6 Quantum search algorithms

Exercises: 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8, 6.9, 6.10, 6.11, 6.12, 6.13, 6.14, 6.15, 6.16, 6.17, 6.18, 6.19, 6.20.

6.1

The operator consists in adding a phase -1 to all states except $|0\rangle$, therefore it has the form

$$\begin{aligned} |0\rangle\langle 0| - \sum_{x=1}^{N-1} |x\rangle\langle x| &= 2\,|0\rangle\langle 0| - \sum_{x=0}^{N-1} |x\rangle\langle x| \\ &= 2\,|0\rangle\langle 0| - I. \end{aligned}$$

6.2

$$\begin{aligned} (2\,|\psi\rangle\langle\psi| - I)\sum_k \alpha_k\,|k\rangle &= \sum_k \left(2\alpha_k\,|\psi\rangle\,\langle\psi|k\rangle - \alpha_k\,|k\rangle\right) \\ &= \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}\sum_k 2\alpha_k\,|x\rangle\,\langle y|k\rangle - \sum_k \alpha_k\,|k\rangle \\ &= \sum_{x=0}^{N-1}\left(\sum_k \frac{\alpha_k}{N}\right) 2\,|x\rangle - \sum_k \alpha_k\,|k\rangle \\ &= \sum_k \Big[-\alpha_k + 2\,\langle\alpha\rangle\Big]\,|k\rangle\,. \end{aligned}$$

6.3

The choice $\sin\theta = 2\sqrt{M(N-M)}/N$ is compatible with Equation (6.10) since

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \quad\Longrightarrow\quad \sin\frac{\theta}{2} = \sqrt{\frac{M}{N}} \text{ and } \cos\frac{\theta}{2} = \sqrt{\frac{N-M}{N}},$$

so we may indeed write the initial state $|\psi\rangle$ as

$$|\psi\rangle = \cos\frac{\theta}{2}\,|\alpha\rangle + \sin\frac{\theta}{2}\,|\beta\rangle\,.$$

Now we must only show that, defining G as in Equation (6.13), we can get the state $\cos\frac{3\theta}{2}\,|\alpha\rangle + \sin\frac{3\theta}{2}\,|\alpha\rangle\,|\beta\rangle$, as can be directly verified

$$G\,|\psi\rangle = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\frac{\theta}{2} - \sin\theta\sin\frac{\theta}{2} \\ \sin\theta\cos\frac{\theta}{2} + \cos\theta\sin\frac{\theta}{2}\end{bmatrix} = \begin{bmatrix}\cos\frac{3\theta}{2} \\ \sin\frac{3\theta}{2}\end{bmatrix}.$$

6.4

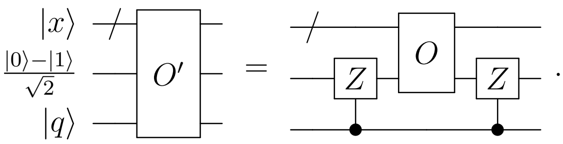
If $1 < M < N/2$ then by the end of step 3 we will have a state which is approximatelly given by

$$\frac{1}{\sqrt{M}}\sum_{j=0}^{M-1}|x_j\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right],$$

that is, a linear combination of the possible solutions $\{|x_0\rangle,\cdots,|x_{M-1}\rangle\}$, each with equal probability of being measured in step 4. So the only difference is that we would need to run the algorithm several times. After the algorithm is executed M times we have a probability of around $M!/M^M$ of having obtained all solutions x_j.

6.5

Notice that if instead of $|x\rangle(|0\rangle-|1\rangle)/\sqrt{2}$ we use $|x\rangle(|0\rangle+|1\rangle)/\sqrt{2}$ then the oracle will not do anything to the state, independently of x being a solution or not. So we may just apply a Z gate, conditioned to qubit $|q\rangle$, to the oracle qubit before and after calling the oracle O, that is



