8 Quantum noise and quantum operations

Exercises: 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11, 8.12, 8.13, 8.14, 8.15, 8.16, 8.17, 8.18, 8.19, 8.20, 8.21, 8.22, 8.23, 8.24, 8.25, 8.26, 8.27, 8.28, 8.29, 8.30, 8.31, 8.32, 8.33, 8.34, 8.35.

8.1

A unitary operator acting on a pure state results in a pure state, that is, $|\psi\rangle \to U|\psi\rangle \equiv |\phi\rangle$. Therefore, if initially we have $\rho = |\psi\rangle\langle\psi|$, then after the process we have

$$\mathcal{E}(\rho) = |\phi\rangle\langle\phi| = \big(U|\psi\rangle\big)\big(\langle\psi|U^\dagger\big) = U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger.$$

8.2

For an initial state $|\psi^j\rangle$, the state immediately after measurement, associated with the measurement outcome m, is given by

$$|\psi_m\rangle = \frac{M_m|\psi^j\rangle}{\sqrt{p(m)}}.$$

Considering density operators, initially we may have any general state $\rho = \sum_j p_j |\psi^j\rangle\langle\psi^j|$. So the final state is obtained considering all possible pure initial states $|\psi^j\rangle$, which occur with probability p_j, that could result in measurement outcome m. It is given by

$$\sum_j p_j \frac{M_m|\psi^j\rangle\langle\psi^j|M_m^\dagger}{p(m)} = \frac{M_m\left(\sum_j p_j|\psi^j\rangle\langle\psi^j|\right)M_m^\dagger}{p(m)} = \frac{M_m\rho M_m^\dagger}{p(m)} = \frac{\mathcal{E}_m(\rho)}{p(m)}.$$

The probability of obtaining the measurement result m is $p(m) = \sum_j p_j \langle\psi^j|M_m M_m^\dagger|\psi^j\rangle$. If we consider a basis $\{|\phi_k\rangle\}$ we may write

$$\begin{aligned}
p(m) &= \sum_j\sum_k p_j \langle\psi^j\,|\,M_m^\dagger\,|\,\phi_k\rangle\langle\phi_k\,|\,M_m\,|\,\psi^j\rangle \\
&= \sum_k\sum_j p_j \langle\phi_k\,|\,M_m\,|\,\psi^j\rangle\langle\psi^j\,|\,M_m^\dagger\,|\,\phi_k\rangle \\
&= \sum_k \langle\phi_k\,|\,\mathcal{E}_m(\rho)\,|\,\phi_k\rangle \\
&= \mathrm{tr}(\mathcal{E}_m(\rho)),
\end{aligned}$$

and the final state is $\mathcal{E}_m(\rho)/\mathrm{tr}(\mathcal{E}_m(\rho))$.

8.3

The resulting state can be written as

$$\mathcal{E}(\rho) = \mathrm{tr}_{AD}\Big(U\Big[\rho\otimes|0\rangle\langle 0|\Big]U^\dagger\Big),$$

where $\rho \in AB$, $|0\rangle\langle 0| \in CD$ and U acts on states living in $ABCD$. Let $\{|\phi_k\rangle\}$ be a basis for subspace AD. Then we can write

$$\begin{aligned}
\mathcal{E}(\rho) &= \sum_k \left\langle \phi_k \middle| U\Big[\rho \otimes |0\rangle\langle 0|\Big]U^\dagger \middle| \phi_k \right\rangle \\
&= \sum_k \langle \phi_k | U | 0 \rangle \, \rho \left\langle 0 \middle| U^\dagger \middle| \phi_k \right\rangle .
\end{aligned}$$

Here we can identify the linear operators $E_k \equiv \langle \phi_k \,|\, U \,|\, 0\rangle$ such that the map $\mathcal{E}(\rho)$ can be written as

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger.$$

These operators are such that

$$\sum_k E_k^\dagger E_k = \sum_k \left\langle 0 \middle| U^\dagger \middle| \phi_k \right\rangle \langle \phi_k \,|\, U \,|\, 0 \rangle .$$

Since the $|\phi_k\rangle$ form a complete basis for subspace AD we have that $\sum_k |\phi_k\rangle\langle\phi_k| = I$ and therefore $\sum_k E_k^\dagger E_k = I$.

8.4

$$\begin{aligned}
E_0 &= \langle 0 \,|\, (P_0 \otimes I + P_1 \otimes X) \,|\, 0 \rangle \\
&= P_0 \langle 0 \,|\, 0 \rangle + P_1 \langle 0 \,|\, 1 \rangle \\
&= P_0,
\end{aligned}$$

$$\begin{aligned}
E_1 &= \langle 1 \,|\, (P_0 \otimes I + P_1 \otimes X) \,|\, 0 \rangle \\
&= P_0 \langle 1 \,|\, 0 \rangle + P_1 \langle 1 \,|\, 1 \rangle \\
&= P_1.
\end{aligned}$$

Therefore

$$\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1.$$

8.5

$$\begin{aligned}
E_0 &= \left\langle 0 \middle| \left(\frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X \right) \middle| 0 \right\rangle \\
&= \frac{X}{\sqrt{2}} \langle 0 \,|\, 0 \rangle + \frac{Y}{\sqrt{2}} \langle 0 \,|\, 1 \rangle \\
&= \frac{X}{\sqrt{2}},
\end{aligned}$$

$$
\begin{aligned}
E_1 &= \left\langle 1 \middle| \left(\frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X \right) \middle| 0 \right\rangle \\
&= \frac{X}{\sqrt{2}} \langle 1 \,|\, 0 \rangle + \frac{Y}{\sqrt{2}} \langle 1 \,|\, 1 \rangle \\
&= \frac{Y}{\sqrt{2}}.
\end{aligned}
$$

Therefore

$$
\mathcal{E}(\rho) = \frac{1}{2} \left(X \rho X + Y \rho Y \right).
$$

8.6

For $\mathcal{E}$ and $\mathcal{F}$ acting on the same space, we consider a principal system A with Hilbert space $\mathscr{H}$ and an environment system B with Hilbert space $\mathscr{H}_{\text{env}}$. Then, for $\rho \in \mathscr{H}$, $\rho_{\text{env}} = \sum_j p_j \left| e_j \middle\rangle\middle\langle e_j \right| \in \mathscr{H}_{\text{env}}$, and unitaries $U_{\mathcal{E}}$ and $U_{\mathcal{F}}$ acting on states of $\mathscr{H} \otimes \mathscr{H}_{\text{env}}$, $\mathcal{E}$ and F can be written as

$$
\begin{aligned}
\mathcal{E}(\rho) &= \operatorname{tr}_B \Big(U_{\mathcal{E}} \rho \otimes \rho_{\text{env}} U_{\mathcal{E}}^{\dagger} \Big), \\
\mathcal{F}(\rho) &= \operatorname{tr}_B \Big(U_{\mathcal{F}} \rho \otimes \rho_{\text{env}} U_{\mathcal{F}}^{\dagger} \Big).
\end{aligned}
$$

In the operation-sum representation they are given by

$$
\begin{aligned}
\mathcal{E}(\rho) &= \sum_k E_k \rho E_k^{\dagger}, \\
\mathcal{F}(\rho) &= \sum_k F_k \rho F_k^{\dagger},
\end{aligned}
$$

where $E_k \equiv \sum_j \sqrt{p_j} \langle e_k \,|\, U_{\mathcal{E}} \,|\, e_j \rangle$ and $F_k \equiv \sum_j \sqrt{p_j} \langle e_k \,|\, U_{\mathcal{F}} \,|\, e_j \rangle$. A composition $\mathcal{F} \circ \mathcal{E}(\rho) \equiv \mathcal{F}(\mathcal{E}(\rho))$ of the operations will be given by

$$
\begin{aligned}
\mathcal{F}(\mathcal{E}(\rho)) &= \sum_k F_k \mathcal{E}(\rho) F_k^{\dagger} \\
&= \sum_k \sum_l F_k E_l \rho E_l^{\dagger} F_k^{\dagger} \\
&= \sum_{k,l} F_k E_l \rho \left(F_k E_l \right)^{\dagger}.
\end{aligned}
$$

Here we can identify linear operators $G_{kl} \equiv F_k E_l$ such that the composition can be written as

$$
\mathcal{F}(\mathcal{E}(\rho)) = \sum_{k,l} G_{kl} \rho G_{kl}^{\dagger},
$$

meaning the composition is also a quantum operation.

