Singular-value decomposition approach to time series modelling

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Abstract: In various signal processing applications, as exemplified by spectral analysis, deconvolution and adaptive filtering, the parameters of a linear recursive model are to be selected so that the model is 'most' representative of a given set of time series observations. For many of these applications, the parameters are known to satisfy a theoretical recursive relationship involving the time series' autocorrelation lags. Conceptually, one may then use this recursive relationship, with appropriate autocorrelation lag estimates substituted, to effect estimates for the operator's parameters. A procedure for carrying out this parameter estimation is given which makes use of the singular-value decomposition (SVD) of an extended-order autocorrelation matrix associated with the given time series. Unlike other SVD modelling methods, however, the approach developed does not require a full-order SVD determination. Only a small subset of the matrix's singular values and associated characteristic vectors need be computed. This feature can significantly alleviate an otherwise overwhelming computational burden that is necessitated when generating a full-order SVD. Furthermore, the modelling performance of this new method has been found empirically to excel that of a near maximum-likelihood SVD method as well as several other more traditional modelling methods.

1 Introduction

In this paper, we shall be concerned with developing a systematic procedure for detecting and identifying rational-type time series which are embedded in additive white noise. The time series $\{x(n)\}$ is said to be of a rational type if its associated autocorrelation sequence satisfies a linear recursive relationship of the form

$$\sum_{k=0}^{p} a_k r_x(n-k) = 0 \qquad \text{for } n \in I \tag{1}$$

in which I is an appropriate integer set associated with the application at hand. Unless otherwise indicated, the time series will be taken to be a complex-valued wide-sense stationary random process with autocorrelation sequence

$$r_x(n) = E\{x(n+k)\bar{x}(k)\} \tag{2}$$

in which the symbols E and the overbar denote the operations of statistical expectation and complex conjugation, respectively. To gain an appreciation for the wide range of relevant signal processing applications which are describable by a homogeneous relationship of the form of eqn. 1, the following abbreviated listing is offered:

(*a*) In modelling a wide-sense stationary time series by an autoregressive moving-average model of order (p, q) [i.e. an ARMA (p, q) model], it can be shown that $I = [q+1, \infty)$ specifies the integer index set.†

(*b*) When the time series is composed of a sum of sinusoidal components of different frequencies that have initial random phase angles uniformly distributed on $(-\pi, \pi)$, the integer index set is $I = (-\infty, \infty)$ in the noise-free case.

(*c*) In transient response analyses associated with linear recursive operations, a homogeneous relationship of the form of eqn. 1 will hold for suitably chosen I.

(*d*) In narrowband interference rejection applications where the a_k coefficients are required to be symmetric so as to achieve zero phase filtering, the relationship is also applicable.

A more detailed description of these and other related applications can be found in the available signal processing literature (for example see References 1–7).

It will now be desirable to provide an algebraic characterisation for the class of rational time series. This is readily accomplished by evaluating eqn. 1 for the set of indices $n_1 \leqslant n \leqslant n_2$, thereby giving rise to the homogeneous system of linear equations

$$\begin{bmatrix} r_x(n_1) & r_x(n_1-1) & \cdots & r_x(n_1-p) \\ r_x(n_1+1) & r(n_1) & \cdots & r_x(n_1-p+1) \\ \vdots & \vdots & & \vdots \\ r_x(n_2) & r_x(n_2-1) & \cdots & r_x(n_2-p) \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3a}$$

or, more compactly,

$$Ra = \mathbf{0} \tag{3b}$$

It is here tacitly assumed that the integers n_1 and n_2 are selected so that $[n_1, n_2] \in I$. In eqns. 3, R is the $(n_2 - n_1 + 1) \times (p+1)$ autocorrelation matrix, a is the $(p+1) \times 1$ model parameter vector and $\mathbf{0}$ is the $(n_2 - n_1 + 1) \times 1$ zero vector. Owing to the manner in which the autocorrelation lags are interrelated through eqn. 1, it follows that the rank of the autocorrelation matrix is specified by

$$\text{rank } [R] = \min(p, n_2 - n_1 + 1)$$

In most practical applications, the integers n_1 and n_2 will be selected so that $n_2 - n_1 > p$. For such a selection, it

†The symbol $[n_1, n_2]$ denotes the set of integers satisfying $n_1 \leqslant n \leqslant n_2$ while $[n_1, \infty)$ specifies the set of integers satisfying $n \geqslant n_1$.

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follows that 'any' vector lying in the one-dimensional null space of the matrix R will provide a solution to eqns. 3. Thus, in the ideal case in which exact autocorrelation lag information is available, the required a_k model parameters may be readily obtained by solving the consistent system of linear homogeneous equations given in eqn. 3. When only time series observations are available for the modelling, however, it will be necessary to suitably adapt this solution procedure. An approach for effecting this adaptation will now be given.

2 Modelling approach

We shall now describe a typical application scenario involving the fitting of a linear model to a given data set. Specifically, it will be assumed that there is available a set of time series observations whose underlying autocorrelation sequence is postulated as satisfying a homogeneous relationship of the form of eqn. 1. Without loss of generality, this observation set will be taken to be composed of the N contiguous measurements

$$x(1), x(2), \ldots, x(N) \tag{4}$$

The task at hand is to use these measurements in some systematic manner so as to generate estimates for the order parameter p and the recursive parameters a_k identifying the model given in eqn. 1. Typically, the recursive parameters are normalised so that $a_0 = 1$. For reasons which will be shortly made apparent, however, this normalisation will not be imposed here.

To effect the required parameter estimation in accordance with eqns. 3, a straightforward procedure is suggested. Namely, one first generates estimates for the underlying autocorrelation lags from the observation set given in expr. 4. Although a variety of procedures are available for this purpose, empirical evidence indicates that the biased estimates as given by

$$\hat{r}_x(n) = \frac{1}{N} \sum_{k=1}^{N-n} x(n+k)\,\bar{x}(k) \qquad 0 \leqslant n \leqslant N-1 \tag{5}$$

typically provide excellent modelling results. Appropriate negative autocorrelation lag estimates may be readily generated from these estimates by appealing to the complex-conjugate symmetric property $r_x(-n) = \bar{r}_x(n)$ which characterises wide-sense stationary processes. These computed autocorrelation lag estimates are next substituted into eqns. 3, and the resultant system of 'estimated' homogeneous linear equations are then solved to obtain the model's parameters. Although this approach has much intuitive appeal, it possesses the following two serious shortcomings:

(i) In most applications, the order of the underlying model, p, is not known *a priori*.

(ii) Owing to errors in the autocorrelation lag estimates, the autocorrelation matrix estimate will have full rank, thereby indicating that the resultant 'estimated' homogeneous equations $Ra = \mathbf{0}$ will generally only have the uninteresting zero-vector solution (i.e. $a = \mathbf{0}$).

