4 Quantum circuits

Exercises: 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11, 4.12, 4.13, 4.14, 4.15, 4.16, 4.17, 4.18, 4.19, 4.20, 4.21, 4.22, 4.23, 4.24, 4.25, 4.26, 4.27, 4.28, 4.29, 4.30, 4.31, 4.32, 4.33, 4.34, 4.35, 4.36, 4.37, 4.38, 4.39, 4.40, 4.41, 4.42, 4.43, 4.44, 4.45, 4.46, 4.47, 4.48, 4.49, 4.50, 4.51.

4.1

$$\begin{aligned}
X\ :\ \text{eigenvectors } &= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \text{ and } \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \\
&= \cos\frac{\pi}{4}|0\rangle + e^{i0}\sin\frac{\pi}{4}|1\rangle \text{ and } \cos\frac{\pi}{4}|0\rangle + e^{i\pi}\sin\frac{\pi}{4}|1\rangle \\
&\Longrightarrow (\theta = \pi/2, \varphi = 0) \text{ and } (\theta = \pi/2, \varphi = \pi) \\
&\Longrightarrow \text{Equator with longitudes } 0^\circ \text{ and } 180^\circ \text{ respectively.}
\end{aligned}$$

$$\begin{aligned}
Y\ :\ \text{eigenvectors } &= \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \text{ and } \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right) \\
&= \cos\frac{\pi}{4}|0\rangle + e^{i\pi/2}\sin\frac{\pi}{4}|1\rangle \text{ and } \cos\frac{\pi}{4}|0\rangle + e^{-i\pi/2}\sin\frac{\pi}{4}|1\rangle \\
&\Longrightarrow (\theta = \pi/2, \varphi = \pi/2) \text{ and } (\theta = \pi/2, \varphi = -\pi/2) \\
&\Longrightarrow \text{Equator with longitudes } 90^\circ \text{ and } -90^\circ \text{ respectively.}
\end{aligned}$$

$$\begin{aligned}
Z\ :\ \text{eigenvectors } &= |0\rangle \text{ and } |1\rangle \\
&= \cos 0\,|0\rangle + e^{i\varphi}\sin 0\,|1\rangle \text{ and } \cos\frac{\pi}{2}|0\rangle + e^{i\varphi}\sin\frac{\pi}{2}|1\rangle \\
&\Longrightarrow (\theta = 0, \varphi) \text{ and } (\theta = \pi, \varphi) \\
&\Longrightarrow \text{North and south poles respectively.}
\end{aligned}$$

4.2

$$\begin{aligned}
\exp(iAx) &= \sum_{j=0}^{\infty} \frac{(iAx)^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j (Ax)^{2j}}{(2j)!} + i\sum_{j=0}^{\infty} \frac{(-1)^j (Ax)^{2j+1}}{(2j+1)!}.
\end{aligned}$$

Since $A^2 = I$ we have $A^{2j} = I$ and $A^{2j+1} = A$, thus

$$\begin{aligned}
\exp(iAx) &= \left(\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!}\right) I + i\left(\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}\right) A \\
&= \cos(x)I + i\sin(x)A.
\end{aligned}$$

From this result, since the Pauli matrices obey $\sigma_j^2 = I$, it follows that

$$\begin{aligned}
e^{-i\theta X/2} &= \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X, \\
e^{-i\theta Y/2} &= \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y, \\
e^{-i\theta Z/2} &= \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z.
\end{aligned}$$

4.3

$$\begin{aligned}
T &= \exp(i\pi/8)\begin{bmatrix} \exp(-i\pi/8) & 0 \\ 0 & \exp(i\pi/8) \end{bmatrix} \\
&= \exp(i\pi/8) R_z(\pi/4).
\end{aligned}$$

4.4

$$\begin{aligned}
H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & -\cos\frac{\pi}{4} \end{bmatrix} \\
&= \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} e^{i\pi/4}\cos\frac{\pi}{4} & e^{i\pi/4}\sin\frac{\pi}{4} \\ e^{-i\pi/4}\sin\frac{\pi}{4} & e^{i3\pi/4}\cos\frac{\pi}{4} \end{bmatrix} \\
&= \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} e^{i\pi/2}\cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & e^{i\pi/2}\cos\frac{\pi}{4} \end{bmatrix}\begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \\
&= e^{i\pi/2}\begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{4} & -i\sin\frac{\pi}{4} \\ -i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}\begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \\
&= e^{i\pi/2} R_z(\pi/2) R_x(\pi/2) R_z(\pi/2).
\end{aligned}$$

4.5

$$\begin{aligned}
(\hat{n}\cdot\vec{\sigma})^2 &= (n_x X + n_y Y + n_z Z)^2 \\
&= n_x^2 X^2 + n_y^2 Y^2 + n_z^2 Z^2 + n_x n_y\{X, Y\} + n_x n_z\{X, Z\} + n_y n_z\{Y, Z\}.
\end{aligned}$$

Since all Pauli matrices obey $\sigma_j^2 = I$ and $\{\sigma_j, \sigma_k\} = 0$ we have

$$(\hat{n}\cdot\vec{\sigma})^2 = \left(n_x^2 + n_y^2 + n_z^2\right) I = I.$$

From this result it follows that

$$\exp(-i\theta\hat{n}\cdot\vec{\sigma}/2) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\left(\hat{n}\cdot\vec{\sigma}\right).$$

4.6

First, let us show that the effects of $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$ upon the Bloch vector correspond to rotations of an angle θ around the x, y and z axis respectively. A qubit in the pure state $|\psi\rangle$ whose Bloch vector is $\vec{\lambda}$ has density operator given by

$$\rho = \frac{I + \vec{\lambda} \cdot \vec{\sigma}}{2}.$$

Applying $R_z(\theta)$ on $|\psi\rangle$ yields

$$R_z(\theta)\rho R_z^\dagger(\theta) = \frac{I + \lambda_x R_z(\theta) X R_z^\dagger(\theta) + \lambda_x R_z(\theta) Y R_z^\dagger(\theta) + \lambda_z Z}{2}.$$

We can expand the terms

$$\begin{aligned}
R_z(\theta) X R_z^\dagger(\theta) &= \left(\cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z\right) X \left(\cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} Z\right) \\
&= \cos\frac{\theta}{2}\cos\frac{\theta}{2} X + i \cos\frac{\theta}{2}\sin\frac{\theta}{2}[X, Z] + \sin\frac{\theta}{2}\sin\frac{\theta}{2} ZXZ \\
&= \left(\cos\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\frac{\theta}{2}\right) X + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} Y \\
&= \cos\theta X + \sin\theta Y, \\
R_z(\theta) Y R_z^\dagger(\theta) &= \left(\cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z\right) Y \left(\cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} Z\right) \\
&= \cos\frac{\theta}{2}\cos\frac{\theta}{2} Y + i \cos\frac{\theta}{2}\sin\frac{\theta}{2}[Y, Z] + \sin\frac{\theta}{2}\sin\frac{\theta}{2} ZYZ \\
&= \left(\cos\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\frac{\theta}{2}\right) Y - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} X \\
&= \cos\theta Y - \sin\theta X,
\end{aligned}$$

yielding

$$\begin{aligned}
R_z(\theta)\rho R_z^\dagger(\theta) &= \frac{I + \lambda_x\left(\cos\theta X + \sin\theta Y\right) + \lambda_y\left(\cos\theta Y - \sin\theta X\right) + \lambda_z Z}{2} \\
&= \frac{I + \vec{\lambda}' \cdot \vec{\sigma}}{2},
\end{aligned}$$

where the new Bloch vector $\vec{\lambda}'$ is given by

$$\begin{aligned}
\vec{\lambda}' &= (\lambda_x \cos\theta - \lambda_y \sin\theta, \lambda_x \sin\theta + \lambda_y \cos\theta, \lambda_z) \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix},
\end{aligned}$$

which is precisely a rotation in 3-dimensional space of an angle θ around the z axis. The process is the same for the $R_x(\theta)$ and $R_y(\theta)$ operators. Now notice that for an arbitrary axis directed by the unit vector $\hat{n} = (\cos\alpha \sin\beta, \sin\alpha \sin\beta, \cos\beta)$ we may write

$$
\begin{aligned}
R_{\hat{n}}(\theta) &= \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} \left(\hat{n} \cdot \vec{\sigma}\right) \\
&= \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} \left(\cos\alpha \sin\beta X + \sin\alpha \sin\beta Y + \cos\beta Z\right) \\
&= \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} \cos\beta Z - i \sin\frac{\theta}{2} \sin\beta \left(\cos\alpha X + \sin\alpha Y\right) \\
&= R_z(\alpha) \left(\cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} \left(\cos\beta Z + \sin\beta X\right)\right) R_z^\dagger(\alpha) \\
&= R_z(\alpha) R_y(\beta) \left(\cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z\right) R_y^\dagger(\beta) R_z^\alpha(\alpha) \\
&= R_z(\alpha) R_y(\beta) R_z(\theta) R_y^\dagger(\beta) R_z^\dagger(\alpha),
\end{aligned}
$$

meaning we can express $R_{\hat{n}}(\theta)$ as a combination of operators R_y and R_z, which were shown to be rotation operators over the Bloch vector, thus $R_{\hat{n}}$ is also a rotation, in particular, around the $\hat{n}$ axis. The physical interpretation is that the operations $R_y^\dagger(\beta) R_z^\dagger(\alpha)$ and $R_z(\alpha) R_y(\beta)$ are just a change in the frame of reference. The Bloch vector is first written in the reference frame where it has the same relative position to the z axis as it had originally with the $\hat{n}$ axis, then the rotation is performed around the z axis and the system is returned to the original frame of reference.

