2 Introduction to quantum mechanics

Exercises: 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21, 2.22, 2.23, 2.24, 2.25, 2.26, 2.27, 2.28, 2.29, 2.30, 2.31, 2.32, 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, 2.39, 2.40, 2.41, 2.42, 2.43, 2.44, 2.45, 2.46, 2.47, 2.48, 2.49, 2.50, 2.51, 2.52, 2.53, 2.54, 2.55, 2.56, 2.57, 2.58, 2.59, 2.60, 2.61, 2.62, 2.63, 2.64, 2.65, 2.66, 2.67, 2.68, 2.69, 2.70, 2.71, 2.72, 2.73, 2.74, 2.75, 2.76, 2.77, 2.78, 2.79, 2.80, 2.81, 2.82.

2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0.$$

2.2

$$\begin{aligned} A\left|0\right\rangle &= \left|1\right\rangle = 0\left|0\right\rangle + 1\left|1\right\rangle, \\ A\left|1\right\rangle &= \left|0\right\rangle = 1\left|0\right\rangle + 0\left|1\right\rangle. \end{aligned}$$

So writing the basis vectors as

$$\left|0\right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \left|1\right\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we have

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

2.3

$$\begin{aligned} BA\left|v_i\right\rangle &= B\left(\sum_j A_{ji}\left|w_j\right\rangle\right) \\ &= \sum_j A_{ji} B\left|w_j\right\rangle \\ &= \sum_j A_{ji} \sum_k B_{kj}\left|x_k\right\rangle \\ &= \sum_j \sum_k B_{kj} A_{ji}\left|x_k\right\rangle, \end{aligned}$$

but $\sum_j B_{kj} A_{ji}$ is precisely B's k-th row multiplied by A's i-th column (matrix multiplication), thus we have

$$BA\left|v_i\right\rangle = \sum_k (BA)_{ki}\left|x_k\right\rangle.$$

2.4

$$I\left|v_i\right\rangle = \sum_j I_{ji}\left|v_j\right\rangle = \left|v_i\right\rangle,$$

that is, I is such that $I_{ji} = 0$ for $j \neq i$ and $I_{ji} = 1$ for $j = i$, meaning I is represented by the identity matrix.

2.5

For $\left|v\right\rangle$ and $\left|w_i\right\rangle \in \mathbb{C}^n$, where

$$\left|v\right\rangle = \left(v_1, \cdots, v_n\right), \left|w_i\right\rangle = \left(w_{i1}, \cdots, w_{in}\right),$$

we have property (1):

$$\begin{aligned}\left(\left|v\right\rangle, \sum_i \lambda_i \left|w_i\right\rangle\right) = \left(\left(v_1, \cdots, v_n\right), \sum_i \lambda_i \left(w_{i1}, \cdots, w_{in}\right)\right) &= \sum_j v_j^* \sum_i \lambda_i w_{ij} \\ &= \sum_i \lambda_i \sum_j v_j^* w_{ij} \\ &= \sum_i \lambda_i \left(\left(v_1, \cdots, v_n\right), \left(w_{i1}, \cdots, w_{in}\right)\right) \\ &= \sum_i \lambda_i \left(\left|v\right\rangle, \left|w_i\right\rangle\right),\end{aligned}$$

property (2):

$$\begin{aligned}\left(\left|v\right\rangle, \left|w_i\right\rangle\right) &= \sum_j v_j^* w_{ij} \\ &= \left(\sum_j w_{ij}^* v_j\right)^* \\ &= \left(\left|w_i\right\rangle, \left|v\right\rangle\right)^*,\end{aligned}$$

and property (3):

$$\begin{aligned}\left(\left|v\right\rangle, \left|v\right\rangle\right) &= \sum_i v_i^* v_i \\ &= \sum_i \left|v_i\right|^2 \geq 0.\end{aligned}$$