For $\mathcal{E}$ and $\mathcal{F}$ acting on different spaces we must consider a system $A^{\mathcal{E}} \otimes B^{\mathcal{E}}$ with Hilbert space $\mathscr{H}^{\mathcal{E}} \otimes \mathscr{H}_{\text{env}}^{\mathcal{E}}$ for $\mathcal{E}$ and another system $A^{\mathcal{F}} \otimes B^{\mathcal{F}}$ with Hilbert space $\mathscr{H}^{\mathcal{F}} \otimes \mathscr{H}_{\text{env}}^{\mathcal{F}}$ for $\mathcal{F}$. The operations have the same form as before with the difference that now each part belongs to its own space. We can naturally construct the spaces $\mathscr{H} \equiv \mathscr{H}^{\mathcal{E}} \otimes \mathscr{H}^{\mathcal{F}}$ and $\mathscr{H}_{\text{env}} \equiv \mathscr{H}_{\text{env}}^{\mathcal{E}} \otimes \mathscr{H}_{\text{env}}^{\mathcal{F}}$ and consider the unitary $U \equiv U_{\mathcal{E}} \otimes U_{\mathcal{F}}$ acting on states of $\mathscr{H} \otimes \mathscr{H}_{\text{env}}$. If we define the quantum operation given by

$$\mathcal{G}(\rho) = \mathrm{tr}_{B^{\mathcal{E}} \otimes B^{\mathcal{F}}}\big(U\rho \otimes \rho_{\text{env}} U^\dagger\big),$$

for $\rho \in \mathscr{H}$ and $\rho_{\text{env}} = \rho_{\text{env}}^{\mathcal{E}} \otimes \rho_{\text{env}}^{\mathcal{F}} \in \mathscr{H}_{\text{env}}$, we see that it can be explicitly written as

$$\begin{aligned}
\mathcal{G}(\rho) &= \mathrm{tr}_{B^{\mathcal{F}}}\Big(\mathrm{tr}_{B^{\mathcal{E}}}\Big([U_{\mathcal{E}} \otimes U_{\mathcal{F}}]\left[\rho \otimes \rho_{\text{env}}^{\mathcal{E}} \otimes \rho_{\text{env}}^{\mathcal{F}}\right]\left[U_{\mathcal{E}}^\dagger \otimes U_{\mathcal{F}}^\dagger\right]\Big)\Big) \\
&= \mathrm{tr}_{B^{\mathcal{F}}}\Big(U_{\mathcal{F}}\mathrm{tr}_{B^{\mathcal{E}}}\Big(U_{\mathcal{E}}\left[\rho \otimes \rho_{\text{env}}^{\mathcal{E}}\right]U_{\mathcal{E}}^\dagger\Big) \otimes \rho_{\text{env}}^{\mathcal{F}} U_{\mathcal{F}}^\dagger\Big) \\
&= \mathrm{tr}_{B^{\mathcal{F}}}\Big(U_{\mathcal{F}}\mathcal{E}(\rho) \otimes \rho_{\text{env}}^{\mathcal{F}} U_{\mathcal{F}}^\dagger\Big) \\
&= \mathcal{F}(\mathcal{E}(\rho)),
\end{aligned}$$

where the unitaries are implicitly understood as $U_{\mathcal{E}} \equiv U_{\mathcal{E}} \otimes I$ and $U_{\mathcal{F}} \equiv I \otimes U_{\mathcal{F}}$. This result shows that the composition of quantum operations acting on different spaces is also a quantum operation.

8.7

After applying a unitary U to the entire system and doing a general measurement with operators M_m, which results in measurement outcome m, the entire system is left on state

$$\frac{M_m U\rho \otimes \sigma U^\dagger M_m^\dagger}{p(m)} = \frac{M_m U\rho \otimes \sum_j q_j \left|e_j\middle\rangle\!\middle\langle e_j\right| U^\dagger M_m^\dagger}{p(m)},$$

where we are considering a basis $\{|e_j\rangle\}$ for the environment such that the initial state σ is diagonal. By tracing it over the environment we get the normalized quantum operation $\mathcal{E}_m(\rho)$ acting on the principal system, that is

$$\frac{\mathcal{E}_m(\rho)}{p(m)} = \frac{\mathrm{tr}_E\Big(M_m U\rho \otimes \sum_j q_j \left|e_j\middle\rangle\!\middle\langle e_j\right| U^\dagger M_m^\dagger\Big)}{p(m)} = \frac{\sum_{j,k} q_j \left\langle e_k \,\middle|\, M_m U\rho \otimes |e_j\rangle\!\langle e_j| U^\dagger M_m^\dagger \,\middle|\, e_k\right\rangle}{p(m)}.$$

Now it is possible to identify the linear operators $E_{jk} \equiv \sqrt{q_j}\,\langle e_k \,|\, M_m U \,|\, e_j\rangle$ such that we can write $\mathcal{E}_m(\rho) = \sum_{j,k} E_{jk}\rho E_{jk}^\dagger$. The probability of obtaining measurement outcome m will be given by

$$\begin{aligned}
p(m) &= \mathrm{tr}\big(M_m^\dagger M_m U\rho \otimes \sigma U^\dagger\big] \\
&= \mathrm{tr}\left(M_m U\rho \otimes \sigma U^\dagger M_m^\dagger \sum_k |e_k\rangle\!\langle e_k|\right) \\
&= \mathrm{tr}\left(\sum_k \left\langle e_k \,\middle|\, M_m U\rho \otimes \sigma U^\dagger M_m^\dagger \,\middle|\, e_k\right\rangle\right) \\
&= \mathrm{tr}\big(\mathrm{tr}_E\big(M_m U\rho \otimes \sigma U^\dagger M_m^\dagger\big)\big) \\
&= \mathrm{tr}(\mathcal{E}_m(\rho)).
\end{aligned}$$

8.8

By following the exact same procedure as the trace-preserving case with the addition of operator $E_\infty \neq 0$, such that, $\sum_k E_k^\dagger E_k + E_\infty^\dagger E_\infty = I$, we will have that $\sum_k E_k^\dagger E_k \leq I$, which characterizes a non-trace-preserving quantum operation.

8.9

Applying U and then measuring with operators P_m in sequence will give the measurement outcome m with probability

$$\begin{aligned}
p(m) &= \mathrm{tr}\big(P_m^\dagger P_m U\rho \otimes |e_0\rangle\langle e_0|\, U^\dagger\big) \\
&= \mathrm{tr}\left(\sum_k |m,k\rangle\langle m,k| \sum_{m',m'',k',k''} E_{m'k'}\rho \otimes |m',k'\rangle\langle m'',k''|\, E_{m''k''}^\dagger\right) \\
&= \mathrm{tr}\left(\sum_{m',m'',k,k',k''} \langle m,k\,|\,m',k'\rangle\, E_{m'k'}\rho E_{m''k''}^\dagger\, \langle m'',k''\,|\,m,k\rangle\right) \\
&= \mathrm{tr}\left(\sum_k E_{mk}\rho E_{mk}^\dagger\right) \\
&= \mathrm{tr}(\mathcal{E}_m(\rho)).
\end{aligned}$$

The principal system, after the process, will therefore be in the state

$$\frac{\mathrm{tr}_E\big(P_m U\rho \otimes |e_0\rangle\langle e_0|\, U^\dagger P_m^\dagger\big)}{p(m)} = \frac{\mathcal{E}_m(\rho)}{\mathrm{tr}(\mathcal{E}_m(\rho))}.$$

8.10

Let $\{|l\rangle\}$ be a basis for the d-dimentional Hilbert space and let $\{E_1, \cdots, E_n\}$, for $n > d^2$, be the set of elements for the operator-sum representation of $\mathcal{E}$. Then we can define an $n \times n$ matrix W such that its entries are given by $W_{jk} \equiv \mathrm{tr}(E_j^\dagger E_k)$, which can be explicitly written as

$$\begin{aligned}
W_{jk} &= \sum_{l=1}^{d} \left\langle l \,\middle|\, E_j^\dagger E_k \,\middle|\, l\right\rangle \\
&= \sum_{l=1}^{d}\sum_{m=1}^{d} \left\langle l \,\middle|\, E_j^\dagger \,\middle|\, m\right\rangle \langle m\,|\,E_k\,|\,l\rangle\,.
\end{aligned}$$

W is clearly hermitian since

$$W_{jk} = \sum_{l=1}^{d} \left\langle l \,\middle|\, E_j^\dagger E_k \,\middle|\, l\right\rangle = \sum_{l=1}^{d} \left\langle l \,\middle|\, E_k^\dagger E_j \,\middle|\, l\right\rangle^\dagger = W_{kj}^*.$$

Furthermore, since there are d independent $|l\rangle$ and d independent $|m\rangle$ there are at most d^2 independent terms $\langle m\,|\,E_k\,|\,l\rangle$. This means that any entry belonging in a row or column in W beyond d^2 can be written as a linear combination of, at most, the first d^2 rows and columns, that is, for any integer

$a > 0$, we have that

$$W_{(d^2+a)k} = \sum_{j=1}^{d^2} b_j W_{jk} \quad \text{and} \quad W_{j(d^2+a)} = \sum_{k=1}^{d^2} c_k W_{jk},$$

meaning $\operatorname{rank}(W) \equiv r \leq d^2$. Since W has rank $r \leq d^2$ and is Hermitian, there exists an $n \times n$ unitary matrix u such that $D \equiv uWu^\dagger$ is diagonal with $r \leq d^2$ non-zero entries. The entries of D are given by $D_{lm} = \delta_{lm}\lambda_m$. Therefore, the relation between W and D can be written as

$$\begin{aligned}\delta_{lm}\lambda_m &= \sum_{j,k=1}^{n} u_{lj} W_{jk} u^*_{km} \\ &= \sum_{j,k=1}^{n} \operatorname{tr}\left(u_{lj} E_j^\dagger E_k u^*_{km}\right)\end{aligned}$$

$$\implies \quad \lambda_l = \sum_{j,k=1}^{n} \operatorname{tr}\left(u_{lj} E_j^\dagger E_k u^*_{kl}\right).$$

