# Rigidity Theory

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# 1. Preliminaries

## 1.1. Basic definitions

### Def 1.1) Framework

> A ***framework*** in $\mathbb{R}^d$ is a pair $(G, p)$, where $G$ is a finite simple graph and $p : V(G) \to \mathbb{R}^d$ is a (not necessarily injective) map. We also say that $p$ is a $d$-dimensional ***realization*** of $G$. Sometimes, we denote $f_p = (G, p)$.

### Def 1.2) Joint, Bar

> For $v_i \in V(G)$, we say that $p_i := p(v_i)$ is the ***joint*** of $(G, p)$ corresponding to $v_i$.
> Also, for $v_i v_j \in E(G)$, we say that $\{p_i, p_j\}$ is the ***bar*** of $(G, p)$ corresponding to $v_i v_j$.
> The ***length of the bar*** $\{p_i, p_j\}$ is defined as $L(\{p_i, p_j\}) := \|p(v_i) - p(v_j)\|_2$.

### Def 1.3) Motion, rigid and flexible framework

> A ***(finite) motion*** of $(G, p)$ is an indexed family $\{P_i\}_{i \in [n]}$ of functions $P_i : [0, 1] \to \mathbb{R}^d$ s.t.
>
> 1. (Initial Condition) $P_i(0) = p(v_i), \quad \forall i \in [n]$.
> 2. (Differentiability) $P_i(t)$ is differentiable on $[0, 1], \quad \forall i \in [n]$.
> 3. (Distance-preserving) $\|P_i(t) - P_j(t)\| = \|p_i - p_j\|$ for all $t \in [0, 1]$ and $v_i v_j \in E(G)$.
>
> A motion $\{P_i\}$ is called ***rigid*** if $\|P_i(t) - P_j(t)\| = \|p_i - p_j\|$ for all $t \in [0, 1]$ and $(i, j) \in [n]^2$.
> Also, $\{P_i\}$ is called ***flexible*** if $\exists (i, j) \in [n]^2 \setminus E(G)$ s.t. $\|P_i(t) - P_j(t)\| \neq \|p_i - p_j\|$ for all $t \in (0, 1]$.
>
> With these definitions, $(G, p)$ is called rigid if there is no flexible motion. Otherwise, $(G, p)$ is called flexible.

Q. Why "for all $t \in (0, 1]$" for a framework to be flexible?

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## 1.2. Equivalence & Congruence

So far, we defined a framework with a tuple of a simple graph and a realization. However, for a formation system(engineering aspect), the set of edges $E(G)$ could be defined by a set of constraints. Here, the constraint means a task or a goal that agents work together to achieve. (e.g. communication network, distance-preserving, ...)

So, we can alternatively describe a formation by $f_p = (\mathbf{V}, \mathbf{C}, p)$ where $\mathbf{V} = [n]$ and $\mathbf{C}$ be a set of constraints. Obviously, $|\mathbf{C}| = e(G)$.

In this aspect, we can introduce more definitions.

### Def 1.4) Equivalence / Congruence of frameworks

> Two frameworks $f_p = (G, p) = (\mathbf{V}, \mathbf{C}, p)$ and $f_q = (\mathbf{V}, \mathbf{C}, q) = (G, q)$ with different realizations $p, q$, are called...
>
> - **Equivalent** if $\mathrm{mag}[c_i(p(x_i))] = \mathrm{mag}[c_i(q(x_i))]$ for all $c_i \in \mathbf{C}$ and denoted by $f_p \sim f_q$ where $x_i \subset \mathbf{V}$ is the set of nodes involved in the $i^{\text{th}}$ constraint $c_i$. In this context, $p(x_i)$ and $q(x_i)$ represent the realized points(coordinates) of these nodes in each framework, and $\mathrm{mag}[\cdot]$ is the exact value of the argument.
> - **Congruent** if $\mathrm{mag}[c_i(p(x_i))] = \mathrm{mag}(c_i(q(x_i)))$ for all $c_i \in \mathbf{C}_K$ and denoted by $f_p \equiv f_q$ where $\mathbf{C}_K$ is the complete constraint set containing constraints for every possible pair of vertices in the graph $G$. This corresponds to the set of all edges in the complete graph $K_n$. In this case, $x_i$ represents the set of nodes involved in each constraint $c_i$ within the expanded set $\mathbf{C}_K$.

For example, if the constraint is given as a distance between each vertices, then $c_i \in \mathbf{C}$ is the distance constraint on the $i^{\text{th}}$ edge and $x_i$ is the set of endpoints of the $i^{\text{th}}$ edge.

So, if $uv \in E(G)$ is the $i^{\text{th}}$ edge, then $x_i = \{u, v\}$ and $c_i(p(x_i)) = \|p(u) - p(v)\|$. In this sense, we can describe the equivalence and congruence for distance constraint only.

Two frameworks $f_p$ and $f_q$ are called...

- **Distance equivalent** if $\|p_i - p_j\| = \|q_i - q_j\|$ for all $v_i v_j \in E(G)$
- **Distance congruent** if $\|p_i - p_j\| = \|q_i - q_j\|$ for all $(i, j) \in [n]^2$.

### Def 1.5) Constraint function

> For a framework $f_p = (G, p) = (\mathbf{V}, \mathbf{C}, p)$, the constraint function $h_G : \mathbb{R}^{dn} \to \mathbb{R}^m$ $(m = e(G))$ is defined by
>
> $$h_G(p) := \begin{bmatrix} \mathrm{mag}[c_1(p(x_1))] \\ \vdots \\ \mathrm{mag}[c_m(p(x_m))] \end{bmatrix}$$
>
> for square of distance constraint,
>
> $$h_G(p) = \begin{bmatrix} \mathrm{mag}[c_1(p(x_1))] \\ \vdots \\ \mathrm{mag}[c_m(p(x_m))] \end{bmatrix} = \begin{bmatrix} \|p_{j_1} - p_{i_1}\|^2 \\ \vdots \\ \|p_{j_m} - p_{i_m}\|^2 \end{bmatrix}$$

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# 2. Rigidity and infinitesimal rigidity

## 2.1. Basic rigidity concepts

Q. What is the "Rigid framework" ?

### Def 2.1) Rigidity (General definition)

> A framework $f_p = (G, p)$ is called ***rigid*** in a given Euclidean space $\mathbb{R}^d$ if there exists a nbd. $U_p$ of $p$ s.t. every framework $f_q = (G, q)$ with $q \in U_p$ satisfy $f_p \sim f_q \Rightarrow f_p \equiv f_q$.