4.7

$$
XYX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -Y.
$$

From this result it follows that

$$
\begin{aligned}
X R_y(\theta) X &= X \left(\cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y\right) X \\
&= \cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} Y \\
&= R_y(-\theta).
\end{aligned}
$$

4.8

We can write any unitary operator as the exponential of some Hermitian operator K as

$$
U = \exp(iK),
$$

and any Hermitian operator can be expanded as

$$
K = \begin{bmatrix} a+d & b-ic \\ b+ic & a-d \end{bmatrix} = aI + bX + cY + dZ.
$$

Choosing $a = \alpha$, $b = -n_x\theta/2$, $c = -n_y\theta/2$, and $d = -n_z\theta/2$ such that $n_x^2 + n_y^2 + n_z^2 = 1$ yields

$$\begin{aligned}
U &= \exp\left(i\alpha I - i\frac{\theta}{2}\left(n_x X + n_y Y + n_z Z\right)\right) \\
&= \exp(i\alpha I)\exp(-i\theta\hat{n}\cdot\vec{\sigma}/2) \\
&= \exp(i\alpha)R_{\hat{n}}(\theta).
\end{aligned}$$

$$\begin{aligned}
H = \frac{X+Z}{\sqrt{2}} &= i\left(-i\left[\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Z\right]\right) \\
&= \exp(i\pi/2)\left(\cos\frac{\pi}{2}I - i\sin\frac{\pi}{2}\left[\frac{1}{\sqrt{2}}X + 0Y + \frac{1}{\sqrt{2}}Z\right]\right) \\
&= \exp(i\pi/2)R_{\frac{1}{\sqrt{2}}(1,0,1)}(\pi) \\
&\implies \alpha = \frac{\pi}{2}, \theta = \pi, \hat{n} = \frac{1}{\sqrt{2}}\left(1,0,1\right).
\end{aligned}$$

$$\begin{aligned}
S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} &= \exp(i\pi/4)\begin{bmatrix} \exp(-i\pi/4) & 0 \\ 0 & \exp(i\pi/4) \end{bmatrix} \\
&= \exp(i\pi/4)R_z(\pi/2) \\
&\implies \alpha = \frac{\pi}{4}, \theta = \frac{\pi}{2}, \hat{n} = \left(0,0,1\right).
\end{aligned}$$

4.9

Since any 3-dimensional rotation around an arbitrary axis can be written in terms of the three Euler angles we can write $R_{\hat{n}}(\theta) = R_z(\beta)R_y(\gamma)R_z(\delta)$. And because any unitary operator can be written as $U = \exp(i\alpha)R_{\hat{n}}(\theta)$ we have

$$\begin{aligned}
U &= \exp(i\alpha)R_z(\beta)R_y(\gamma)R_z(\delta) \\
&= \exp(i\alpha)\begin{bmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{bmatrix}\begin{bmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{bmatrix}\begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{bmatrix} \\
&= \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix}.
\end{aligned}$$

4.10

It is always possible to find three angles β', γ' and δ' analogous to the usual Euler angles such that $R_z(\beta)R_y(\gamma)R_z(\delta) = R_x(\beta')R_y(\gamma')R_x(\delta')$, thus for some α' we may write

$$\begin{aligned}
U &= \exp(i\alpha')R_x(\beta')R_y(\gamma')R_x(\delta') \\
&= \exp(i\alpha')\begin{bmatrix} \cos\frac{\beta'}{2} & -i\sin\frac{\beta'}{2} \\ i\sin\frac{\beta'}{2} & \cos\frac{\beta'}{2} \end{bmatrix}\begin{bmatrix} \cos\frac{\gamma'}{2} & -\sin\frac{\gamma'}{2} \\ \sin\frac{\gamma'}{2} & \cos\frac{\gamma'}{2} \end{bmatrix}\begin{bmatrix} \cos\frac{\delta'}{2} & -i\sin\frac{\delta'}{2} \\ i\sin\frac{\delta'}{2} & \cos\frac{\delta'}{2} \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} A & -B \\ B & A \end{bmatrix},$$

where

$$A = e^{i\alpha'}\left[\frac{1}{2}\cos\left(\frac{\beta'-\delta'-\gamma'}{2}\right) + \frac{1}{2}\cos\left(\frac{\beta'-\delta'+\gamma'}{2}\right) - i\sin\left(\frac{\beta'+\delta'}{2}\right)\sin\left(\frac{\gamma'}{2}\right)\right],$$
$$B = e^{i\alpha'}\left[\cos\left(\frac{\beta'-\delta'}{2}\right)\sin\left(\frac{\gamma'}{2}\right) + i\cos\left(\frac{\gamma'}{2}\right)\sin\left(\frac{\beta'+\delta'}{2}\right)\right].$$

4.11

Writing the operators in terms of Euler angles yields

$$\begin{aligned} R_{\hat{n}}(\beta) &= R_z(\theta_1)R_y(\phi_1)R_z(\psi_1), \\ R_{\hat{m}}(\gamma) &= R_z(\theta_2)R_y(\phi_2)R_z(\psi_2), \\ R_{\hat{n}}(\delta) &= R_z(\theta_3)R_y(\phi_3)R_z(\psi_3), \end{aligned}$$

and considering $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ we have that

$$\begin{aligned} e^{i\alpha}R_{\hat{n}}(\beta)R_{\hat{m}}(\gamma)R_{\hat{n}}(\delta) = &\left[e^{i\alpha_1}R_z(\theta_1)R_y(\phi_1)R_z(\psi_1)\right] \\ &\times\left[e^{i\alpha_2}R_z(\theta_2)R_y(\phi_2)R_z(\psi_2)\right] \\ &\times\left[e^{i\alpha_3}R_z(\theta_3)R_y(\phi_3)R_z(\psi_3)\right]. \end{aligned}$$

According to Theorem 4.1 the three terms between square brackets are unitary operators, and the multiplication of three unitary operators is itself an unitary operator, meaning that for appropriate choice of α, β, γ and δ we may write any unitary as

$$U = e^{i\alpha}R_{\hat{n}}(\beta)R_{\hat{m}}(\gamma)R_{\hat{n}}(\delta).$$

4.12

First let us determine the values of α, β, γ and δ in the Z-Y decomposition through

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix}$$
$$\implies \alpha = \frac{\pi}{2}, \beta = 0, \gamma = \frac{\pi}{2}, \delta = \pi.$$

Therefore the operators are given by

$$A = R_z(0)R_y(\pi/4) = \begin{bmatrix} \cos\frac{\pi}{8} & -\sin\frac{\pi}{8} \\ \sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix},$$
$$B = R_y(-\pi/4)R_z(-\pi/2) = \begin{bmatrix} e^{i\pi/4}\cos\frac{\pi}{8} & e^{-i\pi/4}\sin\frac{\pi}{8} \\ -e^{i\pi/4}\sin\frac{\pi}{8} & e^{-i\pi/4}\cos\frac{\pi}{8} \end{bmatrix},$$

$$C = R_z(\pi/2) = \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}.$$

4.13

$$\begin{aligned} HXH &= \frac{(X+Z)}{\sqrt{2}} X \frac{(X+Z)}{\sqrt{2}} \\ &= \frac{1}{2}(XXX + XXZ + ZXX + ZXZ) \\ &= \frac{1}{2}(X + Z + Z - X) \\ &= Z. \end{aligned}$$

$$\begin{aligned} HYH &= \frac{(X+Z)}{\sqrt{2}} Y \frac{(X+Z)}{\sqrt{2}} \\ &= \frac{1}{2}(XYX + XYZ + ZYX + ZYZ) \\ &= \frac{1}{2}(-Y + iI - iI - Y) \\ &= -Y. \end{aligned}$$

$$\begin{aligned} HZH &= \frac{(X+Z)}{\sqrt{2}} Z \frac{(X+Z)}{\sqrt{2}} \\ &= \frac{1}{2}(XZX + XZZ + ZZX + ZZZ) \\ &= \frac{1}{2}(-Z + X + X + Z) \\ &= X. \end{aligned}$$

4.14

$$\begin{aligned} T &= \exp(i\pi/8) R_z(\pi/4) \\ &= \exp(i\pi/8)\left(\cos\frac{\pi}{8} I - i \sin\frac{\pi}{8} Z\right), \end{aligned}$$

thus

$$\begin{aligned} HTH &= \exp(i\pi/8)\left(\cos\frac{\pi}{8} I - i \sin\frac{\pi}{8} HZH\right) \\ &= \exp(i\pi/8)\left(\cos\frac{\pi}{8} I - i \sin\frac{\pi}{8} X\right) \\ &= \exp(i\pi/8) R_x(\pi/4). \end{aligned}$$

4.15

$$\begin{aligned}
R_{\hat{n}_2}(\beta_2)R_{\hat{n}_1}(\beta_1) &= \exp\left(-i\frac{\beta_2}{2}\hat{n}_2\cdot\vec{\sigma}\right)\exp\left(-i\frac{\beta_1}{2}\hat{n}_1\cdot\vec{\sigma}\right) \\
&= \left(c_2 I - i s_2 \hat{n}_2\cdot\vec{\sigma}\right)\left(c_1 I - i s_1 \hat{n}_1\cdot\vec{\sigma}\right) \\
&= c_1c_2 I - i\left(c_2 s_1\hat{n}_1 + c_1 s_2 \hat{n}_2\right)\cdot\vec{\sigma} - s_1 s_2\left(\hat{n}_1\cdot\vec{\sigma}\right)\left(\hat{n}_2\cdot\vec{\sigma}\right).
\end{aligned}$$

The last term can be expanded to

$$\begin{aligned}
\left(\hat{n}_1\cdot\vec{\sigma}\right)\left(\hat{n}_2\cdot\vec{\sigma}\right) &= \left(n_{1x}X + n_{1y}Y + n_{1z}Z\right)\left(n_{2x}X + n_{2y}Y + n_{2z}Z\right) \\
&= n_{1x}n_{2x}X^2 + n_{1y}n_{2y}Y^2 + n_{1y}n_{2y}Y^2 \\
&\quad + n_{1x}n_{2y}XY + n_{2x}n_{1y}YX + n_{1x}n_{2z}XZ + n_{2x}n_{1z}ZX + n_{1y}n_{2z}YZ + n_{2y}n_{1z}ZY \\
&= \left(n_{1x}n_{2x} + n_{1y}n_{2y} + n_{1z}n_{2z}\right) I \\
&\quad + \left(n_{1x}n_{2y} - n_{2x}n_{1y}\right) iZ - \left(n_{1x}n_{2z} - n_{2x}n_{1z}\right) iY + \left(n_{1y}n_{2z} - n_{2y}n_{1z}\right) iX \\
&= \left(\hat{n}_1\cdot\hat{n}_2\right) I + i\left(\hat{n}_1\times\hat{n}_2\right)\cdot\vec{\sigma}.
\end{aligned}$$

Substituting it back into the equality yields

$$\begin{aligned}
R_{\hat{n}_2}(\beta_2)R_{\hat{n}_1}(\beta_1) &= \left(c_1c_2 - s_1s_2\,\hat{n}_1\cdot\hat{n}_2\right) I - i\left(c_2s_1\hat{n}_1 + c_1s_2\hat{n}_2 - s_1s_2\,\hat{n}_2\times\hat{n}_1\right)\cdot\vec{\sigma} \\
&= c_{12} I - i s_{12}\hat{n}_{12}\cdot\vec{\sigma} \\
&= R_{\hat{n}_{12}}(\beta_{12}).
\end{aligned}$$

For $\beta_1 = \beta_2$ and $\hat{n}_1 = \hat{z}$ we have

$$c_{12} = c^2 - s^2\,\hat{z}\cdot\hat{n}_2,$$

$$\begin{aligned}
s_{12}\hat{n}_{12} &= cs\hat{z} + cs\hat{n}_2 - s^2\,\hat{n}_2\times\hat{z} \\
&= sc\left(\hat{z} + \hat{n}_2\right) - s^2\,\hat{n}_2\times\hat{z}.
\end{aligned}$$