2.6

Using property (2) we have

$$\left(\sum_i \lambda_i \left|w_i\right\rangle, \left|v\right\rangle\right) = \left(\left|v\right\rangle, \sum_i \lambda_i \left|w_i\right\rangle\right)^*,$$

and using properties (1) and (2) yields

$$\begin{aligned}
\left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right)^* &= \left(\sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right)\right)^* \\
&= \sum_i \lambda_i^* \left(|w_i\rangle, |v\rangle\right).
\end{aligned}$$

2.7

$$\langle w \,|\, v\rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0.$$

$$\| |w\rangle \| = \| |v\rangle \| = \sqrt{2},$$

so the normalized forms are given by

$$\frac{|w\rangle}{\| |w\rangle \|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{|v\rangle}{\| |v\rangle \|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

2.8

For $|v_1\rangle$ and $|v_2\rangle$ we have

$$\begin{aligned}
\langle v_1 \,|\, v_2\rangle &= \left(\frac{\langle w_1|}{\| |w_1\rangle \|}\right) \left(\frac{|w_2\rangle \langle v_1 \,|\, w_2\rangle |v_1\rangle}{\| |w_2\rangle \|}\right) \\
&= \frac{\langle w_1 \,|\, w_2\rangle - \langle v_1 \,|\, w_2\rangle \langle w_1 \,|\, v_1\rangle}{\| |w_1\rangle \| \| |w_2\rangle \|} \\
&= \frac{\langle w_1 \,|\, w_2\rangle - \frac{\langle w_1 \,|\, w_2\rangle}{\| |w_1\rangle \|} \|w_1\|}{\| |w_1\rangle \| \| |w_2\rangle \|} \\
&= 0.
\end{aligned}$$

Because it is a structure constructed inductively the same should hold for any $|v_i\rangle$. So by the end of the Gram–Schmidt process we have $\dim(V)$ vectors $|v_i\rangle$ satisfying $\langle v_i \,|\, v_j\rangle = \delta_{ij}$, thus the set $\{|v_i\rangle\}$ is an orthonormal basis for V.

2.9

$$\begin{aligned}
I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|, \\
X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|, \\
Y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|,
\end{aligned}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1| \,.$$

2.10

All matrix elements can be calculated as

$$\langle v_l \,|\, v_j \rangle \langle v_k \,|\, v_m \rangle = \delta_{lj}\delta_{km},$$

thus it is an operator whose matrix representation has value 1 at the j'th row and k'th column and 0 everywhere else.

2.11

$$X \;:\; \det(X - \lambda I) = \lambda^2 - 1 = 0 \quad \Longrightarrow \quad \text{eigenvalues} = \{-1, 1\} \,.$$

For eigenvalue 1 :

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvector} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \,.$$

For eigenvalue -1 :

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -\begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvector} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \,.$$

$$Y \;:\; \det(Y - \lambda I) = \lambda^2 - 1 = 0 \quad \Longrightarrow \quad \text{eigenvalues} = \{-1, 1\} \,.$$

For eigenvalue 1 :

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvector} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + i\,|1\rangle) \,.$$

For eigenvalue -1 :

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -\begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvector} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - i\,|1\rangle) \,.$$

$$Z \;:\; \text{is already diagonal in the computational basis} \quad \Longrightarrow \quad \text{eigenvalues} = \{-1, 1\} \,.$$

For eigenvalue 1 :

$$\text{eigenvector} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \,.$$

For eigenvalue -1 :

$$\text{eigenvector} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \,.$$

2.12

$$\det\begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = 0 \quad \Longrightarrow \quad \text{eigenvalue} = 1 \text{ (degenerate)}.$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad a+b=b \quad \Longrightarrow \quad \text{eigenvector} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle\,.$$

And there is no other eigenvector for the degenerate eigenvalue, thus this matrix is not diagonalizable.