To overcome these deficiencies, it will be necessary to appropriately adapt this modelling procedure in a manner now to be outlined.

2.1 Extended-order model

Since the order parameter is not normally known *a priori*, we shall take the conservative approach by overestimating the model's order. In particular, let us consider a recursive model of the form of eqn. 1 with order p_e (i.e. p is replaced by p_e) in which p_e is selected to be much larger than the expected value of the underlying unknown order p of the time series being analysed. This extended-order model would be characterised by

$$\sum_{k=0}^{pe} a_k r(n-k) = 0 \tag{6}$$

Using appropriate lag estimates (for example the biased estimates given in eqn. 5) as entries, this extended-order model is next evaluated for $n_1 \leqslant n \leqslant n_2$ to yield the estimated homogeneous relationship

$$\hat{R}_e a = \mathbf{0} \tag{7}$$

in which a is the $(p_e + 1) \times 1$ extended-order model parameter vector and R_e is the $(n_2 - n_1 + 1) \times (p_e + 1)$ extended-order autocorrelation matrix estimate given by

$$\hat{R}_e = \begin{bmatrix} \hat{r}_x(n_1) & \hat{r}_x(n_1-1) & \cdots & \hat{r}_x(n_1-p_e) \\ \hat{r}_x(n_1+1) & \hat{r}_x(n_1) & \cdots & \hat{r}_x(n_1-p_e+1) \\ \vdots & \vdots & & \vdots \\ \hat{r}_x(n_2) & \hat{r}_x(n_2-1) & & \hat{r}_x(n_2-p_e) \end{bmatrix} \tag{8}$$

It should be noted that, by taking this extended-order model approach, the resultant modelling as represented by eqn. 7 will be less sensitive to the statistical quirks of the data being analysed than would be the case for the minimal-order choice p.

If the autocorrelation lag estimates used in the matrix $\hat{R}_e$ were exact, it would follow that the rank of $\hat{R}_e$ would be p. This of course assumes that these autocorrelation lags entries would satisfy a pth-order recursive relationship of the form of eqn. 1 and that $p_e \geqslant p$ and $n_2 - n_1 > p$. In the more realistic case in which the autocorrelation lag estimates are in error, however, it follows that the rank of the extended autocorrelation matrix estimate of eqn. 8 will always be full [i.e. min $(p_e + 1, n_2 - n_1 + 1)$]. This in turn implies that there will not exist a nontrivial parameter vector solution to the homogeneous relationship given in eqn. 7. Nonetheless, it is still desirable to select a nontrivial parameter vector a which will cause this homogeneous relationship to be best satisfied in some sense. With this objective in mind, we shall seek a parameter vector which will minimise the l_2 norm of the equation error vector $\hat{R}_e a$, in which the mth component of a shall be constrained to be one. This constrained minimisation problem is formally specified by

$$\min_{a_m = 1} \; a^* \hat{R}_e^* \hat{R}_e a \tag{9}$$

Empirical evidence has demonstrated that the parameter vector solution to this problem will typically provide excellent modelling performance. In this expression, the asterisk denotes the operation of complex-conjugate transposition, and m is a fixed integer in $[0, p_e]$. The selection of the integer m is critical, and fundamentally different modelling behaviour will arise as a function of m. In the standard approach to spectral estimation, m is taken to be zero [1]. We shall say more about this important issue in later Sections.

Using straightforward methods, the parameter vector a which minimises the functional given in expr. 9 is found to be

$$a = \lambda [\hat{R}_e^* \hat{R}_e]^{-1} e_{m+1} \tag{10}$$

where e_{m+1} is the $(p_e + 1) \times 1$ standard basis vector whose components are all zero except for its $(m+1)$th, which is one, and λ is a normalising scalar selected so that the $(m+1)$th component of a is one as required. The main computational requirement in generating this parameter vector estimate is seen to be the inversion of the $(p_e + 1) \times (p_e + 1)$ matrix $\hat{R}_e^* \hat{R}_e$. Although the parameter vector estimate given in eqn. 10 typically provides satisfactorily good performance, it is possible to obtain a further performance improvement by taking full advantage of the algebraic properties associated with the theoretical extended-order autocorrelation matrix. The following Section will provide the analytical tools for this purpose.

3 SVD representation algorithm

The extended-order autocorrelation matrix of eqn. 8 has a theoretical rank of p when its entries corresponded to the actual autocorrelation lags interrelated according to the pth-order recursion of eqn. 1. As suggested earlier, when lag estimates are incorporated, the actual rank of R_e will be full or near full. Nonetheless, the 'effective rank' of R_e will still tend to be near p for even moderately low signal/noise ratios. The ambiguous terminology 'effective rank of a matrix' can be quantified by appealing to the concept of singular-value decomposition (SVD). In particular, the SVD representation for the extended-order autocorrelation matrix given in eqn. 8 will be of the form [8]

$$\hat{R}_e = \sum_{k=1}^{h} \sigma_k u_k v_k^* \tag{11a}$$

in which

$$h = \min (p_e + 1, n_2 - n_1 + 1) \tag{11b}$$

The terms composing this SVD representation may be obtained by solving the following two eigenvalue-eigenvector problems:

$$\left.\begin{aligned} \hat{R}_e \hat{R}_e^* u_k &= \sigma_k^2 u_k \\ \hat{R}_e^* \hat{R}_e v_k &= \sigma_k^2 v_k \end{aligned}\right\} \quad 1 \leqslant k \leqslant h \tag{12}$$

From these Hermitian eigenvector relationships, it is apparent that the normalised $(n_2 - n_1 + 1) \times 1$ and $(p_e + 1) \times 1$ characteristic vector sets $\{u_k\}$ and $\{v_k\}$, respectively, are each orthonormal. Moreover, the non-negative 'singular' values σ_k are here ordered in the conventional nonincreasing-size manner (i.e. $\sigma_k \geqslant \sigma_{k+1}$).

To obtain the effective rank of the matrix $\hat{R}_e$ using this SVD representation, let us now consider the related problem of finding that $(n_2 - n_1 + 1) \times (p_e + 1)$ matrix of rank p which will best approximate R_e in the least-squares sense. It is readily shown that the best rank p approximation is obtained by simply truncating the SVD representation to its largest p singular-value components [8], i.e.

$$\hat{R}_e^{(p)} = \sum_{k=1}^{p} \sigma_k u_k v_k^* \tag{13}$$

where it is assumed that $p \leqslant h$. The degree to which this rank p matrix approximates R_e is conveniently given by the norm ratio

$$\frac{\|\hat{R}_e^{(p)}\|}{\|\hat{R}_e\|} = \sqrt{\frac{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_p^2}{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_h^2}} \qquad 1 \leqslant p \leqslant h \tag{14}$$

in which the matrix norm symbol $\|\cdot\|$ designates the standard l_2 matrix norm. This norm ratio is seen to equal its largest value of one at $p = h$, where $\hat{R}_e^{(h)} = \hat{R}_e$. The extended-order autocorrelation matrix estimate $\hat{R}_e$ is said to have a low effective rank if this ratio is near one for small values of p relative to h. Otherwise, $\hat{R}_e$ is said to have a high effective rank.