4.16

The operators shown in the circuits can be written, respectively, as

$$I\otimes H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

4.17

Since $HZH = X$ we have that a CNOT gate is equivalent to a controlled-(HZH) gate, that is

$$C(HZH) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = CX$$

4.18

Taking two generic states $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$ and $|\phi\rangle = b_0 |0\rangle + b_1 |1\rangle$ we have the joint state

$$|\psi\rangle \otimes |\phi\rangle = a_0 b_0 |0\rangle \otimes |0\rangle + a_0 b_1 |0\rangle \otimes |1\rangle + a_1 b_0 |1\rangle \otimes |0\rangle + a_1 b_1 |1\rangle \otimes |1\rangle .$$

Applying $CZ_{(1,2)}$ and $CZ_{(2,1)}$ over the joint state, where the first index is the control qubit and the second one the target, yields

$$\begin{aligned} CZ_{(1,2)} |\psi\rangle \otimes |\phi\rangle &= a_0 b_0 |0\rangle \otimes |0\rangle + a_0 b_1 |0\rangle \otimes |1\rangle + a_1 b_0 |1\rangle \otimes |0\rangle - a_1 b_1 |1\rangle \otimes |1\rangle , \\ CZ_{(2,1)} |\psi\rangle \otimes |\phi\rangle &= a_0 b_0 |0\rangle \otimes |0\rangle + a_0 b_1 |0\rangle \otimes |1\rangle + a_1 b_0 |1\rangle \otimes |0\rangle - a_1 b_1 |1\rangle \otimes |1\rangle . \end{aligned}$$

4.19

Considering a generic joint state $|\psi\rangle \otimes |\phi\rangle = (a_0 |0\rangle + a_1 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle)$, the density operator and the action of CNOT upon it are given, respectively, by

$$\begin{aligned} \rho = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| &= \begin{bmatrix} |a_0|^2 & a_0 a_1^* \\ a_0^* a_1 & |a_1|^2 \end{bmatrix} \otimes \begin{bmatrix} |b_0|^2 & b_0 b_1^* \\ b_0^* b_1 & |b_1|^2 \end{bmatrix} \\ &= \begin{bmatrix} |a_0|^2 |b_0|^2 & |a_0|^2 b_0 b_1^* & a_0 a_1^* |b_0|^2 & a_0 a_1^* b_0 b_1^* \\ |a_0|^2 b_0^* b_1 & |a_0|^2 |b_1|^2 & a_0 a_1^* b_0^* b_1 & a_0 a_1^* |b_1|^2 \\ a_0^* a_1 |b_0|^2 & a_0^* a_1 b_0 b_1^* & |a_1|^2 |b_0|^2 & |a_1|^2 b_0 b_1^* \\ a_0^* a_1 b_0^* b_1 & a_0^* a_1 |b_1|^2 & |a_1|^2 b_0^* b_1 & |a_1|^2 |b_1|^2 \end{bmatrix}, \end{aligned}$$

$$(CX)\,\rho\,(CX)^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} |a_0|^2 |b_0|^2 & |a_0|^2 b_0 b_1^* & a_0 a_1^* |b_0|^2 & a_0 a_1^* b_0 b_1^* \\ |a_0|^2 b_0^* b_1 & |a_0|^2 |b_1|^2 & a_0 a_1^* b_0^* b_1 & a_0 a_1^* |b_1|^2 \\ a_0^* a_1 |b_0|^2 & a_0^* a_1 b_0 b_1^* & |a_1|^2 |b_0|^2 & |a_1|^2 b_0 b_1^* \\ a_0^* a_1 b_0^* b_1 & a_0^* a_1 |b_1|^2 & |a_1|^2 b_0^* b_1 & |a_1|^2 |b_1|^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} |a_0|^2|b_0|^2 & |a_0|^2 b_0 b_1^* & a_0 a_1^* b_0 b_1^* & a_0 a_1^* |b_0|^2 \\ |a_0|^2 b_0^* b_1 & |a_0|^2 |b_1|^2 & a_0 a_1^* |b_1|^2 & a_0 a_1^* b_0^* b_1 \\ a_0^* a_1 b_0^* b_1 & a_0^* a_1 |b_1|^2 & |a_1|^2 |b_1|^2 & |a_1|^2 b_0^* b_1 \\ a_0^* a_1 |b_0|^2 & a_0^* a_1 b_0 b_1^* & |a_1|^2 b_0 b_1^* & |a_1|^2 |b_0|^2 \end{bmatrix},$$

which is a permutation between the third and fourth columns, and third and fourth rows.

4.20

The circuit in the image can be written as

$$\begin{aligned}(H\otimes H)\, CX_{(1,2)}\, (H\otimes H) &= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = CX_{(2,1)}.\end{aligned}$$

$$\begin{aligned}
CX_{(1,2)}\,|+\rangle\,|+\rangle &= (H\otimes H)\,(H\otimes H)\,CX_{(1,2)}\,(H\otimes H)\,|0\rangle\,|0\rangle \\
&= (H\otimes H)\,CX_{(2,1)}\,|0\rangle\,|0\rangle \\
&= (H\otimes H)\,|0\rangle\,|0\rangle \\
&= |+\rangle\,|+\rangle\,, \\
CX_{(1,2)}\,|-\rangle\,|+\rangle &= (H\otimes H)\,(H\otimes H)\,CX_{(1,2)}\,(H\otimes H)\,|1\rangle\,|0\rangle \\
&= (H\otimes H)\,CX_{(2,1)}\,|1\rangle\,|0\rangle \\
&= (H\otimes H)\,|1\rangle\,|0\rangle \\
&= |-\rangle\,|+\rangle\,, \\
CX_{(1,2)}\,|+\rangle\,|-\rangle &= (H\otimes H)\,(H\otimes H)\,CX_{(1,2)}\,(H\otimes H)\,|0\rangle\,|1\rangle \\
&= (H\otimes H)\,CX_{(2,1)}\,|0\rangle\,|1\rangle \\
&= (H\otimes H)\,|1\rangle\,|1\rangle \\
&= |-\rangle\,|-\rangle\,, \\
CX_{(1,2)}\,|-\rangle\,|-\rangle &= (H\otimes H)\,(H\otimes H)\,CX_{(1,2)}\,(H\otimes H)\,|1\rangle\,|1\rangle \\
&= (H\otimes H)\,CX_{(2,1)}\,|1\rangle\,|1\rangle \\
&= (H\otimes H)\,|0\rangle\,|1\rangle \\
&= |+\rangle\,|-\rangle\,.
\end{aligned}$$

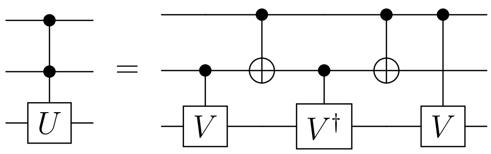
4.21

Considering the initial state as a (normalized) superposition of all four possibilities for the control qubits c_1 and c_2 with the target qubit t, given by $|\Psi\rangle = (|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)|\psi\rangle$, we have

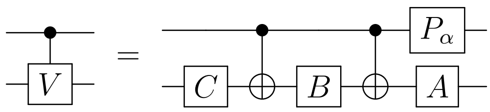
$$
\begin{aligned}
|\Psi\rangle &\xrightarrow{CV_{(c_2,t)}} |0\rangle|0\rangle|\psi\rangle + |0\rangle|1\rangle\Big(V|\psi\rangle\Big) + |1\rangle|0\rangle|\psi\rangle + |1\rangle|1\rangle\Big(V|\psi\rangle\Big) \\
&\xrightarrow{CX_{(c_1,c_2)}} |0\rangle|0\rangle|\psi\rangle + |0\rangle|1\rangle\Big(V|\psi\rangle\Big) + |1\rangle|1\rangle|\psi\rangle + |1\rangle|0\rangle\Big(V|\psi\rangle\Big) \\
&\xrightarrow{CV^\dagger_{(c_2,t)}} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle\Big)|\psi\rangle + |1\rangle|1\rangle\Big(V^\dagger|\psi\rangle\Big) + |1\rangle|0\rangle\Big(V|\psi\rangle\Big) \\
&\xrightarrow{CX_{(c_2,c_1)}} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle\Big)|\psi\rangle + |1\rangle|0\rangle\Big(V^\dagger|\psi\rangle\Big) + |1\rangle|1\rangle\Big(V|\psi\rangle\Big) \\
&\xrightarrow{CV_{(c_1,t)}} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle\Big)|\psi\rangle + |1\rangle|1\rangle\Big(V^2|\psi\rangle\Big) \\
&= \Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle\Big)|\psi\rangle + |1\rangle|1\rangle\Big(U|\psi\rangle\Big) = C^2(U)|\Psi\rangle\,.
\end{aligned}
$$

4.22

We have the following relations:



,

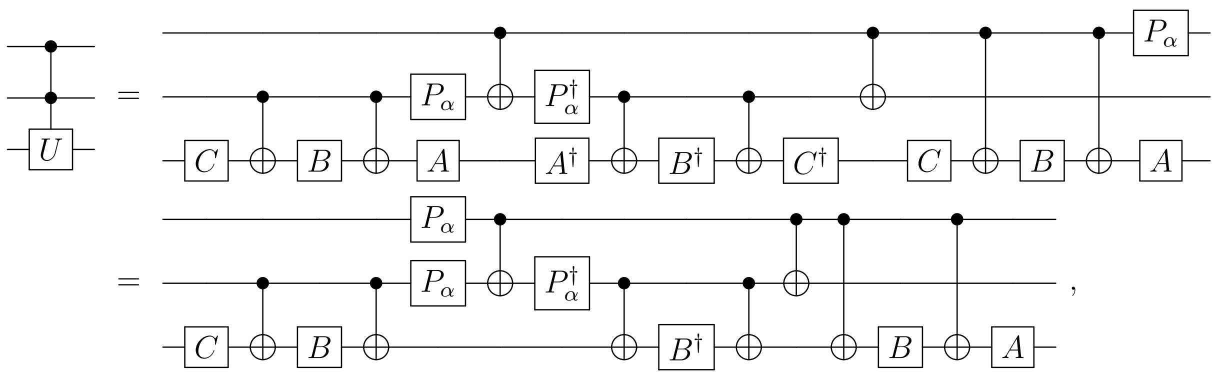


,

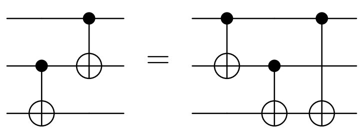
where P_α denotes the phase gate given by

$$
P_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}.
$$

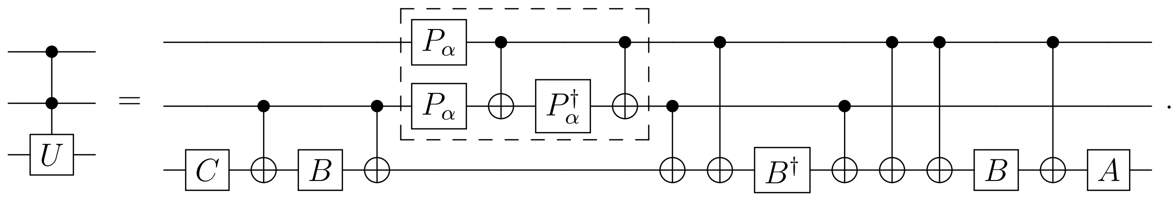
Using the second relation we can decompose $C^2(U)$ as