2.13

$$\begin{aligned}
\Big(|a\rangle\,,(|w\rangle\langle v|)^\dagger\,|b\rangle\Big) &= (\langle v\,|\,a\rangle\,|w\rangle\,,|b\rangle) \\
&= \langle v\,|\,a\rangle^*\,(|w\rangle\,,|b\rangle) \\
&= \langle a\,|\,v\rangle\,\langle w\,|\,b\rangle \\
&= (|a\rangle\,,(|v\rangle\langle w|)\,|b\rangle)\,.
\end{aligned}$$

2.14

$$\begin{aligned}
\left(\left(\sum_i a_i A_i\right)^\dagger |v\rangle\,,|w\rangle\right) &= \left(|v\rangle\,,\sum_i a_i A_i\,|w\rangle\right) \\
&= \sum_i a_i\,(|v\rangle\,,A_i\,|w\rangle) \\
&= \sum_i a_i\,\Big(A_i^\dagger\,|v\rangle\,,|w\rangle\Big) \\
&= \left(\sum_i a_i^* A_i^\dagger\,|v\rangle\,,|w\rangle\right).
\end{aligned}$$

2.15

$$\begin{aligned}
\Big(\big(A^\dagger\big)^\dagger\,|v\rangle\,,|w\rangle\Big) &= \big(|v\rangle\,,A^\dagger\,|w\rangle\big) \\
&= \big(A^\dagger\,|w\rangle\,,|v\rangle\big)^* \\
&= (|w\rangle\,,A\,|v\rangle)^* \\
&= (A\,|v\rangle\,,|w\rangle)\,.
\end{aligned}$$

2.16

$$P^2 = \left(\sum_{i=1}^k |i\rangle\langle i|\right)\left(\sum_{j=1}^k |j\rangle\langle j|\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{k}\sum_{j=1}^{k} |i\rangle \langle i \,|\, j\rangle \,|j\rangle \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} |i\rangle\langle j|\, \delta_{ij} \\
&= \sum_{i=1}^{k} |i\rangle\langle i| = P.
\end{aligned}
$$

2.17

Let H be a normal operator. Then there exists an operator M that diagonalizes H to H_d, that is

$$
\begin{aligned}
H &= M^\dagger H_d M, \\
H^\dagger &= M^\dagger H_d^\dagger M.
\end{aligned}
$$

If H has real eigenvalues then $H_d^\dagger = H_d$, thus

$$
H^\dagger = M^\dagger H_d^\dagger M = M^\dagger H_d M = H.
$$

Conversely, if H is Hermitian then $H_d = H_d^\dagger$, which means all eigenvalues are real.

2.18

Let $|v\rangle$ be an eigenvector of U. Then

$$
\begin{aligned}
U\,|v\rangle &= \lambda\,|v\rangle\,, \\
\langle v|\,U^\dagger &= \lambda^*\,\langle v|\,.
\end{aligned}
$$

But we must have

$$
\left\langle v \,\middle|\, U^\dagger U \,\middle|\, v\right\rangle = \lambda^*\lambda\,\langle v \,|\, v\rangle = \langle v \,|\, v\rangle \quad\Longrightarrow\quad \lambda^*\lambda = 1 \quad\Longrightarrow\quad \lambda = e^{i\theta}.
$$

2.19

$$
\begin{aligned}
X\ :X^\dagger &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X \\
X^\dagger X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\
Y\ :Y^\dagger &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^\dagger = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y \\
Y^\dagger Y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
\end{aligned}
$$

$$Z\ : Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$
$$Z^\dagger Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

2.20

$$I = \sum_i |v_i\rangle\langle v_i| = \sum_i |w_i\rangle\langle w_i| \, .$$

Then A'_{ij} can be written as

$$\begin{aligned} A'_{ij} = \langle v_i \,|\, A \,|\, v_j\rangle &= \sum_k \sum_l \langle v_i \,|\, w_k\rangle \langle w_k \,|\, A \,|\, w_l\rangle \langle w_l \,|\, v_j\rangle \\ &= \sum_k \sum_l \langle v_i \,|\, w_k\rangle \, A''_{kl} \, \langle w_l \,|\, v_j\rangle \, . \end{aligned}$$