6.6

Considering that the two qubits are in a (normalized) superposition of all four possible states, given by $|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, the action of the circuit is

$$\begin{aligned}
|\Psi\rangle &\xrightarrow{X\otimes X} a|11\rangle + b|10\rangle + c|01\rangle + d|00\rangle \\
&\xrightarrow{I\otimes H} a\frac{|10\rangle-|11\rangle}{\sqrt{2}} + b\frac{|10\rangle+|11\rangle}{\sqrt{2}} + c\frac{|00\rangle-|01\rangle}{\sqrt{2}} + d\frac{|00\rangle+|01\rangle}{\sqrt{2}} \\
&\xrightarrow{CX_{(1,2)}} a\frac{|11\rangle-|10\rangle}{\sqrt{2}} + b\frac{|11\rangle+|10\rangle}{\sqrt{2}} + c\frac{|00\rangle-|01\rangle}{\sqrt{2}} + d\frac{|00\rangle+|01\rangle}{\sqrt{2}} \\
&\xrightarrow{I\otimes H} -a|11\rangle + b|10\rangle + c|01\rangle + d|00\rangle \\
&\xrightarrow{X\otimes X} -a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \\
&\qquad = (-1)\Big(a|00\rangle - b|01\rangle - c|10\rangle - d|11\rangle\Big) \\
&\qquad = (-1)\Big(2a|00\rangle - |\Psi\rangle\Big) \\
&\qquad = (-1)\Big(2|00\rangle\langle 00| - I\Big)|\Psi\rangle .
\end{aligned}$$

6.7

*It seems the phase gates should be with $-i\Delta t$ instead of $i\Delta t$.

First let us write the operations $\exp(-i\,|x\rangle\langle x|\,\Delta t)$ and $\exp(-i\,|\psi\rangle\langle\psi|\,\Delta t)$ explicitly:

$$\begin{aligned}\exp(-i\,|x\rangle\langle x|\,\Delta t) &= I + \sum_{j=1}^{\infty} \frac{(-i\Delta t)^j}{j!}\,|x\rangle\langle x| \\ &= I - |x\rangle\langle x| + \sum_{j=0}^{\infty} \frac{(-i\Delta t)^j}{j!}\,|x\rangle\langle x| \\ &= |y\rangle\langle y| + e^{-i\Delta t}\,|x\rangle\langle x|\,,\end{aligned}$$

$$\begin{aligned}\exp(-i\,|\psi\rangle\langle\psi|\,\Delta t) &= I + \sum_{j=1}^{\infty} \frac{(-i\Delta t)^j}{j!}\,|\psi\rangle\langle\psi| \\ &= I - |\psi\rangle\langle\psi| + \sum_{j=0}^{\infty} \frac{(-i\Delta t)^j}{j!}\,|\psi\rangle\langle\psi| \\ &= I + \left(e^{-i\Delta t} - 1\right)|\psi\rangle\langle\psi|\,.\end{aligned}$$

Now we can verify that the circuit in Figure 6.4 acts as

$$\begin{aligned}|y\rangle\,|0\rangle &= \Big(\,|x\rangle\langle x| + |y\rangle\langle y|\,\Big)\,|y\rangle\,|0\rangle \\ &= \langle x|y\rangle\,|x\rangle\,|0\rangle + |y\rangle\,|0\rangle \\ &\xrightarrow{\text{Oracle}} \langle x|y\rangle\,|x\rangle\,|1\rangle + |y\rangle\,|0\rangle \\ &\xrightarrow{I^{\otimes n}\otimes P_{-\Delta t}} e^{-i\Delta t}\,\langle x|y\rangle\,|x\rangle\,|1\rangle + |y\rangle\,|0\rangle \\ &\xrightarrow{\text{Oracle}} e^{-i\Delta t}\,\langle x|y\rangle\,|x\rangle\,|0\rangle + |y\rangle\,|0\rangle \\ &= \Big(\,|y\rangle\langle y| + e^{-i\Delta t}\,|x\rangle\langle x|\,\Big)\,|y\rangle\,|0\rangle\,,\end{aligned}$$