Upon examination of the above norm ratio as a function of p, one is typically able to form a good estimate for the order integer p characterising the underlying recursive relationship given in eqn. 1. Namely, the p largest singular values will tend to dominate the remaining $h - p$ smaller sized singular values. Once this order selection has been obtained, the parameter vector a is next estimated by considering the associated reduced-order homogeneous relationship

$$\hat{R}_e^{(p)} a = 0 \tag{15}$$

Unlike the original expression given in eqn. 7, this system of equations is guaranteed to be consistent provided that $p < h$. From this expression, the following fundamental result applies.

Lemma 1

The set of model parametric vectors which satisfy the homogeneous relationship given in eqn. 15 lie in the $(h - p)$-dimensional null space associated with the reduced-order matrix $\hat{R}_e^{(p)}$.

The implications of this lemma are profound since it implies that there will exist an infinite number of candidates for the model's parameter vector. Clearly, from this infinity of choices, we wish to select one which will yield an acceptably good modelling representation. With this objective in mind, extensive empirical evidence has suggested that the solution to the following problem will yield such a choice:‡

$$\min_{a \in A} \quad a^* a \tag{16a}$$

where

$$A = \{a : \hat{R}_e^{(p)} a = 0 \quad \text{and} \quad a_m = 1\} \tag{16b}$$

Namely, we seek that $(p_e + 1) \times 1$ parameter vector of minimum l_2 norm which lies in the null space of $\hat{R}_e^{(p)}$ and has its $(m+1)$th component constrained to be one. Using standard procedures, it is found that the solution to this constrained minimisation problem is given by the $(p_e + 1) \times 1$ parameter vector

$$a = \frac{\left[e_m - \sum_{k=1}^{p} \bar{v}_k(m) v_k\right]}{\left[1 - \sum_{k=1}^{p} |v_k(m)|^2\right]} \tag{17a}$$

$$= \frac{\left[\sum_{k=p+1}^{h} \bar{v}_k(m) v_k\right]}{\left[\sum_{k=p+1}^{h} |v_k(m)|^2\right]} \tag{17b}$$

where the $(p_e + 1) \times 1$ vectors v_k are those used in the SVD

‡ A more general modelling procedure is to be found in the Appendix.

representation in eqn. 11. Although the two parameter vector relationships specified by eqn. 17 are mathematically equivalent, one may be preferable to the other when computational considerations are taken into account. Namely, the first representation, eqn. 17*a* is seen to entail knowledge of the p 'largest' v_k characteristic vectors, while the latter expression, eqn. 17*b*, requires knowledge of the $h - p$ 'smallest' v_k characteristic vectors. Typically, p is much smaller than h so that one need only compute a relatively small subset of the full set of eigenvectors v_k in order to generate the desired parameter vector. An appealing method for taking advantage of this feature is outlined in Reference 2.

In summary, the modelling approach taken in this paper for representing rational-type data entails the four-step procedure listed in Table 1. How one is to interpret step (iv) may be dependent on the particular application at hand, as will be demonstrated in Sections 4 and 5. An important consideration in using this approach is the selection of the extended-order parameter p_e and the reduced-order parameter p. *Truly exceptional* modelling performance can be achieved by a proper choice of these parameters. This selection process may require empirical experimentation in some applications.

Table 1: Basic steps of an SVD modelling approach

Step	
(i)	Generate autocorrelation lag estimates from the given time series observations.
(ii)	To reduce the model's sensitivity to autocorrelation lag errors, one forms the extended-order autocorrelation matrix estimate given in eqn. 8.
(iii)	The effective rank p of this extended-order autocorrelation matrix estimate is next obtained by generating the matrix's SVD and evaluating its associated singular-value behaviour using the norm ratio eqn. 14.
(iv)	The required parameter vector is then generated by solving the associated reduced pth-ordered system of equations.

4 Sinusoids in white noise example

In order to illustrate how one might apply the SVD representation approach as developed in the preceding Section, let us consider the important practical problem of detecting the presence of sinusoids in additive white noise. The time series under consideration is then taken to be specified by

$$x(n) = \sum_{k=1}^{p} A_k \exp\left[j(\omega_k n + \phi_k)\right] + w(n) \tag{18}$$

in which $\{w(n)\}$ is a zero-mean white-noise process whose statistically independent real and imaginary components each have a variance of $0.5\,\sigma^2$, and the p complex sinusoids have unknown real amplitudes A_k, frequencies ω_k, and independent random variable phases ϕ_k which are uniformly distributed on $[-\pi, \pi]$. Under these assumptions, it is readily shown that the autocorrelation sequence associated with the time series given by eqn. 18 is given by

$$r_x(n) = \sum_{k=1}^{p} A_k^2 \exp(j\omega_k n) + \sigma^2 \delta(n) \tag{19}$$

in which $\delta(n)$ designates the Kronecker delta sequence. Appealing to linear systems theory, it can be established that the autocorrelation sequence given in eqn. 19 will satisfy the homogeneous linear recursion expression

$$\sum_{k=0}^{p} a_k r_x(n-k) = 0 \qquad \text{for } |n| \geqslant p+1 \tag{20}$$

in which the required a_k coefficients are theoretically linked to the sinusoidal frequency parameters according to

$$\sum_{k=0}^{p} a_k z^{-k} = a_0 \prod_{k=1}^{p} (1 - \exp(j\omega_k) z^{-1}) \tag{21}$$

Thus there is seen to be a one-to-one relationship between the recursive a_k parameters and the sinusoidal frequencies (provided that the ω_k are restricted to the Nyquist interval). If one is able to effect evaluations of the a_k parameters, it follows that the corresponding values for the sinusoidal frequencies may then be obtained through the root relationship given in eqn. 21 and vice versa.

In the ideal case in which exact autocorrelation lag information is available, the procedure for obtaining the required a_k parameters is conceptually straightforward. This simply entails evaluating relationship given in eqn. 20 over any set of p or more indices satisfying $n \geqslant p + 1$. The resultant consistent system of homogeneous equations is then solved to yield the required unique (to within a scalar factor) parameter vector. It is, of course, assumed here that the autocorrelation lags which are used do in fact satisfy a recursive relationship of the form of eqn. 20. Finally, the polynomial given in eqn. 21 associated with this parameter vector is factored to determine the sinusoidal frequencies ω_k.