If $|v_i\rangle$ and $|w_i\rangle$ are both orthonormal bases then there exists a unitary U such that $|w_i\rangle = U\,|v_i\rangle$, thus

$$A'_{ij} = \sum_k \sum_l U_{ik} A''_{kl} U^\dagger_{lj}.$$

2.21

Let $|v\rangle \in V(\text{dim} = 1)$ be an eigenvector of $M = M^\dagger$ with eigenvalue λ and also element of subspace P. Then

$$\begin{aligned} M &= (P+Q)\, M\, (P+Q) \\ &= PMP + PMQ + QMP + QMQ. \end{aligned}$$

Because M is Hermitian we have that $PMP = \lambda P$ and QMQ are normal and diagonal with respect to an orthonormal basis for the subspaces P and Q respectively. Also

$$\begin{aligned} &QMP = 0, \\ &PMQ = (QMP)^\dagger = 0. \end{aligned}$$

Thus $M = PMP + QMQ$ is diagonal with respect to some orthonormal basis for space V. And by induction, this must be true for higher dimensional Hilbert spaces.

2.22

Let $|v\rangle$ and $|w\rangle$ be two eigenvectors of a Hermitian operator H with different eigenvalues. Then

$$\begin{aligned} H\,|v\rangle &= \alpha\,|v\rangle \, , \\ H\,|w\rangle &= \beta\,|w\rangle \, . \end{aligned}$$

But we must have

$$\langle w \,|\, H \,|\, v\rangle = \alpha \langle w \,|\, v\rangle = \beta \langle w \,|\, v\rangle \quad \Longrightarrow \quad (\alpha - \beta) \langle w \,|\, v\rangle = 0.$$

Since $\alpha \neq \beta$, $|v\rangle$ and $|w\rangle$ must be orthogonal.

2.23

Let $|v\rangle$ be an eigenvector of the projector P. Then

$$\begin{aligned} P\,|v\rangle &= \lambda\,|v\rangle\,, \\ P^2\,|v\rangle &= \lambda P\,|v\rangle = \lambda^2\,|v\rangle\,. \end{aligned}$$

Since P is a projector, $P = P^2$, thus

$$\lambda = \lambda^2 \quad \Longrightarrow \quad \lambda = 0 \text{ or } 1.$$

2.24

Defining

$$B := \frac{A + A^\dagger}{2}, C := \frac{A - A^\dagger}{2i},$$

we can write operator A as $A = B + iC$. If A is positive then

$$(|v\rangle\,, A\,|v\rangle) = (|v\rangle\,, B\,|v\rangle) + i\,(|v\rangle\,, C\,|v\rangle) \geq 0.$$

But that is only possible if $C = 0 \;\Rightarrow\; A = A^\dagger$.

2.25

$$\begin{aligned} A\,|v\rangle &= |w\rangle\,, \\ \langle v|\,A^\dagger &= \langle w|\,, \end{aligned}$$

$$\Longrightarrow \quad \left\langle v \,\middle|\, A^\dagger A \,\middle|\, v\right\rangle = \langle w \,|\, w\rangle \geq 0.$$

2.26

$$|\psi\rangle^{\otimes 2} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{|0\rangle\,|0\rangle + |0\rangle\,|1\rangle + |1\rangle\,|0\rangle + |1\rangle\,|1\rangle}{2}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix},$$

$$\begin{aligned}
|\psi\rangle^{\otimes 3} &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{|0\rangle|0\rangle|0\rangle + |0\rangle|0\rangle|1\rangle + |0\rangle|1\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |1\rangle|0\rangle|0\rangle + |1\rangle|0\rangle|1\rangle + |1\rangle|1\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{2\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix},
\end{aligned}$$

2.27

$$X \otimes Z = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{bmatrix},$$

$$I \otimes X = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \otimes \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix},$$

$$X \otimes I = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}.$$

The last two examples show that the tensor product is non-commutative.

