ASSIGNMENT-1

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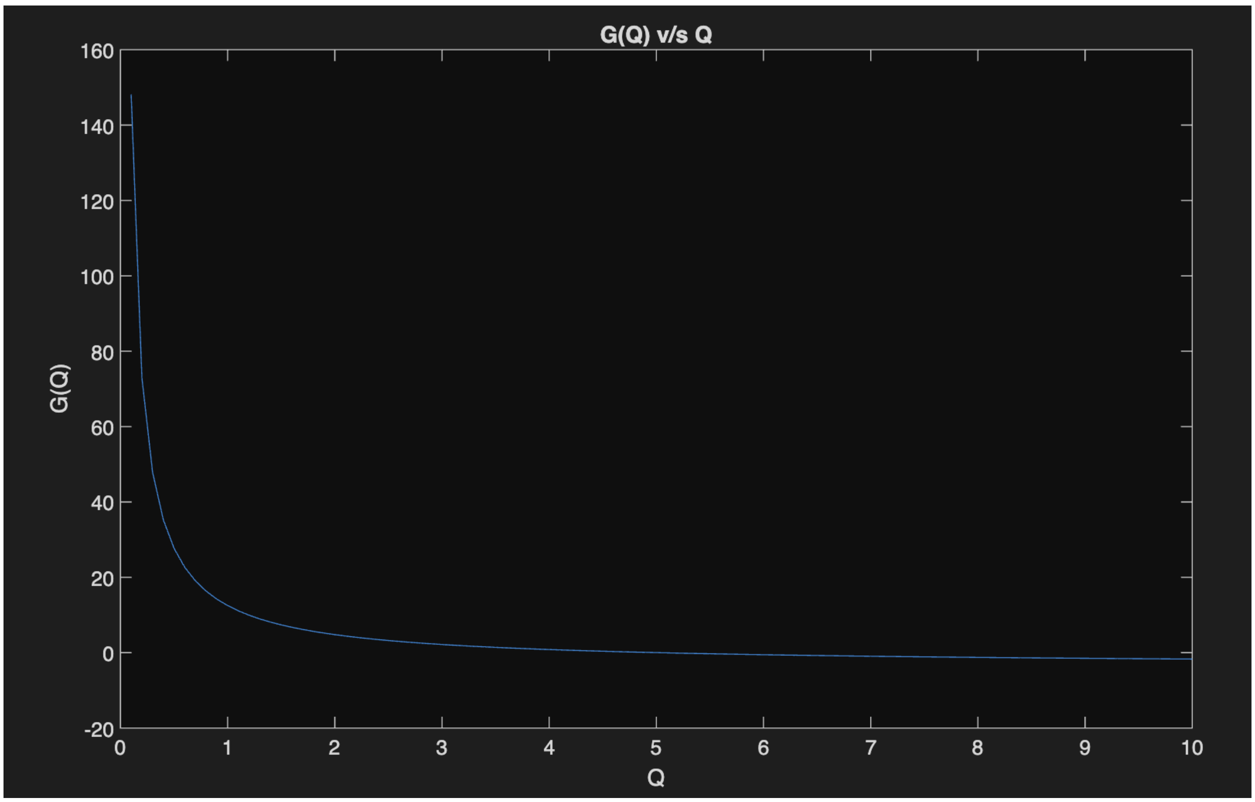
Q1.a)



b)

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root = 5.026143e+00>>
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c)