Algebraically, $f_p = (G, p)$ is rigid in $\mathbb{R}^d$ if there exists a nbd. $U_p$ of $p$ s.t.

$$h_G^{-1}(h_G(p)) \cap U_p = h_K^{-1}(h_K(p)) \cap U_p$$

Here,

- LHS = the set of frameworks in $U_p$ satisfying the constraint $\mathbf{C}$.
- RHS = the set of frameworks in $U_p$ satisfying the constraint $\mathbf{C}_K$.

Obviously, there is only one congruent framework satisfying $\mathbf{C}_K$. Also, since $\mathbf{C} \subset \mathbf{C}_K$, $\mathrm{LHS} \supset \mathrm{RHS}$. So, this expression means that, if $\mathrm{LHS} \neq \mathrm{RHS}$, then there is at least one framework that is distinct (non-congruent) from that of in RHS and hence $f_p$ is not rigid.

### Def 2.2) Distance rigidity

> A framework $f_p$ is called ***distance rigid*** in $\mathbb{R}^d$ if there exists $\varepsilon > 0$ s.t. for any framework $f_q$ that is locally perturbed from $f_p$ (i.e. $q_i \in B_\varepsilon(p_i),\ \forall i$), $f_p$ and $f_q$ are distance equivalent only if $f_p$ and $f_q$ are distance congruent.

This means that $f_p$ is distance rigid if every framework satisfying same edge-distance constraints to $f_p$ near $p$ is congruent to $f_p$ (Locally unique embedding satisfying edge-distance constraint).

Q. Are there two definition of distance rigidity? $\rightarrow$ No.

### Thm 2.3) Equivalent of definitions of rigidity

> Two definitions of distance rigidity (in the view of motion and set-theoretic definition) of framework is equivalent.

**Pf)**

### Def 2.4) Global distance rigidity

> A framework $(G, p)$ is called ***globally distance rigid*** in $\mathbb{R}^d$ if "every" framework $(G, q)$ equivalent to $(G, p)$ is also congruent to $(G, p)$ in $\mathbb{R}^d$.

Henceforth, unless otherwise specified, the term 'rigidity' will refer exclusively to **'distance rigidity'.**

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## 2.2. Infinitesimal rigidity

**Motivation:** It is in general very difficult to determine whether a given framework is rigid or not since it requires solving a system of quadratic equations. It is therefore common to **linearize** the problem by differentiating the equations.

Let $\{P_i\}$ be a finite motion of a 2-dimensional framework $(G, p)$ and let $\{p_i, p_j\}$ be a bar of length $l$. Then,

$$\|P_i(t) - P_j(t)\|^2 = \left[P_i^{(1)}(t) - P_j^{(1)}(t)\right]^2 + \left[P_i^{(2)}(t) - P_j^{(2)}(t)\right]^2 = l^2 \quad \forall t \in [0,1]$$

where $P_i^{(k)}(t)$ is the $k^{\text{th}}$ coordinate of $p_i$ at $t$.

By differentiating both sides w.r.t. $t$,

$$\left(P_1^{(1)}(t) - P_j^{(1)}(t), P_1^{(2)}(t) - P_j^{(2)}(t)\right) \cdot \left(\left[P_1^{(1)}(t) - P_j^{(1)}(t)\right]', \left[P_1^{(2)}(t) - P_j^{(2)}(t)\right]'\right) = 0$$

Evaluating at $t = 0$, we obtain

$$\left(p_i^{(1)} - p_j^{(1)}, p_i^{(2)} - p_j^{(2)}\right) \cdot \left(u_i^{(1)} - u_j^{(1)}, u_i^{(2)} - u_j^{(2)}\right) = 0$$

or in short $(p_i - p_j) \cdot (u_i - u_j)$. Here, $u_i, u_j$ are initial velocity vectors of $p_i$ and $p_j$, respectively. So, this implies that the difference between initial velocity vectors is orthogonal to the line through $p_i$ and $p_j$.

These vectors preserve the length of $\{p_i, p_j\}$ at first order.

### Def 2.5) Constricted framework

> A $d$-dimensional framework $(G, p)$ is called ***constricted*** if all points in $\mathcal{P} = \{p_1, \cdots, p_n\}$ lie in $k \geq d - 1$ dimensional affine subspace. Otherwise, $(G, p)$ is called ***normal***.

Thus, a framework is constricted if it is geometrically degenerated.

### Def 2.6) Infinitesimal motion / Infinitesimal rigidity

> An ***infinitesimal motion*** of a framework $(G, p)$ is a function $u : V(G) \to \mathbb{R}^d$ s.t.
>
> $$(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \forall v_i v_j \in E(G)$$
>
> That is, for a distance constrained graph $G$, a function consists of $dn$ instantaneous velocity vectors of the nodes that do not change the equivalence conditions between nodes.
>
> Here, an infinitesimal motion is called ***rigid*** if
>
> $$(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \forall (i, j) \in [n]^2$$
>
> Also, an infinitesimal motion is called ***flex*** if
>
> $$(p_i - p_j) \cdot (u_i - u_j) \neq 0 \quad \exists v_i v_j \notin E(G)$$
>
> In this sense, $(G, p)$ is called ***infinitesimally rigid*** if every infinitesimal motion of $(G, p)$ is an infinitesimal rigid motion. Otherwise, $(G, p)$ is said to be ***infinitesimaly flexible***.

**Note:** In some context, infinitesimal rigid motion also can be called by a "Trivial motion" or a "Special Euclidean motion".