2.28

Let A be represented by an $m \times n$ matrix. Then

$$(A \otimes B)^* = \begin{bmatrix}A_{11}B & \cdots & A_{1n}B\\ \vdots & \ddots & \vdots\\ A_{m1}B & \cdots & A_{mn}B\end{bmatrix}^* = \begin{bmatrix}A_{11}^*B^* & \cdots & A_{1n}^*B^*\\ \vdots & \ddots & \vdots\\ A_{m1}^*B^* & \cdots & A_{mn}^*B^*\end{bmatrix} = A^* \otimes B^*,$$

$$(A \otimes B)^T = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T = \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} = A^T \otimes B^T,$$

$$(A \otimes B)^\dagger = ((A \otimes B)^*)^T = (A^* \otimes B^*)^T = (A^*)^T \otimes (B^*)^T = A^\dagger \otimes B^\dagger.$$

2.29

$$\begin{aligned} (U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) &= \left(U_1^\dagger \otimes U_2^\dagger\right)(U_1 \otimes U_2) \\ &= U_1^\dagger U_1 \otimes U_2^\dagger U_2 \\ &= I_1 \otimes I_2 \end{aligned}$$

2.30

$$\begin{aligned} H_1 \otimes H_2 &= H_1^\dagger \otimes H_2^\dagger \\ &= (H_1 \otimes H_2)^\dagger \end{aligned}$$

2.31

$$\begin{aligned} (|v\rangle \otimes |w\rangle, (A \otimes B)|v\rangle \otimes |w\rangle) &= (|v\rangle \otimes |w\rangle, A|v\rangle \otimes B|w\rangle) \\ &= \langle v | A | v\rangle \langle w | B | w\rangle. \end{aligned}$$

Since $\langle v | A | v\rangle \geq 0$ and $\langle w | B | w\rangle \geq 0$ it follows that

$$(|v\rangle \otimes |w\rangle, (A \otimes B)|v\rangle \otimes |w\rangle) \geq 0.$$

2.32

$$\begin{aligned} (P_1 \otimes P_2)^2 &= (P_1 \otimes P_2)(P_1 \otimes P_2) \\ &= P_1^2 \otimes P_2^2 \\ &= P_1 \otimes P_2. \end{aligned}$$

2.33

$$\begin{aligned} H &= \frac{1}{\sqrt{2}}\big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\big] \\ &= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\ &= \frac{1}{\sqrt{2}}\sum_{x,y=0}^{1}(-1)^{x\cdot y}|x\rangle\langle y|. \end{aligned}$$

Thus

$$\begin{aligned}
H^{\otimes n} &= \frac{1}{\sqrt{2}} \sum_{x_1,y_1=0}^{1} (-1)^{x_1\cdot y_1} |x_1\rangle\langle y_1| \otimes \cdots \otimes \frac{1}{\sqrt{2}} \sum_{x_n,y_n=0}^{1} (-1)^{x_n\cdot y_n} |x_n\rangle\langle y_n| \\
&= \frac{1}{\sqrt{2^n}} \sum_{x_1,y_1=0}^{1} \cdots \sum_{x_n,y_n=0}^{1} (-1)^{x_1\cdot y_1} \cdots (-1)^{x_n\cdot y_n} |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_n\rangle\langle y_n| .
\end{aligned}$$

If we define $|x\rangle$ and $|y\rangle$ as the bit sequence states

$$\begin{aligned}
|x\rangle &\coloneqq \bigotimes_{i=1}^{n} |x_i\rangle , \\
|y\rangle &\coloneqq \bigotimes_{i=1}^{n} |y_i\rangle ,
\end{aligned}$$

and $x \cdot y$ as the bitwise product

$$x \cdot y \coloneqq \bigoplus_{i=1}^{n} x_i \cdot y_i,$$

where $\oplus$ denotes sum modulo-2 (or the XOR logic operation), then we can write