and the circuit in Figure 6.5 performs

$$\begin{aligned}|y\rangle\,|0\rangle &= \left(\sum_{j=0}^{N} |j\rangle\langle j|\right)|y\rangle\,|0\rangle \\ &= \sum_{j=0}^{N-1} \langle j|y\rangle\,|j\rangle\,|0\rangle \\ &\xrightarrow{H^{\otimes n}\otimes I} \sum_{j=0}^{N-1} \langle j|y\rangle\,H^{\otimes n}\,|j\rangle\,|0\rangle \\ &= \left(\sum_{j=0}^{N-1} \langle j|y\rangle\,H^{\otimes n}\,|j\rangle - \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\langle j|y\rangle\,|0\rangle\right)|0\rangle + \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\langle j|y\rangle\,|0\rangle\,|0\rangle \\ &\xrightarrow{C^nX_{(\neg 1,2)}\otimes I} \left(\sum_{j=0}^{N-1} \langle j|y\rangle\,H^{\otimes n}\,|j\rangle - \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\langle j|y\rangle\,|0\rangle\right)|0\rangle + \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\langle j|y\rangle\,|0\rangle\,|1\rangle \\ &\xrightarrow{I^{\otimes n}\otimes P_{-\Delta t}} \left(\sum_{j=0}^{N-1} \langle j|y\rangle\,H^{\otimes n}\,|j\rangle - \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\langle j|y\rangle\,|0\rangle\right)|0\rangle + \frac{e^{-i\Delta t}}{\sqrt{N}}\sum_{j=0}^{N-1}\langle j|y\rangle\,|0\rangle\,|1\rangle\end{aligned}$$

$$
\begin{aligned}
&\xrightarrow{C^n X_{(\neg 1,2)}\otimes I} \left(\sum_{j=0}^{N-1} \langle j|y\rangle \, H^{\otimes n} |j\rangle - \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \langle j|y\rangle \, |0\rangle \right) |0\rangle + \frac{e^{-i\Delta t}}{\sqrt{N}} \sum_{j=0}^{N-1} \langle j|y\rangle \, |0\rangle \, |0\rangle \\
&\xrightarrow{H^{\otimes n}\otimes I} \left(\sum_{j=0}^{N-1} \langle j|y\rangle \, |j\rangle - \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \langle j|y\rangle \, \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle \right) |0\rangle + \frac{e^{-i\Delta t}}{\sqrt{N}} \sum_{j=0}^{N-1} \langle j|y\rangle \, \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle \, |0\rangle \\
&= \left(\sum_{j=0}^{N-1} |j\rangle\langle j| - |\psi\rangle\langle\psi| \right) |y\rangle \, |0\rangle + \left(e^{-i\Delta t} \, |\psi\rangle\langle\psi| \right) |y\rangle \, |0\rangle \\
&= \left[I + \left(e^{-i\Delta t} - 1 \right) |\psi\rangle\langle\psi| \right] |y\rangle \, |0\rangle
\end{aligned}
$$

6.8

If we have accuracy of $O(\Delta t^r)$ for each step the cumulative error is given by $O\left(\Delta t^r \sqrt{N}/\Delta t\right) = O\left(\Delta t^{r-1}\sqrt{N}\right)$. We need the error to be $O(1)$ in order to simulate H with high accuracy, that is

$$
\Delta t^{r-1}\sqrt{N} = 1 \quad \Longrightarrow \quad \Delta t = \left(\frac{1}{\sqrt{N}}\right)^{1/(r-1)} = N^{-1/2(r-1)}.
$$

Therefore the number of required oracle calls is

$$
O\left(\sqrt{N}/\Delta t\right) = O\left(N^{1/2} N^{1/2(r-1)}\right) = O\left(N^{r/2(r-1)}\right).
$$

6.9

$$
U(\Delta t) = \exp\left[-i\Delta t\left(I + \vec{\psi}\cdot\vec{\sigma}\right)/2\right] \exp\left[-i\Delta t\left(I + \hat{z}\cdot\vec{\sigma}\right)/2\right].
$$

Let us expand each exponential explicitly, and for notation simplicity, let us define the quantities $c \coloneqq \cos(\Delta t/2)$ and $s \coloneqq \sin(\Delta t/2)$. The first exponential yields

$$
\begin{aligned}
\exp\left[-i\Delta t\left(I + \vec{\psi}\cdot\vec{\sigma}\right)/2\right] &= \exp\left[-i\frac{\Delta t}{2} I\right] \exp\left[-i\frac{\Delta t}{2} \vec{\psi}\cdot\vec{\sigma}\right] \\
&= (c - is)\, I \left(cI - is\vec{\psi}\cdot\vec{\sigma} \right) \\
&= c^2 I - s^2 \vec{\psi}\cdot\vec{\sigma} - ics\left(I + \vec{\psi}\cdot\vec{\sigma} \right),
\end{aligned}
$$

and the second one yields

$$
\begin{aligned}
\exp\left[-i\Delta t\left(I + \hat{z}\cdot\vec{\sigma}\right)/2\right] &= \exp\left[-i\frac{\Delta t}{2} I\right] \exp\left[-i\frac{\Delta t}{2} \hat{z}\cdot\vec{\sigma}\right] \\
&= (c - is)\, I \left(cI - is\hat{z}\cdot\vec{\sigma} \right) \\
&= c^2 I - s^2 \hat{z}\cdot\vec{\sigma} - ics\left(I + \hat{z}\cdot\vec{\sigma} \right).
\end{aligned}
$$