- **Trivial motions** is an infinitesimal motions include only the translational and rotational motions. (infinitesimal rigid motion)
- **Special Euclidean motion** is defined by the motions of nodes that keep the congruence conditions among the nodes. It contains the translations and rotations, and hence Special Euclidean motion is equivalent to the **direct isometries.**

In short,

**Equivalence of definitions**

- Infinitesimal rigid motion = Trivial motion
- Special Euclidean motion = Direct isometry

However, **in the infinitesimal viewpoint, the four of them are all equivalent.**

**Remark:** All constricted frameworks infinitesimally flexible.

- Now, we can come up with a very natural question.

Q. Is there any relationship between the rigidity and the infinitesimal rigidity?

### Thm 2.7) Infinitesimal rigidity is stronger than rigidity

> If a framework $f_p$ is infinitesimally rigid, then $f_p$ is rigid.

**Proof)**

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## 2.2.1. Rigidity Matrix

### Def 2.8) Edge function / Rigidity Matrix

> Let $(G, p)$ be a framework with graph $G$ and realization $p$. Then, the ***edge function*** $g_G : \mathbb{R}^{dn} \to \mathbb{R}^m$ where $m = e(G)$ is defined by
>
> $$g_G(p) = \frac{1}{2}\left[\|p_i - p_j\|^2\right]_{(i,j)\in E(G)}$$
>
> Then, the ***rigidity matrix*** is defined by
>
> $$\mathbf{R}(G,p) = \frac{\partial g_G(p)}{\partial p} \in \mathbb{R}^{m\times dn}$$
>
> Here, each row and column corresponds to each edges and nodes of the framework, respectively.

Explicitly, the rigidity matrix is expressed as

$$\mathbf{R}(G,p) = \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \dots & 0 & p_i - p_j & 0 & \dots & 0 & p_j - p_i & 0 & \dots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix}$$

Let $e_k = v_i v_j \in E(G)$ is the $k^{\text{th}}$ edge with endpoints $v_i$ and $v_j$. Then, it occupies the $d(i-1)+1^{\text{th}}$ to $di^{\text{th}}$ columns and $d(j-1)+1^{\text{th}}$ to $dj^{\text{th}}$ columns of the $k^{\text{th}}$ row.

Here, $\left(p_i^{(1)} - p_j^{(1)}\right), \cdots, \left(p_i^{(d)} - p_j^{(d)}\right)$ are in the $d(i-1)+1^{\text{th}}$ to $di^{\text{th}}$ columns of the $k^{\text{th}}$ row.

Similarly, $\left(p_j^{(1)} - p_i^{(1)}\right), \cdots, \left(p_j^{(d)} - p_i^{(d)}\right)$ are in the $d(j-1)+1^{\text{th}}$ to $dj^{\text{th}}$ columns of the $k^{\text{th}}$ row.

**Example (Tetrahedron)**

For a formation $(G,p)$ of regular tetrahedron in $\mathbb{R}^3$,
$V(G)=\{v_1,v_2,v_3,v_4\}$ and $E(G)=\{v_1v_2,v_1v_3,v_1v_4,v_2v_3,v_2v_4,v_3v_4\}$ and
$p=\left[p_1^\top \; p_2^\top \; p_3^\top \; p_4^\top\right]^\top$ where $p_i=[x_i \; y_i \; z_i]$.

Note that $|V(G)|=n(G)=4,\ |E(G)|=e(G)=6,\ d=3.$

Then, by definition, the rigidity matrix be

$$\mathbf{R}(p)=\begin{bmatrix} (p_1-p_2)^\top & (p_2-p_1)^\top & 0^\top & 0^\top \\ (p_1-p_3)^\top & 0^\top & (p_3-p_1)^\top & 0^\top \\ (p_1-p_4)^\top & 0^\top & 0^\top & (p_4-p_1)^\top \\ 0^\top & (p_2-p_3)^\top & (p_3-p_2)^\top & 0^\top \\ 0^\top & (p_2-p_4)^\top & 0^\top & (p_4-p_2)^\top \\ 0^\top & 0^\top & (p_3-p_4)^\top & (p_4-p_3)^\top \end{bmatrix} \in \mathbb{R}^{6\times(3\times4)}$$

Let $p_1=(1,1,1),\ p_2=(1,-1,-1),\ p_3=(-1,1,-1),\ p_4=(-1,-1,1)$ be coordinates of each vertex. Then, the rigidity matrix be

$$\mathbf{R}(p)=\left[\begin{array}{ccc|ccc|ccc|ccc} 0 & 2 & 2 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 0 \\ \hline 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 \\ \hline 0 & 0 & 0 & 2 & -2 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & -2 & 2 \end{array}\right]$$

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Note that infinitesimal motion is a function of velocity vectors of each nodes. So, there are $d$ velocity components for each nodes and hence one infinitesimal motion can be identified as a $dn$-dimensional column vector $\mathbf{u}$.

- Recall: $u:V(G)\to\mathbb{R}^d$ is an infinitesimal motion if $(p_i-p_j)\cdot(u_i-u_j)=0$ for all $v_iv_j\in E(G)$.

From here, we can find some useful property.

### Prop 2.9) Infinitesimal motion with rigidity matrix

> The set of infinitesimal motion of framework $(G,p)$ is the kernel of $\mathbf{R}(G,p)$.

**Proof**

**Proof)** Let $\mathcal{M}(G,p)$ be the set of infinitesimal motion of $(G,p)$ and $\mathbf{u}\in\mathcal{M}(G,p)$.

Let $e_k=v_iv_j$. Then, the $k^{\text{th}}$ row of $\mathbf{R}(G,p)\mathbf{u}$ be

$$[\mathbf{R}(G,p)\mathbf{u}]_k=(p_i-p_j)\cdot u_i+(p_j-p_i)\cdot u_j=(p_i-p_j)\cdot(u_i-u_j)$$

Here, since $\mathbf{u}$ is a column vector representation of an infinitesimal motion, $\mathrm{RHS}=0$ and hence $\mathbf{u}\in\mathrm{Ker}(R(G,p))$.

Thus, $\mathcal{M}(G,p)\subset\mathrm{Ker}(R(G,p))$.

Conversely, given $\mathbf{x}\in\mathrm{Ker}(\mathbf{R}(G,p))$, $[\mathbf{R}(G,p)\mathbf{x}]_k=0$ for each $k$ and hence $(p_i-p_j)\cdot(x_i-x_j)=0$.