Then, by defining a set of n operators $F_j = \sum_{k=1}^{n} E_k u^*_{kj}$ we have

$$\lambda_l = \operatorname{tr}\left(F_l^\dagger F_l\right) = \sum_{p=1}^{d} \left\langle p \,\middle|\, F_l^\dagger F_l \,\middle|\, p\right\rangle = \sum_{p=1}^{d}\sum_{q=1}^{d} \left\langle p \,\middle|\, F_l^\dagger \,\middle|\, q\right\rangle \langle q \,|\, F_l \,|\, p\rangle = \sum_{p,q=1}^{d} \left|\langle p \,|\, F_l \,|\, q\rangle\right|^2,$$

and since $\lambda_l = 0$ for $l > r$, it follows that $\sum_{p,q=1}^{d} \langle p \,|\, F_l \,|\, q\rangle = 0$ for $l > r$, and because this must hold for any arbitrary choice of basis, we have that $F_l = 0$ for $l > r$. Since the operators F_j and E_k are related by a unitary transformation then, by Theorem 8.2, we can write

$$\mathcal{E}(\rho) = \sum_{j=1}^{r} F_j \rho F_j^\dagger.$$

8.11

Let $\{|j\rangle\}$ be a basis for the d-dimentional Hilbert space and $\{|j'\rangle\}$ be a basis for the d'-dimentional one. We can write the elements of the operator-sum representation as

$$E_k = \sum_{j=1}^{d}\sum_{j'=1}^{d'} |j'\rangle\langle j'|\, E_k \,|j\rangle\langle j| = \sum_{j=1}^{d}\sum_{j'=1}^{d'} \langle j' \,|\, E_k \,|\, j\rangle \,|j'\rangle\langle j|\,.$$

Since there are d independent $|j\rangle$ and d' independent $|j'\rangle$, there are $r \leq dd'$ independent $\langle j' \,|\, E_k \,|\, j\rangle$, meaning that for $k > r$ it holds

$$\langle j' \,|\, E_k \,|\, j\rangle = \sum_{l=1}^{r} c_l \,\langle j' \,|\, E_l \,|\, j\rangle\,.$$

This is the same as saying that, in a basis where the matrix $\mathrm{tr}\left(E_j^\dagger E_k\right)$ is diagonal, there are $r \leq dd'$ non-zero eigenvalues. Then in such basis, for $k > r$ we have

$$0 = \mathrm{tr}\left(E_k^\dagger E_k\right) = \sum_{l=1}^{d} \left\langle l \,\middle|\, E_k^\dagger E_k \,\middle|\, l \right\rangle = \sum_{l=1}^{d} \sum_{l'=1}^{d'} \left\langle l \,\middle|\, E_k^\dagger \,\middle|\, l' \right\rangle \langle l' \,|\, E_k \,|\, l \rangle = \sum_{l=1}^{d} \sum_{l'=1}^{d'} |\langle l' \,|\, E_k \,|\, l \rangle|^2,$$

and since it must hold for any choice of basis for the two Hilbert spaces, it follows that $E_k = 0$ for $k > r$ and the quantum operation can be described by

$$\mathcal{E}(\rho) = \sum_{k=1}^{r} E_k \rho E_k^\dagger.$$

8.12

All orthogonal matrices are such that $\det(O) = \pm 1$. The subgroup of orthogonal matrices whose determinant is -1 do not include the identity matrix. Since the set of possible matrices O must include the identity matrix, for when M is symmetric, it follows that $\det(O) = 1$.

8.13

From theorem 4.1, any unitary operator can be written as $U = e^{i\alpha} R_z(\theta) R_y(\phi) R_z(\psi)$, which corresponds to active rotations of the Bloch vector. Alternatively, it can be interpreted as passive rotations of the Bloch sphere.

8.14

M is a real matrix and thus, can have a negative determinant. We have that $\det(M) = \det(OS) = \det(O)\det(S) = \det(S)$, and therefore $\det(S)$ can be negative.

8.15

Just like the measurement in the $\{|0\rangle, |1\rangle\}$ basis (or equivalently, the phase-flip channel with $p = 1/2$), where the entire Bloch sphere collapses on the Z axis, a measurement in the $\{|+\rangle, |-\rangle\}$ causes the Bloch sphere to collapse on the X axis. To see this explicitly let us consider an initial state $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$, where $\vec{r} = \{r_x, r_y, r_z\}$ with $\|\vec{r}\| \leq 1$, and $\vec{\sigma} = \{X, Y, Z\}$. The evolved state will be

$$\begin{aligned}
\mathcal{E}(\rho) &= \frac{1}{2} |+\rangle\langle+| (I + r_x X + r_y Y + r_z Z) |+\rangle\langle+| + \frac{1}{2} |-\rangle\langle-| (I + r_x X + r_y Y + r_z Z) |-\rangle\langle-| \\
&= \frac{1}{2} (1 + r_x + 0 + 0) |+\rangle\langle+| + \frac{1}{2} (1 - r_x + 0 + 0) |-\rangle\langle-| \\
&= \frac{1 + r_x}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1 - r_x}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & r_x \\ r_x & 1 \end{bmatrix} = \frac{I + r_x X}{2}.
\end{aligned}$$

That is, the Bloch vector is given by $\{r_x, 0, 0\}$, meaning the Bloch sphere collapsed on the X axis.

8.16

Consider $\mathcal{E}_0(\rho) = |0\rangle\langle 0| \rho |0\rangle\langle 0|$, that is, a quantum operation that describes the outcome $|0\rangle$ for a measurement in the computational basis. This cannot be seen as a deformation of the Bloch sphere. To see this, consider an initial state $|\psi\rangle = a|0\rangle + b|1\rangle$. The action of the quantum operation yields

$$\begin{aligned}
\mathcal{E}_0(\rho) &= \frac{1}{2} |0\rangle\langle 0| \left(|a|^2 |0\rangle\langle 0| + ab^* |0\rangle\langle 1| + a^* b |1\rangle\langle 0| + |b|^2 |1\rangle\langle 1|\right) |0\rangle\langle 0| \\
&= \frac{|a|^2}{2} |0\rangle\langle 0| = \begin{bmatrix} |a|^2 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Since this is not a density operator for general a it cannot be associated with a Bloch sphere.

8.17

$$\begin{aligned}
\mathcal{E}(I) &= \frac{I + XIX + YIY + ZIZ}{4} = \frac{I + I + I + I}{4} = I, \\
\mathcal{E}(X) &= \frac{X + XXX + YXY + ZXZ}{4} = \frac{X + X - X - X}{4} = 0, \\
\mathcal{E}(Y) &= \frac{Y + XXX + YYY + ZYZ}{4} = \frac{Y - Y + Y - Y}{4} = 0, \\
\mathcal{E}(Z) &= \frac{Z + XZX + YZY + ZZZ}{4} = \frac{Z - Z - Z + Z}{4} = 0.
\end{aligned}$$

Any density operator can be written as $\rho = (I + \vec{r} \cdot \vec{\sigma})/2$ for $\vec{r} = \{r_x, r_y, r_z\}$, with $\|\vec{r}\| \leq 1$, and $\vec{\sigma} = \{X, Y, Z\}$, thus

$$\mathcal{E}(\rho) = \frac{1}{2}\mathcal{E}(I) + \frac{r_x}{2}\mathcal{E}(X) + \frac{r_y}{2}\mathcal{E}(Y) + \frac{r_z}{2}\mathcal{E}(Z) = \frac{I}{2}.$$

Therefore, using the definition of $\mathcal{E}(\rho)$, we have that

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}.$$

8.18

Let $\{|j\rangle\}$ be a basis for the d-dimensional Hilbert space such that ρ is diagonal, that is,

$$\rho = \sum_{j=1}^{d} q_j |j\rangle\langle j| \quad \Longrightarrow \quad \operatorname{tr}\left(\rho^k\right) = \sum_{j=1}^{k} q_j^k.$$

This trace satisfies $1/d^{k-1} \leq \operatorname{tr}\left(\rho^k\right) \leq 1$. Now let $\mathcal{E}(\rho) \equiv \sigma$, then

$$\sigma = \frac{pI}{d} + (1-p)\rho$$

$$
\begin{aligned}
&= \frac{p}{d}\sum_{j=1}^{k} |j\rangle\langle j| + (1-p)\sum_{j=1}^{k} q_j\, |j\rangle\langle j| \\
&= \sum_{j=1}^{k} \left[p\left(\frac{1}{d} - q_j\right) + q_j \right] |j\rangle\langle j|
\end{aligned}
$$

$$
\implies \quad \mathrm{tr}\big(\sigma^k\big) = \sum_{j=1}^{k} \left[p\left(\frac{1}{d} - q_j\right) + q_j \right]^k .
$$

Notice that treating the term inside square brackets as a function of p, its derivative is constant and equal to $1/d - q_j$, meaning its extreme values always occur for $p = 0$ and $p = 1$. Therefore $\mathrm{tr}\big(\sigma^k\big)$ is bounded by the cases $p = 0$ and $p = 1$. Calculating each case yields

$$
\begin{aligned}
&\text{for } p = 0: \quad \mathrm{tr}\big(\sigma^k\big) = \sum_{j=1}^{k} q_j^k = \mathrm{tr}\big(\rho^k\big) \\
&\text{for } p = 1: \quad \mathrm{tr}\big(\sigma^k\big) = \sum_{j=1}^{k} \frac{1}{d^k} = \frac{1}{d^{k-1}} \leq \mathrm{tr}\big(\rho^k\big).
\end{aligned}
$$

Therefore $\mathrm{tr}\big(\sigma^k\big) \leq \mathrm{tr}\big(\rho^k\big)$, with equality holding if $p = 0$ or if ρ is the maximum mixed state.