and using the fact that



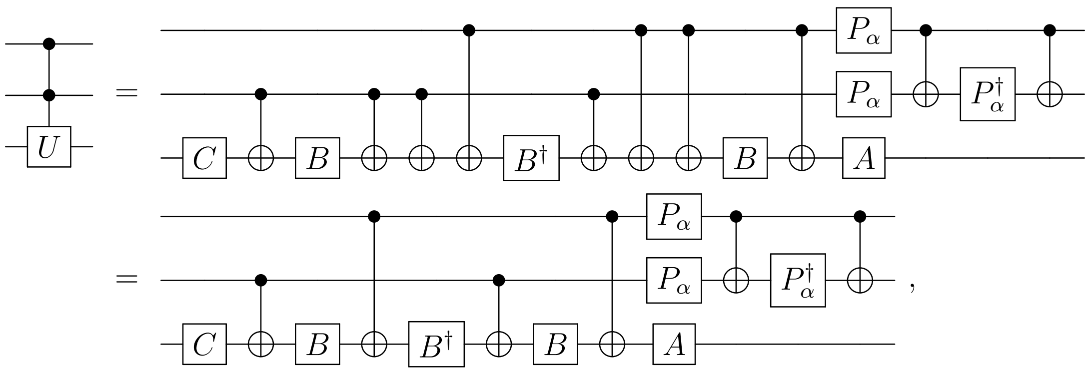
we can write the sixth CNOT gate before the fourth and fifth ones as



Notice that the operator inside the box, given by

$$CX_{(c_1,c_2)}\left(I\otimes P_\alpha^\dagger\right)CX_{(c_1,c_2)}\left(P_\alpha\otimes P_\alpha\right)=\begin{bmatrix}I & 0\\ 0 & X\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & P_\alpha^\dagger\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & X\end{bmatrix}\begin{bmatrix}P_\alpha & 0\\ 0 & e^{i\alpha}P_\alpha\end{bmatrix}$$
$$=\begin{bmatrix}P_\alpha & 0\\ 0 & e^{i\alpha}XP_\alpha^\dagger XP_\alpha\end{bmatrix},$$

is diagonal and can, therefore, be written in any part of the circuit since it involves only the two control qubits, thus

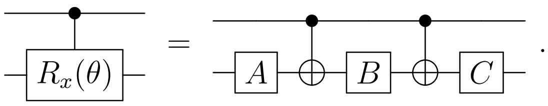


4.23

For the $R_x(\theta)$ gate we have

$$R_x(\theta)=\begin{bmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix}=\begin{bmatrix}e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2}\\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2}\end{bmatrix}$$
$$\implies \alpha=0,\beta=-\frac{\pi}{2},\gamma=\theta,\delta=\frac{\pi}{2},$$

meaning that with the operators $A=R_z(-\pi/2)R_y(\theta/2), B=R_y(-\theta/2)$ and $C=R_z(\pi/2)$ we can write $R_x(\theta)=AXBXC$, and thus create the circuit

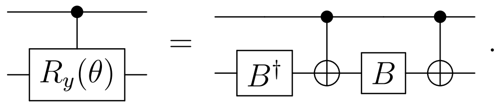


It is impossible to reduce the number of single qubit gates to two for general θ since that would require the possibility of writing $R_x(\theta)=e^{i\omega}XD^\dagger XD$ for some operator D and phase ω, which is equivalent to $D^\dagger XD=R_x(\theta+\pi)$ up to a global phase that would be corrected by ω. But it is a fact that $(D^\dagger XD)^2=I$, and that implies $(R_x(\theta+\pi))^2=R_x(2\theta)=I$, which is only true for $\theta=0$ or π.

On the other hand, for the $R_y(\theta)$ gate we can write

$$\begin{aligned} R_y(\theta) &= R_y(\theta/2)R_y(\theta/2) \\ &= \left(\cos\frac{\theta}{4}I - i\sin\frac{\theta}{4}Y\right)R_y(\theta/2) \\ &= \left(\cos\frac{\theta}{4}I + i\sin\frac{\theta}{4}XYX\right)R_y(\theta/2) \\ &= X\left(\cos\frac{\theta}{4}I + i\sin\frac{\theta}{4}Y\right)XR_y(\theta/2) \\ &= XR_y(-\theta/2)XR_y(\theta/2), \end{aligned}$$

meaning that, using the operator B defined above, we can create the circuit



4.24

Considering the initial state as a (normalized) superposition of all four possibilities for the control qubits c_1 and c_2 with the target qubit t, given by $|\Psi\rangle = (|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)|\psi\rangle$, we have

$$\begin{aligned} |\Psi\rangle &\xrightarrow{I\otimes I\otimes H} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\Big)H|\psi\rangle \\ &\xrightarrow{CX_{(c_2,t)}} |0\rangle|0\rangle\Big(H|\psi\rangle\Big) + |0\rangle|1\rangle\Big(XH|\psi\rangle\Big) + |1\rangle|0\rangle\Big(H|\psi\rangle\Big) + |1\rangle|1\rangle\Big(XH|\psi\rangle\Big) \\ &\xrightarrow{I\otimes I\otimes T^\dagger} |0\rangle|0\rangle\Big(T^\dagger H|\psi\rangle\Big) + |0\rangle|1\rangle\Big(T^\dagger XH|\psi\rangle\Big) + |1\rangle|0\rangle\Big(T^\dagger H|\psi\rangle\Big) + |1\rangle|1\rangle\Big(T^\dagger XH|\psi\rangle\Big) \\ &\xrightarrow{CX_{(c_1,t)}} |0\rangle|0\rangle\Big(T^\dagger H|\psi\rangle\Big) + |0\rangle|1\rangle\Big(T^\dagger XH|\psi\rangle\Big) + |1\rangle|0\rangle\Big(XT^\dagger H|\psi\rangle\Big) \\ &\qquad + |1\rangle|1\rangle\Big(XT^\dagger XH|\psi\rangle\Big) \\ &\xrightarrow{I\otimes I\otimes T} |0\rangle|0\rangle\Big(H|\psi\rangle\Big) + |0\rangle|1\rangle\Big(XH|\psi\rangle\Big) + |1\rangle|0\rangle\Big(TXT^\dagger H|\psi\rangle\Big) \\ &\qquad + |1\rangle|1\rangle\Big(TXT^\dagger XH|\psi\rangle\Big) \\ &\xrightarrow{CX_{(c_2,t)}} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle\Big)H|\psi\rangle + |1\rangle|0\rangle\Big(TXT^\dagger H|\psi\rangle\Big) + |1\rangle|1\rangle\Big(XTXT^\dagger XH|\psi\rangle\Big) \\ &\xrightarrow{I\otimes I\otimes T^\dagger} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle\Big)T^\dagger H|\psi\rangle + |1\rangle|0\rangle\Big(XT^\dagger H|\psi\rangle\Big) + |1\rangle|1\rangle\Big(T^\dagger XTXT^\dagger XH|\psi\rangle\Big) \\ &\xrightarrow{CX_{(c_1,t)}} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle\Big)T^\dagger H|\psi\rangle + |1\rangle|1\rangle\Big(XT^\dagger XTXT^\dagger XH|\psi\rangle\Big) \\ &\xrightarrow{I\otimes T^\dagger\otimes T} \Big(|0\rangle|0\rangle + e^{-i\pi/4}|0\rangle|1\rangle + |1\rangle|0\rangle\Big)H|\psi\rangle + e^{-i\pi/4}|1\rangle|1\rangle\Big(\big(TXT^\dagger X\big)^2 H|\psi\rangle\Big) \\ &\xrightarrow{CX_{(c_1,c_2)}\otimes H} \Big(|0\rangle|0\rangle + e^{-i\pi/4}|0\rangle|1\rangle + |1\rangle|1\rangle\Big)|\psi\rangle + e^{-i\pi/4}|1\rangle|0\rangle\Big(H\big(TXT^\dagger X\big)^2 H|\psi\rangle\Big) \\ &\xrightarrow{I\otimes T^\dagger\otimes I} \Big(|0\rangle|0\rangle + e^{-i\pi/2}|0\rangle|1\rangle + e^{-i\pi/4}|1\rangle|1\rangle\Big)|\psi\rangle + e^{-i\pi/4}|1\rangle|0\rangle\Big(H\big(TXT^\dagger X\big)^2 H|\psi\rangle\Big) \\ &\xrightarrow{CX_{(c_1,c_2)}} \Big(|0\rangle|0\rangle + e^{-i\pi/2}|0\rangle|1\rangle + e^{-i\pi/4}|1\rangle|0\rangle\Big)|\psi\rangle + e^{-i\pi/4}|1\rangle|1\rangle\Big(H\big(TXT^\dagger X\big)^2 H|\psi\rangle\Big) \\ &\xrightarrow{T\otimes S\otimes I} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle\Big)|\psi\rangle + i|1\rangle|1\rangle\Big(H\big(TXT^\dagger X\big)^2 H|\psi\rangle\Big). \end{aligned}$$

But expanding the operator acting on the last term yields

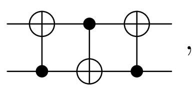
$$
\begin{aligned}
TXT^\dagger X &= R_z(\pi/4) X R_z(-\pi/4) X \\
&= R_z(\pi/2) \\
\implies \quad iH\left(TXT^\dagger X\right)^2 H &= iHR_z(\pi)H \\
&= iR_x(\pi) \\
&= X,
\end{aligned}
$$

thus the action of the circuit is effectively

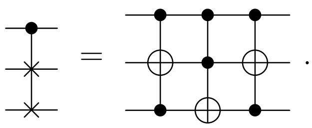
$$
|\Psi\rangle \longrightarrow \Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle \Big)|\psi\rangle + |1\rangle|1\rangle \Big(X|\psi\rangle \Big) = CCX\,|\Psi\rangle\,.
$$

4.25

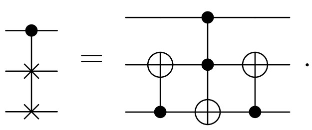
It is a fact that the SWAP gate can be built with CNOT gates as



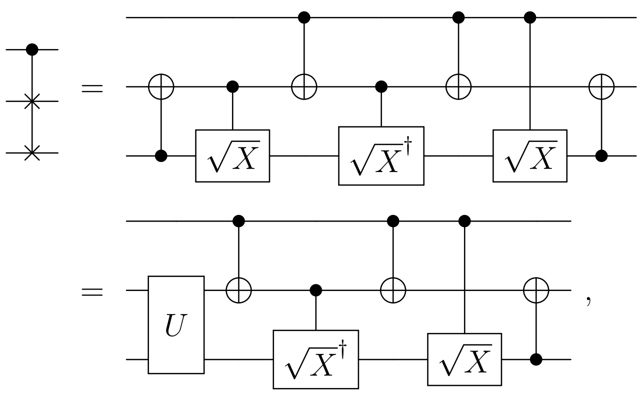
thus a controlled-SWAP gate can be written as