$$Q^{(1)} = 1$$

$$Q^{(n+1)} = \cancel{G(Q^{n})} \quad \frac{\Delta p}{A\log(Q^{(n)}) + B}$$

$$Q^{(2)} = \frac{15}{2.5} = 6$$

$$Q^{(3)} = \frac{15}{0.3\log(6) + 2.5} = \cancel{5.49} \quad 4.94$$

$$\cancel{Q^{(4)} = \frac{15}{0.3\log(5.49) + 2.5} =}$$

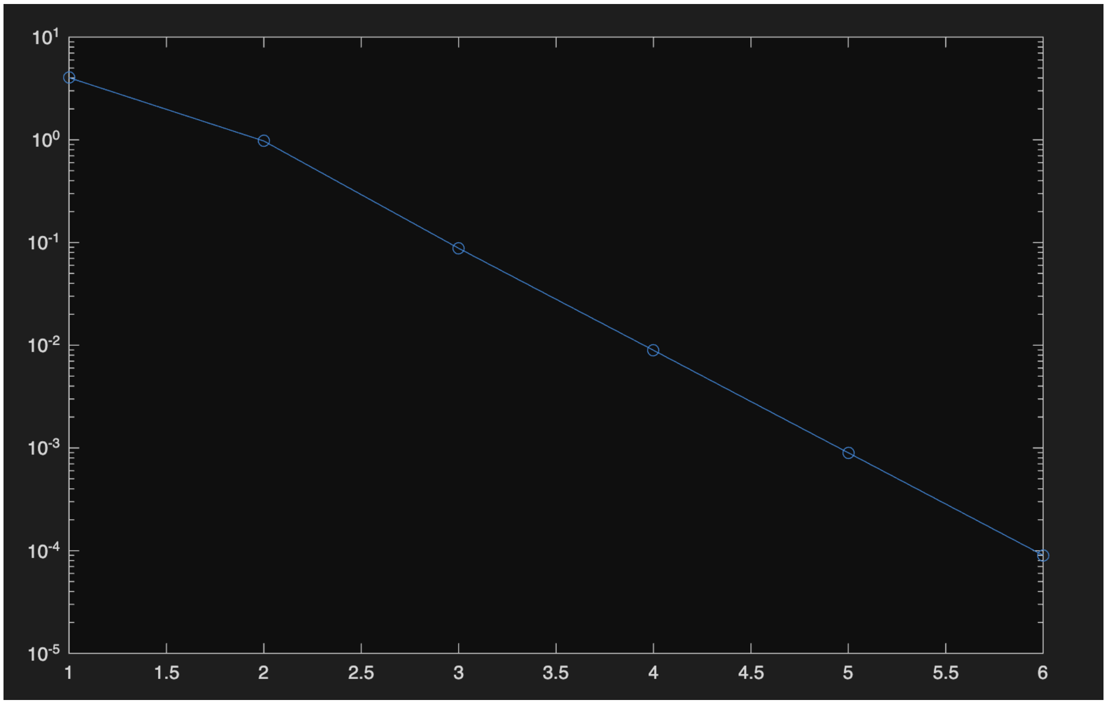
$$Q^{(4)} = \frac{15}{0.3\log(4.94) + 2.5} = 5.03$$

$$Q^{(5)} = 5.025$$

$$Q^{(6)} = 5.026$$

Q6 = 5.02623

d)



e)

As G*(Q) < 1 denotes convergence, we cannot say that it converges for every starting point, as the sign may change.

Hence, there exists at least one starting point for which it does not converge.

f)

$$e^{(n)} = |\varphi^{(n)} - \varphi^*|$$
$$e^{(n+1)} = |\varphi^{(n+1)} - \varphi^*|$$

$$\frac{e^{(n+1)}}{e^{(n)}} = \frac{|\varphi^{(n+1)} - \varphi^*|}{|\varphi^{(n)} - \varphi^*|}$$

$$f(x) = f(x_0) + (x - x_0) f'(x_0) \longrightarrow \text{1st order TSA}$$

$$\varphi^{(n+1)} = F(\varphi^{(n)}) = F(\varphi^*) + (\varphi^{(n)} - \varphi^*) F'(\varphi^*)$$
$$\varphi^{(n+1)} - \varphi^* \approx (\varphi^{(n)} - \varphi^*) F'(\varphi^*)$$

$$\rightarrow \quad \frac{\varphi^{(n+1)} - \varphi^*}{\varphi^{(n)} - \varphi^*} \approx F'(\varphi^*)$$

$$\Rightarrow \quad \frac{e^{(n+1)}}{e^{(n)}} = \frac{|\varphi^{(n+1)} - \varphi^*|}{|\varphi^{(n)} - \varphi^*|} \approx |F'(\varphi^*)|$$

Q2)

Handwritten Solutions

(a) Analytical Solution of the given ODE with the given Conditions.