In the more realistic case in which autocorrelation lag estimates (for example the biased lag estimates given in eqn. 5) are used in this solution process, however, the errors inherent in the lag entries will give rise to inaccurate estimates for the a_k parameters. As suggested earlier, the deleterious effects caused by the lag errors can be significantly reduced by considering an extended-order model such as

$$\sum_{k=0}^{p_e} a_k r_x(n-k) = 0 \qquad \text{for } |n| \geqslant p_e + 1 \tag{22}$$

in which p_e is normally taken to be significantly larger than p. To generate an estimate for the $p_e + 1$ recursive a_k parameters, one might then evaluate eqn. 22 for $p_e + 1 \leqslant n \leqslant 2p_e + 1$ to yield

$$\begin{bmatrix} \hat{r}_x(p_e+1) & \hat{r}_x(p_e) & \dots & \hat{r}_x(1) \\ \hat{r}_x(p_e+2) & \hat{r}_x(p_e+1) & \dots & \hat{r}_x(2) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hat{r}_x(2p_e+1) & \hat{r}_x(2p_e) & \dots & \hat{r}_x(p_e+1) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{p_e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{23a}$$

or equivalently

$$\hat{R}_e a = \mathbf{0} \tag{23b}$$

in which autocorrelation lag estimates have been used as matrix entries. Since the rank of the $(p_e + 1) \times (p_e + 1)$ extended-order autocorrelation matrix estimate is generally full (i.e. $p_e + 1$), it will be then necessary to appeal to a solution procedure such as represented by eqn. 10 or eqns. 17 in order to obtain a suitable nontrivial parameter vector estimate. In using this extended-order approach, the resultant parameter estimates will be generally found to be less sensitive to the errors contained in the autocorrelation lag estimates than for the minimal ideal order choice of p. Moreover, the spectral estimate which is generated from the extended-order estimate typically will have a much more sharply defined peak behaviour at the p frequencies ω_k than will the minimum-

order choice, and the resolution capabilities will also be better.

4.1 Improved method

Although the above approach for generating spectral estimates of the sinusoids in white noise will generally provide satisfactory results, the method now to be outlined can yield a significant performance improvement. It is predicated on first observing that in the noise-free case (i.e. $\sigma^2 = 0$), the autocorrelation lags given in eqn. 19 will satisfy the homogeneous relationship

$$\sum_{k=0}^{p} a_k r_x(n-k) = 0 \tag{24}$$

for *all* integer choices of n. The ability to use the specific indices $0 \leqslant n \leqslant p$ for the noise-free case (in contrast to the noisy case) will have profound computational and algebraic consequences. To illustrate one such effect, let us now evaluate the theoretical relationship given in eqn. 24 for the indices $0 \leqslant n \leqslant p$ to obtain

$$\begin{bmatrix} r_x(0) & r_x(-1) & \dots & r_x(-p) \\ r_x(1) & r_x(0) & \dots & r_x(-p+1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ r_x(p) & r_x(p-1) & \dots & r_x(0) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{25}$$

The matrix involved in this system of homogeneous equations is seen to have a complex-conjugate symmetric Toeplitz structure. This structure is of particular interest since it may be effectively used in algebraically characterising the required parameter vector solution.

As suggested earlier, when lag estimates are used in eqn. 25, the resultant system of homogeneous equations will generally not have a nontrivial parameter vector solution. To overcome the deleterious effects caused by the errors inherent in the lag estimates, it will then be desirable to consider the extended-order version of eqn. 25 in which p is replaced by p_e and p_e is taken to be much larger than p. The corresponding system of extended-order homogeneous relationships with autocorrelation lag entries then becomes

$$\begin{bmatrix} \hat{r}_x(0) & \hat{r}_x(-1) & \dots & \hat{r}_x(-p_e) \\ \hat{r}_x(1) & \hat{r}_x(0) & \dots & \hat{r}_x(1-p_e) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \hat{r}_x(p_e) & \hat{r}_x(p_e-1) & \dots & \hat{r}_x(0) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{p_e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{26a}$$

or

$$\hat{R}_e \boldsymbol{a} = \boldsymbol{0} \tag{26b}$$

The $(p_e + 1) \times (p_e + 1)$ extended-order autocorrelation matrix estimate $\hat{R}_e$ is again seen to have a complex-conjugate symmetric Toeplitz structure. Although the theoretical rank of matrix R_e is p, its estimate $\hat{R}_e$ will almost always have a full rank (i.e. $p_e + 1$) because of errors inherent in the autocorrelation lag estimate process.

Owing to the full-rank nature of matrix R_e, it will be advisable to take the approach described in the preceding Section so as to obtain a suitably good parameter vector estimate. Specifically, an SVD representation for R_e is first made so as to determine its effective rank p. The required choice for the order parameter p will be based on an examination of the singular values associated with the matrix R_e. In most situations, it will be found that there will exist p readily identifiable dominant singular values and consequently $p_e - p + 1$ relatively insignificant singular values. Once the determination of p has been made, the optimum $(p_e + 1) \times 1$ parameter vector $\boldsymbol{a}$ is next obtained using eqns. 17, and this set of parameters in turn ultimately gives rise to the factored polynomial

$$A(z) = \sum_{k=0}^{p_e} a_k z^{-k} = a_0 \prod_{k=1}^{p_e} (1 - p_k z^{-1}) \tag{27}$$

The presence of sinusoids in the time series will be indicated whenever any of the roots p_k are sufficiently close to the unit circle (i.e. $|p_k| \simeq 1$), as suggested by the theoretical expression given in eqn. 21. For the time series given in eqn. 18, it will generally be found that p of the p_e roots will tend to be located close to the values $\exp(j\omega_k)$ for $1 \leqslant k \leqslant p$, while the remaining $p_e - p$ roots will tend to be sufficiently removed from the unit circle. A particularly convenient method for displaying the sinusoidal content of a given time series is then obtained by making a plot of $1/(A(\exp(j\omega))$ against ω. It will be found that this spectral function will produce sharply defined peaks for values of ω in which $\exp(j\omega)$ is close to roots located near the unit circle and will otherwise be more smoothly behaved. The location and the sharpness of these peaks then provides a mechanism for detecting sinusoidal components.

In using eqns. 17 for estimating the model parameter vector $\boldsymbol{a}$, the selection of the parameter $m \in [0, p_e]$ is of fundamental importance. Typically, this parameter is selected to be $m = 0$, as exemplified in linear prediction methods and other procedures [1]. For the sinusoids in white noise time series given in eqn. 18, however, other choices may be more preferable. This will be a consequence of the complex-conjugate symmetric Toeplitz structure of the matrix R_e. Specifically, because of this structure, it has been shown in Reference 2 that the SVD characteristic vectors $\boldsymbol{v}_k$ which appear in the parameter vector expression given in eqns. 17 will be either complex-conjugate symmetric or complex-conjugate skew-symmetric; i.e.

$$\bar{\boldsymbol{v}}_k = J\boldsymbol{v}_k \quad \text{or} \quad \bar{\boldsymbol{v}}_k = -J\boldsymbol{v}_k \tag{28}$$

where J is the $[(p_e + 1) \times (p_e + 1)]$-order reversal matrix whose components are all zero except for ones which appear along its main antidiagonal. This characteristic will now be used to obtain a parameteric vector $\boldsymbol{a}$ which itself will be complex-conjugate symmetric.