Thus, $\mathbf{x}$ be an infinitesimal motion of $(G,p)$ and hence $\mathrm{Ker}(\mathbf{R}(G,p)) \subset \mathcal{M}(G,p)$.

$\therefore \mathcal{M}(G,p) = \mathrm{Ker}(\mathbf{R}(G,p))$. □

From this fact, by letting $\mathcal{R}(G,p)$ be the set of all infinitesimal rigid motion,

$$\mathcal{R}(G,p) \subset \mathcal{M}(G,p)$$

by definition.

Also, $(G,p)$ is infinitesimally rigid if and only if $\mathcal{R}(G,p) = \mathcal{M}(G,p)$.

By the previous proposition, the following fact directly follows.

### Lem 2.10) Dimension and infinitesimal rigidity

> Let $(G,p)$ be a normal framework in $\mathbb{R}^d$. Then,
>
> $$\dim \mathcal{R}(G,p) \leq \dim \mathcal{M}(G,p) = \mathrm{nullity}(\mathbf{R}(G,p))$$
>
> Equivalently, by rank-nullity theorem,
>
> $$\mathrm{rank}(\mathbf{R}(G,p)) \leq dn - \dim \mathcal{R}(G,p)$$
>
> The equality holds if and only if $(G,p)$ is infinitesimally rigid.

Here, while we know $d, n$ but we don't know about $\dim \mathcal{R}(G,p)$ and hence we need to find it.

To find it, we assume **2-dimensional general framework** for the following arguments. They can be directly generalized to an arbitrary dimension.

### Def 2.11) General Position

> $\mathcal{P} = \{v_1, \cdots, v_n\} \subseteq \mathbb{R}^d$ is in ***general position*** if for $1 \leq m \leq d$, no $m+1$ points in $\mathcal{P}$ lie in an $(m-1)$-dimensional affine subspace of $\mathbb{R}^d$. A framework whose joints are in general position is called a ***general framework***.

The following definition explains about two types of infinitesimal motions.

### Def 2.12) Infinitesimal translation / Infinitesimal rotation ($\mathbb{R}^2$ ver.)

> An ***infinitesimal translation of the plane*** is of the form
>
> $$\tau_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}^2 \; ; \quad \tau_{(u,v)}(x,y) = (u,v)$$
>
> This implies that every point $(x,y)$ in $\mathbb{R}^2$ moves with same velocity $(u,v)$.
>
> An ***infinitesimal rotation of the plane*** is of the form
>
> $$a\rho_{(x_0,y_0)} : \mathbb{R}^2 \to \mathbb{R}^2 \; ; \quad a\rho_{(x_0,y_0)}(x,y) = a(y_0 - y, x - x_0)$$

Let $r$ be a radius vector from a center $(x_0, y_0)$ to $(x,y)$, i.e. $r = (x - x_0, y - y_0)$. Then, the rotation vector $v$ of $(x,y)$ must satisfy $r \perp v$ and hence $r \cdot v = 0$. So, $v$ must be of the form $a(-(y-y_0), x - x_0) = a(y_0 - y, x - x_0)$ where $a \in \mathbb{R}$.

### Lem 2.13) Dimension of infinitesimal translations and rotations

> The set of all infinitesimal translations and rotations of $\mathbb{R}^2$ forms a 3-dimensional vector space.

**Proof)**

Now, recall that for infinitesimal viewpoint, infinitesimal rigid motion is equal to direct isometry and hence **infinitesimal rigid motion consists of infinitesimal translation and rotation only.**

The following lemma describes about this fact.

### Lem 2.14) Classifications fo infinitesimal rigid motions

> Let $(G, p)$ be a 2-dimensional general framework with $V(G) = \{v_1, \cdots, v_n\}$, $n \geq 2$. Then $\mathcal{R}(G, p)$ is the 3-dimensional space of infinitesimal translations and rotations restricted to the points $\mathcal{P} = \{v_1, \cdots, v_n\}$ of $(G, p)$.

**Proof**

The proof consists of two steps.

For a restriction map $(\cdot)|_{\mathcal{P}} : I'_d(\mathbb{R}^2) \to \mathcal{R}(G, p); \quad f \mapsto f|_{\mathcal{P}}$, (where $I'_d(\mathbb{R}^2)$ is a set of infinitesimal direct isometries, i.e. infinitesimal translations and rotations)

- **Surjectivity:** Given $u \in \mathcal{R}(G, p)$, $\exists \bar{u} \in I'_d(\mathbb{R}^2)$ s.t. $\bar{u}|_{\mathcal{P}} = u$.
- **Injectivity:** Given $\kappa, \lambda \in I'_d(\mathbb{R}^2)$ and corresponding restriction $\kappa|_{\mathcal{P}}, \lambda|_{\mathcal{P}} \in \mathcal{R}(G, p)$, $\kappa|_{\mathcal{P}} = \lambda|_{\mathcal{P}} \Rightarrow \kappa = \lambda$.

**Surjectivity**

Let $u = (u_1, \cdots, u_n) \in \mathcal{R}(G, p)$ with $u_i = (u_i^{(1)}, u_i^{(2)})$ for all $i = 1, \cdots, n$.
Clearly, the restriction of $\tau_{u_1}$ to $\mathcal{P}$ is an infinitesimal rigid motion of $(G, p)$.

Hence $\hat{u} = u - \tau_{u_1}|_{\mathcal{P}} \in \mathcal{R}(G, p)$.

Here, note that $\hat{u} = (u_1 - u_1, u_2 - u_1, \cdots, u_n - u_1)$ and hence $\hat{u}_1 = (0, 0)$. (fixing $p_1$)

Next, from the definition of infinitesimally rigid motion $(p_i - p_j) \cdot (u_i - u_j) = 0$ for all $(i, j) \in [n]^2$, we have

$$(p_2 - p_1) \cdot (\hat{u}_2 - \hat{u}_1) = (p_2 - p_1) \cdot \hat{u}_2 = 0$$

Since $(p_2 - p_1) \cdot \hat{u}_2 = \left(p_2^{(1)} - p_1^{(1)}, p_2^{(2)} - p_1^{(2)}\right) \cdot \hat{u}_2 = 0$,
$\hat{u}_2$ must be of the form $\hat{u}_2 = a\left(p_2^{(1)} - p_1^{(1)}, -(p_2^{(2)} - p_1^{(2)})\right)$.