Substituting both results in the expression for $U(\Delta t)$ we obtain

$$
U(\Delta t) = \left[c^2 I - s^2\vec{\psi}\cdot\vec{\sigma} - icsI - ics\vec{\psi}\cdot\vec{\sigma}\right]\left[c^2 I - s^2\hat{z}\cdot\vec{\sigma} - icsI - ics\hat{z}\cdot\vec{\sigma}\right]
$$

$$= \left[c^2 - 2ics - s^2\right] \left(c^2 I - s^2 \left(\vec{\psi} \cdot \vec{\sigma}\right)\left(\hat{z} \cdot \vec{\sigma}\right)\right) - ics \left[c^2 - 2ics - s^2\right] \left(\vec{\psi} + \hat{z}\right) \cdot \vec{\sigma}.$$

Now we use the fact that (see Exercise **??**)

$$\left(\vec{\psi} \cdot \vec{\sigma}\right)\left(\hat{z} \cdot \vec{\sigma}\right) = \left(\vec{\psi} \cdot \hat{z}\right) I + i \left(\vec{\psi} \times \hat{z}\right) \cdot \vec{\sigma}.$$

Substituting it in the expression for $U(\Delta t)$ yields

$$\begin{aligned} U(\Delta t) &= \left[c^2 - 2ics - s^2\right] \left(c^2 I - s^2 \left(\vec{\psi} \cdot \hat{z}\right) I - is^2 \left(\vec{\psi} \times \hat{z}\right) \cdot \vec{\sigma}\right) - ics \left[c^2 - 2ics - s^2\right] \left(\vec{\psi} + \hat{z}\right) \cdot \vec{\sigma} \\ &= \left[c^2 - 2ics - s^2\right] \left(c^2 - s^2 \left(\vec{\psi} \cdot \hat{z}\right)\right) I + \left[c^2 - 2ics - s^2\right] (-2is) \left(s \frac{\vec{\psi} \times \hat{z}}{2} + c \frac{\vec{\psi} + \hat{z}}{2}\right) \cdot \vec{\sigma}. \end{aligned}$$

Now notice that the term

$$\left[c^2 - 2ics - s^2\right] = \left[c - is\right]^2 = \exp(-i\Delta t),$$

multiplying all terms is just a global phase factor and can therefore be eliminated from the expression, and what is left is the result shown in Equation (6.25)

$$\begin{aligned} & U(\Delta t) = \left(c^2 - s^2 \left(\vec{\psi} \cdot \hat{z}\right)\right) I - 2is \left(c \frac{\vec{\psi} + \hat{z}}{2} + s \frac{\vec{\psi} \times \hat{z}}{2}\right) \cdot \vec{\sigma} \\ \Longrightarrow \quad & U(\Delta t) = \left(\cos^2\left(\frac{\Delta t}{2}\right) - \sin^2\left(\frac{\Delta t}{2}\right) \vec{\psi} \cdot \hat{z}\right) I \\ & \qquad - 2i \sin\left(\frac{\Delta t}{2}\right) \left(\cos\left(\frac{\Delta t}{2}\right) \frac{\vec{\psi} + \hat{z}}{2} + \sin\left(\frac{\Delta t}{2}\right) \frac{\vec{\psi} \times \hat{z}}{2}\right) \cdot \vec{\sigma}. \end{aligned}$$

6.10

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6.11

We may write a Hamiltonian analog the one shown in Equation (6.18), that is

$$H = |\chi\rangle\langle\chi| + |\psi\rangle\langle\psi|,$$

where we define the state $|\chi\rangle$ as

$$|\chi\rangle := \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} |x_j\rangle.$$

Just like the process for the case of a single solution, if we choose $|\psi\rangle = \sum_{j=0}^{N-1} |x_j\rangle / \sqrt{N}$ then this Hamiltonian can be used to rotate the state $|\psi\rangle$ to the state $|\chi\rangle$, that is, we are sending our state to a superposition of solution states that, when measured, will give us one of the M possible solutions. As for the simulation of such Hamiltonian, we can simulate the Hamiltonians $H_1 = |\chi\rangle\langle\chi|$

and $H_2 = |\psi\rangle\langle\psi|$ for time increments Δt, just like it was done for the case of a single solution.