A2(a)

$$\frac{dT}{dt} + T = 0 \qquad 0 \le t \le 1 \quad \& \quad T(0) = 1$$

$$\frac{dT}{dt} = -T$$

$$\Rightarrow \frac{dT}{T} = -dt$$

integrating both sides

$$\ln(T) = -t + C$$

$$T(t) = e^{-t+c} = e^{-t} \cdot e^{c} = A \cdot e^{-t}$$

$$\Rightarrow \boxed{T(t) = A \cdot e^{-t}}$$

$$\because T(0) = 1$$

$$\Rightarrow 1 = A \cdot e^{-0} \Rightarrow A = 1$$

$$\Rightarrow \boxed{T_*(t) = e^{-t}}$$

(b) RMSE Calculations for different dt values.

dt	Exp Euler	Imp Euler	Crank-Nic
0.10	0.014851	0.013451	0.000235
0.05	0.007262	0.006909	0.000059
0.01	0.001428	0.001413	0.000002

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(b)&(c) $\Delta t = 0.1,\ 0.05,\ 0.01$, $t^n = n\Delta t$ & $0 \le n \le N$ $\left(N = \frac{1}{\Delta t}\right)$

Time marching methods

(i) Explicit Euler (T_{exp})

$$\frac{T^{(n+1)} - T^n}{\Delta t} + T^n = 0$$

$$\Rightarrow T^{(n+1)} = -T^n \cdot \Delta t + T^n$$

$$\Rightarrow \boxed{T^{(n+1)} = T^n(1-\Delta t)}$$

$$\boxed{\frac{T^{n+1}}{T^n} = 1-\Delta t}$$

(ii) Implicit Euler

$$\frac{T^{n+1} - T^n}{\Delta t} + T^{n+1} = 0$$

$$\Rightarrow T^{n+1} + T^{n+1}\Delta t = T^n$$

$$\cancel{\Delta t} \; T^{n+1}(1+\Delta t) = T^n$$

$$\Rightarrow \boxed{T^{n+1} = \frac{T^n}{1+\Delta t}}$$

$$\boxed{\frac{T^{n+1}}{T^n} = \frac{1}{1+\Delta t}}$$

(iii)