We have that $\text{CNOT}^2 = I \otimes I$, so we can just eliminate the second control from the first and third Toffoli gates because, for the $|0\rangle$ component of the controlled-SWAP's control bit, the action of the circuit will be only the first and third CNOTs, which will cancel each other, and for the $|1\rangle$ component, the SWAP operation will be performed naturally. So

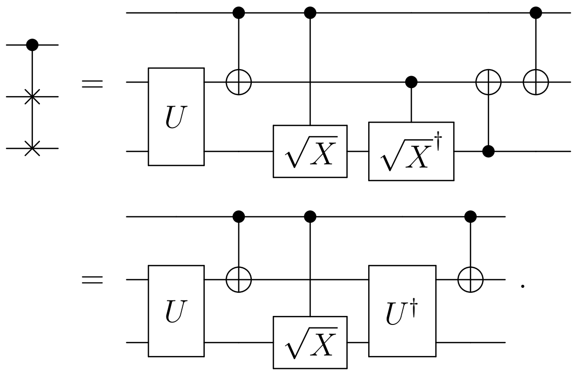


Decomposing the Toffoli gate using only two-qubit gates yields



where we have defined the two-qubit gate $U \coloneqq C\sqrt{X}_{(t_1,t_2)}CX_{(t_2,t_1)}$.

The $\sqrt{X}$ gate commutes with the two previous gates and the last two CNOT gates also commutes, meaning we can write



4.26

Considering the initial state as a (normalized) superposition of all four possibilities for the control qubits c_1 and c_2 with the target qubit t, given by $|\Psi\rangle = (|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)|\psi\rangle$, we have

$$
\begin{aligned}
|\Psi\rangle &\xrightarrow{I\otimes I\otimes R_y(\theta)} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\Big)R_y(\theta)|\psi\rangle \\
&\xrightarrow{CX_{(c_2,t)}} |0\rangle|0\rangle\Big(R_y(\theta)|\psi\rangle\Big) + |0\rangle|1\rangle\Big(XR_y(\theta)|\psi\rangle\Big) + |1\rangle|0\rangle\Big(R_y(\theta)|\psi\rangle\Big) \\
&\qquad + |1\rangle|1\rangle\Big(XR_y(\theta)|\psi\rangle\Big) \\
&\xrightarrow{I\otimes I\otimes R_y(\theta)} |0\rangle|0\rangle\Big(R_y(2\theta)|\psi\rangle\Big) + |0\rangle|1\rangle\Big(R_y(\theta)XR_y(\theta)|\psi\rangle\Big) + |1\rangle|0\rangle\Big(R_y(2\theta)|\psi\rangle\Big) \\
&\qquad + |1\rangle|1\rangle\Big(R_y(\theta)XR_y(\theta)|\psi\rangle\Big) \\
&\xrightarrow{CX_{(c_1,t)}} |0\rangle|0\rangle\Big(R_y(2\theta)|\psi\rangle\Big) + |0\rangle|1\rangle\Big(R_y(\theta)XR_y(\theta)|\psi\rangle\Big) + |1\rangle|0\rangle\Big(XR_y(2\theta)|\psi\rangle\Big) \\
&\qquad + |1\rangle|1\rangle\Big((XR_y(\theta))^2|\psi\rangle\Big) \\
&\xrightarrow{I\otimes I\otimes R_y(-\theta)} |0\rangle|0\rangle\Big(R_y(\theta)|\psi\rangle\Big) + |0\rangle|1\rangle\Big(XR_y(\theta)|\psi\rangle\Big) + |1\rangle|0\rangle\Big(R_y(-\theta)XR_y(2\theta)|\psi\rangle\Big) \\
&\qquad + |1\rangle|1\rangle\Big(R_y(-\theta)\,(XR_y(\theta))^2|\psi\rangle\Big) \\
&\xrightarrow{CX_{(c_2,t)}} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle\Big)R_y(\theta)|\psi\rangle + |1\rangle|0\rangle\Big(R_y(-\theta)XR_y(2\theta)|\psi\rangle\Big) \\
&\qquad + |1\rangle|1\rangle\Big(XR_y(-\theta)\,(XR_y(\theta))^2|\psi\rangle\Big) \\
&\xrightarrow{I\otimes I\otimes R_y(-\theta)} \Big(|0\rangle|0\rangle + |0\rangle|1\rangle\Big)|\psi\rangle + |1\rangle|0\rangle\Big(R_y(-2\theta)XR_y(2\theta)|\psi\rangle\Big) \\
&\qquad + |1\rangle|1\rangle\Big(R_y(-\theta)XR_y(-\theta)\,(XR_y(\theta))^2|\psi\rangle\Big).
\end{aligned}
$$

Setting $\theta = \pi/4$ we have, for the remaining operators:

$$
\begin{aligned}
R_y(-\pi/2)XR_y(\pi/2) &= XXR_y(-\pi/2)XR_y(\pi/2) \\
&= XR_y(\pi) \\
&= Z,
\end{aligned}
$$

$$
\begin{aligned}
R_y(-\pi/4)XR_y(-\pi/4)\left(XR_y(\pi/4)\right)^2 &= \Big(R_y(-\pi/4)XR_y(-\pi/4)\Big)X\Big(R_y(\pi/4)XR_y(\pi/4)\Big) \\
&= \Big(XR_y(0)\Big)X\Big(XR_y(0)\Big) \\
&= X.
\end{aligned}
$$

Thus the action of the circuit is effectively

$$
|\Psi\rangle \longrightarrow \Big(\,|0\rangle\,|0\rangle + |0\rangle\,|1\rangle\,\Big)\,|\psi\rangle + |1\rangle\,|0\rangle\,\Big(Z\,|\psi\rangle\Big) + |1\rangle\,|1\rangle\,\Big(X\,|\psi\rangle\Big),
$$

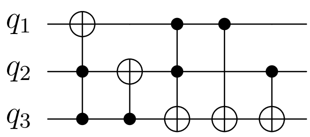
that is, the Toffoli gate with the addition of a relative phase π to $|\Psi\rangle$'s $|1\rangle\,|0\rangle\,|1\rangle$ component.

4.27

The transformation consists of the following permutations:

$$
\begin{aligned}
|0\rangle\,|0\rangle\,|0\rangle &\longrightarrow |0\rangle\,|0\rangle\,|0\rangle\,, \\
|0\rangle\,|0\rangle\,|1\rangle &\longrightarrow |0\rangle\,|1\rangle\,|0\rangle\,, \\
|0\rangle\,|1\rangle\,|0\rangle &\longrightarrow |0\rangle\,|1\rangle\,|1\rangle\,, \\
|0\rangle\,|1\rangle\,|1\rangle &\longrightarrow |1\rangle\,|0\rangle\,|0\rangle\,, \\
|1\rangle\,|0\rangle\,|0\rangle &\longrightarrow |1\rangle\,|0\rangle\,|1\rangle\,, \\
|1\rangle\,|0\rangle\,|1\rangle &\longrightarrow |1\rangle\,|1\rangle\,|0\rangle\,, \\
|1\rangle\,|1\rangle\,|0\rangle &\longrightarrow |1\rangle\,|1\rangle\,|1\rangle\,, \\
|1\rangle\,|1\rangle\,|1\rangle &\longrightarrow |0\rangle\,|0\rangle\,|1\rangle\,.
\end{aligned}
$$

If we treat each component of the three-qubit state as a binary number, the action is effectively $+1$ mod 8 ($+2$ for $|1\rangle\,|1\rangle\,|1\rangle$), with the exception of the $|0\rangle\,|0\rangle\,|0\rangle$ component, which is left invariant. Notice that the first qubit, q_1, flips when both the second and third ones are set to $|1\rangle$, the second qubit, q_2, flips when the third one is set to $|1\rangle$ and the third qubit, q_3, always flips except when the first and second ones both end up as $|0\rangle$. So we can construct the circuit



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4.28

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4.29

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4.30

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4.31

Considering a generic state $|\psi\rangle = (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$ we have:

$$\begin{aligned}
CX_1C|\psi\rangle &= CX_1\Big(a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle + a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle\Big) \\
&= C\Big(a_0b_0|1\rangle|0\rangle + a_0b_1|1\rangle|1\rangle + a_1b_0|0\rangle|1\rangle + a_1b_1|0\rangle|0\rangle\Big) \\
&= a_0b_0|1\rangle|1\rangle + a_0b_1|1\rangle|0\rangle + a_1b_0|0\rangle|1\rangle + a_1b_1|0\rangle|0\rangle \\
&= X_1X_2|\psi\rangle,
\end{aligned}$$

$$\begin{aligned}
CY_1C|\psi\rangle &= CY_1\Big(a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle + a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle\Big) \\
&= iC\Big(a_0b_0|1\rangle|0\rangle + a_0b_1|1\rangle|1\rangle - a_1b_0|0\rangle|1\rangle - a_1b_1|0\rangle|0\rangle\Big) \\
&= i\Big(a_0b_0|1\rangle|1\rangle + a_0b_1|1\rangle|0\rangle - a_1b_0|0\rangle|1\rangle - a_1b_1|0\rangle|0\rangle\Big) \\
&= Y_1X_2|\psi\rangle,
\end{aligned}$$

$$\begin{aligned}
CZ_1C|\psi\rangle &= CZ_1\Big(a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle + a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle\Big) \\
&= C\Big(a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle - a_1b_0|1\rangle|1\rangle - a_1b_1|1\rangle|0\rangle\Big) \\
&= a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle - a_1b_0|1\rangle|0\rangle - a_1b_1|1\rangle|1\rangle \\
&= Z_1|\psi\rangle,
\end{aligned}$$

$$\begin{aligned}
CX_2C|\psi\rangle &= CX_2\Big(a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle + a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle\Big) \\
&= C\Big(a_0b_0|0\rangle|1\rangle + a_0b_1|0\rangle|0\rangle + a_1b_0|1\rangle|0\rangle + a_1b_1|1\rangle|1\rangle\Big) \\
&= a_0b_0|0\rangle|1\rangle + a_0b_1|0\rangle|0\rangle + a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle \\
&= X_2|\psi\rangle,
\end{aligned}$$

$$\begin{aligned}
CY_2C|\psi\rangle &= CY_2\Big(a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle + a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle\Big) \\
&= iC\Big(a_0b_0|0\rangle|1\rangle - a_0b_1|0\rangle|0\rangle - a_1b_0|1\rangle|0\rangle + a_1b_1|1\rangle|1\rangle\Big) \\
&= i\Big(a_0b_0|0\rangle|1\rangle - a_0b_1|0\rangle|0\rangle - a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle\Big) \\
&= Z_1Y_2|\psi\rangle,
\end{aligned}$$