6.12

(1) The evolution associated with this Hamiltonian is $\exp(-iHt)$. If we restrict ourselves to the space spanned by $|x\rangle$ and $|\psi\rangle$ we can write $|\psi\rangle = \alpha\,|x\rangle + \beta\,|y\rangle$, where $\{|x\rangle\,, |y\rangle\}$ is an orthonormal basis for this space, and therefore

$$H = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix} = \begin{bmatrix} 2\alpha & \beta \\ \beta & 0 \end{bmatrix} = \alpha(I+Z) + \beta X.$$

The evolution is then given by

$$\begin{aligned} \exp(-iHt) &= \exp(-i\alpha t I)\exp(-it\,(\beta X + \alpha Z)) \\ &= e^{-i\alpha t} I \left[\cos(t) I - i\sin(t)\,(\beta X + \alpha Z)\right]. \end{aligned}$$

The global phase factor $e^{-i\alpha t}$ can be ignored. Now applying this evolution to $|\psi\rangle$ yields

$$\begin{aligned} \exp(-iHt)\,|\psi\rangle &= \cos(t)\,|\psi\rangle - i\sin(t)\,(\beta X + \alpha Z)\,|\psi\rangle \\ &= \cos(t)\,|\psi\rangle - i\sin(t)\,|x\rangle\,. \end{aligned}$$

As one can notice, this evolution takes the state $|\psi\rangle$ to the state $|x\rangle$ in a time interval $t = \pi/2$, which is a constant, so it takes time $O(1)$ to be performed.

(2) **Incomplete...**

6.13

Let us denote $\mathbf{E}(S)$ as the expectation of S, then we have

$$\Delta S = \sqrt{\mathbf{E}(S^2) - \mathbf{E}(S)^2}.$$

S is just the mean of the list $\{X_1, \cdots, X_k\}$, that we denote $\overline{X} \coloneqq \sum_j X_j/k$, multiplied by a constant N. If we use the fact that $\mathbf{E}(cx) = c\mathbf{E}(x)$ for constant c we have that

$$\begin{aligned} \Delta S &= \sqrt{N^2\mathbf{E}\left(\overline{X}^2\right) - N^2\mathbf{E}\left(\overline{X}\right)^2} \\ &= N\sqrt{\mathbf{E}\left(\overline{X}^2\right) - \mathbf{E}\left(\overline{X}\right)^2}. \end{aligned}$$

Each X_j has probability M/N of being 1 so $\mathbf{E}(X_j) = M/N$, and since $X_j^2 = X_j$ for all j it is a fact that $\mathbf{E}(X_j^2) = \mathbf{E}(X_j)$. The necessary expected values can then be calculated as

$$\mathbf{E}\left(\overline{X}^2\right) = \mathbf{E}\left(\frac{1}{k^2}\sum_{j,l} X_j X_l\right)$$

$$
\begin{aligned}
&= \frac{1}{k^2}\left(\sum_{j\neq l}\mathbf{E}(X_jX_l) + \sum_j \mathbf{E}(X_j^2)\right) \\
&= \frac{1}{k^2}\left(\sum_{j\neq l}\mathbf{E}(X_j)\mathbf{E}(X_l) + \sum_j \mathbf{E}(X_j)\right) \\
&= \frac{1}{k^2}\left(k(k-1)\frac{M^2}{N^2} + k\frac{M}{N}\right) \\
&= \frac{M}{kN} - \frac{M^2}{kN^2} + \frac{M^2}{N^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(\overline{X})^2 &= \mathbf{E}\left(\frac{1}{k}\sum_j X_j\right)\mathbf{E}\left(\frac{1}{k}\sum_l X_l\right) \\
&= \frac{1}{k^2}\sum_{j,l}\mathbf{E}(X_j)\mathbf{E}(X_l) = \frac{M^2}{N^2}.
\end{aligned}
$$

Substituting yields

$$
\begin{aligned}
\Delta S &= N\sqrt{\frac{M}{kN} - \frac{M^2}{kN^2} + \frac{M^2}{N^2} - \frac{M^2}{N^2}} \\
&= \sqrt{\frac{M(N-M)}{k}}.
\end{aligned}
$$

Now, in order to obtain M within an accuracy $\sqrt{M}$ with probability 3/4 we need that $\Delta S \leq \alpha\sqrt{M}$ for some constant α. But this can only happen if $k \geq (N-M)/\alpha^2$ which means k must be $\Omega(N)$.