It will now be assumed that we have chosen the extended-order parameter p_e to be even. Furthermore, let us take the integer m used in the model parameter vector expression given in eqns. 17 to be equal to $0.5p_e$ (i.e. the integer midway between 0 and p_e). For this selection of m, it follows that the $\boldsymbol{v}_k(m)$ as required in eqns. 17 will be zero for all complex-conjugate skew-symmetric vectors $\boldsymbol{v}_k$ therein appearing (a requirement so that $\bar{\boldsymbol{v}}_k = -J\boldsymbol{v}_k$). Thus the parameter vector given in eqns. 17 is simply equal to a linear combination of complex-conjugate symmetric vectors, which implies that $\boldsymbol{a}$ is itself similarly characterised; i.e.

$$\bar{\boldsymbol{a}} = J\boldsymbol{a} \tag{29}$$

Since $\boldsymbol{a}$ is complex-conjugate symmetric, it then follows that the roots of the associated polynomial $A(z)$ in eqn. 27 will appear in conjugate reciprocal pairs, i.e.

$$\text{if} \quad A(p_k) = 0 \quad \text{then} \quad A(\bar{p}_k^{-1}) = 0 \tag{30}$$

A little thought will convince one of the desirability of this property when seeking to detect the presence of sinusoids. Specifically, a sinusoid will have its root in $A(z)$ located on the unit circle [i.e. $p_k = \exp(j\omega_k)$] and the complex conjugate reciprocal of this root will be equal to itself [i.e. $\bar{p}_k^{-1} = \exp(j\omega_k)$]. Empirical evidence has demonstrated that there will be a marked tendency to position a pole *exactly* on the unit circle whenever a sinusoidal component is present in the signal. For the situation in which there are p sinusoids as specified in eqn. 18, the autoregressive parameter vector will typically produce p roots on the unit circle at or near the desired locations $\exp(j\omega_k)$ for $1 \leqslant k \leqslant p$, and the remaining $p_e - p$ roots will tend to be evenly located on two concentric circles located inside and outside the unit circle such that the reciprocal relationship given in eqn. 30 is satisfied.

Although the approach just described was developed for the noise-free case in which $w(n) = 0$, its utilisation for situations in which additive noise is present also gives rise to satisfactory spectral estimation performance. The reason for this is readily explained by noting that the additive white noise will only directly effect the zero-lag term $r_x(0)$ in eqns. 23. Thus only the $p_e + 1$ diagonal terms in the extended autocorrelation matrix given in eqns. 23 will be primarily affected by the presence of additive white noise. It can then be argued that the SVD representation of R_e for a sufficiently large selection of the extended-order parameter p_e will be typically only moderately influenced by the additive white noise since only the diagonal elements of R_e are explicitly influenced. This behaviour has been empirically observed for even low-signal/noise (i.e. $A_k^2/2\sigma^2$) environments, as the numerical example given in Section 6 will demonstrate.

5 Autoregressive modelling

It is possible to straightforwardly adapt the philosophy behind the SVD approach taken in Section 3 to evolve ARMA model representations for satationary time series. To demonstrate this, let us now consider the task of estimating the parameters of a hypothesised AR(p) model as specified by

$$x(n) + \sum_{k=1}^{p} a_k x(n-k) = b_0 \epsilon(n) \tag{31}$$

with this estimation being based on a finite set of time series observations. In this expression, the excitation $\{\epsilon(n)\}$ is composed of a set of zero-mean unit-variance uncorrelated random variables (i.e. white noise). It is well known that the autocorrelation sequence of this AR(p) model satisfies the homogeneous relationship [1]

$$r_x(n) + \sum_{k=1}^{p} a_k r_x(n-k) = 0 \qquad \text{for } n \geqslant 1 \tag{32}$$

If the underlying time series is an AR(p) process, then the required a_k parameters of the model given in eqn. 31 may be conceptually obtained by solving the consistent system of eqn. 32 for any set of p or more indices satisfying $n \geqslant 1$.

When one only has available time series observations to effect the modelling, it will then be necessary to compute autocorrelation lag estimates from these observations and use these estimates in eqn. 32. Owing to errors inherent in the lag estimate procedure, however, the resultant system of homogeneous equations given in eqn. 32 with lag estimates will not, in general, be consistent. To overcome the deleterious effects due to these errors, it will again be beneficial to consider an extended order AR model, which will give rise to the following relationships in which lag estimates are incorporated:

$$\hat{r}_x(n) + \sum_{k=1}^{p_e} a_k \hat{r}_x(n-k) = 0 \qquad \text{for } n \geqslant 1 \tag{33}$$

The extended-order parameter p_e is here selected to satisfy $p_e > p$. An evaluation of these equations for the indices $1 \leqslant n \leqslant p_e$ is seen to yield the system of equations

$$\begin{bmatrix} \hat{r}_x(0) & \hat{r}_x(-1) & \dots & \hat{r}_x(1-p_e) \\ \hat{r}_x(1) & \hat{r}_x(0) & & \hat{r}_x(2-p_e) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \hat{r}_x(p_e-1) & \hat{r}_x(p_e-2) & \dots & \hat{r}_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_{p_e} \end{bmatrix} = - \begin{bmatrix} \hat{r}_x(1) \\ \hat{r}_x(2) \\ . \\ . \\ . \\ \hat{r}_x(p_e) \end{bmatrix} \tag{34a}$$

or in the more compact matrix form

$$\hat{R}_e a = -r \tag{34b}$$

We have here elected to separate out the vector r so as to cause the matrix R_e to have the desired complex-conjugate symmetric Toeplitz structure. This will enable us to take advantage of the algebraic and computational properties associated with such matrices. Whatever the case, because of the aforementioned lag estimate errors, the parameter vector solution to this system of generally consistent equations as specified by

$$a = -\hat{R}_e^{-1} r \tag{35}$$

will not, in general, adequately represent the dynamics of the underlying theoretical process given in eqn. 33. By choosing p_e to be much larger than p, however, there will be a tendency to reduce the sensitivity to lag estimate errors, and thereby to produce a more representative model.

