9 Distance measures for quantum information

Exercises: 9.1, 9.2, 9.3, 9.4, 9.5, 9.6, 9.7, 9.8, 9.9, 9.10, 9.11, 9.12, 9.13, 9.14, 9.15, 9.16, 9.17, 9.18, 9.19, 9.20, 9.21, 9.22, 9.23.

9.1

For probability distributions $p_x = (1, 0)$ and $q_x = (1/2, 1/2)$:

$$D(p_x, q_x) = \frac{1}{2}\left(\left|1 - \frac{1}{2}\right| + \left|0 - \frac{1}{2}\right|\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2};$$

For probability distributions $p_x = (1/2, 1/3, 1/6)$ and $q_x = (3/4, 1/8, 1/8)$:

$$D(p_x, q_x) = \frac{1}{2}\left(\left|\frac{1}{2} - \frac{3}{4}\right| + \left|\frac{1}{3} - \frac{1}{8}\right| + \left|\frac{1}{6} - \frac{1}{8}\right|\right) = \frac{1}{2}\left(\frac{1}{4} + \frac{5}{24} + \frac{1}{24}\right) = \frac{1}{4}.$$

9.2

$$\begin{aligned}
D(p_x, q_x) &= \frac{1}{2}\left(|p - q| + |(1-p) - (1-q)|\right) \\
&= \frac{1}{2}\left(|p - q| + |q - p|\right) \\
&= |p - q|.
\end{aligned}$$

9.3

For probability distributions $p_x = (1, 0)$ and $q_x = (1/2, 1/2)$:

$$F(p_x, q_x) = \sqrt{1 \times \frac{1}{2}} + \sqrt{0 \times \frac{1}{2}} = \sqrt{\frac{1}{2}} + 0 = \frac{\sqrt{2}}{2};$$

For probability distributions $p_x = (1/2, 1/3, 1/6)$ and $q_x = (3/4, 1/8, 1/8)$:

$$F(p_x, q_x) = \sqrt{\frac{1}{2} \times \frac{3}{4}} + \sqrt{\frac{1}{3} \times \frac{1}{8}} + \sqrt{\frac{1}{6} \times \frac{1}{8}} = \sqrt{\frac{3}{8}} + \sqrt{\frac{1}{24}} + \sqrt{\frac{1}{48}} = \frac{4\sqrt{6} + \sqrt{3}}{12}.$$

9.4

$$\begin{aligned}
D(p_x, q_x) &= \frac{1}{2}\sum_x |p_x - q_x| \\
&= \frac{1}{2}\left[\sum_{p_x > q_x} (p_x - q_x) + \sum_{p_x < q_x} (q_x - p_x)\right].
\end{aligned}$$

Using the fact that $\sum_x p_x = \sum_x q_x = 1$, the second sum can be rewritten as

$$\sum_{p_x < q_x} (q_x - p_x) = \sum_{p_x < q_x} q_x - \sum_{p_x < q_x} p_x$$

$$
\begin{aligned}
&= \left(\sum_x q_x - \sum_{p_x > q_x} q_x\right) - \left(\sum_x p_x - \sum_{p_x > q_x} p_x\right) \\
&= \sum_{p_x > q_x} (p_x - q_x),
\end{aligned}
$$

thus, we obtain that

$$
D(p_x, q_x) = \sum_{p_x > q_x} (p_x - q_x) = \sum_{p_x > q_x} p_x - \sum_{p_x > q_x} q_x.
$$

If we consider all possible subsets S of the indices $\{x\}$, the set for which we have $p_x > q_x$ for all indices corresponds to the maximum of the sum. To see that, notice that if we remove any element $p_x - q_x$ the value of the sum will decrease, and if we add any element, it will be one such that $p_x - q_x$ is negative and the value of the sum will also decrease, therefore it is the maximum, so

$$
D(p_x, q_x) = \max_S \left(\sum_{x \in S} p_x - \sum_{x \in S} q_x\right).
$$

Since the maximum value of the sum will always be positive, it makes no difference to write it with absolute value signs, so we conclude Eq. (9.3)

$$
D(p_x, q_x) = \max_S \left|\sum_{x \in S} p_x - \sum_{x \in S} q_x\right|.
$$

9.5

See Exercise 9.4.