6.14

We consider that all N elements have probability M/N of being a solution, that is an uniform distribution. So the average of a sample of such set of elements is the best estimate we can have for M/N and hence M. In other words, $N \times \sum_j X_j/k$ where all the X_j are sampled uniformly and independently, is the best estimate we can make for M. And since the algorithm based upon it that guesses M correctly within accuracy $\sqrt{M}$ requires $\Omega(N)$ oracle calls, any other classical algorithm will require at least the same number of calls.

6.15

$$
\begin{aligned}
\sum_x \|\psi - x\|^2 &\geq \|\psi\|^2 + \sum_x \|x\|^2 - 2\sum_x \|\psi\|\|x\| \\
&= 1 + N - 2\sum_x \|\psi\|\|x\|.
\end{aligned}
$$

Cauchy-Schwarz inequality gives us $\sum_x \|\psi\|\|x\| \leq \sqrt{\|\psi\|^2}\sqrt{\sum_x \|x\|^2} = \sqrt{N}$. Substituting we get

$$
\sum_x \|\psi - x\|^2 \geq 1 + N - 2\sqrt{N}
$$

$$\geq 2N - 2\sqrt{N}.$$

6.16

In this case, instead of $|\langle x|\psi_k^x\rangle|^2 \geq 1/2$, we suppose $\sum_x |\langle x|\psi_k^x\rangle|^2/N \geq 1/2 \Rightarrow \sum_x |\langle x|\psi_k^x\rangle|^2 \geq N/2$. Without loss of generality, we may choose $\langle x|\psi_k^x\rangle = |\langle x|\psi_k^x\rangle|$, so

$$\sum_x \|\psi_k^x - x\|^2 = 2N - 2\sum_x |\langle x|\psi_k^x\rangle|.$$

Since $0 \leq |\langle x|\psi_k^x\rangle| \leq 1$ for all x, we clearly have $\sum_x |\langle x|\psi_k^x\rangle|^2 \leq \sum_x |\langle x|\psi_k^x\rangle|$, so

$$\sum_x \|\psi_k^x - x\|^2 \leq 2N - 2\frac{N}{2} = N.$$

So the only difference between this case and the one where we impose that $|\langle x|\psi_k^x\rangle|^2 \geq 1/2$, is that, instead of having $E_k \leq (2-\sqrt{2})N$, we have $E_k \leq N$. Now, if E_k is still $O(N)$ then D_k is still $O(N)$ which means k is still $O(\sqrt{N})$, meaning $O(\sqrt{N})$ oracle calls are still required.

6.17

If the objective is to detect only one solution among the M possible ones, then the average time to do so is the same as finding the only solution of a N/M search space. Therefore it would take $O(\sqrt{N/M})$ oracle applications to find a solution.

6.18

Suppose there are two distinct minimum degree polynomials $p_1(X)$ and $p_2(X)$ representing some Boolean function $F(X)$. If they are distinct there is at least one number $X_0 \in \{0,1\}^n$ such that $p_1(X_0) - p_2(X_0) \neq 0$, but if they both represent $F(X)$ it is also true that $p_1(X_0) = p_2(X_0) = F(X_0)$, meaning $p_1(X_0) - p_2(X_0) = 0$, which is a contradiction. So the minimum degree polynomial representing $F(X)$ is unique.

6.19

The OR operation should return 1 if at least one of the X_i equals 1, and return 0 only if all X_i equal 0. It is straightforward to conclude that the function

$$P(X) = 1 - \prod_{i=0}^{N-1}(1 - X_i)$$

only returns 0 if $\prod_{i=0}^{N-1}(1 - X_i) = 1$, which in turn can only happen if all X_i equal 0. So this is a representation of OR.

6.20

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