8.19

If ρ is a density operator it always holds that $\mathrm{tr}(\rho) = 1$. So similarly to the qubit case, we can write the identity matrix in terms of ρ as

$$
\frac{I}{d} = \frac{I}{d}\,\mathrm{tr}(\rho) = \frac{1}{d}\sum_{j=1}^{d} |j\rangle\langle j| \sum_{k=1}^{d} \langle k\,|\,\rho\,|\,k\rangle = \frac{1}{d}\sum_{j=1}^{d}\sum_{k=1}^{d} |j\rangle\langle k|\,\rho\,|k\rangle\langle j| .
$$

Substituting it in the depolarizing quantum operation yields

$$
\begin{aligned}
\mathcal{E}(\rho) &= \frac{p}{d}\sum_{j=1}^{d}\sum_{k=1}^{d} |j\rangle\langle k|\,\rho\,|k\rangle\langle j| + (1-p)\rho \\
&= \sum_{j,k=1}^{d} \left(\sqrt{\frac{p}{d}}\,|j\rangle\langle k|\right) \rho \left(\sqrt{\frac{p}{d}}\,|k\rangle\langle j|\right) + \sqrt{1-p}I\rho\sqrt{1-p}I.
\end{aligned}
$$

Now we can identify $E_0 \equiv \sqrt{1-p}I$ and $E_{jk} \equiv \sqrt{p/d}\,|j\rangle\langle k|$, for j and k ranging from 1 to d, as a set of elements for the operator-sum representation of the generalized depolarizing channel.

8.20

Let the principal system start at state $\rho_{\text{in}} \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $|\psi_i\rangle = \alpha_i |0\rangle + \beta_i |1\rangle$, meaning the total state is $\rho = \sum_i p_i (\alpha_i |00\rangle + \beta_i |10\rangle)(\alpha_i^* \langle 00| + \beta_i^* \langle 10|)$. Then the action of the circuit will be

$$
\begin{aligned}
\rho &\xrightarrow{CR_y(\theta)_{(1,2)}} \sum_i p_i \left(\alpha_i |00\rangle + \beta_i \cos\frac{\theta}{2} |10\rangle + \beta_i \sin\frac{\theta}{2} |11\rangle\right)\left(\alpha_i^* \langle 00| + \beta_i^* \cos\frac{\theta}{2} \langle 10| + \beta_i^* \sin\frac{\theta}{2} \langle 11|\right) \\
&\xrightarrow{CX_{(2,1)}} \sum_i p_i \left(\alpha_i |00\rangle + \beta_i \cos\frac{\theta}{2} |10\rangle + \beta_i \sin\frac{\theta}{2} |01\rangle\right)\left(\alpha_i^* \langle 00| + \beta_i^* \cos\frac{\theta}{2} \langle 10| + \beta_i^* \sin\frac{\theta}{2} \langle 01|\right).
\end{aligned}
$$

We may call this state ρ'. Then, a measurement of the environment is made, which yields

$$
\begin{aligned}
\rho_{\text{out}} &= \operatorname{tr}_{\text{env}}(\rho') \\
&= \sum_i p_i \left[\left(\alpha_i |0\rangle + \beta_i \cos\frac{\theta}{2} |1\rangle\right)\left(\alpha_i^* \langle 0| + \beta_i^* \cos\frac{\theta}{2} \langle 1|\right) + \left(\beta_i \sin\frac{\theta}{2} |0\rangle\right)\left(\beta_i^* \sin\frac{\theta}{2} \langle 0|\right)\right] \\
&= \sum_i p_i \left[\left(|\alpha_i|^2 + |\beta_i|^2 \sin^2\frac{\theta}{2}\right) |0\rangle\langle 0| + \alpha_i \beta_i^* \cos\frac{\theta}{2} |0\rangle\langle 1| + \alpha_i^* \beta_i \cos\frac{\theta}{2} |1\rangle\langle 0| + |\beta_i|^2 \cos^2\frac{\theta}{2} |1\rangle\langle 1|\right] \\
&= \sum_i p_i \begin{bmatrix} |\alpha_i|^2 + |\beta_i|^2 \sin^2\frac{\theta}{2} & \alpha_i \beta_i^* \cos\frac{\theta}{2} \\ \alpha_i^* \beta_i \cos\frac{\theta}{2} & |\beta_i|^2 \cos^2\frac{\theta}{2} \end{bmatrix}.
\end{aligned}
$$

Applying the amplitude damping channel to the same initial state yields

$$
\begin{aligned}
\mathcal{E}_{\text{AD}}(\rho_{\text{in}}) &= E_0 \rho_{\text{in}} E_0^\dagger + E_1 \rho_{\text{in}} E_1^\dagger \\
&= \sum_i p_i \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}\begin{bmatrix} |\alpha_i|^2 & \alpha_i \beta_i^* \\ \alpha_i^* \beta_i & |\beta_i|^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} + \sum_i p_i \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} |\alpha_i|^2 & \alpha_i \beta_i^* \\ \alpha_i^* \beta_i & |\beta_i|^2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix} \\
&= \sum_i p_i \begin{bmatrix} |\alpha_i|^2 + |\beta_i|^2 \gamma & \alpha_i \beta_i^* \sqrt{1-\gamma} \\ \alpha_i^* \beta_i \sqrt{1-\gamma} & |\beta_i|^2 (1-\gamma) \end{bmatrix}.
\end{aligned}
$$

From this result we see that setting $\sin^2\frac{\theta}{2} = \gamma$, both quantities coincide.

8.21

In general, if $\{|m\rangle\}$ is a basis for the principal system then E_k can be written as

$$
E_k = \langle k_b \,|\, U \,|\, 0_b\rangle \equiv \sum_{m,n} \langle m, k_b \,|\, U \,|\, n, 0_b\rangle \, |m\rangle\langle n| \,,
$$

and considering the basis $\{|m\rangle\}$ as the eigenstates of $a^\dagger a$, we have

$$
\begin{aligned}
E_k &= \sum_{m,n} \left\langle m, k_b \,\middle|\, U \frac{(a^\dagger)^n}{\sqrt{n!}} \,\middle|\, 0, 0_b \right\rangle |m\rangle\langle n| \\
&= \sum_{m,n} \frac{1}{\sqrt{n!}} \langle m, k_b \,|\, U \left(a^\dagger\right)^n \,|\, 0, 0_b\rangle \, |m\rangle\langle n| \,.
\end{aligned}
$$

Notice that we may write $U\left(a^\dagger\right)^n = U\left(a^\dagger U^\dagger U\right)^n = \left(U a^\dagger U^\dagger\right)^n U$, thus

$$E_k = \sum_{m,n} \frac{1}{\sqrt{n!}} \left\langle m, k_b \,\middle|\, \left(U a^\dagger U^\dagger\right)^n U \,\middle|\, 0, 0_b \right\rangle |m\rangle\langle n| \,.$$

The unitary operator is given by

$$U = \exp\left[-i\chi\left(a^\dagger b + a b^\dagger\right)\Delta t\right] = \sum_{n=0}^{\infty} \frac{(-i\chi\Delta t)^n}{n!} \left(a^\dagger b + a b^\dagger\right)^n .$$

Given that $a\,|0\rangle = 0$ and $b\,|0_b\rangle = 0$, it follows that, when acting on the vacuum state of both modes, the unitary operator yields $U\,|0, 0_b\rangle = |0, 0_b\rangle$, and we get

$$E_k = \sum_{m,n} \frac{1}{\sqrt{n!}} \left\langle m, k_b \,\middle|\, \left(U a^\dagger U^\dagger\right)^n \,\middle|\, 0, 0_b \right\rangle |m\rangle\langle n| \,.$$

The physical interpretation is, of course, that since it is the vacuum state, there are no particles to be created or annihilated in any mode, and the state remains at the vacuum. Using the Baker-Campbell-Hausdorff formula, we obtain

$$U a^\dagger U^\dagger = \sum_{j=0}^{\infty} \frac{(-i\chi\Delta t)^j}{j!} C_j ,$$

where $C_0 \equiv a^\dagger$ and $C_j \equiv \left[a^\dagger b + a b^\dagger, C_{j-1}\right]$. From this recursive relation we calculate

$$\begin{aligned} C_1 &= \left[a^\dagger b + a b^\dagger, a^\dagger\right] = \left[a, a^\dagger\right] b^\dagger = b^\dagger , \\ C_2 &= \left[a^\dagger b + a b^\dagger, b^\dagger\right] = a^\dagger \left[b, b^\dagger\right] = a^\dagger , \end{aligned}$$

that is,

$$C_j = \begin{cases} a^\dagger & \text{for even } j; \\ b^\dagger & \text{for odd } j. \end{cases}$$

Substituting yields

$$\begin{aligned} U a^\dagger U^\dagger &= \sum_{j=0}^{\infty} \frac{(-i\chi\Delta t)^{2j}}{(2j)!} a^\dagger + \sum_{j=0}^{\infty} \frac{(-i\chi\Delta t)^{2j+1}}{(2j+1)!} b^\dagger \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{(\chi\Delta t)^{2j}}{(2j)!} a^\dagger - i \sum_{j=0}^{\infty} (-1)^j \frac{(\chi\Delta t)^{2j+1}}{(2j+1)!} b^\dagger \\ &= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger , \end{aligned}$$

and defining $\gamma \equiv 1 - \cos^2(\chi\Delta t)$ we write $U a^\dagger U^\dagger = \sqrt{1-\gamma}\, a^\dagger - i\sqrt{\gamma}\, b^\dagger$. The quantity that appears in the expression for E_k can therefore be calculated as

$$\left(U a^\dagger U^\dagger\right)^n |0, 0_b\rangle = \left(\sqrt{1-\gamma}\, a^\dagger - i\sqrt{\gamma}\, b^\dagger\right)^n |0, 0_b\rangle$$

$$
\begin{aligned}
&= \sum_{j=0}^{n} \binom{n}{j} \left(\sqrt{1-\gamma}\right)^{n-j} (-i\sqrt{\gamma})^j \left(a^\dagger\right)^{n-j} \left(b^\dagger\right)^j |0, 0_b\rangle \\
&= \sum_{j=0}^{n} (-i)^j \frac{n!}{j!(n-j)!} \sqrt{(1-\gamma)^{n-j}\gamma^j} \sqrt{(n-j)!j!}\, |n-j, j_b\rangle \\
&= \sum_{j=0}^{n} (-i)^j \frac{n!}{\sqrt{j!(n-j)!}} \sqrt{(1-\gamma)^{n-j}\gamma^j}\, |n-j, j_b\rangle .
\end{aligned}
$$