9.6

For density operators $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$ and $\sigma = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$:

$$
\begin{aligned}
&|\rho - \sigma| = \left|\frac{3}{4} - \frac{2}{3}\right| |0\rangle\langle 0| + \left|\frac{1}{4} - \frac{1}{3}\right| |1\rangle\langle 1| = \frac{1}{12}|0\rangle\langle 0| + \frac{1}{12}|1\rangle\langle 1| \\
\implies \quad & D(\rho, \sigma) = \frac{1}{2}\sum_{i=0}^{1} \left\langle i \left| \frac{1}{12}|0\rangle\langle 0| + \frac{1}{12}|1\rangle\langle 1| \right| i \right\rangle = \frac{1}{2}\left(\frac{1}{12} + \frac{1}{12}\right) = \frac{1}{12}.
\end{aligned}
$$

For density operators $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$ and $\sigma = \frac{2}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -|$:
First, writing σ in the computational basis yields

$$
\begin{aligned}
\sigma &= \frac{2}{3}\left(\frac{|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|}{2}\right) + \frac{1}{3}\left(\frac{|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|}{2}\right) \\
&= \left(\frac{1}{3} + \frac{1}{6}\right)|0\rangle\langle 0| + \left(\frac{1}{3} - \frac{1}{6}\right)|0\rangle\langle 1| + \left(\frac{1}{3} - \frac{1}{6}\right)|1\rangle\langle 0| + \left(\frac{1}{3} + \frac{1}{6}\right)|1\rangle\langle 1| \\
&= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) + \frac{1}{6}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right).
\end{aligned}
$$

Thus

$$\rho - \sigma = \begin{bmatrix} \frac{3}{4} - \frac{1}{2} & 0 - \frac{1}{6} \\ 0 - \frac{1}{6} & \frac{1}{4} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{4} \end{bmatrix}.$$

Its eigenvalues can be calculated as

$$\begin{aligned} \det(\rho - \sigma - \lambda I) = \lambda^2 - \frac{13}{144} = 0 \quad &\implies \quad \text{eigenvalues} = \pm\frac{\sqrt{13}}{12}. \\ &\implies \quad D(\rho, \sigma) = \frac{1}{2}\left(\left|\frac{\sqrt{13}}{12}\right| + \left|-\frac{\sqrt{13}}{12}\right|\right) = \frac{\sqrt{13}}{12}. \end{aligned}$$

9.7

All density operators are Hermitian, therefore $\rho - \sigma$ is Hermitian, meaning it can be diagonalized through some unitary operator U, that is, $\rho - \sigma = UDU^\dagger$. We may write the diagonal operador as

$$D = \sum_i \lambda_i \left|i\right\rangle\!\left\langle i\right| = \sum_{\lambda_i > 0} \lambda_i \left|i\right\rangle\!\left\langle i\right| - \sum_{\lambda_i < 0} |\lambda_i| \left|i\right\rangle\!\left\langle i\right|.$$

Defining operators $Q \equiv \sum_{\lambda_i > 0} \lambda_i U \left|i\right\rangle\!\left\langle i\right| U^\dagger$ and $S \equiv \sum_{\lambda_i < 0} |\lambda_i| U \left|i\right\rangle\!\left\langle i\right| U^\dagger$ we see that both are positive operators, and since the $\{\left|i\right\rangle\}$ form an orthonormal basis, they have orthogonal support, meaning we can always write $\rho - \sigma = Q - S$.

9.8

There exists some projector P such that

$$\begin{aligned} D\left(\sum_i p_i \rho_i, \sigma\right) &= \operatorname{tr}\left[P\left(\sum_i p_i \rho_i - \sigma\right)\right] \\ &= \sum_i p_i \operatorname{tr}(P\rho_i) - \operatorname{tr}(P\sigma) \end{aligned}$$

Since $\sum_i p_i = 1$ we may write

$$D\left(\sum_i p_i \rho_i, \sigma\right) = \sum_i p_i \operatorname{tr}[P(\rho_i - \sigma)] \leq \sum_i p_i D(\rho_i, \sigma).$$

9.9

The set of all density operators is a convex and compact Hilbert space, and all trace preserving quantum operations are continuous transformations over such space. Thus it follows from Schauder's fixed point theorem that all trace preserving quantum operations have a fixed point.

9.10

Let us suppose that $\rho \neq \sigma$ are both fixed points of a strictly contractive trace-preserving map $\mathcal{E}$. So we have

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = D(\rho, \sigma),$$

which contradicts the fact that $\mathcal{E}$ is strictly contractive. So either $\rho = \sigma$ or one of them is not a fixed point. Either way, the conclusion is that $\mathcal{E}$ has a unique fixed point.