$$\begin{aligned}
CZ_2C|\psi\rangle &= CZ_2\Big(a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle + a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle\Big) \\
&= C\Big(a_0b_0|0\rangle|0\rangle - a_0b_1|0\rangle|1\rangle - a_1b_0|1\rangle|1\rangle + a_1b_1|1\rangle|0\rangle\Big) \\
&= a_0b_0|0\rangle|0\rangle - a_0b_1|0\rangle|1\rangle - a_1b_0|1\rangle|0\rangle + a_1b_1|1\rangle|1\rangle \\
&= Z_1Z_2|\psi\rangle,
\end{aligned}$$

$$
\begin{aligned}
R_{z,1}(\theta)C\,|\psi\rangle &= R_{z,1}(\theta)\Big(a_0b_0\,|0\rangle\,|0\rangle + a_0b_1\,|0\rangle\,|1\rangle + a_1b_0\,|1\rangle\,|1\rangle + a_1b_1\,|1\rangle\,|0\rangle\Big) \\
&= e^{-i\theta/2}a_0b_0\,|0\rangle\,|0\rangle + e^{-i\theta/2}a_0b_1\,|0\rangle\,|1\rangle + e^{i\theta/2}a_1b_0\,|1\rangle\,|1\rangle + e^{i\theta/2}a_1b_1\,|1\rangle\,|0\rangle \\
&= CR_{z,1}(\theta)\,|\psi\rangle\,,
\end{aligned}
$$

$$
\begin{aligned}
R_{x,2}(\theta)C\,|\psi\rangle &= R_{x,2}(\theta)\Big(a_0b_0\,|0\rangle\,|0\rangle + a_0b_1\,|0\rangle\,|1\rangle + a_1b_0\,|1\rangle\,|1\rangle + a_1b_1\,|1\rangle\,|0\rangle\Big) \\
&= a_0b_0\,|0\rangle\left(\cos\frac{\theta}{2}\,|0\rangle - i\sin\frac{\theta}{2}\,|1\rangle\right) + a_0b_1\,|0\rangle\left(\cos\frac{\theta}{2}\,|1\rangle - i\sin\frac{\theta}{2}\,|0\rangle\right) \\
&\quad + a_1b_0\,|1\rangle\left(\cos\frac{\theta}{2}\,|1\rangle - i\sin\frac{\theta}{2}\,|0\rangle\right) + a_1b_1\,|1\rangle\left(\cos\frac{\theta}{2}\,|0\rangle - i\sin\frac{\theta}{2}\,|1\rangle\right) \\
&= a_0\left(b_0\cos\frac{\theta}{2} - ib_1\sin\frac{\theta}{2}\right)|0\rangle\,|0\rangle + a_0\left(b_1\cos\frac{\theta}{2} - ib_0\sin\frac{\theta}{2}\right)|0\rangle\,|1\rangle \\
&\quad + a_1\left(b_0\cos\frac{\theta}{2} - ib_1\sin\frac{\theta}{2}\right)|1\rangle\,|1\rangle + a_1\left(b_1\cos\frac{\theta}{2} - ib_0\sin\frac{\theta}{2}\right)|1\rangle\,|0\rangle \\
&= CR_{x,2}(\theta)\,|\psi\rangle\,.
\end{aligned}
$$

4.32

Let us consider the first qubit initially in state $\rho_1 = |\psi\rangle\!\langle\psi|$ and the second one initially in $\rho_2 = |\phi\rangle\!\langle\phi|$. After a projective measurement is performed on the second qubit, its state becomes

$$
\begin{aligned}
\rho_2' &= \sum_{i=0}^{1} p(i)\,|i\rangle\!\langle i| \\
&= \sum_{i=0}^{1} \mathrm{tr}(P_i\,|\phi\rangle\!\langle\phi|\,P_i)\frac{P_i\,|\phi\rangle\!\langle\phi|\,P_i}{\mathrm{tr}(P_i\,|\phi\rangle\!\langle\phi|\,P_i)} \\
&= \sum_{i=0}^{1} P_i\,|\phi\rangle\!\langle\phi|\,P_i,
\end{aligned}
$$

while for the first qubit we have $\rho_1' = \rho_1$. Thus the post-measurement density operator of the entire system is given by

$$
\begin{aligned}
\rho' = \rho_1' \otimes \rho_2' &= \sum_{i=0}^{1} |\psi\rangle\!\langle\psi| \otimes P_i\,|\phi\rangle\!\langle\phi|\,P_i \\
&= \sum_{i=0}^{1} (I \otimes P_i)\,\rho\,(I \otimes P_i) \\
&= P_0\rho P_0 + P_1\rho P_1,
\end{aligned}
$$

where the projectors are to be understood as acting on the second qubit only.

$$
\begin{aligned}
\mathrm{tr}_2(\rho') &= \sum_{i=0}^{1} |\psi\rangle\!\langle\psi|\,\mathrm{tr}_2\left(P_i\,|\phi\rangle\!\langle\phi|\,P_i\right) \\
&= \sum_{i=0}^{1}\sum_{j=0}^{1} |\psi\rangle\!\langle\psi|\,\langle j\,|\,i\rangle\,\langle i\,|\,\phi\rangle\,\langle\phi\,|\,i\rangle\,\langle i\,|\,j\rangle
\end{aligned}
$$

$$= \sum_{j=0}^{1} |\psi\rangle\langle\psi| \langle j \,|\, \phi\rangle \langle \phi \,|\, j\rangle = \mathrm{tr}_2(\rho).$$

4.33

Considering a generic two-qubit state written in the Bell basis $|\psi\rangle = a_+ |\Phi^+\rangle + b_+ |\Psi^+\rangle + a_- |\Phi^-\rangle + b_- |\Psi^-\rangle$, the circuit performs

$$\begin{aligned}
|\psi\rangle &\xrightarrow{CX_{(1,2)}} a_+ \left(\frac{|0\rangle|0\rangle + |1\rangle|0\rangle}{\sqrt{2}}\right) + b_+ \left(\frac{|0\rangle|1\rangle + |1\rangle|1\rangle}{\sqrt{2}}\right) + a_- \left(\frac{|0\rangle|0\rangle - |1\rangle|0\rangle}{\sqrt{2}}\right) \\
&\qquad + b_- \left(\frac{|0\rangle|1\rangle - |1\rangle|1\rangle}{\sqrt{2}}\right) \\
&\xrightarrow{H\otimes I} a_+ |0\rangle|0\rangle + b_+ |0\rangle|1\rangle + a_- |1\rangle|0\rangle + b_- |1\rangle|1\rangle ,
\end{aligned}$$

and after the measurements in the computational basis we obtain

$$\begin{aligned}
&|0\rangle|0\rangle \quad \text{with probability } |a_+|^2, \\
&|0\rangle|1\rangle \quad \text{with probability } |b_+|^2, \\
&|1\rangle|0\rangle \quad \text{with probability } |a_-|^2, \\
&|1\rangle|1\rangle \quad \text{with probability } |b_-|^2.
\end{aligned}$$

These are the same probabilities of obtaining $|\Phi^+\rangle$, $|\Psi^+\rangle$, $|\Phi^-\rangle$ and $|\Psi^-\rangle$ respectively if we had performed the measurement directly using the Bell projectors. So effectively, the circuit is a measurement in the Bell basis, meaning that obtaining $|0\rangle|0\rangle$ is equivalent to obtaining $|\Phi^+\rangle$, $|0\rangle|1\rangle$ is equivalent to $|\Psi^+\rangle$, $|1\rangle|0\rangle$ is equivalent to $|\Phi^-\rangle$ and $|1\rangle|1\rangle$ is equivalent to $|\Psi^-\rangle$.

4.34

Let us consider that $U|\psi_+\rangle = |\psi_+\rangle$ and $U|\psi_-\rangle = -|\psi_-\rangle$ are the two eigenvalue relations for operator U. Now, if we consider a generic two-qubit state given by $|\Psi\rangle = |0\rangle|\psi_{\text{in}}\rangle = |0\rangle(a|\psi_+\rangle + b|\psi_-\rangle)$ the circuit performs

$$\begin{aligned}
|\Psi\rangle &\xrightarrow{H\otimes I} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)(a|\psi_+\rangle + b|\psi_-\rangle) \\
&\xrightarrow{CU_{(1,2)}} |0\rangle \left(\frac{a|\psi_+\rangle + b|\psi_-\rangle}{\sqrt{2}}\right) + |1\rangle \left(\frac{a|\psi_+\rangle - b|\psi_-\rangle}{\sqrt{2}}\right) \\
&\xrightarrow{H\otimes I} a|0\rangle|\psi_+\rangle + b|1\rangle|\psi_-\rangle .
\end{aligned}$$

After measuring the first qubit we have

$$\begin{aligned}
&|0\rangle \quad \text{with probability } |a|^2, \\
&|1\rangle \quad \text{with probability } |b|^2,
\end{aligned}$$

and because this is an entangled state, if we obtain $|0\rangle$ then the second qubit is in state $|\psi_+\rangle$, meaning the result for observable U is $+1$, and if we obtain $|1\rangle$ then the second qubit is in state $|\psi_-\rangle$, meaning the result for observable U is -1. Thus this circuit implements a measurement of U.