We can ignore the global phase $(-i)^j$ and substitute in the expression for E_k, finally yielding

$$
\begin{aligned}
E_k &= \sum_{m,n} \sum_{j=0}^{\infty} \frac{1}{\sqrt{n!}} \frac{n!}{\sqrt{j!(n-j)!}} \sqrt{(1-\gamma)^{n-j}\gamma^j}\, \langle m, k_b \,|\, n-j, j_b\rangle\, |m\rangle\!\langle n| \\
&= \sum_{m,n} \sum_{j=0}^{\infty} \sqrt{\binom{n}{j}} \sqrt{(1-\gamma)^{n-j}\gamma^j}\, \delta_{m(n-j)} \delta_{kj}\, |m\rangle\!\langle n| \\
&= \sum_{n} \sqrt{\binom{n}{k}} \sqrt{(1-\gamma)^{n-k}\gamma^k}\, |n-k\rangle\!\langle n| .
\end{aligned}
$$

$$
\begin{aligned}
\sum_k E_k^\dagger E_k &= \sum_k \sum_{m,n} \sqrt{\binom{m}{k}\binom{n}{k}} \sqrt{(1-\gamma)^{m+n-2k}\gamma^{2k}}\, |m\rangle \langle m-k \,|\, n-k\rangle \langle n| \\
&= \sum_n \sum_k \binom{n}{k} (1-\gamma)^{n-k}\gamma^k\, |n\rangle\!\langle n| \\
&= \sum_n ((1-\gamma)+\gamma)^n\, |n\rangle\!\langle n| \\
&= \sum_n |n\rangle\!\langle n| = I,
\end{aligned}
$$

meaning the elements E_k define a trace-preserving quantum operation.

8.22

$$
\begin{aligned}
\mathcal{E}_{\mathrm{AD}}(\rho) &= E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \\
&= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix} \\
&= \begin{bmatrix} a + c\gamma & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c\,(1-\gamma) \end{bmatrix},
\end{aligned}
$$

and since $a + c = 1$

$$
\mathcal{E}_{\mathrm{AD}}(\rho) = \begin{bmatrix} a + \gamma(1-a) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c\,(1-\gamma) \end{bmatrix} = \begin{bmatrix} 1 - (1-\gamma)(1-a) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c\,(1-\gamma) \end{bmatrix}.
$$

8.23

$$\rho \equiv |\psi\rangle\langle\psi| = (a\,|01\rangle + b\,|10\rangle)\,(a^*\,\langle 01| + b^*\,\langle 10|)\,.$$

We may write $(\mathcal{E}_{\mathrm{AD}} \otimes \mathcal{E}_{\mathrm{AD}})(\rho)$ as a composition, that is, $\mathcal{E}_{\mathrm{AD}} \otimes \mathcal{E}_{\mathrm{AD}} = (I \otimes \mathcal{E}_{\mathrm{AD}})(\mathcal{E}_{\mathrm{AD}} \otimes I)$. Therefore

$$\begin{aligned}
\mathcal{E}_{\mathrm{AD}}(\rho) &= (I \otimes \mathcal{E}_{\mathrm{AD}})\left((E_0 \otimes I)\,|\psi\rangle\langle\psi|\left(E_0^\dagger \otimes I\right) + (E_1 \otimes I)\,|\psi\rangle\langle\psi|\left(E_1^\dagger \otimes I\right)\right)\\
&= (I \otimes \mathcal{E}_{\mathrm{AD}})\left(\left(a\,|01\rangle + \sqrt{1-\gamma}b\,|10\rangle\right)\left(a^*\,\langle 01| + \sqrt{1-\gamma}b^*\,\langle 10|\right) + (\sqrt{\gamma}b\,|00\rangle)\,(\sqrt{\gamma}b^*\,\langle 00|)\right)\\
&= \left(\sqrt{1-\gamma}a\,|01\rangle + \sqrt{1-\gamma}b\,|10\rangle\right)\left(\sqrt{1-\gamma}a^*\,\langle 01| + \sqrt{1-\gamma}b^*\,\langle 10|\right)\\
&\quad + (\sqrt{\gamma}b\,|00\rangle)\,(\sqrt{\gamma}b^*\,\langle 00|) + (\sqrt{\gamma}a\,|00\rangle)\,(\sqrt{\gamma}a^*\,\langle 00|)\\
&= \sqrt{1-\gamma}\,(a\,|01\rangle + b\,|10\rangle)\,(a^*\,\langle 01| + b^*\,\langle 10|)\,\sqrt{1-\gamma} + \sqrt{\gamma}\,(a+b)\,|00\rangle\langle 00|\,(a^* + b^*)\,\sqrt{\gamma}\\
&= \sqrt{1-\gamma}I\rho\sqrt{1-\gamma}I + \sqrt{\gamma}\,(|00\rangle\langle 01| + |00\rangle\langle 10|)\,\rho\sqrt{\gamma}\,(|01\rangle\langle 00| + |10\rangle\langle 00|)\,.
\end{aligned}$$

Here we can identity the two elements of the operator-sum representation $E_0^{\mathrm{dr}} \equiv \sqrt{1-\gamma}I$ and $E_1^{\mathrm{dr}} \equiv \sqrt{\gamma}\,(|00\rangle\langle 01| + |00\rangle\langle 10|)$

8.24

The unitary operation resulting from the Jaynes-Cummings interaction with detuning $\delta = 0$ is

$$U = |00\rangle\langle 00| + \cos\Omega t\,(|01\rangle\langle 01| + |10\rangle\langle 10|) - i\sin\Omega t\,(|01\rangle\langle 10| + |10\rangle\langle 01|)\,.$$

The corresponding quantum operation for when we trace over the field system, considering it to initially be the vacuum state, will have elements $E_k = \langle k\,|\,U\,|\,0\rangle$. This gives us

$$\begin{aligned}
E_0 &= \langle 0\,|\,U\,|\,0\rangle = |0\rangle\langle 0| + \cos\Omega t\,|1\rangle\langle 1|\,,\\
E_1 &= \langle 1\,|\,U\,|\,0\rangle = -i\sin\Omega t\,|0\rangle\langle 1|\,.
\end{aligned}$$

Ignoring the global phase $-i$ and defining $\gamma \equiv \sin^2\Omega t$ we get

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \quad \text{and} \quad E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix},$$

which correspond to the amplitude damping operation elements.

8.25

The state with temperature T is given by

$$\rho = \frac{e^{-E_0/k_{\mathrm{B}}T}\,|0\rangle\langle 0| + e^{-E_1/k_{\mathrm{B}}T}\,|1\rangle\langle 1|}{e^{-E_0/k_{\mathrm{B}}T} + e^{-E_1/k_{\mathrm{B}}T}}.$$

Comparing with ρ_∞, we see that its temperature, which we may denote T_∞, is such that

$$p = \frac{e^{-E_0/k_\mathrm{B}T_\infty}}{e^{-E_0/k_\mathrm{B}T_\infty} + e^{-E_1/k_\mathrm{B}T_\infty}} \quad \text{and} \quad 1-p = \frac{e^{-E_1/k_\mathrm{B}T_\infty}}{e^{-E_0/k_\mathrm{B}T_\infty} + e^{-E_1/k_\mathrm{B}T_\infty}}.$$

Dividing both relations we obtain

$$\frac{p}{1-p} = \frac{e^{-E_0/k_\mathrm{B}T_\infty}}{e^{-E_1/k_\mathrm{B}T_\infty}} \quad \Longrightarrow \quad e^{(E_1-E_0)/k_\mathrm{B}T_\infty} = \frac{p}{1-p}.$$

And then we must simply solve for T_∞

$$\frac{E_1 - E_0}{k_\mathrm{B}T_\infty} = \ln\left(\frac{p}{1-p}\right) \quad \Longrightarrow \quad T_\infty = \frac{E_1 - E_0}{k_\mathrm{B}\ln\left(\frac{p}{1-p}\right)}.$$

Notice that in the limit for $p \to 1$, which corresponds to the usual amplitude damping, the temperature tends to zero, as expected.