Although the overextended parameter vector selection given in eqn. 35 will typically yield satisfactory modelling, a significant improvement in modelling performance can still be achieved by a suitable modification of this approach. This improvement takes advantage of the fact that the order of the underlying AR process is p. This in turn implies that the theoretical rank of the $p_e \times p_e$ matrix R_e (i.e. with exact autocorrelation lag entries) appearing in eqns. 34 is also p. Thus, by obtaining the best rank p approximation to the matrix estimate R_e through an SVD representation, the following system of equations will more properly reflect the dynamics of the underlying process:

$$\hat{R}_e^{(p)} a = -r \tag{36}$$

The parameter vector which provides the least-squares solution to this generally inconsistent system of equations is next found by using the expression

$$\begin{aligned} a &= -[\hat{R}_e^{(p)}]^{\#} r \\ &= -\sum_{k=1}^{p} \sigma_k^{-1} (v_k^* r) v_k \end{aligned} \tag{37}$$

where $[R_e^{(p)}]^{\#}$ denotes the pseudoinverse of matrix $R_e^{(p)}$, and the pairs (σ_k, v_k) for $1 \leqslant k \leqslant p$ correspond to the p 'largest' singular-value-characteristic-vector pairs associated with the SVD of the matrix R_e.[§] On examination of eqn. 37,

[§] The complex-conjugate symmetric Toeplitz matrix R_e will have an SVD of the form $\sum_{k=1}^{p_e} \sigma_k v_k v_k^*$, and its reduced pth-order pseudoinverse will then be given by $[R_e^{(p)}]^{\#} = \sum_{k=1}^{p} \sigma_k^{-1} v_k v_k^*$, as shown in Reference 2.

it is important to note that *only* the p 'largest' singular-value-characteristic pairs (σ_k, v_k) need be calculated when determining the optimum parameter vector. This observation can result in a significant computational saving whenever $p << p_e$, since one may suitably adapt the method of exhaustion to this situation [2].

In most practical applications, one does not know the value of the order parameter p *a priori*. Fortunately, it is possible to effect an estimate for p by examining the singular-value behaviour of the matrix $\hat{R}_e$. Since the theoretical rank of $\hat{R}_e$ is p, it follows that the effective rank of the $p_e \times p_e$ matrix estimate $\hat{R}_e$ will be close to p. This implies that the p largest singular values of $\hat{R}_e$ will tend to dominate its remaining $p_e - p$ singular values. The degree to which this is so is measured by the norm ratio

$$\frac{\sum_{k=1}^{p} \sigma_k^2}{\sum_{k=1}^{p_e} \sigma_k^2} = \frac{\sum_{k=1}^{p} \sigma_k^2}{\|\hat{R}_e\|^2} \tag{38}$$

where $\|\hat{R}_e\|$ designates the standard l_2 norm of the matrix $\hat{R}_e$. This ratio is bounded above by one (achieved at $p = p_e$), and it provides a measure for the effective rank of $\hat{R}_e$. The matrix $\hat{R}_e$ is said to have an effective rank of p if this ratio is satisfactorily close to one. Thus, when p is not known *a priori*, one simply applies the adapted method of exhaustion [2] to the matrix $\hat{R}_e$ to compute the pair (σ_k, v_k) as k is incrementally increased from one. Once the right-hand side of eqn. 38 is determined to be suitably close to one, that value of k is taken to be the required value of p.

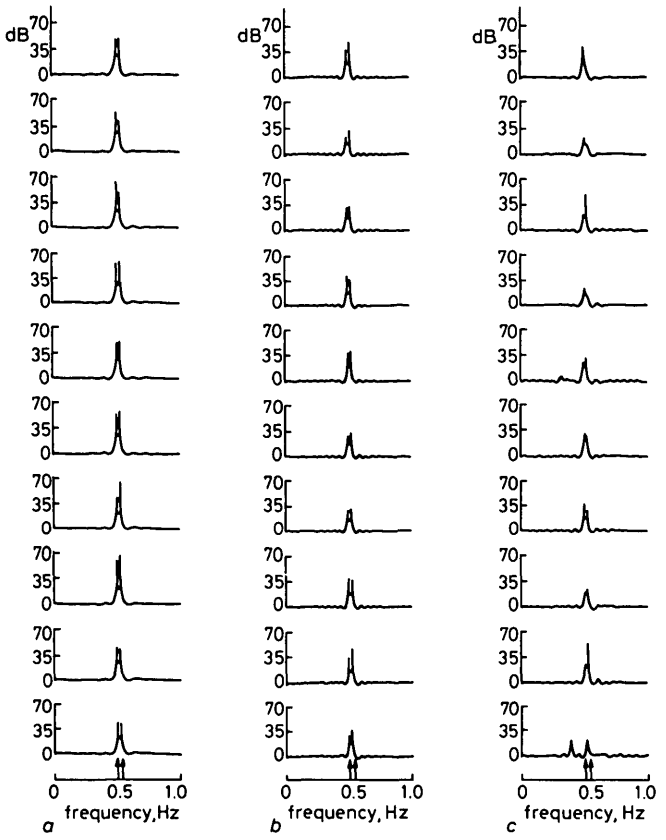

Fig. 1 *Spectral estimates corresponding to ten statistically independent trials of the time series given in eqn. 39 at a signal/noise ratio of 10 dB*

a Paper's method: $p_e = 20, p = 2, m = 10$
b Paper's method: $p_e = 20, p = 2, m = 0$
c Tufts-Kumaresan method: $L = 18, M = 2$

Table 2: Norm ratio for this paper's method and the Tufts-Kumaresan method time series given in eqn. 39

p	1	2	3	4	5
Norm ratio (paper's method)	0.982	0.998	0.999	0.999	0.999
Norm ratio (Tufts-Kumaresan method)	0.980	0.983	0.986	0.989	0.992

6 Numerical example

To illustrate the effectiveness of the method presented in Section 4, we shall consider examples which involve both real- and complex-valued sinusoidal data. The resultant spectral estimates are compared with those obtained using the Tufts-Kumaresan method [9] for the complex-valued data example, a procedure claimed to perform in a near maximum-likelihood fashion. The Tufts-Kumaresan method serves as an excellent bench mark in that it has been found to provide exceptional spectral estimates for the sinusoids in white noise case.

Example 1

In the first example, we shall use the numerical example considered by Tufts-Kumaresan [9], i.e.

$$x(n) = \exp\{j(2\pi f_1 n + \pi/4)\} + \exp(j2\pi f_2 n) + w(n) \qquad 1 \leqslant n \leqslant 25 \tag{39}$$

where $f_1 = 0.52$ Hz, $f_2 = 0.5$ Hz and the $w(n)$ are independent complex Gaussian random variables with variance σ^2 for each of the uncorrelated real and imaginary components. The variance σ^2 is selected to be equal to 0.05 so as to give a signal /noise ratio [as measured by $10\log(1/2\sigma^2)$] of 10 dB. Ten statistically independent trials of the time series given in eqn. 39 were made next with each trial being of length 25. Finally, the individual spectral estimates and superimposed plots of the roots of $A(z)$ for the ten trials obtained using this paper's method, as represented by eqns. 17 with $p_e = 20, p = 2$ and $m = 10$, and $p_e = 20, p = 2$ and $m = 0$, were obtained and are shown displayed in Figs. 1*a* and *b* and 2*a* and *b*, respectively. From the plots, it is apparent that a spectral resolution of the two complex sinusoids was achieved in an impressive manner for each of the ten trials. When the Tufts-Kumaresan method was applied to the same data for a parameter selection of $L = 18$ (their declared optimum selection) and $M = 2$ (the Tufts-Kumaresan parameter M corresponds to our p), it produced the individual spectral estimates and superimposed root plots shown in Figs. 1*c* and 2*c*, respectively. The Tufts-Kumaresan estimate was able to produce a distinctive resolution of the two complex sinusoids in only five of the ten trials. Furthermore, an examination of the associated root plots reveals that this paper's method produced an almost exact overlay in position location (a desirable feature) for each of the ten statistically independent trials, while the Tufts-Kumaresan method provided a somewhat more scattered root location behaviour.