$$\frac{T^{n+1} - T^n}{\Delta t} + \frac{T^n + T^{n+1}}{2} = 0$$

$$2T^{n+1} - 2T^n + \Delta t \cdot T^n + \Delta t\, T^{n+1} = 0$$

$$T^{n+1}(2+\Delta t) - T^n(2-\Delta t) = 0$$

$$\Rightarrow T^{n+1} = \frac{T^n(2-\Delta t)}{(2+\Delta t)}$$

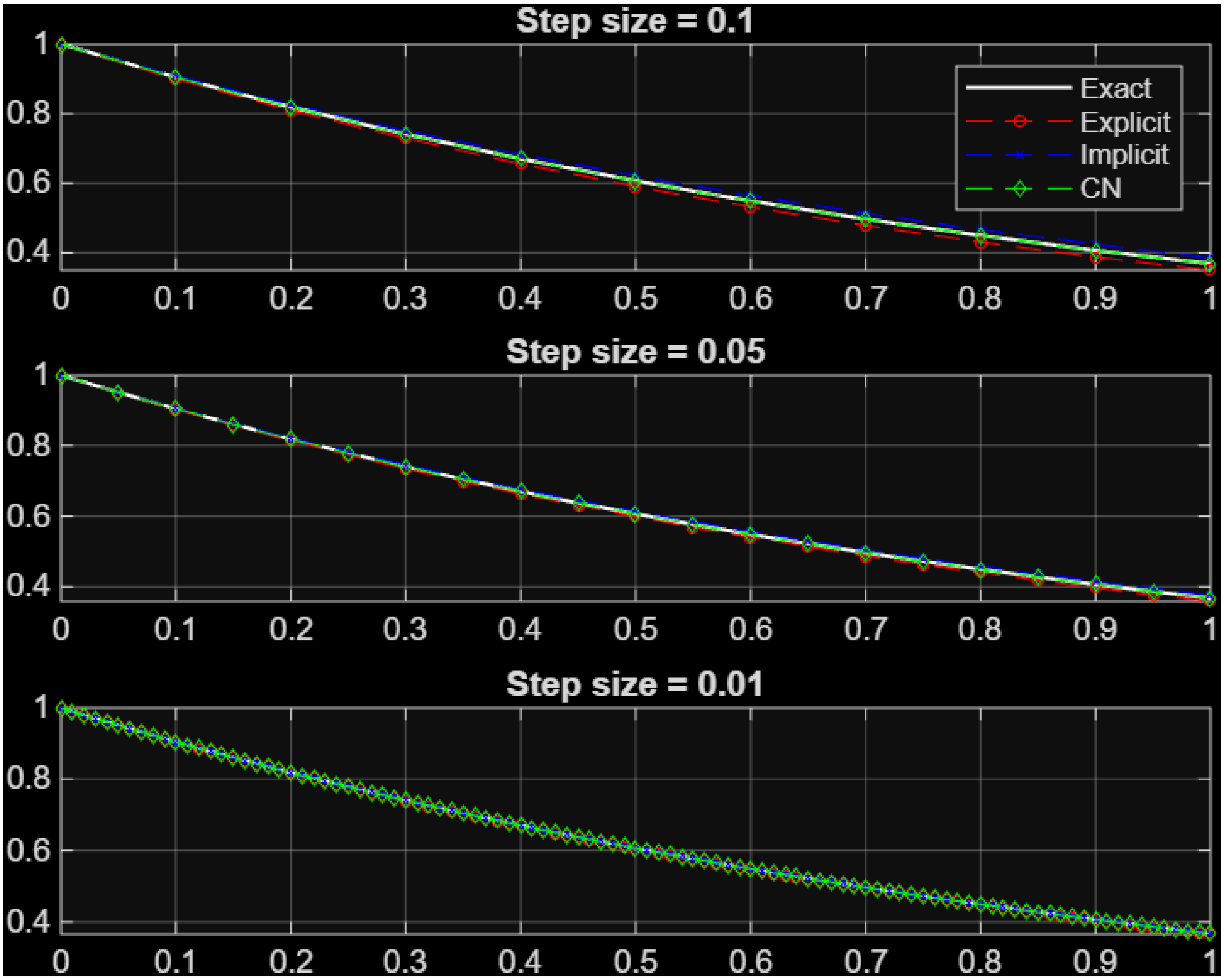
$$\Rightarrow \boxed{T^{n+1} = \frac{T^n(1-\Delta t/2)}{(1+\Delta t/2)}}$$

$$\boxed{\frac{T^{n+1}}{T^n} = \frac{1-\Delta t/2}{1+\Delta t/2}}$$

(In notebook Δt is represented as h)

$$\boxed{RMSE = \sqrt{\left(\frac{1}{N+1}\right) \cdot \sum_{n=0}^{N} \left(T^n - T_*(t^n)\right)^2}}$$

Time-Marching:



(c) Analysis showing why Implicit CN is the best Fit.

Implicit Crank-Nicolson is the best method to be used for this problem.

(b) $\because \frac{T^{(n+1)}}{T^{(n)}} \approx \frac{e^{-\Delta t(n+1)}}{e^{-\Delta t n}} \approx 1-\Delta t+\frac{(\Delta t)^2}{2}-\frac{(\Delta t)^3}{6}+\ldots$

(i) Explicit Euler

$\because \frac{T^{n+1}}{T^n}=1-\Delta t$

$\Rightarrow$ Accuracy matches for the first two terms of Taylor expansion

(ii) Implicit Euler

$\frac{T^{n+1}}{T^n}=\frac{1}{1+\Delta t}$ $\left[\because \frac{1}{1+x}=1-x+x^2-x^3+\ldots\right]$