8.26

Let the principal system start at state $\rho_\mathrm{in} \equiv \sum_i p_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right|$, with $\left|\psi_i\right\rangle = \alpha_i \left|0\right\rangle + \beta_i \left|1\right\rangle$, meaning the total state is $\rho = \sum_i p_i \left(\alpha_i \left|00\right\rangle + \beta_i \left|10\right\rangle\right)\left(\alpha_i^* \left\langle 00\right| + \beta_i^* \left\langle 10\right|\right)$. Then the action of the circuit will be

$$\rho \xrightarrow{CR_y(\theta)_{(1,2)}} \sum_i p_i \left(\alpha_i \left|00\right\rangle + \beta_i \cos\frac{\theta}{2}\left|10\right\rangle + \beta_i \sin\frac{\theta}{2}\left|11\right\rangle\right)\left(\alpha_i^* \left\langle 00\right| + \beta_i^* \cos\frac{\theta}{2}\left\langle 10\right| + \beta_i^* \sin\frac{\theta}{2}\left\langle 11\right|\right).$$

We may call this state ρ'. Then, a measurement of the environment is made, which yields

$$\begin{aligned}
\rho_\mathrm{out} &= \mathrm{tr}_\mathrm{env}(\rho') \\
&= \sum_i p_i \left[\left(\alpha_i \left|0\right\rangle + \beta_i \cos\frac{\theta}{2}\left|1\right\rangle\right)\left(\alpha_i^* \left\langle 0\right| + \beta_i^* \cos\frac{\theta}{2}\left\langle 1\right|\right) + \left(\beta_i \sin\frac{\theta}{2}\left|1\right\rangle\right)\left(\beta_i^* \sin\frac{\theta}{2}\left\langle 1\right|\right)\right] \\
&= \sum_i p_i \left[\left|\alpha_i\right|^2 \left|0\right\rangle\!\left\langle 0\right| + \alpha_i\beta_i^* \cos\frac{\theta}{2}\left|0\right\rangle\!\left\langle 1\right| + \alpha_i^*\beta_i \cos\frac{\theta}{2}\left|1\right\rangle\!\left\langle 0\right| + \left|\beta_i\right|^2 \left|1\right\rangle\!\left\langle 1\right|\right] \\
&= \sum_i p_i \begin{bmatrix} \left|\alpha_i\right|^2 & \alpha_i\beta_i^*\cos\frac{\theta}{2} \\ \alpha_i^*\beta_i\cos\frac{\theta}{2} & \left|\beta_i\right|^2 \end{bmatrix}.
\end{aligned}$$

Applying the phase damping channel to the same initial state yields

$$\begin{aligned}
\mathcal{E}_\mathrm{PD}(\rho_\mathrm{in}) &= E_0\rho_\mathrm{in}E_0^\dagger + E_1\rho_\mathrm{in}E_1^\dagger \\
&= \sum_i p_i \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix}\begin{bmatrix} \left|\alpha_i\right|^2 & \alpha_i\beta_i^* \\ \alpha_i^*\beta_i & \left|\beta_i\right|^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} + \sum_i p_i \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}\begin{bmatrix} \left|\alpha_i\right|^2 & \alpha_i\beta_i^* \\ \alpha_i^*\beta_i & \left|\beta_i\right|^2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \\
&= \sum_i p_i \begin{bmatrix} \left|\alpha_i\right|^2 & \alpha_i\beta_i^*\sqrt{1-\lambda} \\ \alpha_i^*\beta_i\sqrt{1-\lambda} & \left|\beta_i\right|^2 \end{bmatrix}.
\end{aligned}$$

From this result, we see that choosing θ such that $\cos\frac{\theta}{2} = \sqrt{1-\lambda}$, both quantities coincide.

8.27

We have the relations $\widetilde{E}_0 = u_{00}E_0 + u_{01}E_1$ and $\widetilde{E}_1 = u_{10}E_0 + u_{11}E_1$. Writing in their explicit matrix representations yields

$$\begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha} \end{bmatrix} = u_{00} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} + u_{01} \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix},$$

$$\begin{bmatrix} \sqrt{1-\alpha} & 0 \\ 0 & -\sqrt{1-\alpha} \end{bmatrix} = u_{10} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} + u_{11} \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}.$$

This gives us a system with the equations

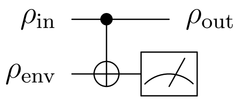
$$\begin{cases} u_{00} = \sqrt{\alpha}, \\ u_{00}\sqrt{1-\lambda} + u_{01}\sqrt{\lambda} = \sqrt{\alpha}, \\ u_{10} = \sqrt{1-\alpha}, \\ u_{10}\sqrt{1-\lambda} + u_{11}\sqrt{\lambda} = -\sqrt{1-\alpha}, \end{cases} \implies \begin{cases} u_{00} = \sqrt{\alpha}, \\ u_{01} = \sqrt{\dfrac{\alpha}{\lambda}}\left(1-\sqrt{1-\lambda}\right), \\ u_{10} = \sqrt{1-\alpha}, \\ u_{11} = -\sqrt{\dfrac{1-\alpha}{\lambda}}\left(1+\sqrt{1-\lambda}\right). \end{cases}$$

Using the fact that $1+\sqrt{1-\lambda} = 2\alpha$, we can write the resulting unitary operator as

$$u = \begin{bmatrix} \sqrt{\alpha} & 2(1-\alpha)\sqrt{\frac{\alpha}{\lambda}} \\ \sqrt{1-\alpha} & -2\alpha\sqrt{\frac{1-\alpha}{\lambda}} \end{bmatrix}.$$

8.28

We wish to verify whether the circuit



can be used to model phase damping for some appropriate choice of ρ_{env}. Let the principal system start at state $\rho_{\text{in}} \equiv \sum_i q_i |\psi_i\rangle\langle\psi_i|$, with $|\psi_i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle$, and the environment at the mixed state $\rho_{\text{env}} = p|+\rangle\langle+| + (1-p)|-\rangle\langle-|$, with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, that is,

$$\begin{aligned} \rho_{\text{env}} &= \frac{p}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right) + \frac{1-p}{2}\left(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\right) \\ &= \frac{1}{2}|0\rangle\langle 0| + \left(p-\frac{1}{2}\right)|0\rangle\langle 1| + \left(p-\frac{1}{2}\right)|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|. \end{aligned}$$

This results in the total state

$$\begin{aligned} \rho &= \sum_i q_i \left(|\alpha_i|^2 |0\rangle\langle 0| + \alpha_i\beta_i^* |0\rangle\langle 1| + \alpha_i^*\beta_i |1\rangle\langle 0| + |\beta_i|^2 |1\rangle\langle 1|\right) \otimes \rho_{\text{env}} \\ &= \sum_i q_i \left[\frac{|\alpha_i|^2}{2}\left(|00\rangle\langle 00| + |01\rangle\langle 01|\right) + |\alpha_i|^2\left(p-\frac{1}{2}\right)\left(|00\rangle\langle 01| + |01\rangle\langle 00|\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\alpha_i\beta_i^*}{2}\left(|00\rangle\langle 10| + |01\rangle\langle 11|\right) + \alpha_i\beta_i^*\left(p-\frac{1}{2}\right)\left(|00\rangle\langle 11| + |01\rangle\langle 10|\right)\\
&+ \frac{\alpha_i^*\beta_i}{2}\left(|10\rangle\langle 00| + |11\rangle\langle 01|\right) + \alpha_i^*\beta_i\left(p-\frac{1}{2}\right)\left(|10\rangle\langle 01| + |11\rangle\langle 00|\right)\\
&+ \frac{|\beta_i|^2}{2}\left(|10\rangle\langle 10| + |11\rangle\langle 11|\right) + |\beta_i|^2\left(p-\frac{1}{2}\right)\left(|10\rangle\langle 11| + |11\rangle\langle 10|\right)\Bigg].
\end{aligned}
$$

Then the action of the circuit will be

$$
\begin{aligned}
\rho \xrightarrow{CX_{(1,2)}} \sum_i q_i \Bigg[&\frac{|\alpha_i|^2}{2}\left(|00\rangle\langle 00| + |01\rangle\langle 01|\right) + |\alpha_i|^2\left(p-\frac{1}{2}\right)\left(|00\rangle\langle 01| + |01\rangle\langle 00|\right)\\
&+ \frac{\alpha_i\beta_i^*}{2}\left(|00\rangle\langle 11| + |01\rangle\langle 10|\right) + \alpha_i\beta_i^*\left(p-\frac{1}{2}\right)\left(|00\rangle\langle 10| + |01\rangle\langle 11|\right)\\
&+ \frac{\alpha_i^*\beta_i}{2}\left(|11\rangle\langle 00| + |10\rangle\langle 01|\right) + \alpha_i^*\beta_i\left(p-\frac{1}{2}\right)\left(|11\rangle\langle 01| + |10\rangle\langle 00|\right)\\
&+ \frac{|\beta_i|^2}{2}\left(|11\rangle\langle 11| + |10\rangle\langle 10|\right) + |\beta_i|^2\left(p-\frac{1}{2}\right)\left(|11\rangle\langle 10| + |10\rangle\langle 11|\right)\Bigg].
\end{aligned}
$$

We may call this state ρ'. Then, a measurement of the environment is made, which yields

$$
\begin{aligned}
\rho_{\text{out}} &= \operatorname{tr}_{\text{env}}(\rho')\\
&= \sum_i q_i\left(|\alpha_i|^2|0\rangle\langle 0| + \alpha_i\beta_i^*(2p-1)|0\rangle\langle 1| + \alpha_i^*\beta_i(2p-1)|1\rangle\langle 0| + |\beta_i|^2|1\rangle\langle 1|\right)\\
&= \begin{bmatrix} |\alpha_i|^2 & \alpha_i\beta_i^*(2p-1)\\ \alpha_i^*\beta_i(2p-1) & |\beta_i|^2\end{bmatrix}.
\end{aligned}
$$

Applying the phase damping channel to the same initial state yields

$$
\begin{aligned}
\mathcal{E}_{\text{PD}}(\rho_{\text{in}}) &= E_0\rho_{\text{in}}E_0^\dagger + E_1\rho_{\text{in}}E_1^\dagger\\
&= \sum_i p_i \begin{bmatrix}1 & 0\\ 0 & \sqrt{1-\lambda}\end{bmatrix}\begin{bmatrix}|\alpha_i|^2 & \alpha_i\beta_i^*\\ \alpha_i^*\beta_i & |\beta_i|^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \sqrt{1-\lambda}\end{bmatrix} + \sum_i p_i\begin{bmatrix}0 & 0\\ 0 & \sqrt{\lambda}\end{bmatrix}\begin{bmatrix}|\alpha_i|^2 & \alpha_i\beta_i^*\\ \alpha_i^*\beta_i & |\beta_i|^2\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \sqrt{\lambda}\end{bmatrix}\\
&= \sum_i p_i\begin{bmatrix}|\alpha_i|^2 & \alpha_i\beta_i^*\sqrt{1-\lambda}\\ \alpha_i^*\beta_i\sqrt{1-\lambda} & |\beta_i|^2\end{bmatrix}.
\end{aligned}
$$

Comparing both results we conclude that the circuit is indeed a model for phase damping, and that given the probability λ of the scattering occuring, we must choose $p = \left(1+\sqrt{1-\lambda}\right)/2$.