To illustrate the effectiveness of using the singular values of the SVD of matrix R_e for model order determination, the norm ratio given in eqn. 14 for a typical trial is shown in Table 2. From this Table, it is apparent that both this paper's method and the Tufts-Kumaresan method provide convincing evidence for selecting a low-order effective rank. This paper's method, however, produced a better order behaviour in that its norm ratio at the correct order $p = 2$ was much closer to one than the Tufts-Kumaresan norm ratio (i.e. 0.99 against 0.983).

Using the same time series description as in eqn. 39, the additive noise variance was next increased to 0.5 to give a

0 dB signal/noise ratio. Applying this paper's method with $p_e = 20$, $p = 2$ and $m = 10$, and $p_e = 20$, $p = 2$ and $m = 0$, and the Tufts-Kumaresan method with $L = 18$ and $M = 2$ to the ten trials, each of length 25, produced the ten individual spectral plots shown in Fig. 3 and the superimposed root plots shown in Figs. 2*d–f*. The Tufts-Kumaresan method did not resolve the two sinusoids in any of the ten trails, while this paper's method produced a resolution for each of the ten trials for $m = 10$, and a resolution in seven out of the ten trials for $m = 0$. Moreover, although somewhat deteriorated because of the low signal/noise ratio, the root pattern behaviour continued to display a desirable uniformity.

Example 2

In the second example, the case of real sinusoids in white noise as specified by

$$x(n) = \cos(2\pi f_1 n + \pi/4) + \cos(2\pi f_2 n) + w(n) \qquad 1 \leqslant n \leqslant 16 \tag{40}$$

was treated, in which $f_1 = 0.2$ and $f_2 = 0.25$, where the $w(n)$ are independent real Gaussian random variables with variance σ^2. The variance was taken to be 0.5, thereby giving a signal/noise ratio of 0 dB. Ten statistically independent trials of the time series given in eqn. 40 were then made, with each trial being of the relatively *short* length 16. The spectral estimates obtained using the method given in Section 4 with parameter selections $p_e = 14$, $p = 4$, $m = 7$ are shown in Fig. 4*a*. A resolution of the two real-valued sinusoids was achieved in each of the ten trials. The Tufts-Kumaresan method was next applied to the same data and produced the spectral estimates shown in Fig. 4*b* for the parameter selection $L = 12$ (their optimal selection) and $M = 4$. In only five of the ten trials did the Tufts-Kumaresan method yield a resolution of the two sinusoids, and these resolutions were not of a sharply defined nature. Next, the periodogram method was applied to the ten sets of data length 16. To avoid loss of resolution due to sampling, each of these data sets was padded with 496 zeros. The 512-point FFT spectral plots for the ten padded data sets are shown in Fig. 4*c*. A resolution of the two sinusoids was not achieved in a sharply defined manner for any of the ten FFT spectral estimates. Finally, the forward data version of the Burg algorithm was applied to the same ten sets of data length 16 with an order choice of eight. The ten resultant spectral plots are shown in Fig. 4*d*, in which a frequency resolution was achieved in only four estimates.

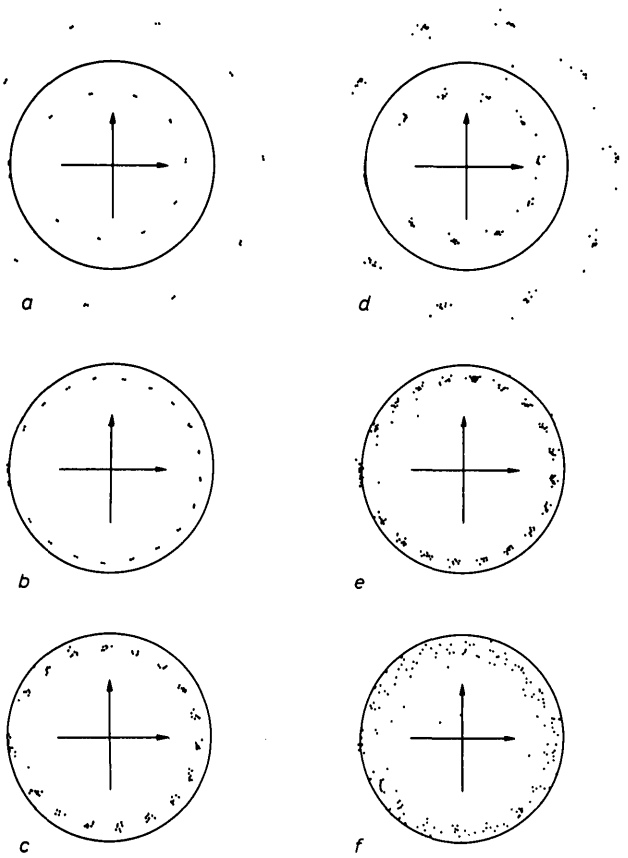

Fig. 2 *Superimposed roots corresponding to ten statistically independent trials of the time series given in eqn. 39 at signal/noise ratios of 10 and 0 dB*

a 10 dB, paper's method $p_e = 20, p = 2, m = 10$
b 10 dB, paper's method $p_e = 20, p = 2, m = 0$
c 10 dB, Tufts-Kumaresan method $L = 18, M = 2$
d 0 dB, paper's method $p_e = 20, p = 2, m = 10$
e 0 dB, paper's method $p_e = 20, p = 2, m = 0$
f 0 dB, Tufts-Kumaresan method $L = 18, M = 2$

7 Conclusion

An SVD approach for estimating the model parameters characterising important classes of time series has been presented. This method entails forming an extended-order autocorrelation matrix estimate and then determining the 'best' rank p approximation of that matrix from its SVD representation. Finally, the model's parameters are obtained by finding the least-squares solution to a system of equations associated with the reduced-order autocorrelation matrix estimate.