$\Rightarrow \frac{T^{n+1}}{T^n}=1-\Delta t+(\Delta t)^2-(\Delta t)^3+\ldots$

$\Rightarrow$ Accuracy matches upto first two terms

(iii) Implicit Crank-Nicolson

$\because \frac{T^{n+1}}{T^n}=\frac{(1-\Delta t/2)}{(1+\Delta t/2)}=\left(1-\frac{\Delta t}{2}\right)\left(1-\frac{\Delta t}{2}+\frac{\Delta t^2}{4}-\frac{\Delta t^3}{8}+\ldots\right)$

$\Rightarrow \frac{T^{n+1}}{T^n}\approx 1-\Delta t+\frac{(\Delta t)^2}{2}-\frac{(\Delta t)^3}{4}+\ldots$

$\therefore$ Accuracy matches upto first 3 terms

Thus, Implicit CN is likely to be better than the other two as it is a second order accurate method ($O(\Delta t^3)$ = Error) whereas the other two are first order accurate ($O(\Delta t^2)$ = Error).

Q3)

Handwritten solution

Q.3)

(a) The equation is -

$$\frac{dh}{dt} + \sqrt{h} = q(t) \quad ; \quad h(t) > 0, \; h(t=0) = H$$

where $H = 2$; $q(t) = 0$

$$\Rightarrow \int \frac{dh}{\sqrt{h}} = -\int dt$$

$$\Rightarrow 2\sqrt{h} = -t + C$$

Applying initial condition: (at $t=0 \Rightarrow h = H = 2$)

$$\Rightarrow \boxed{2\sqrt{2} = C}$$

So, $2\sqrt{h} = -t + 2\sqrt{2} \Rightarrow \boxed{h = \left(\sqrt{2} - t/2\right)^2}$

(b)

(i) Explicit Euler: $\sqrt{h} \simeq \sqrt{h^{(n)}}$

$$\frac{h^{n+1} - h^n}{\Delta t} = -\sqrt{h^n}$$

$$\Rightarrow \boxed{h^{n+1} = h^n - \Delta t \sqrt{h^n}}$$

(ii) Implicit Euler: $\sqrt{h} = \sqrt{H^{n+1}}$

$$\frac{h^{n+1} - h^n}{\Delta t} = -\sqrt{h^{n+1}}$$

$$\Rightarrow h^{n+1} + \Delta t \sqrt{h^{n+1}} - h^n = 0$$

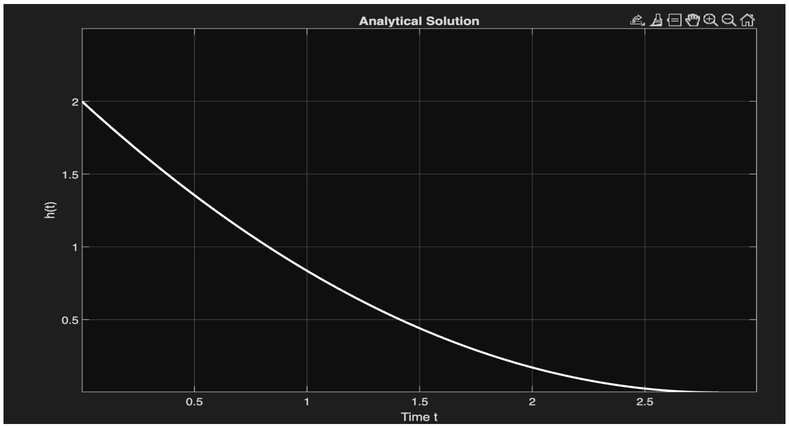
Let $y = \sqrt{h^{n+1}}$

$$\Rightarrow y^2 + \Delta t\, y - h^n = 0$$

$$\Rightarrow y^2 = h^{n+1} = \left(\frac{-\Delta t + \sqrt{\Delta t^2 + 4h^n}}{2}\right)^2$$

Results

Analytical Sol



Comparison