8.29

Any quantum process applied to the identity can be written as $\mathcal{E}(I) = \sum_k E_k I E_k^\dagger = \sum_k E_k E_k^\dagger$. Therefore, a process is unital if $\sum_k E_k E_k^\dagger = I$. Using the operation elements of the depolarizing, phase damping, and amplitude damping channels we get

$$
\mathcal{E}_{\text{D}}(I) = (1-p)I^2 + \frac{p}{3}X^2 + \frac{p}{3}Y^2 + \frac{p}{3}Z^2 = I,
$$

$$\mathcal{E}_{\mathrm{PD}}(I) = \begin{bmatrix} 1 & 0 \\ 0 & 1-\lambda \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = I,$$

$$\mathcal{E}_{\mathrm{AD}}(I) = \begin{bmatrix} 1 & 0 \\ 0 & 1-\gamma \end{bmatrix} + \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix} = I + \gamma Z.$$

8.30

**From errata:* T_2 should be replaced by $2T_2$.

Given a general density operator, the amplitude damping channel produces (see Exercise 8.22)

$$\mathcal{E}_{\mathrm{AD}}(\rho) = \begin{bmatrix} 1-(1-a)(1-\gamma) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & (1-a)(1-\gamma) \end{bmatrix}.$$

The decay of the elements in the diagonal is given by the term $1-\gamma$ while for the off-diagonal ones is $\sqrt{1-\gamma}$. This means that $1-\gamma = e^{-t/T_1}$, and $\sqrt{1-\gamma} = e^{-t/2T_2}$ and thus

$$e^{-t/T_1} = \left(e^{-t/2T_2}\right)^2 = e^{-t/T_2} \quad \Longrightarrow \quad T_2 = T_1.$$

If phase damping is also applied then the final state will be

$$\mathcal{E}_{\mathrm{PD}}(\mathcal{E}_{\mathrm{AD}}(\rho)) = \begin{bmatrix} 1-(1-a)(1-\gamma) & b\sqrt{1-\gamma}\sqrt{1-\lambda} \\ b^*\sqrt{1-\gamma}\sqrt{1-\lambda} & (1-a)(1-\gamma) \end{bmatrix},$$

and now we have $1-\gamma = e^{-t/T_1}$, and $\sqrt{1-\gamma}\sqrt{1-\lambda} = e^{-t/2T_2}$. Since $1-\lambda \leq 1$, we have

$$e^{-t/T_1} \geq (1-\gamma)(1-\lambda) = e^{-t/T_2} \quad \Longrightarrow \quad T_2 \leq T_1.$$

8.31

The unitary operator will be

$$U \equiv e^{-i\chi a^\dagger a(b+b^\dagger)\Delta t} = e^{-iA(b+b^\dagger)},$$

where we have defined the Hermitian operator $A \equiv (\chi\Delta t)a^\dagger a$. Using the Baker-Campbell-Hausdorff formula, we can rewrite it as (see Exercise **??**)

$$U = e^{-iAb^\dagger} e^{-iAb} e^{\frac{A^2}{2}\left[b^\dagger, b\right]} + O(A^3).$$

But since $\left[b^\dagger, b\right] = -1$, and the terms of $O(A^3)$ involves commutation relations with this quantity, they all vanish because it is a constant. Therefore, the unitary operator can be written exactly as

$$U = e^{-iAb} e^{-iAb^\dagger} e^{-\frac{A^2}{2}}.$$

We are considering the total state to be $\rho \otimes |0\rangle\langle 0|$, thus the action of the unitary operator yields

$$U\left(\rho \otimes |0\rangle\langle 0|\right)U^{\dagger} = \sum_{n,m} \rho_{nm} e^{-iAb^{\dagger}} e^{-iAb} e^{-\frac{A^2}{2}} \left(|n\rangle\langle m| \otimes |0\rangle\langle 0|\right) e^{-\frac{A^2}{2}} e^{iAb^{\dagger}} e^{iAb}.$$

We have that

$$\begin{aligned}
e^{-iAb^{\dagger}} e^{-iAb} e^{-\frac{A^2}{2}} |n\rangle \otimes |0\rangle &= e^{-\frac{(\chi\Delta t)^2}{2}n^2} |n\rangle \otimes e^{-i(\chi\Delta t)nb^{\dagger}} e^{-i(\chi\Delta t)nb} |0\rangle \\
&= e^{-\frac{(\chi\Delta t)^2}{2}n^2} |n\rangle \otimes \sum_{j,k=0}^{\infty} \frac{(-in\chi\Delta t)^{j+k}}{j!k!} \left(b^{\dagger}\right)^k b^j |0\rangle \\
&= e^{-\frac{(\chi\Delta t)^2}{2}n^2} |n\rangle \otimes \sum_{k=0}^{\infty} \frac{(-in\chi\Delta t)^{k}}{k!} \left(b^{\dagger}\right)^k |0\rangle \\
&= e^{-\frac{(\chi\Delta t)^2}{2}n^2} |n\rangle \otimes \sum_{k=0}^{\infty} \frac{(-in\chi\Delta t)^{k}}{\sqrt{k!}} |k\rangle .
\end{aligned}$$

Substituting this result back we obtain

$$U\left(\rho \otimes |0\rangle\langle 0|\right)U^{\dagger} = \sum_{n,m} \rho_{nm} e^{-\frac{(\chi\Delta t)^2}{2}(n^2+m^2)} |n\rangle\langle m| \otimes \sum_{k,k'=0}^{\infty} \frac{(-in\chi\Delta t)^{k}}{\sqrt{k!}} \frac{(im\chi\Delta t)^{k'}}{\sqrt{k'!}} |k\rangle\langle k'| .$$

In order to get the resulting density operator of the harmonic oscillator we must trace out the environment, that is,

$$\begin{aligned}
\mathrm{tr}_{\mathrm{env}}\big(U(\rho \otimes |0\rangle\langle 0|)U^{\dagger}\big) &= \sum_{n,m} \rho_{nm} e^{-\frac{(\chi\Delta t)^2}{2}(n^2+m^2)} |n\rangle\langle m| \sum_{l} \sum_{k,k'=0}^{\infty} \frac{(-in\chi\Delta t)^{k}}{\sqrt{k!}} \frac{(im\chi\Delta t)^{k'}}{\sqrt{k'!}} \langle l \,|\, k\rangle \langle k' \,|\, l\rangle \\
&= \sum_{n,m} \rho_{nm} e^{-\frac{(\chi\Delta t)^2}{2}(n^2+m^2)} |n\rangle\langle m| \sum_{k=0}^{\infty} \frac{(-in\chi\Delta t)^{k}}{\sqrt{k!}} \frac{(im\chi\Delta t)^{k}}{\sqrt{k!}} \\
&= \sum_{n,m} \rho_{nm} e^{-\frac{(\chi\Delta t)^2}{2}(n^2+m^2)} |n\rangle\langle m| \sum_{k=0}^{\infty} \frac{(nm\chi^2\Delta t^2)^{k}}{k!} \\
&= \sum_{n,m} \rho_{nm} e^{-\frac{(\chi\Delta t)^2}{2}(n^2+m^2)} e^{(\chi\Delta t)^2 nm} |n\rangle\langle m| \\
&= \sum_{n,m} \rho_{nm} e^{-\frac{(\chi\Delta t)^2}{2}(n-m)^2} |n\rangle\langle m| .
\end{aligned}$$

Defining the constant $\lambda \equiv (\chi\Delta t)^2/2$, we see that the element ρ_{nm} decays as $e^{-\lambda(n-m)^2}$.