So, $\hat{u}_2^{(1)} = a\left(p_2^{(1)} - p_1^{(1)}\right)$ and $\hat{u}_2^{(2)} = a\left(p_1^{(2)} - p_2^{(2)}\right)$ and hence

$$a = \frac{\hat{u}_2^{(1)}}{p_2^{(2)} - p_1^{(2)}} = \frac{\hat{u}_2^{(2)}}{p_1^{(1)} - p_2^{(1)}} \quad \text{where } p_1^{(1)} \neq p_2^{(1)} \text{ or } p_1^{(2)} \neq p_2^{(2)}$$

Now, define another motion $\bar{u} = \hat{u} - a\rho_{p_1}|_{\mathcal{P}} = u - \tau_{u_1}|_{\mathcal{P}} - a\rho_{p_1}|_{\mathcal{P}}$.
Here, since $p_1$ is fixed by $\hat{u}$ and the rotation $a\rho_{p_1}|_{\mathcal{P}}$ is a rotation through $p_1$, $\bar{u}$ fixes $p_1$ as well.

Also, by choosing proper rotation, we can make $\bar{u}$ fixes $p_2$. Then, since $\bar{u}$ fixes both $p_1$ and $p_2$, $\bar{u}_1 = \bar{u}_2 = (0, 0)$.

Now, claim: **If $p_1$ and $p_2$ be fixed, then $p_i$ are fixed for all $i > 2$.**

From the definition of infinitesimal rigid motion, we have

$$(p_i - p_1) \cdot (\bar{u}_i - \bar{u}_1) = (p_i - p_1) \cdot \bar{u}_i = 0 \quad \forall i > 2 \ \ \cdots \ (*)$$
$$(p_i - p_2) \cdot (\bar{u}_i - \bar{u}_2) = (p_i - p_2) \cdot \bar{u}_i = 0 \quad \forall i > 2 \ \ \cdots \ (**)$$

Since we assumed the general position, $p_i$ is not on the line through $p_1$ and $p_2$ and hence the vectors $p_i - p_1$ and $p_i - p_2$ are not parallel.

However, since $\bar{u}_i$ is perpendicular to both of them by $(*)$ and $(**)$.

Note that in the two-dimensional plane, the only vector that is perpendicular to two vectors that are not parallel to each other is the zero vector $(0, 0)$ and hence $\bar{u}_i = (0, 0)$ for all $i > 2$.

Thus, $\bar{u} = u - \tau_{u_1}|_{\mathcal{P}} - a\rho_{p_1}|_{\mathcal{P}} = \mathbf{0}$ and hence $u = \tau_{u_1}|_{\mathcal{P}} + a\rho_{p_1}|_{\mathcal{P}} = (\tau_{u_1} + a\rho_{p_1})|_{\mathcal{P}}$.
This implies that any infinitesimal rigid motion $u$ of framework $(G, p)$ is the restriction of combination of infinitesimal translation and rotation of $\mathbb{R}^2$ to $\mathcal{P}$.

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**Injectivity**

Suppose that $\kappa$ and $\lambda$ are two infintiesimal rigid motions of $\mathbb{R}^2$ s.t.

$$\kappa|_{\mathcal{P}} = \lambda|_{\mathcal{P}} = u$$

Let $x \in \mathbb{R}^2$ that is not on the line through $p_1$ and $p_2$.
Consider $\mu = \kappa - \lambda$. Then, obviously, $\mu$ is an infinitesimal rigid motion of $\mathbb{R}^2$.

Then, by the definition of infinitesimal rigid motion,

$$(\mu(x) - \mu(p_1)) \cdot (x - p_1) = 0$$
$$(\mu(x) - \mu(p_2)) \cdot (x - p_2) = 0$$

Here, since $\kappa|_{\mathcal{P}} = \lambda|_{\mathcal{P}}$, for any joint $p_i$, $\kappa(p_i) = \lambda(p_i)$ and hence $\mu(p_i) = (0, 0)$ for all $i$.
So,

$$\mu(x) \cdot (x - p_1) = 0$$
$$\mu(x) \cdot (x - p_2) = 0$$

By assumption, $x$ is not on the line through $p_1$ and $p_2$ and hence two vectors $x - p_1$ and $x - p_2$ are not parallel. Again, since the only vector that is perpendicular to two vectors that are not parallel to each other is the zero vector $(0, 0)$, we can find that $\mu(x) = (0, 0)$.

So far, we've shown that $\mu(x) = (0, 0)$ for $x \in \mathbb{R}^2 \setminus L(p_1, p_2)$ where $L(p_1, p_2)$ is the straight line through $p_1$ and $p_2$.

Finally, let $y \in L(p_1, p_2)$. Then, obviously, $y$ does not lie on a line through $x$ and $p_1$ or $x$ and $p_2$ (or both).

WLOG, let $y$ is not on a line through $x$ and $p_1$. By the definition of infinitesimal rigid motion,

$$(\mu(y) - \mu(p_1)) \cdot (y - p_1) = 0$$
$$(\mu(y) - \mu(x)) \cdot (y - x) = 0$$

and since $\mu(p_1) = \mu(x) = 0$, $\mu(y)$ is perpendicular to both $y - p_1$ and $y - x$ that are not in parallel. Thus, $\mu(y) = (0, 0)$.

Therefore, we proved that $\mu(x) = 0$ for all $x \in \mathbb{R}^2$ and hence $\kappa = \lambda$.

Hence, proved. $\square$

Using previous lemma, we can obatain an important result.
Before observing it, introduce a definition first.

### Def 2.15) Degree of freedom

> For a framework $(G, p)$, a ***degree of freedom*** of $(G, p)$ is $\text{nullity}\ R(G, p)$.
> Also, $\text{nullity}(\mathbf{R}(G, p)) - \dim \mathcal{R}(G, p)$ is called an ***internal degrees of freedom*** of $(G, p)$.