4.35

Considering the initial state as a generic superposition, given by $|\Psi\rangle = (a|0\rangle + b|1\rangle)|\psi\rangle$, we have, for the left-hand side

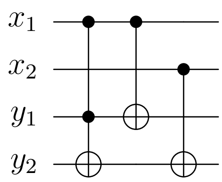
$$
\begin{aligned}
|\Psi\rangle &\xrightarrow{CU_{(c,t)}} a|0\rangle|\psi\rangle + b|1\rangle\Big(U|\psi\rangle\Big) \\
&\xrightarrow{\text{meas.}} \begin{cases} |0\rangle|\psi\rangle & \text{with probability } |a|^2 \\ |1\rangle\big(U|\psi\rangle\big) & \text{with probability } |b|^2 \end{cases},
\end{aligned}
$$

and for the right-hand side

$$
\begin{aligned}
|\Psi\rangle &\xrightarrow{\text{meas.}} \begin{cases} |0\rangle|\psi\rangle & \text{with probability } |a|^2 \\ |1\rangle|\psi\rangle & \text{with probability } |b|^2 \end{cases} \\
&\xrightarrow{CU_{(c,t)}} \begin{cases} |0\rangle|\psi\rangle & \text{with probability } |a|^2 \\ |1\rangle\big(U|\psi\rangle\big) & \text{with probability } |b|^2 \end{cases}.
\end{aligned}
$$

4.36

Index 1 indicates the least significant bit and 2 indicates the most significant bit:



4.37

Labeling the transform as T we have

$$
\begin{aligned}
U_1 T &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \sqrt{2} & \frac{1+i}{\sqrt{2}} & 0 & \frac{1-i}{\sqrt{2}} \\ 0 & \frac{1-i}{\sqrt{2}} & \sqrt{2} & \frac{1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix},
\end{aligned}
$$

$$
U_2 U_1 T = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} \sqrt{2} & \frac{1+i}{\sqrt{2}} & 0 & \frac{1-i}{\sqrt{2}} \\ 0 & \frac{1-i}{\sqrt{2}} & \sqrt{2} & \frac{1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}
$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ 0 & \frac{1-i}{\sqrt{2}} & \sqrt{2} & \frac{1+i}{\sqrt{2}} \\ 0 & \frac{3+i}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{3-i}{\sqrt{6}} \\ 1 & -i & -1 & i \end{bmatrix},$$

$$U_3U_2U_1T = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{3} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ 0 & \frac{1-i}{\sqrt{2}} & \sqrt{2} & \frac{1+i}{\sqrt{2}} \\ 0 & \frac{3+i}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{3-i}{\sqrt{6}} \\ 1 & -i & -1 & i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{1-i}{\sqrt{2}} & \sqrt{2} & \frac{1+i}{\sqrt{2}} \\ 0 & \frac{3+i}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{3-i}{\sqrt{6}} \\ 0 & \frac{2i}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{2i}{\sqrt{3}} \end{bmatrix},$$

$$U_4U_3U_2U_1T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}(1+i)}{4} & \frac{3-i}{4} & 0 \\ 0 & \frac{3+i}{4} & -\frac{\sqrt{3}(1-i)}{4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{1-i}{\sqrt{2}} & \sqrt{2} & \frac{1+i}{\sqrt{2}} \\ 0 & \frac{3+i}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{3-i}{\sqrt{6}} \\ 0 & \frac{2i}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{2i}{\sqrt{3}} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\sqrt{\frac{2}{3}} & i\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & \sqrt{2} & i\sqrt{2} \\ 0 & \frac{2i}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{2i}{\sqrt{3}} \end{bmatrix},$$

$$U_5U_4U_3U_2U_1T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & -\frac{i}{\sqrt{3}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{i}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\sqrt{\frac{2}{3}} & i\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & \sqrt{2} & i\sqrt{2} \\ 0 & \frac{2i}{\sqrt{3}} & \frac{2}{\sqrt{3}} & -\frac{2i}{\sqrt{3}} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{2} & i\sqrt{2} \\ 0 & 0 & -\sqrt{2} & i\sqrt{2} \end{bmatrix},$$

$$U_6U_5U_4U_3U_2U_1T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{2} & i\sqrt{2} \\ 0 & 0 & -\sqrt{2} & i\sqrt{2} \end{bmatrix}$$

$$= I.$$

4.38

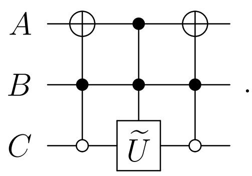
Any $d \times d$ unitary matrix whose first column does not contain a zero will require at least $d - 1$ two-level unitary matrices. The reason is that one two-level unitary matrix is required in order to turn one entry of the first column to zero, and there are $d-1$ entries in the first column that needs to be eliminated, meaning at least $d-1$ two-level unitary matrices are required for the decomposition.

4.39

The transformation acts on states $|0\rangle|1\rangle|0\rangle$ and $|1\rangle|1\rangle|1\rangle$, and a Gray code connecting them is

$$\begin{array}{ccc} A & B & C \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}.$$

So a circuit that implements the transformation is



4.40

$$\begin{aligned} E(R_{\hat{n}}(\alpha), R_{\hat{n}}(\alpha+\beta)) &= \max_{|\psi\rangle} \left\| \left(R_{\hat{n}}(\alpha) - R_{\hat{n}}(\alpha+\beta)\right)|\psi\rangle \right\| \\ &= \max_{|\psi\rangle} \left\| R_{\hat{n}}(\alpha)\left(I - R_{\hat{n}}(\beta)\right)|\psi\rangle \right\| \\ &= \max_{|\psi\rangle} \sqrt{\left\langle \psi \middle| \left(I - R_{\hat{n}}^\dagger(\beta)\right) R_{\hat{n}}^\dagger(\alpha) R_{\hat{n}}(\alpha)\left(I - R_{\hat{n}}(\beta)\right) \middle| \psi \right\rangle} \\ &= \max_{|\psi\rangle} \sqrt{\left\langle \psi \middle| \left(2I - R_{\hat{n}}^\dagger(\beta) - R_{\hat{n}}(\beta)\right) \middle| \psi \right\rangle} \\ &= \max_{|\psi\rangle} \sqrt{2 - \left\langle \psi \middle| \left(e^{i\frac{\beta}{2}\hat{n}\cdot\vec{\sigma}} + e^{-i\frac{\beta}{2}\hat{n}\cdot\vec{\sigma}}\right) \middle| \psi \right\rangle}. \end{aligned}$$

The operator in the last terms can be rewritten as

$$\begin{aligned} e^{i\frac{\beta}{2}\hat{n}\cdot\vec{\sigma}} + e^{-i\frac{\beta}{2}\hat{n}\cdot\vec{\sigma}} &= \cos\frac{\beta}{2} I - i\sin\frac{\beta}{2}\hat{n}\cdot\vec{\sigma} + \cos\frac{\beta}{2} I + i\sin\frac{\beta}{2}\hat{n}\cdot\vec{\sigma} \\ &= 2\cos\frac{\beta}{2} I, \end{aligned}$$

and the result therefore does not depend on $|\psi\rangle$. Thus

$$\begin{aligned} E(R_{\hat{n}}(\alpha), R_{\hat{n}}(\alpha+\beta)) &= \sqrt{2 - 2\cos\frac{\beta}{2}} \\ &= \sqrt{2 - e^{i\beta/2} - e^{-i\beta/2}} \\ &= \sqrt{\left(1 - e^{i\beta/2}\right)\left(1 - e^{-i\beta/2}\right)} \\ &= |1 - \exp(i\beta/2)|. \end{aligned}$$

Given a tolerance $\epsilon > 0$ we can always find β such that $3|1 - \exp(i\beta/2)| < \epsilon$, and once an appropriate β is calculated we can always approximate $R_{\hat{n}}(\theta)^n \approx R_{\hat{n}}(\alpha + \beta)$ with arbitrary accuracy for some integer n, justifying Equation (4.76).

4.41

The action of the circuit is

$$\begin{aligned}
|0\rangle|0\rangle|\psi\rangle &\xrightarrow{H\otimes H\otimes I} \frac{1}{2}\Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\Big)|\psi\rangle \\
&\xrightarrow{CCX} \frac{1}{2}\Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle\Big)|\psi\rangle + \frac{1}{2}|1\rangle|1\rangle\Big(X|\psi\rangle\Big) \\
&\xrightarrow{I\otimes I\otimes S} \frac{1}{2}\Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle\Big)S|\psi\rangle + \frac{1}{2}|1\rangle|1\rangle\Big(SX|\psi\rangle\Big) \\
&\xrightarrow{CCX} \frac{1}{2}\Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle\Big)S|\psi\rangle + \frac{1}{2}|1\rangle|1\rangle\Big(XSX|\psi\rangle\Big) \\
&\xrightarrow{H\otimes H\otimes I} \frac{1}{4}\Big(|0\rangle|0\rangle\big(3S + XSX\big)|\psi\rangle + |0\rangle|1\rangle\big(S - XSX\big)|\psi\rangle + |1\rangle|0\rangle\big(S - XSX\big)|\psi\rangle \\
&\qquad + |1\rangle|1\rangle\big(-S + XSX\big)|\psi\rangle\Big),
\end{aligned}$$

meaning that, after the measurement, if we obtain $|0\rangle|0\rangle$ then

$$\begin{aligned}
3S + XSX &= 3\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} + \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \\
&= \sqrt{10}\begin{bmatrix}\frac{3+i}{\sqrt{10}} & 0\\ 0 & \frac{1+3i}{\sqrt{10}}\end{bmatrix}
\end{aligned}$$

is applied to the third qubit. We can rewrite this operator as

$$\begin{aligned}
3S + XSX &= \sqrt{10}e^{i\phi}\begin{bmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{bmatrix} \\
&\implies \phi = \frac{1}{4}, \theta = \arccos\frac{3}{5} \\
&\implies \sqrt{10}e^{i\pi/4}R_z(\theta)
\end{aligned}$$

that is, $R_z(\theta)$ is applied to the third qubit up to a global phase. Meanwhile, if we obtain any other result then the applied operator is

$$\begin{aligned}
\pm(S - XSX) &= \pm\left(\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} - \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right) \\
&= \pm\sqrt{2}\begin{bmatrix}\frac{1-i}{\sqrt{2}} & 0\\ 0 & -\frac{1-i}{\sqrt{2}}\end{bmatrix} \\
&= \pm\sqrt{2}e^{-i\pi/4}Z,
\end{aligned}$$

that is, Z up to a global phase. The probability of obtaining $|0\rangle|0\rangle$ is

$$\left|\frac{\sqrt{10}e^{i\pi/4}}{4}\right|^2 \langle 0 \,|\, 0\rangle \langle 0 \,|\, 0\rangle \langle \psi \,|\, R_z^\dagger(\theta) R_z(\theta) \,|\, \psi\rangle = \frac{5}{8}.$$

After the measurement is performed the result can be $|0\rangle|0\rangle (R_z(\theta)|\psi\rangle)$, $|0\rangle|1\rangle (Z|\psi\rangle)$, $|1\rangle|0\rangle (Z|\psi\rangle)$ or $|1\rangle|1\rangle (Z|\psi\rangle)$, so in order to guarantee that $R_z(\theta)$ is applied to the third qubit we must simply apply Z to the third qubit and X (remember that $X = HZH$) to any ancilla qubits that end up in state $|1\rangle$, and apply the same circuit again until we get $|0\rangle|0\rangle$ as result. The probability of not obtaining $|0\rangle|0\rangle$ after n repetitions of the circuit is $(3/8)^n$, which approaches zero for large n.

4.42

If θ is a rational multiple of 2π then it can be written as

$$\theta = \frac{a}{m} \times 2\pi,$$

where $a \in \mathbb{Z}$ and m is some positive integer. Thus

$$\begin{aligned} e^{i\theta m} = e^{ia\times 2\pi} = 1 \quad &\Longrightarrow \quad \frac{(3+4i)^m}{5^m} = 1 \\ &\Longrightarrow \quad (3+4i)^m = 5^m. \end{aligned}$$

We have that $(3+4i)^2 = -7+24i \equiv 3+4i \pmod 5$. By induction, it follows that $(3+4i)^m \equiv 3+4i \pmod 5$ for all integer $m > 0$, meaning there is no m that satisfies $(3+4i)^m = 5^m$, therefore θ must be an irrational multiple of 2π.