9.11

$$\begin{aligned} D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= D(p\rho_0 + (1-p)\mathcal{E}'(\rho), p\rho_0 + (1-p)\mathcal{E}'(\sigma)) \\ &\leq pD(\rho_0, \rho_0) + (1-p)D(\mathcal{E}'(\rho), \mathcal{E}'(\sigma)) \\ &\leq (1-p)D(\rho, \sigma). \end{aligned}$$

Since $0 < p \leq 1$, we have that $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$, meaning $\mathcal{E}$ is strictly contractive, and thus has a unique fixed point.

9.12

$$\begin{aligned} D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= \frac{1}{2} \operatorname{tr} \left| \frac{pI}{2} + (1-p)\rho - \frac{pI}{2} - (1-p)\sigma \right| \\ &= (1-p)\frac{1}{2} \operatorname{tr} |\rho - \sigma| \\ &= (1-p)D(\rho, \sigma). \end{aligned}$$

We must assume p strictly larger than zero, otherwise $\mathcal{E}$ would be simply the trivial identity channel. Therefore, the depolarizing channel is strictly contractive.

9.13

The bit-flip channel is given by $\mathcal{E}(\rho) = p\rho + (1-p)X\rho X$. Therefore we have

$$\begin{aligned} D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= \frac{1}{2} \operatorname{tr} |p\rho + (1-p)X\rho X - p\sigma - (1-p)X\sigma X| \\ &= \frac{1}{2} \operatorname{tr} |p(\rho - \sigma) + (1-p)X(\rho - \sigma)X| \\ &\leq \frac{p}{2} \operatorname{tr} |\rho - \sigma| + \frac{1-p}{2} \operatorname{tr} |X(\rho - \sigma)X|. \end{aligned}$$

The trace operation is invariant under cyclic permutations, meaning $\operatorname{tr} |X(\rho - \sigma)X| = \operatorname{tr} |X^2(\rho - \sigma)| = \operatorname{tr} |\rho - \sigma|$, therefore

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq \frac{1}{2} \operatorname{tr} |\rho - \sigma| = D(\rho, \sigma),$$

meaning the bit-flip channel is contractive but not strictly contractive.

The fixed points are the ones such that $X\rho X = \rho$. If we consider $\rho = (I + \vec{r}\cdot\vec{\sigma})/2$ we may write

$$\rho = \frac{I + r_x X + r_y Y + r_z Z}{2}, \quad \text{and} \quad X\rho X = \frac{I + r_x X - r_y Y - r_z Z}{2},$$

and we see that the condition is that $r_y = r_z = 0$. Therefore, all density operators of the form $\rho = (I + r_x X)/2$, for $0 \leq r_x \leq 1$, are fixed points.

9.14

$$\begin{aligned} F\left(U\rho U^\dagger, U\sigma U^\dagger\right) &= \operatorname{tr}\sqrt{\left(U\rho U^\dagger\right)^{1/2} U\sigma U^\dagger \left(U\rho U^\dagger\right)^{1/2}} \\ &= \operatorname{tr}\sqrt{U\rho^{1/2}U^\dagger U\sigma U^\dagger U\rho^{1/2}U^\dagger} \\ &= \operatorname{tr}\left(U\sqrt{\rho^{1/2}\sigma\rho^{1/2}}U^\dagger\right) \\ &= \operatorname{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} \\ &= F(\rho,\sigma). \end{aligned}$$

9.15

**From errata:* $A^\dagger$ should be A^{T} in Eq. (9.73), therefore Eq. (9.69) must be modified accordingly resulting in $U \equiv V_Q V_R^{\mathrm{T}} U_R^* U_Q^\dagger$ in Eq. (9.70).