Then, we can observe a degree of freedom by the following theorem.

### Thm 2.16) Lower bound of DOF

> Let $(G, p)$ be a general framework in $\mathbb{R}^d$ with $n(G) \geq d$. Then,
>
> $$\text{nullity}(\mathbf{R}(G, p)) \geq d + \binom{d}{2} = \binom{d+1}{2}$$
>
> or equivalently,
>
> $$\text{rank}(R(G, p)) \leq dn - \binom{d+1}{2}$$
>
> Here, equality holds if and only if $(G, p)$ is infinitesimally rigid.

Proof

**Proof)** Recall that $\dim \mathcal{R}(G, p) \leq \dim \mathcal{M}(G, p) = \text{nullity}(\mathbf{R}(G, p))$ where $\mathcal{R}(G, p)$ is the set of infinitesimally rigid motions of $(G, p)$ and $\mathcal{M}(G, p)$ is the set of infinitesimal motions of $(G, p)$ by lemma 2.10.

By lemma 2.14, a set of infinitesimal rigid motion consists of infinitesimal translations and rotations and hence $\dim \mathcal{R}(G, p)$ is the number of basis, i.e. the number of independent translations and rotations.

Note that there are $d$ translations $\tau_{e_1}, \cdots, \tau_{e_d}$ where $e_i$ is a standard basis vector and there are $\binom{d}{2}$ rotations $R_{ij}$ where for all $i, j \in [d]$, $R_{ij} : \mathcal{P} \to \mathbb{R}^d;\ \ R_{ij}(p_k) = p_k^{(i)} e_j - p_k^{(j)} e_i$ for all $k$.

So, the total number of basis is $d + \binom{d}{2} = \binom{d+1}{2}$ and hence

$$\binom{d+1}{2} \leq \text{nullity}(\mathbf{R}(G, p))$$

Hence, proved. $\square$

### Cor 2.17) Maximum number of edges of flexible framework

> Let $(G, p)$ be a general framework in $\mathbb{R}^d$ with $n(G) \geq d$. If $e(G) < dn - \binom{d+1}{2}$, then $(G, p)$ is infiinitesimally flexible.

This corollary provides the lower bound of number of edges for a framework to be rigid. **The following definition explains about a framework that has exactly this minimum threshold of edges required for rigidity.**

### Def 2.18) Minimally rigid framework

> A rigid framework $(G, p)$ is called ***minimally rigid*** if the removal of any edges makes it flexible. Similarly, an infinitesimally rigid framework $(G, p)$ is called ***minimally infinitesimally rigid*** if the removal of any edges makes it not infinitesimally rigid.

### Def 2.19) Independent / Isostatic framework

> A framework $(G, p)$ is ***independent*** if the rows of its rigidity matrix $\mathbf{R}(G, p)$ are linearly independent. $(G, p)$ is ***isostatic*** if it ts infinitesimally rigid and independent.

### Thm 2.20) Equivalent conditions of isostatic fraemwork

> For a $d$-dimensional general framework $(G, p)$ with $n(G) \geq d$, TFAE:
>
> 1. $(G, p)$ is isostatic.
> 2. $(G, p)$ is infinitesimally rigid and $e(G) = dn - \binom{d+1}{2}$.
> 3. $(G, p)$ is independent and $e(G) = dn - \binom{d+1}{2}$.
> 4. $(G, p)$ is minimal infinitesimally rigid.

### Def 2.21) Regular points

> Given a framework $(G, p)$, a realization $p$ is called ***regular*** if is satisfies
>
> $$\operatorname{rank}(\mathbf{R}(G, p)) = \max_{p \in \mathbb{R}^{dn}} \operatorname{rank}(\mathbf{R}(G, p))$$
>
> While this definition might be confused with General position, they represent fundamentally different perspectives on a framework:

**General position** is a purely geometric property of the point coordinates, whereas **Regularity** is an algebraic property concerning the rank of the rigidity matrix.

To break this down more precisely:

- **General position** focuses on the **geometry of the joints**; it requires that no $m + 1$ points lie in an $(m - 1)$-dimensional affine subspace, a condition that depends only on the point locations and is independent of the graph's edges.
- **Regular realization** focuses on the **algebraic rank of the structure**; a realization $p$ is regular if its rigidity matrix $R_p$ attains the maximum possible rank $k$ for that specific graph, meaning it inherently

depends on the graph's connectivity.

Therefore, a "general framework" is defined by its well-distributed geometry, while a "regular realization" is defined by its ability to reach the full structural potential (maximum rank) allowed by its specific graph topology.

## 3. Combinatorial aspects of rigidity

#### Motivation

Up to this point, our analysis has **focused on the rigidity of frameworks based on their specific geometric configurations**—that is, the exact coordinates of vertices in $\mathbb{R}^d$.

Now, a fundamental question arises: to what extent can these **properties be characterized solely by the underlying graph's connectivity**, independent of its specific realization?

So, we'll investigate the possibility of **treating rigidity as an intrinsic property** of the underlying graph, leading us to the study of generic frameworks where infinitesimal rigidity becomes purely combinatorial.

Introduct some definitions first.

#### Def 3.1) Indeterminate rigidity matrix

> Let $K_n$ be the complete graph on $n$ vertices. For each $i = 1, 2, \cdots, n$, we introduce a $d$-tuple of variables $p_i' = \left((p_i^{(1)})', \cdots, (p_i^{(d)})'\right)$ ($d$ unkown variables).
>
> Then, let
>
> $$\mathbf{R}(n,d) = \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \ldots & 0 & p_i' - p_j' & 0 & \ldots & 0 & p_j' - p_i' & 0 & \ldots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix}$$
>
> be the matrix that is obtained from the rigidity matrix $\mathbf{R}(K_n, p)$ by replacing each $p_i^{(j)} \in \mathbb{R}$ with the unknown variable $(p_i^{(j)})'$. We call $\mathbf{R}(n,d)$ the $d$-dimensional ***indeterminate rigidity matrix of*** $K_n$.

