** Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's Lemma?
Solution. Fatou's Lemma says:

If $f_n : X \to [0, \infty]$ is measurable for each positive integer n,then

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$

To find the relationship, we should calculate the left hand and the right hand for this f_n. This is possible since $f_n \geq 0$.
Since $\liminf_{n\to\infty} f_n = 0$, the left hand should therefore be 0. Notice that

$$\int_X f_n d\mu = \begin{cases} \mu(E), & n \text{ is odd} \\ \mu(X) - \mu(E), & n2 \text{ d is even} \end{cases}$$

So for the right hand, we obtain

$$\liminf_{n\to\infty} \int_X f_n d\mu = \min(\mu(E), \mu(X) - \mu(E))$$

If $\mu(E)$ and $\mu(X - E)$ is not equal to 0, then this f is an example that the strict inequality holds.

9. **Exercise.** Suppose μ is a positive measure on X, $f : X \to [0, \infty]$ is measurable, $\int_X f d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to\infty} \int_X n \log[1 + (f/n)^\alpha] d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ c & \text{if } \alpha = 1 \\ 0 & \text{if } 1 < \alpha < \infty \end{cases}$$

Hint: If $\alpha \geq 1$, prove that the integrands are dominated by αf. If $\alpha < 1$, Fatou's lemma can be applied.
Solution.

(a) $\alpha \geq 1$:
We shall prove that

$$\log\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) \leq \alpha \frac{f(x)}{n}$$

holds for all n. It suffices to show that

$$\log(1 + x^\alpha) \leq \alpha x$$

for all $x > 0$ and $\alpha \geq 1$.
Define $g(x) = \log(1 + x^\alpha) - \alpha x$, then $g(0) = 0$. Computing derivative of g, we have

$$g'(x) = \frac{\alpha x^{\alpha-1}}{1 + x^\alpha} - \alpha = \alpha(\frac{x^{\alpha-1}}{1 + x^\alpha} - 1)$$

When $x \geq 1$, we have

$$\frac{x^{\alpha-1}}{1 + x^\alpha} \leq \frac{x^{\alpha-1}}{x^\alpha} = \frac{1}{x} \leq 1 \tag{1}$$

When $0 \leq x \leq 1$, we have

$$(x^{\alpha-1} - 1) - x^\alpha \leq 0 \tag{2}$$

which implies that $\frac{x^{\alpha-1}}{1+x^\alpha} \leq 1$.
So $g'(x) \leq 0$ when $x \geq 0$. Consequently, $g(x) \leq 0$ holds for all x. The inequality at the beginning is proved.
Define $h_n(x) = n \log[1 + (f/n)^\alpha]$, then $h_n(x) \leq \alpha f(x)$ holds for all n. Also notice that $\int_X \alpha f d\mu = c\alpha < \infty$, hence Lebesgue's Dominated Convergence Theorem can be applied here. We therefore should discuss the limit of $h_n(x)$.
$h_n(x)$ can be rewritten as

$$h_n(x) = \log[1 + (f/n)^\alpha]^n \to \frac{f^\alpha}{n^{\alpha-1}}$$

When $\alpha = 1$, the limit is $f(x)$, hence the integral is c. When $\alpha > 1$, the limit is 0, so is the integral. (The set where $f(x) = \infty$ should be ignored since $f(x) < \infty$ a.e. on X.)

(b) $0 < \alpha < 1$:
In this case, $h_n(x) \to \infty$ a.e. on X since $n^{\alpha-1} \to 0$. Since $h_n : X \to [0, \infty]$ is measurable for each n, Fatou's Lemma can be applied here, hence we obtain

$$\int_X \left(\liminf_{n\to\infty} h_n\right) d\mu = \infty \leq \liminf_{n\to\infty} \int_X h_n d\mu \leq \lim_{n\to\infty} \int_X h_n d\mu$$

10. **Exercise.** Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu,$$

and show that the hypothesis "$\mu(x) < \infty$" cannot be omitted.
Solution. Since $f_n \to f$ uniformly on X, for every $\varepsilon > 0$, there exists a positive integer n_0 such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0$$

Since $|f_n(x) - f(x)| \geq |f_n(x)| - |f(x)|$ and $|f_{n_0}(x) - f(x)| \geq |f(x)| - |f_{n_0}(x)|$, we obtain

$$\begin{cases} |f_n(x)| < |f(x)| + \varepsilon \\ |f(x)| < |f_{n_0}(x)| + \varepsilon \end{cases}$$

Hence

$$|f_n(x)| < |f_{n_0}(x)| + 2\varepsilon$$

Define $g(x) = \max\{|f_1(x)|, \cdots, |f_{n_0-1}(x)|, |f_{n_0}(x)| + 2\varepsilon\}$, then $|f_n(x)| \leq |g(x)|$ holds for all n. Also $g(x)$ is bounded since all $f_n(x)$ are. Since $\mu(X) < \infty$, $g \in L^1(\mu)$. Now the equation is an example of Lebesgue's Dominated Convergence Theorem.
To show that $\mu(X) < \infty$ is a necessary condition, it suffices to show a counterexample with $\mu(X) = \infty$. Suppose $f_n(x) = \frac{1}{n}$, then $f(x) = 0$. In this case, $\lim_{n\to\infty} \int_X f_n d\mu = \infty$, while $\int_X f d\mu = 0$.