Let $|\psi\rangle = \left(U_R \otimes \sqrt{\rho}U_Q\right)|m\rangle$ and $|\varphi\rangle = \left(V_R \otimes \sqrt{\rho}V_Q\right)|m\rangle$ be purifications of ρ and σ respectively, where U_R and U_Q are fixed. We know that

$$|\langle\psi|\varphi\rangle| = \left|\operatorname{tr}\left(\sqrt{\rho}\sqrt{\sigma}U\right)\right| \leq \operatorname{tr}\left|\sqrt{\rho}\sqrt{\sigma}\right| = F(\rho,\sigma),$$

where $U = V_Q V_R^{\mathrm{T}} U_R^* U_Q^\dagger$. Suppose $\left|\sqrt{\rho}\sqrt{\sigma}\right|V$ is the polar decomposition of $\sqrt{\rho}\sqrt{\sigma}$. Then if we choose $V_Q = V^\dagger U_Q$ and $V_R = U_R$ we obtain

$$|\langle\psi|\varphi\rangle| = \left|\operatorname{tr}\left(\left|\sqrt{\rho}\sqrt{\sigma}\right|VV^\dagger U_Q U_R^{\mathrm{T}} U_R^* U_Q^\dagger\right)\right| = \operatorname{tr}\left|\sqrt{\rho}\sqrt{\sigma}\right| = F(\rho,\sigma),$$

showing that equality is always attainable and therefore

$$F(\rho,\sigma) = \max_{|\varphi\rangle} |\langle\psi|\varphi\rangle|.$$

9.16

**From errata:* $A^\dagger$ should be A^{T} in Eq. (9.73).

Notice that $\left(A^{\mathrm{T}}B\right)_{jk} = \sum_i A_{ij}B_{ik}$, thus

$$\begin{aligned} \operatorname{tr}\left(A^{\mathrm{T}}B\right) &= \sum_{i,j} A_{ij}B_{ij} \\ &= \sum_{i,j} \langle i_R \,|\, A \,|\, j_R\rangle \langle i_Q \,|\, B \,|\, j_Q\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} \langle i_R| \langle i_Q| (A \otimes B) |j_R\rangle |j_Q\rangle \\
&= \langle m| (A \otimes B) |m\rangle .
\end{aligned}
$$

9.17

$$
0 \leq F(\rho, \sigma) \leq 1 \quad \Longrightarrow \quad \arccos 0 \geq \arccos F(\rho, \sigma) \geq \arccos 1.
$$

We have that $\arccos 0 = \pi/2$ and $\arccos 1 = 0$, and by definition $A(\rho, \sigma) \equiv \arccos F(\rho, \sigma)$, thus

$$
0 \leq A(\rho, \sigma) \leq \pi/2.
$$

We know that $F(\rho, \sigma) = 1$ if and only if $\rho = \sigma$, meaning $A(\rho, \sigma) = 0$ if and only if $\rho = \sigma$.

9.18

$$
F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma) \quad \Longrightarrow \quad \arccos F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq \arccos F(\rho, \sigma).
$$

From the definition $A(\rho, \sigma) \equiv \arccos F(\rho, \sigma)$ it follows that $A(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq A(\rho, \sigma)$.

9.19

Let $|\psi_i\rangle$ and $|\varphi_i\rangle$ be the purifications of, respectively, ρ_i and σ_i such that $F(\rho_i, \sigma_i) = \langle \psi_i | \varphi_i \rangle$. If we introduce an auxiliary system with orthonormal basis $|i\rangle$ we can define the purifications $|\psi\rangle \equiv \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$ and $|\varphi\rangle \equiv \sum_i \sqrt{p_i} |\varphi_i\rangle |i\rangle$ of the states $\sum_i p_i \rho_i$ and $\sum_i p_i \sigma_i$ respectively. From Uhlmann's formula we conclude

$$
F\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i \right) \geq |\langle \psi | \varphi \rangle| = \sum_{i,j} \sqrt{p_i} \sqrt{p_j} \langle \psi_i | \varphi_j \rangle \langle i | j \rangle = \sum_i p_i F(\rho_i, \sigma_i).
$$

9.20

Let $|\psi_i\rangle$ and $|\varphi\rangle$ be the purifications of, respectively, ρ_i and σ such that $F(\rho_i, \sigma) = \langle \psi_i | \varphi \rangle$. If we introduce an auxiliary system with orthonormal basis $|i\rangle$ we can define the purifications $|\psi\rangle \equiv \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$ and $|\Phi\rangle \equiv \sum_i \sqrt{p_i} |\varphi\rangle |i\rangle$ of the states $\sum_i p_i \rho_i$ and σ respectively. From Uhlmann's formula we conclude

$$
F\left(\sum_i p_i \rho_i, \sigma \right) \geq |\langle \psi | \Phi \rangle| = \sum_{i,j} \sqrt{p_i} \sqrt{p_i} \langle \psi_i | \varphi \rangle \langle i | j \rangle = \sum_i p_i F(\rho_i, \sigma).
$$