#### Def 3.2) Generic framework

> Let $V = \{v_1, \cdots, v_n\}$ and $p : V \to \mathbb{R}^d$ be a map.
>
> We say that $p$ is ***generic realization*** if the determinant of any submatrix of $\mathbf{R}(K_n, p)$ is zero only if the determinant of the corresponding submatrix of $\mathbf{R}(n,d)$ is (identically) zero.
>
> A framework $(G,p)$ is said to be ***generic*** if $p$ is a generic map.

#### Lem 3.3) Rank of generic frameworks

> Let $p : V(G) \to \mathbb{R}^d$ be a generic map, Then, the rank of the rigidity matrix $\mathbf{R}(G,p)$ is equal to the maximum possible rank attainable by any realization of $G$ in $\mathbb{R}^d$. That is,
>
> $$\operatorname{rank}(\mathbf{R}(G,p)) = \max_{q \in \mathbb{R}^{dn}} \operatorname{rank}(R(G,q))$$

**Proof**

Let $k = \max_{q\in\mathbb{R}^{dn}} \mathrm{rank}(R(G,q))$ be the maximum rank.
By the definition of matrix rank, there exists at least one realization $q_0 \in \mathbb{R}^{dn}$ s.t. $\mathbf{R}(G,q_0)$ contains a $k\times k$ submatrix $S$ with a non-zero determinant, i.e. $\det S(q_0) \neq 0$.

The determinant $\det S$ is a polynomial in the coordinates of the joints. Since there exists a point $q_0$ where $\det S(q_0) \neq 0$, this polynomial is not identically zero.

IAW definition 3.2, a map $p$ is generic if the determinant of any submatrix is zero only if the corresponding determinant polynomial is identically zero.

Since our $k\times k$ determinant polynomial is not identically zero, it must be non-zero for any generic map $p$. Thus, $\mathrm{rank}(\mathbf{R}(G,p)) \geq k$.

As $k$ is defined as the maximum possible rank, we can conclude that $\mathrm{rank}(\mathbf{R}(G,p) = k$. □

Then, there is a result directly follows.

### Cor) Genericity implies Regularity

> Every generic realization $p$ of a graph $G$ in $\mathbb{R}^d$ is a regular point.

**Example (Genericity of triangle frameworks)**

We want to determine whether frameworks $(G,p),(G,q)$ in $\mathbb{R}^2$ is generic where

$$V(G) = \{v_1, v_2, v_3\}$$

with distinct realizations $p, q$ where

$$p_1 = (0,0),\ p_2 = (1,0),\ p_3 = (2,0) \quad \text{and} \quad q_1 = (0,0),\ q_2 = (1,0),\ q_3 = (0,1)$$

Before algebraically determining them, we can guess.

1. $p$ seems NOT generic because three points are colinear.
2. $q$ seems generic because three points are not colinear.

Let's check.

First, construct an indeterminate rigidity matrix with $n = 3$.

$$\mathbf{R}(3,2) = \left(\begin{array}{cc|cc|cc} x_1-x_2 & y_1-y_2 & x_2-x_1 & y_2-y_1 & 0 & 0 \\ \hline x_1-x_3 & y_1-y_3 & 0 & 0 & x_3-x_1 & y_3-y_1 \\ \hline 0 & 0 & x_2-x_3 & y_2-y_3 & x_3-x_2 & y_3-y_2 \end{array}\right)$$

and construct $\mathbf{R}(K_3,p)$.

$$\mathbf{R}(K_3,p) = \left(\begin{array}{cc|cc|cc} 0-1 & 0-0 & 1-0 & 0-0 & 0 & 0 \\ \hline 0-2 & 0-0 & 0 & 0 & 2-0 & 0-0 \\ \hline 0 & 0 & 1-2 & 0-0 & 2-1 & 0-0 \end{array}\right) = \left(\begin{array}{cc|cc|cc} -1 & 0 & 1 & 0 & 0 & 0 \\ \hline -2 & 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right)$$

Taking submatrix $\mathbf{R}_{(145)}$of $\mathbf{R}(K_3,p)$ by taking column 1, 4 and 5. Then,

$$\det \mathbf{R}_{(145)} = \det \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

However, a determinant of corresponding submatrix $\mathbf{R}(3,2)_{145}$ is

$$\begin{aligned} \det \mathbf{R}(3,2)_{(145)} &= \det \begin{pmatrix} x_1 - x_2 & y_2 - y_1 & 0 \\ x_1 - x_3 & 0 & x_3 - x_1 \\ 0 & y_2 - y_3 & x_3 - x_2 \end{pmatrix} \\ &= (x_1 - x_3)[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \end{aligned}$$

This is not necessarily zero (may not be zero by choosing coordinates properly) and hence, $p$ **is NOT a generic realization.** ($\rightarrow$ We correctly guessed)

Now, again, to determine whether $q$ is generic or not, construct $\mathbf{R}(K_3, q)$.

$$\mathbf{R}(K_3, q) = \left(\begin{array}{cc|cc|cc} 0-1 & 0-0 & 1-0 & 0-0 & 0 & 0 \\ \hline 0-0 & 0-1 & 0 & 0 & 0-0 & 1-0 \\ \hline 0 & 0 & 1-0 & 0-1 & 0-1 & 1-0 \end{array}\right) = \left(\begin{array}{cc|cc|cc} -1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Taking submatrix $\mathbf{R}_{(145)}$of $\mathbf{R}(K_3, q)$ by taking column 1, 4 and 5. Then,

$$\det \mathbf{R}_{(145)} = \det \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix} = 0$$

However, a determinant of corresponding submatrix $\mathbf{R}(3,2)_{145}$ is

$$\begin{aligned} \det \mathbf{R}(3,2)_{(145)} &= \det \begin{pmatrix} x_1 - x_2 & y_2 - y_1 & 0 \\ x_1 - x_3 & 0 & x_3 - x_1 \\ 0 & y_2 - y_3 & x_3 - x_2 \end{pmatrix} \\ &= (x_1 - x_3)[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \end{aligned}$$

This is not necessarily zero (may not be zero by choosing coordinates properly) and hence, $q$ **is NOT a generic realization.** ($\rightarrow$ Our guess was Wrong!)