8.32

-

8.33

Given a single qubit density matrix ρ, we can always write it as

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} = \frac{I}{2} + \frac{r_x}{2}X + \frac{r_y}{2}Y + \frac{r_z}{2}Z,$$

where $r_x^2 + r_y^2 + r_z^2 \leq 1$. This can be rewritten as

$$\begin{aligned}
\rho &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{r_x}{2} \\ \frac{r_x}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i\frac{r_y}{2} \\ i\frac{r_y}{2} & 0 \end{bmatrix} + \begin{bmatrix} \frac{r_z}{2} & 0 \\ 0 & -\frac{r_z}{2} \end{bmatrix} \\
&= \frac{1+r_z}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1-r_z}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + r_x \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + r_y \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} - \frac{r_x + r_y}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \frac{1 - r_x - r_y + r_z}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1 - r_x - r_y - r_z}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + r_x \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + r_y \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}.
\end{aligned}$$

These four matrices are linearly independent and can be written in terms of the four pure states: $|0\rangle$, $|1\rangle$, $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$, and $|i_+\rangle \equiv (|0\rangle + i|1\rangle)/\sqrt{2}$. Thus any single qubit density matrix can be written as a combination of the four pure states

$$\rho = \frac{1 - r_x - r_y + r_z}{2}|0\rangle\langle 0| + \frac{1 - r_x - r_y - r_z}{2}|1\rangle\langle 1| + r_x|+\rangle\langle +| + r_y|i_+\rangle\langle i_+|.$$

This decomposition is, of course, not unique. Even mixed states could be used for writing the decomposition, however, the description using pure states exclusively is necessary since we can only get complete information about a quantum operation when it acts on a fully known state, that is, a pure state. Thus the action of a quantum operation will be

$$\mathcal{E}(\rho) = \frac{1 - r_x - r_y + r_z}{2}\mathcal{E}(|0\rangle\langle 0|) + \frac{1 - r_x - r_y - r_z}{2}\mathcal{E}(|1\rangle\langle 1|) + r_x\mathcal{E}(|+\rangle\langle +|) + r_y\mathcal{E}(|i_+\rangle\langle i_+|).$$

So a quantum operation $\mathcal{E}$ can be fully specified if we know how it acts on a set of at least four points on the Bloch sphere.

8.34

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8.35

From the four given density operators we can conclude that $\mathcal{E}_1$ is the amplitude damping channel (see Exercise 8.22). We shall use it to check the final answer. We are using

$$\rho_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \rho_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \rho_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \rho_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

as the 2×2 matrix basis, and

$$\widetilde{E}_0 = I, \widetilde{E}_1 = X, \widetilde{E}_2 = -iY, \widetilde{E}_3 = Z$$

as basis for the operation elements. We can determine the elements λ_{jk} with the relation $\mathcal{E}_1(\rho_j) \equiv \rho'_j = \sum_k \lambda_{jk}\rho_k$. Using the experimentally obtained density matrices we have

$$\begin{bmatrix} \rho'_1 \\ \rho'_2 \\ \rho'_3 \\ \rho'_4 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \sqrt{1-\gamma}\rho_2 \\ \sqrt{1-\gamma}\rho_3 \\ \gamma\rho_1 + (1-\gamma)\rho_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ \gamma & 0 & 0 & 1-\gamma \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{bmatrix},$$

$$\implies \quad \lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ \gamma & 0 & 0 & 1-\gamma \end{bmatrix}.$$

The elements β^{mn}_{jk} can be obtained with the relation $\widetilde{E}_m\rho_j\widetilde{E}^\dagger_n = \sum_k \beta^{mn}_{jk}\rho_k$. β can be thought of as a 4×4 matrix with elements β_{jk}, where each β_{jk} is in turn a 4×4 matrix with elements β^{mn}_{jk}. To avoid working with too many nested matrices, let us consider the relation for each row j of β separately. Starting with $j=1$, we obtain

$$\begin{bmatrix} I\rho_1 I & I\rho_1 X & I\rho_1 iY & I\rho_1 Z \\ X\rho_1 I & X\rho_1 X & X\rho_1 iY & X\rho_1 Z \\ -iY\rho_1 I & -iY\rho_1 X & -iY\rho_1 iY & -iY\rho_1 Z \\ Z\rho_1 I & Z\rho_1 X & Z\rho_1 iY & Z\rho_1 Z \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 & \rho_2 & \rho_1 \\ \rho_3 & \rho_4 & \rho_4 & \rho_3 \\ \rho_3 & \rho_4 & \rho_4 & \rho_3 \\ \rho_1 & \rho_2 & \rho_2 & \rho_1 \end{bmatrix} = \sum_k \beta_{1k}\rho_k,$$

meaning

$$\beta_{11} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \beta_{12} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \beta_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $j=2$ we obtain

$$\begin{bmatrix} I\rho_2 I & I\rho_2 X & I\rho_2 iY & I\rho_2 Z \\ X\rho_2 I & X\rho_2 X & X\rho_2 iY & X\rho_2 Z \\ -iY\rho_2 I & -iY\rho_2 X & -iY\rho_2 iY & -iY\rho_2 Z \\ Z\rho_2 I & Z\rho_2 X & Z\rho_2 iY & Z\rho_2 Z \end{bmatrix} = \begin{bmatrix} \rho_2 & \rho_1 & -\rho_1 & -\rho_2 \\ \rho_4 & \rho_3 & -\rho_3 & -\rho_4 \\ \rho_4 & \rho_3 & -\rho_3 & -\rho_4 \\ \rho_2 & \rho_1 & -\rho_1 & -\rho_2 \end{bmatrix} = \sum_k \beta_{2k}\rho_k,$$

meaning

$$\beta_{21} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \beta_{22} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \beta_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $j = 3$ we obtain

$$\begin{bmatrix} I\rho_3 I & I\rho_3 X & I\rho_3 iY & I\rho_3 Z \\ X\rho_3 I & X\rho_3 X & X\rho_3 iY & X\rho_3 Z \\ -iY\rho_3 I & -iY\rho_3 X & -iY\rho_3 iY & -iY\rho_3 Z \\ Z\rho_3 I & Z\rho_3 X & Z\rho_3 iY & Z\rho_3 Z \end{bmatrix} = \begin{bmatrix} \rho_3 & \rho_4 & \rho_4 & \rho_3 \\ \rho_1 & \rho_2 & \rho_2 & \rho_1 \\ -\rho_1 & -\rho_2 & -\rho_2 & -\rho_1 \\ -\rho_3 & -\rho_4 & -\rho_4 & -\rho_3 \end{bmatrix} = \sum_k \beta_{3k}\rho_k,$$

meaning

$$\beta_{31} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_{33} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, \beta_{34} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}.$$

And finally, for $j = 4$ we obtain

$$\begin{bmatrix} I\rho_4 I & I\rho_4 X & I\rho_4 iY & I\rho_4 Z \\ X\rho_4 I & X\rho_4 X & X\rho_4 iY & X\rho_4 Z \\ -iY\rho_4 I & -iY\rho_4 X & -iY\rho_4 iY & -iY\rho_4 Z \\ Z\rho_4 I & Z\rho_4 X & Z\rho_4 iY & Z\rho_4 Z \end{bmatrix} = \begin{bmatrix} \rho_4 & \rho_3 & -\rho_3 & -\rho_4 \\ \rho_2 & \rho_1 & -\rho_1 & -\rho_2 \\ -\rho_2 & -\rho_1 & \rho_1 & \rho_2 \\ -\rho_4 & -\rho_3 & \rho_3 & \rho_4 \end{bmatrix} = \sum_k \beta_{4k}\rho_k,$$

meaning

$$\beta_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_{34} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \beta_{44} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

In order to obtain the entries of the χ matrix we use the relation $\sum_{mn} \beta_{jk}^{mn}\chi_{mn} = \lambda_{jk}$, but we need to invert the β matrix. Since we are treating it as a 4×4 block matrix of 4×4 matrices this is an inconvenient task. However, notice that the relation shows that each entry λ_{jk}, which is a number, is uniquely related to β_{jk}, which is a 4×4 matrix. The number is calculated by summing up every entry of the β_{jk} matrix multiplied by each element of the χ matrix. Therefore, it can be interpreted as if λ and χ were column vectors of 16 entries each, where we append each row of the 4×4 matrix forming a single column, while β can be thought of as a 16×16 matrix where each row is built by flattening each β_{jk} into a single row. In practice, we are considering

$$\lambda = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{1-\gamma} \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{1-\gamma} \\ 0 \\ \gamma \\ 0 \\ 0 \\ 1-\gamma \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Calculating the inverse of β yields the matrix

$$\kappa = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Calculating $\chi = \kappa\lambda$ and rewriting χ in a 4×4 matrix format results in

$$\chi = \frac{1}{4} \begin{bmatrix} 2-\gamma+2\sqrt{1-\gamma} & 0 & 0 & \gamma \\ 0 & \gamma & -\gamma & 0 \\ 0 & -\gamma & \gamma & 0 \\ \gamma & 0 & 0 & 2-\gamma-2\sqrt{1-\gamma} \end{bmatrix}.$$

In order to check this result, we can calculate the operation elements E_i from it. As it can be readily checked, this matrix has only two non-zero eigenvalues, given by $1-\gamma/2$ and $\gamma/2$, meaning $\mathcal{E}_1$ is described by only two operation elements. The two respective normalized eigenvectors are

$$u = \frac{1}{2\sqrt{1-\frac{\gamma}{2}}}\begin{bmatrix} 1+\sqrt{1-\gamma} \\ 0 \\ 0 \\ 1-\sqrt{1-\gamma} \end{bmatrix} \quad \text{and} \quad v = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

yielding operation elements

$$E_0 = \sqrt{1-\frac{\gamma}{2}}\sum_j u_j \widetilde{E}_j = \frac{1}{2}\left[\left(1+\sqrt{1-\gamma}\right) I + \left(1-\sqrt{1-\gamma}\right) Z\right] = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix},$$

$$E_1 = \sqrt{\frac{\gamma}{2}}\sum_j v_j \widetilde{E}_j = \frac{\sqrt{\gamma}}{2}(X+iY) = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}.$$

These are the operation elements for the amplitude damping channel, confirming that the obtained χ matrix is indeed correct.