11. **Exercise.** Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in theorem 1.41, and hence prove the theorem without any reference to integration.
Solution. Let $\{E_k\}$ be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty,$$

then A is the set of all x which lie in infinitely many E_k.
Let $B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. If $x \in A$, then $x \in \bigcup_{k=n}^{\infty} E_k$ holds for all $k > 0$. That is, $x \in B$. Conversely, if $x \notin A$, then this x is contained by finitely many sets $\{E_{i_1}, E_{i_2}, \cdots, E_{i_r}\}(i_1 < i_2 < \cdots < i_r)$ such that $x \notin \bigcup_{k>i_r} E_k$. In this case, $x \in \bigcup_{k=n}^{\infty} E_k$ does not hold for $k = i_r + 1$, which implies that $x \notin B$. Hence $A = B$.
The second part is an application of Theorem 1.19. Since $\sum \mu(E_k) < \infty$, $\mu(E_k) \to 0$(Cauthy criteria). Since $\{E_k\}$ are not necessarily pairwise disjoint, $\mu(\bigcup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} E_k$. Let $A_n = \bigcup_{k=n}^{\infty} E_k$, then $\mu(A_n) \to 0$. Since $A = \bigcap_{n=1}^{\infty} A_n$, according to theorem 1.19(e), $\mu(A_n) \to \mu(A)$. Hence, $\mu(A) = 0$.

12. **Exercise.** Suppose $f \in L_1(\mu)$. Prove that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \varepsilon$ whenever $\mu(E) < \delta$.
Solution. Let $(X, \mathfrak{M}, \mu)$ be the measure space. Suppose the statement is false, therefore there exists a $\varepsilon > 0$ such that for each $\delta > 0$, there exists a $E_\delta < \mathfrak{M}$ such that $\mu(E_\delta) < \delta$ while $\int_{E_\delta} |f| d\mu > \varepsilon$. Let $\delta_n = \frac{1}{2^n}$, there exists a sequence of measurable sets $\{E_{\delta_n}\}$ satisfying the inequalities mentioned above.
Define $A_k = \bigcup_{n=k}^{\infty} E_{\delta_n}$ and $A = \bigcap_{k=1}^{\infty} A_k$, then $A_1 \supset A_2 \supset A_3 \supset \cdots$, and $\mu(A_1) = \mu(\bigcap_{n=1}^{\infty} E_{\delta_n}) \leq \sum \mu(E_{\delta_n}) < \sum \frac{1}{2^n} = 1 < \infty$, therefore according to theorem 1.19(e), $\mu(A_k) \to \mu(A)$.
Define $\varphi : \mathfrak{M} \to [0, \infty]$ such that $\varphi(E) = \int_E |f| d\mu$. By theorem 1.29, φ is a measure. Since $f(x) \in L^1(\mu)$, $\varphi(A_k) < \infty$ for every k. By theorem 1.19(e), we get $\varphi(A_k) \to \varphi(A)$. According to exercise 11, $\mu(A) = 0$. Therefore $\varphi(A) = 0$. On the other hand,

$$\varphi(A_k) = \varphi(\bigcup_{n=k}^{\infty} E_{\delta_n}) \geq \varphi(E_{\delta_k}) > \varepsilon.$$

Therefore a contradiction is obtained. Hence the result.

13. **Exercise.** Show that proposition 1.24(c) is also true when $c = \infty$.
Solution. We have to show

$$\int_X cf d\mu = c \int_X f d\mu, \text{ when } c = \infty \text{ and } f \geq 0$$

holds for both $\int_X f d\mu = 0$ and $\int_X f d\mu > 0$.

(a) $\int_X f d\mu = 0$.

In this case, $f = 0$ a.e. on X and $cf = 0$ a.e. on X. Therefore

$$\int_X cf d\mu = 0 = c \int_X f d\mu$$

(b) $\int_X f d\mu > 0$.

In this case, there exist some $\varepsilon > 0$ and a measurable set E where $\mu(E) > 0$ and $f(x) \geq \varepsilon$ whenever $x \in E$. (If not, $f(x) < \varepsilon$ a.e. for all $\varepsilon > 0$; making $\varepsilon \to 0$, we get $f(x) = 0$ a.e. on X. It's of the first case.) Then

$$\int_X cf d\mu \geq \int_E cf d\mu \geq \varepsilon \int_E c d\mu = \infty.$$

That is,

$$c \int_X f d\mu = \int_X cf d\mu = \infty.$$