9.21

Let $\{E_m\}$ be a POVM such that $p_m \equiv \operatorname{tr}(|\psi\rangle\langle\psi| E_m)$ and $q_m \equiv \operatorname{tr}(\sigma E_m)$ are the probabilities of obtaining outcome m for the pure state $|\psi\rangle$ and for the mixed state σ respectively. We can always choose the POVM to be such that, for $m = k$, we have $E_k = |\psi\rangle\langle\psi|$. The consequence is that $p_k = 1$ and $p_j = 0$ for all $j \neq k$. With this we have the inequality

$$D(|\psi\rangle, \sigma) \geq D(p_m, q_m) = \frac{1}{2}\sum_j |p_j - q_j| = \frac{1}{2}\left(1 - q_k + \sum_{j\neq k} q_j\right) = 1 - q_k.$$

Using Eq. (9.60) we have

$$F(|\psi\rangle, \sigma)^2 = \langle\psi|\sigma|\psi\rangle .$$

If we choose a basis $|\phi_i\rangle$ that contains $|\psi\rangle$ as one of its elements we obtain

$$F(|\psi\rangle, \sigma)^2 = \sum_i \langle\phi_i|\sigma|\psi\rangle \langle\psi|\phi_i\rangle = \mathrm{tr}(\sigma\, |\psi\rangle\langle\psi|) = \mathrm{tr}(\sigma E_k) = q_k.$$

Substituting this result in the inequality yields

$$1 - F(|\psi\rangle, \sigma)^2 \leq D(|\psi\rangle, \sigma).$$

9.22

By the triangle inequality we have

$$d\big(VU\rho U^\dagger V^\dagger, \mathcal{F}\circ\mathcal{E}(\rho)\big) \leq d\big(VU\rho U^\dagger V^\dagger, \mathcal{F}\big(U\rho U^\dagger\big)\big) + d\big(\mathcal{F}\big(U\rho U^\dagger\big), \mathcal{F}\circ\mathcal{E}(\rho)\big),$$

and by the contractive property of trace-preserving maps we have

$$d\big(\mathcal{F}\big(U\rho U^\dagger\big), \mathcal{F}\circ\mathcal{E}(\rho)\big) \leq d\big(U\rho U^\dagger, \mathcal{E}(\rho)\big).$$

Since these two inequalities are true for any density operator, we may choose ρ to be the one that maximizes the left-hand side of the first inequality, that is, we choose ρ such that

$$E(VU, \mathcal{F}\circ\mathcal{E}) = d\big(VU\rho U^\dagger V^\dagger, \mathcal{F}\circ\mathcal{E}(\rho)\big).$$

Combining this with the two inequalities we obtain

$$E(VU, \mathcal{F}\circ\mathcal{E}) \leq d\big(VU\rho U^\dagger V^\dagger, \mathcal{F}\big(U\rho U^\dagger\big)\big) + d\big(U\rho U^\dagger, \mathcal{E}(\rho)\big).$$

Now notice that, since $U\rho U^\dagger$ is a density operator, we have that $d\big(VU\rho U^\dagger V^\dagger, \mathcal{F}\big(U\rho U^\dagger\big)\big) \leq E(V, \mathcal{F})$ and $d(U\rho U^\dagger, \mathcal{E}(\rho)) \leq E(U, \mathcal{E})$, thus

$$E(VU, \mathcal{F}\circ\mathcal{E}) \leq E(U, \mathcal{E}) + E(V, \mathcal{F}).$$

9.23

If $\mathcal{E}(\rho_j) = \rho_j$ for all j such that $p_j \neq 0$ then $F(\rho_j, \mathcal{E}(\rho_j)) = 1$ for all j such that $p_j \neq 0$, meaning $\bar{F} = \sum_j p_j = 1$. Conversely, if $\bar{F} = 1$, then we have that

$$\begin{aligned}
\sum_j p_j F(\rho_j, \mathcal{E}(\rho_j))^2 = 1 = \sum_j p_j \quad &\implies \quad F(\rho_j, \mathcal{E}(\rho_j))^2 = 1 \quad \text{for all } j \text{ such that } p_j \neq 0 \\
&\implies \quad F(\rho_j, \mathcal{E}(\rho_j)) = 1 \quad \text{for all } j \text{ such that } p_j \neq 0.
\end{aligned}$$