One of the more important applications of this approach is

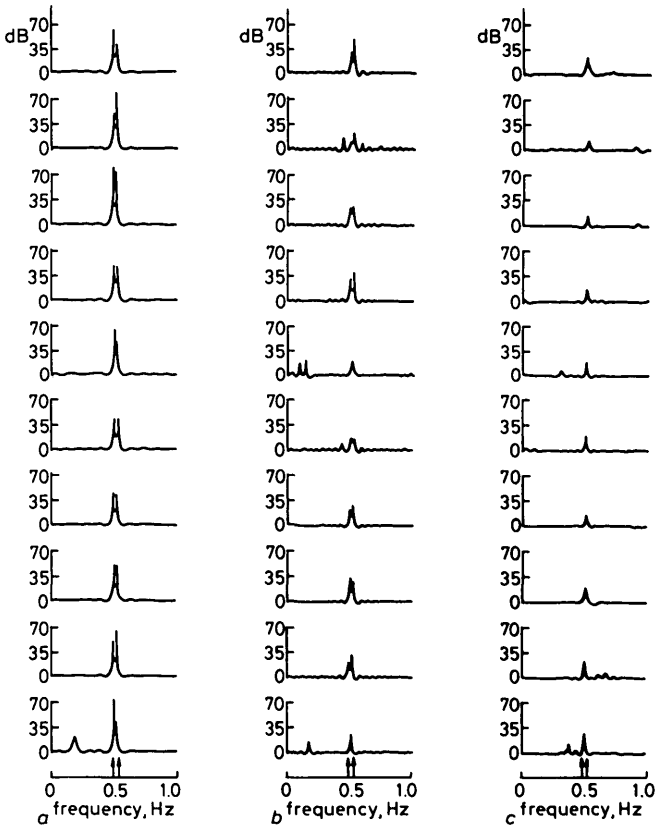

Fig. 3 *Spectral estimates corresponding to ten statistically independent trials of the time series given in eqn. 39 at a signal/noise ratio of 0 dB*

a Paper's method: $p_e = 20, p = 2, m = 10$
b Paper's method: $p_e = 20, p = 2, m = 0$
c Tufts-Kumaresan method: $L = 18, M = 2$

that of detecting the presence of sinusoids in noise-contaminated data. In this case, the solution of the reduced-order system of equations is sought in which the $(m+1)$th component of the parameter vector is constrained to be one. By appropriately selecting the integer m, the resultant spectral estimate may be made to take on different characteristics. For example, the selection of $m=0$ will tend to cause the nonsinusoidal roots of $A(z)$ to lie within the unit circle. On the other hand, a selection of $m=0.5p_e$ will tend to cause the nonsinusoidal roots to be equally distributed on concentric circles located inside and outside the unit circle. Moreover, this selection will create a strong tendency to locate 'sinusoidal' roots exactly on the unit circle owing to the complex-conjugate symmetrical property possessed by the resultant parameter vector. Clearly, this is a desirable feature to have when seeking to detect the presence of sinusoidal components in additive noise. Moreover, the associated polynomial $A(z)$ will have a linear phase characterisation, which can be of importance in such applications as interference rejection filtering. It is further noted that the nonsinusoidal roots tend to lie further from the unit circle than in the $m=0$ case (a desirable quality). In a similar fashion, it can be empirically demonstrated that a choice of $m=p_e$ will tend to result in all the nonsinusoidal roots being located outside the unit circle.

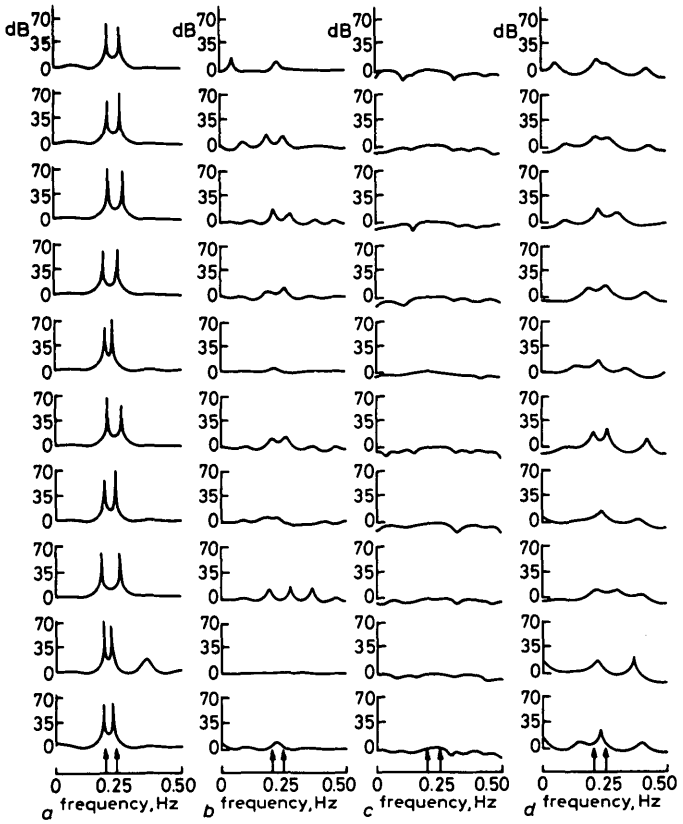

Fig. 4 *Ten trial estimates of the time series given in eqn. 40 at a signal/noise ratio of 0 dB*

a Paper's method: $p_e = 14, p = 4, m = 8$
b Tufts-Kumaresan method: $L = 12, M = 4$
c 512-point padded periodogram
d Burg method: $p = 8$

Although we have here concentrated our attention on the sinusoids in white noise case, the spectral modelling procedure presented herein is applicable to a much broader class of problems. For example, it has produced excellent performance when generating AR and ARMA models, in detecting rational type signals and in performing deconvolution operations.

8 Acknowledgment

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10 Appendix

Other rational modelling procedures can be obtained by considering more general functionals than those utilised in Section 3. In particular, let it be desired to select the model parameter vector $\boldsymbol{a}$ to solve the constrained minimisation problem

$$\min_{\boldsymbol{a}\in A} \boldsymbol{a}^*\boldsymbol{a} \tag{41}$$

where

$$A = \{\boldsymbol{a}: R^{(p)}\boldsymbol{a} = \boldsymbol{0} \text{ and } \boldsymbol{a}^*\boldsymbol{h} = 1\}$$

in which $\boldsymbol{h}$ is given a $(p_e+1)\times 1$ vector characterising a hyperplane in $l_{p_e}+1$ space. It is readily shown that the solution to this problem is given by

$$\boldsymbol{a} = \alpha\boldsymbol{h} - \sum_{k=1}^{p} \bar{v}_k(m)\, v_k \tag{42}$$

where normalising scalar α is selected to ensure that $\boldsymbol{a}^*\boldsymbol{h}=1$ as required. It is to be noted that, on setting $\boldsymbol{h}=\boldsymbol{e}_m$, we obtain the solution procedure found in Section 3.