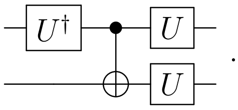
12 Quantum information theory

Exercises: 12.1, 12.2, 12.3, 12.4, 12.5, 12.6, 12.7, 12.8, 12.9, 12.10, 12.11, 12.12, 12.13, 12.14, 12.15, 12.16, 12.17, 12.18, 12.19, 12.20, 12.21, 12.22, 12.23, 12.24, 12.25, 12.26, 12.27, 12.28, 12.29, 12.30, 12.31, 12.32, 12.33, 12.34, 12.35, 12.36, 12.37, 12.38.

12.1

If $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal states, there is a unitary operator U such that $U|0\rangle = |\psi\rangle$ and $U|1\rangle = |\varphi\rangle$. The following circuit will output either $|\psi\rangle|\psi\rangle$ or $|\varphi\rangle|\varphi\rangle$ depending on the state of the data qubit.



We can verify this explicitly as

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\begin{aligned}
|\psi\rangle|0\rangle \text{ or } |\varphi\rangle|0\rangle &\xrightarrow{U^\dagger\otimes I} |0\rangle|0\rangle \text{ or } |1\rangle|0\rangle \\
&\xrightarrow{CX} |0\rangle|0\rangle \text{ or } |1\rangle|1\rangle \\
&\xrightarrow{U\otimes U} |\psi\rangle|\psi\rangle \text{ or } |\varphi\rangle|\varphi\rangle .
\end{aligned}
$$

12.2

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12.3

The Holevo bound implies that $H(X:Y) \leq H(X)$. For n bits, we have $H(X) \leq n$, with equality when the bits are sampled from a uniform distribution. Therefore, using n qubits, we can transmit at most n bits of information.

12.4

Denoting $\rho_i \equiv |X_i\rangle\langle X_i|$, the state sent by Alice is

$$
\begin{aligned}
\rho &= \frac{1}{4}(\rho_1 + \rho_2 + \rho_3 + \rho_4) \\
&= \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3}e^{-2\pi i/3} \\ \frac{\sqrt{2}}{3}e^{2\pi i/3} & \frac{2}{3} \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3}e^{2\pi i/3} \\ \frac{\sqrt{2}}{3}e^{-2\pi i/3} & \frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix},
\end{aligned}
$$

and so $S(\rho) = 1$. Since the four states are all pure we have $S(|X_i\rangle\langle X_i|) = 0$ for all i, and so $H(X:Y) \leq 1$. But since the four states do not have orthogonal support, the inequality is strict, and thus the mutual information is less than one bit.

The four states $|X_i\rangle$ form the vertices of a regular tetrahedron in the Bloch sphere. The POVM that maximizes the mutual information is constructed using the four states forming another regular tetrahedron whose vertices are maximally far away from the vertices of the tetrahedron formed by the $|X_i\rangle$. Such states are given by

$$
\begin{aligned}
|Y_1\rangle &= |1\rangle\,, \\
|Y_2\rangle &= \sqrt{\frac{1}{3}}\left[\sqrt{2}\,|0\rangle - |1\rangle\right] \\
|Y_3\rangle &= \sqrt{\frac{1}{3}}\left[\sqrt{2}\,|0\rangle + e^{i\pi/3}\,|1\rangle\right] \\
|Y_4\rangle &= \sqrt{\frac{1}{3}}\left[\sqrt{2}\,|0\rangle + e^{-i\pi/3}\,|1\rangle\right].
\end{aligned}
$$

We then define the POVM elements as $E_i \equiv \frac{1}{2}\,|Y_i\rangle\langle Y_i|$. Now, we may calculate the mutual information as $H(X:Y) = H(X) + H(Y) - H(X,Y)$. Using "$p$" to denote probability and using the fact that $p(X_j) = 1/4$ for all j, we obtain

$$
H(X) = -\sum_{j=1}^{4} p(X_j)\log(p(X_j)) = 2.
$$

We have that $H(Y) = -\sum_i p(Y_i)\log(p(Y_i))$, where $p(Y_i) = \sum_j p(X_j)p(Y_i|X_j)$, with $p(Y_i|X_j) = \mathrm{tr}(E_i\rho_j)$, therefore

$$
\begin{aligned}
H(Y) &= -\sum_{i=1}^{4}\left(\sum_{j=1}^{4} p(X_j)p(Y_i|X_j)\right)\log\left(\sum_{j=1}^{4} p(X_j)p(Y_i|X_j)\right) \\
&= -\sum_{i=1}^{4}\frac{1}{4}\left(\sum_{j=1}^{4}\mathrm{tr}(E_i\rho_j)\right)\log\left(\frac{1}{4}\sum_{j=1}^{4}\mathrm{tr}(E_i\rho_j)\right) = 2.
\end{aligned}
$$

And finally, we use the fact that $p(X_j, Y_i) = p(X_j)p(Y_i|X_j)$ to obtain

$$
\begin{aligned}
H(X,Y) &= -\sum_{i,j=1}^{4} p(X_j, Y_i)\log(p(X_j, Y_i)) \\
&= -\sum_{i,j=1}^{4}\frac{1}{4}\,\mathrm{tr}(E_i\rho_j)\log\left(\frac{1}{4}\,\mathrm{tr}(E_i\rho_j)\right) \approx 3.585.
\end{aligned}
$$

With these results, we get $H(X:Y) \approx 2 + 2 - 3.585 = 0.415$.

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