4.43

We can write any unitary as

$$U = R_z(\beta) R_x(\gamma) R_z(\delta),$$

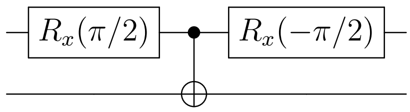
and analogously to the proof that Hadamard, phase, CNOT and $\pi/8$ form an universal set of gates, we have that it is possible to approximate any rotation $R_z(\alpha)$ with arbitrary precision as $R_z(\theta)^n$ for some integer n. Besides, notice that the circuit in Figure 4.17 can be used to perform $R_x(\theta)$ if we apply a Hadamard gate to the third qubit at the beggining and end of the circuit. So using only Hadamard, phase, CNOT and Toffoli gates we can perform any unitary

$$U = R_z(\theta)^{n_1} R_x(\theta)^{n_2} R_z(\theta)^{n_3}$$

for integers n_1, n_2 and n_3, meaning they form a set of universal gates.

4.44

We can approximate any rotation $R_x(\theta)$ with arbitrary precision as $R_x(\pi\alpha)^n$ for some integer n and irrational α. So, with gate G and ancilla qubits set to $|1\rangle$, we can generate $R_x(\theta)$, CNOT and Toffoli gates up to global phases. So from an initial two-qubit state $|0\rangle\,|0\rangle$ we may apply the circuit



,

with arbitrary precision, obtaining the transformation

$$
\begin{aligned}
|0\rangle\,|0\rangle &\xrightarrow{R_x\otimes I} \frac{|0\rangle\,|0\rangle - i\,|1\rangle\,|0\rangle}{\sqrt{2}} \\
&\xrightarrow{CX} \frac{|0\rangle\,|0\rangle - i\,|1\rangle\,|1\rangle}{\sqrt{2}} \\
&\xrightarrow{R_x^\dagger\otimes I} \frac{|0\rangle\,|0\rangle + i\,|1\rangle\,|0\rangle + |0\rangle\,|1\rangle - i\,|1\rangle\,|1\rangle}{2} \\
&\phantom{\xrightarrow{R_x^\dagger\otimes I}} = \frac{1}{\sqrt{2}}\Big(\,|0\rangle\,|+\rangle + i\,|1\rangle\,|-\rangle\,\Big).
\end{aligned}
$$

If we measure the first qubit, the second one is left in either the state $|+\rangle$ or $|-\rangle$, meaning we can indirectly perform Hadamard gates, and since $HR_x(\theta)H = R_z(\theta)$, we can perform any rotations $R_z(\theta)$. Thus we are able to reproduce any one-qubit gate as well as CNOT and Toffoli gates with arbitrary precision, so G is an universal gate.

4.45

H, S, CNOT and Toffoli gates, in matricial representation, all consist of matrices with complex integer entries. Any unitary U involving n qubits is represented by an $SU(2^n)$ matrix, meaning it is represented by a $2^n \times 2^n$ matrix, and since it is constructed with a tensor product of the four gates, all of its entries are also complex integers. And lastly, all four gates only contains 1, -1 and i, the exception is the Hadamard gate, which is multiplied by $1/\sqrt{2} = 2^{-1/2}$. Thus if k Hadamard gates are necessary to create U, then it will have the form $2^{-k/2}M$, where M is a $2^n \times 2^n$ matrix with complex integer entries. And since it is possible to build the Toffoli gate using H, CNOT and the $\pi/8$ gate, this result should hold if we replace the Toffoli gate by the $\pi/8$ gate.

4.46

For one qubit, we need two real numbers to describe its amplitudes, that is

$$
|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle\,.
$$

For n qubits we naturally need 2^n real numbers, so the $2^n \times 2^n$ density matrix will have $2^{2n} = 4^n$ independent real numbers. Lastly, we need to consider the normalization condition, that is, $\mathrm{tr}(\rho) = 1$, and this will account for $4^n - 1$ independent real numbers.

4.47

Let us start with the simplest case of $L = 2$, then

$$\begin{aligned} e^{-iHt} &= e^{-i(H_1+H_2)t} \\ &= \sum_{j=0}^{\infty} \frac{(-i)^j (H_1 + H_2)^j t^j}{j!}. \end{aligned}$$

If $[H_1, H_2] = 0$, then we have

$$(H_1 + H_2)^j = \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} H_1^k H_2^{j-k}.$$

Substituting this result yields

$$\begin{aligned} e^{-iHt} &= \sum_{j=0}^{\infty} (-i)^j t^j \sum_{k=0}^{j} \frac{H_1^k}{k!} \frac{H_2^{j-k}}{(j-k)!} \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-i)^k H_1^k t^k}{k!} \frac{(-i)^{j-k} H_2^{j-k} t^{j-k}}{(j-k)!}. \end{aligned}$$

Since j runs from 0 to ∞, k also varies from 0 to ∞, as well as the difference $j - k \equiv l$, so we may rewrite this result as

$$\begin{aligned} e^{-iHt} &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-i)^k H_1^k t^k}{k!} \frac{(-i)^l H_2^l t^l}{l!} \\ &= e^{-iH_1 t} e^{-iH_2 t}. \end{aligned}$$

If we make the replacement $H_1 + H_2 \to \sum_k^L H_k$ such that $[H_j, H_k]$ $\forall$ j and k, this result should still hold by induction. Thus

$$e^{-iHt} = e^{-iH_1 t} e^{-iH_2 t} \cdots e^{-iH_L t}.$$

4.48

If all H_k involves at most c of the total n qubits, then there is roughly $n!/(n-c)! \leq n^c$ different possible H_k, meaning L is $O(n^c)$, that is, a polynomial in n.

4.49

$$\begin{aligned} e^{(A+B)\Delta t} &= I + (A+B)\Delta t + \frac{(A+B)^2}{2!}\Delta t^2 + O\big(\Delta t^3\big) \\ &= \left[I + A\Delta t + \frac{A^2}{2!}\Delta t^2\right]\left[I + B\Delta t + \frac{B^2}{2!}\Delta t^2\right]\left[I - \frac{[A,B]}{2}\Delta t^2\right] + O\left(\Delta t^3\right) \\ &= \left[e^{A\Delta t} + O\big(\Delta t^3\big)\right]\left[e^{B\Delta t} + O\big(\Delta t^3\big)\right]\left[e^{-\frac{1}{2}[A,B]\Delta t^2} + O\big(\Delta t^4\big)\right] + O\big(\Delta t^3\big) \\ &= e^{A\Delta t} e^{B\Delta t} e^{-\frac{1}{2}[A,B]\Delta t^2} + O\big(\Delta t^3\big), \end{aligned}$$

this is, Equation (4.105). Now, for Equation (4.103) we have

$$\begin{aligned}
e^{i(A+B)\Delta t} &= I + i(A+B)\Delta t + O\big(\Delta t^2\big) \\
&= [I + iA\Delta t]\,[I + iB\Delta t] + O\big(\Delta t^2\big) \\
&= \big[e^{iA\Delta t} + O\big(\Delta t^2\big)\big]\,\big[e^{iB\Delta t} + O\big(\Delta t^2\big)\big] + O(\Delta t^2) \\
&= e^{iA\Delta t}e^{iB\Delta t} + O\big(\Delta t^2\big).
\end{aligned}$$

And for Equation (4.104) we have

$$\begin{aligned}
e^{i(A+B)\Delta t} &= I + i(A+B)\Delta t + \frac{i^2(A+B)^2}{2!}\Delta t^2 + O\big(\Delta t^3\big) \\
&= \left[I + \frac{iA}{2}\Delta t + \frac{\frac{i^2A^2}{4}}{2!}\Delta t^2\right]\left[I + iB\Delta t + \frac{i^2B^2}{2!}\Delta t^2\right]\left[I + \frac{iA}{2}\Delta t + \frac{\frac{i^2A^2}{4}}{2!}\Delta t^2\right] + O\big(\Delta t^3\big) \\
&= \big[e^{iA\Delta t/2} + O\big(\Delta t^3\big)\big]\,\big[e^{iB\Delta t} + O\big(\Delta t^3\big)\big]\,\big[e^{iA\Delta t/2} + O\big(\Delta t^3\big)\big] + O\big(\Delta t^3\big) \\
&= e^{iA\Delta t/2}e^{iB\Delta t}e^{iA\Delta t/2} + O\big(\Delta t^3\big).
\end{aligned}$$

4.50

Multiplying both terms inside the square brackets yields

$$U_{\Delta t} = e^{-iH_1\Delta t}\cdots e^{-iH_{L-1}\Delta t}e^{-2iH_L\Delta t}e^{-iH_{L-1}\Delta t}\cdots e^{-iH_1\Delta t}.$$

Now we use the fact that

$$e^{-iH_{L-1}\Delta t}e^{-2iH_L\Delta t}e^{-iH_{L-1}\Delta t} = e^{-2i(H_L+H_{L-1})} + O\big(\Delta t^3\big)$$

$$\implies \quad U_{\Delta t} = e^{-iH_1\Delta t}\cdots e^{-iH_{L-2}\Delta t}\left[e^{-2i(H_L+H_{L-1})} + O\big(\Delta t^3\big)\right]e^{-iH_{L-2}\Delta t}\cdots e^{-iH_1\Delta t}.$$

If we apply this relation until we reach H_1, we get the result

$$\begin{aligned}
U_{\Delta t} &= e^{-2i\sum_k^L H_k\Delta t} + O\big(\Delta t^3\big) \\
&= e^{-2iH\Delta t} + O\big(\Delta t^3\big).
\end{aligned}$$

The error has an upper limit given by

$$E\big(U_{\Delta t}^m, e^{-2miH\Delta t}\big) \le mE\big(U_{\Delta t}, e^{-2iH\Delta t}\big).$$

From the result obtained above we know that $E\big(U_{\Delta t}, e^{-2iH\Delta t}\big)$ is $O(\Delta t^3)$, that is, there is some constant α such that $E\big(U_{\Delta t}, e^{-2iH\Delta t}\big) = \alpha\Delta t^3$, meaning that

$$E\big(U_{\Delta t}^m, e^{-2miH\Delta t}\big) \le m\alpha\Delta t^3.$$

4.51

Using the fact that $X = HZH$ and $Y = S^\dagger X S$, the Hamiltonian can be rewritten as

$$
\begin{aligned}
H &= H_1 Z_1 H_1 \otimes S_2^\dagger H_2 Z_2 H_2 S_2 \otimes Z_3 \\
&= \Big(H_1 \otimes S_2^\dagger H_2 \otimes I_3\Big)\Big(Z_1 \otimes Z_2 \otimes Z_3\Big)\Big(H_1 \otimes H_2 S_2 \otimes I_3\Big).
\end{aligned}
$$

That is, if we apply the H, $S^\dagger$ and S gates to the first and second qubits at the beginning and end of the circuit, the effective Hamiltonian that must be implemented is the $Z_1 \otimes Z_2 \otimes Z_3$. Thus, a possible circuit to implement the Hamiltonian H is