As anticipated, the realization $p$ with collinear points proved to be non-generic. However, realization $q$ - which we initially presumed to be generic because it is a general framework -was also found to be non-generic upon verification.

This leads us to an important question: **What is the fundamental distinction between being 'general' and 'generic'?**

**Remark) Generic is stronger than general**

Generic frameworks are general. But the converse doesn't hold.

There are some fundamental facts regarding this definition of generic.

**Thm 3.3) Set theoretic properties of non-generic realizations**

> The set of all non-generic realizations $p : V \to \mathbb{R}^d$ where $V = \{v_1, \cdots, v_n\}$ is a **closed set of measure zero.**

### Thm 3.4) Combinatorial Rigidity Theorem

> For a fixed dimension $d$ and a graph $G$ with $n(G) \geq d$, TFAE:
>
> 1. $(G, p)$ is infinitesimally rigid (or independent, or isostatic) for at least one injective map $p : V(G) \to \mathbb{R}^d$.
> 2. $(G, p)$ is infinitesimally rigid (or independent, or isostatic) for every $d$-dimensional generic realization $p$.

Intuitively, success in a single normal configuration implies success in almost every possible configuration.

The three structural properties (**infinitesimal rigidity**, **independence**, and **isostaticity**) only depend on the **underlying graph** $G$; specifically its **combinatorial structure**.

**Proof**

$(1) \Rightarrow (2)$: Suppose that $(G, p)$ is infinitesimally rigid for an injective $p : V(G) \to \mathbb{R}^d$ (not necessarily generic).

By theorem 2.16, for infinitesimally rigid framework,

$$\operatorname{rank}(\mathbf{R}(G, p)) = dn - \binom{d+1}{2}$$

where $n(G) = n$.

Let $\hat{p}$ be a $d$-dimensional generic framework for a graph $G$. By lemma 3.3, the rigidity matrix of $\hat{p}$ has the maximum rank and hence $\operatorname{rank}(\mathbf{R}(G, \hat{p})) \geq \operatorname{rank}(\mathbf{R}(G, p))$.

Again, by theorem 2.16, a rank of rigidity matrix of $d$-dimensional framework cannot exceed $dn - \binom{d+1}{2}$ and hence $\operatorname{rank}(\mathbf{R}(G, \hat{p})) \leq dn - \binom{d+1}{2}$.

Combining the equation and inequalities, we obtain

$$dn - \binom{d+1}{2} = \operatorname{rank}(\mathbf{R}(G, p)) \leq \operatorname{rank}(\mathbf{R}(G, \hat{p})) \leq dn - \binom{d+1}{2}$$

Thus, $\operatorname{rank}(\mathbf{R}(G, \hat{p})) = dn - \binom{d+1}{2}$ and hence we can conclude that $(G, \hat{p})$ is infinitesimally rigid by theorem 2.16.

$(2) \Rightarrow (1)$: Suppose that $(G, p)$ is infinitesimally rigid for every generic realization $p$.

**Existence of $p$:** Since the set of generic realizations is open dense set in $\mathbb{R}^{dn}$, there exists a map $p : V(G) \to \mathbb{R}^d$ s.t. $(G, p)$ is infinitesimally rigid.

**Injectivity:** Also, since generic framework is general, the vertices never overlap and hence $p$ must be injective.

Thus, for any generic realization $p$, it is an injective realization that satisfies $(G, p)$ is infinitesimally rigid.

This proof can be applied directly for independence or isostaticity instead of infinitesimal rigidity. □

### Thm 3.5) Equivalence of rigidity

> For generic frameworks, infinitesimal rigidity is equivalent to rigidity.

### Def 3.6) Generically rigid/independent/isostatic

> A graph $G$ is called ***generically rigid/independent/isostatic in dimension*** $d$ or generically $d$-rigid/independent/isostatic if $d$-dimensional generic realizations of $G$ are infinitesimally rigid/independent/isostatic.

Here, by theorem 3.5, A graph $G$ is generically rigid if a generic realizations of $G$ is rigid ($\because$ inifintiesimally rigid $\equiv$ rigid for generic).

**Note:** Generically $d$-isostatic graphs can be constructed using inductive construction techniques. The following definition 3.7 and theorem 3.8 describe about it.

Introduce a definition first.

### Def 3.7) Vertex addition

> Let $G$ be a graph, $U \subseteq V(G)$ with $|U| = d$ and $v \notin V(G)$. Then the graph $\widehat{G}$ with $V(\widehat{G}) = V(G) \cup \{v\}$ and $E(\widehat{G}) = E(G) \cup \{\{v, u\} : u \in U\}$ is called ***a vertex*** $d$***-addition*** (by $v$) of $G$.

Intuitively, a vertex $d$-addition is the process of expanding a graph by introducing **one new vertex** and connecting it to exactly $d$ **distinct existing vertices** via $d$ new edges.

This operation serves as a fundamental building block for inductive constructions because it preserves the structural properties of the graph, as stated in the following theorem.

### Thm 3.8) Vertex Addition Theorem

> A vertex $d$-addition of a generically $d$-isostatic graph is generically $d$-isostatic. Conversely, deleting a vertex of degree $d$ from a generically $d$-isostatic graph results in a generically $d$-isostatic graph.

**Proof)** By definition 2.19, to show that a vertex $d$-addition of a generically $d$-isostatic graph is generically $d$-isostatic, we need to show that it is infinitesimally rigid and independent.

First, show the infinitesimal rigidity.

Now, show the independence.

### Def 3.9) Edge split

> Let $G$ be a graph, and let $U \subseteq V(G)$ with $|U| = d + 1$ and $\{u_1, u_2\} \in E(G)$ for some $u_1, u_2 \in U$. Further, let $v \notin V(G)$. Then, the graph $\hat{G}$ with $V(\hat{G}) = V(G) \cup \{v\}$ and $E(\hat{G}) = (E(G) \setminus \{u_1 u_2\}) \cup \{vu : u \in U\}$ is called an ***edge*** $d$***-split (on*** $u_1, u_2; v$***)*** of $G$.