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x\cdot y} |x\rangle\langle y| .$$

$$\begin{aligned}
H^{\otimes 2} = \frac{1}{\sqrt{2^2}} \big[&(-1)^{0\oplus 0} |00\rangle\langle 00| + (-1)^{0\oplus 0} |01\rangle\langle 00| + (-1)^{0\oplus 0} |10\rangle\langle 00| + (-1)^{0\oplus 0} |11\rangle\langle 00| \\
&+(-1)^{0\oplus 0} |00\rangle\langle 01| + (-1)^{0\oplus 1} |01\rangle\langle 01| + (-1)^{0\oplus 0} |10\rangle\langle 01| + (-1)^{0\oplus 1} |11\rangle\langle 01| \\
&+(-1)^{0\oplus 0} |00\rangle\langle 10| + (-1)^{0\oplus 0} |01\rangle\langle 10| + (-1)^{1\oplus 0} |10\rangle\langle 10| + (-1)^{1\oplus 0} |11\rangle\langle 10| \\
&+(-1)^{0\oplus 0} |00\rangle\langle 11| + (-1)^{0\oplus 1} |01\rangle\langle 11| + (-1)^{1\oplus 0} |10\rangle\langle 11| + (-1)^{1\oplus 1} |11\rangle\langle 11| \big]
\end{aligned}$$

$$\implies \quad H^{\otimes 2} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

2.34

$$\det \begin{bmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{bmatrix} = \lambda^2 - 8\lambda + 7 = 0 \quad \implies \quad \text{eigenvalues} = \{1, 7\}.$$

For eigenvalue 1 :

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \implies \quad \text{eigenvector} = |v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For eigenvalue 7 :

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 7\begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvector} = |v_7\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So labeling the matrix as M we have

$$\sqrt{M} = \sqrt{1}\,|v_1\rangle\langle v_1| + \sqrt{7}\,|v_7\rangle\langle v_7| = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\sqrt{7}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7} \end{bmatrix},$$

$$\log(M) = \log(1)\,|v_1\rangle\langle v_1| + \log(7)\,|v_7\rangle\langle v_7| = \frac{\log(7)}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

2.35

$$\begin{aligned} \vec{v}\cdot\vec{\sigma} &= v_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}. \end{aligned}$$

$$\det(\vec{v}\cdot\vec{\sigma} - \lambda I) = \lambda^2 - \left(v_1^2 + v_2^2 + v_3^2\right) = 0.$$

Since $\vec{v}$ is a unit vector, $v_1^2 + v_2^2 + v_3^2 = 1$ and the eigenvalues are $\{-1, 1\}$. Labeling the respective eigenvectors as $|e_-\rangle$ and $|e_+\rangle$ we have

$$\begin{aligned} I &= |e_+\rangle\langle e_+| + |e_-\rangle\langle e_-|\,, \\ \vec{v}\cdot\vec{\sigma} &= |e_+\rangle\langle e_+| - |e_-\rangle\langle e_-|\,. \end{aligned}$$

$$\begin{aligned} \exp(i\theta\vec{v}\cdot\vec{\sigma}) &= \exp(i\theta)\,|e_+\rangle\langle e_+| + \exp(-i\theta)\,|e_-\rangle\langle e_-| \\ &= \cos(\theta)\,|e_+\rangle\langle e_+| + i\sin(\theta)\,|e_+\rangle\langle e_+| + \cos(\theta)\,|e_-\rangle\langle e_-| - i\sin(\theta)\,|e_-\rangle\langle e_-| \\ &= \cos(\theta)\,(|e_+\rangle\langle e_+| + |e_-\rangle\langle e_-|) + i\sin(\theta)\,(|e_+\rangle\langle e_+| - |e_-\rangle\langle e_-|) \\ &= \cos(\theta) I + i\sin(\theta)\vec{v}\cdot\vec{\sigma}. \end{aligned}$$

2.36

$$\begin{aligned} \mathrm{tr}(X) &= 0 + 0 = 0, \\ \mathrm{tr}(Y) &= 0 + 0 = 0, \\ \mathrm{tr}(Z) &= 1 + (-1) = 0. \end{aligned}$$

2.37

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_i \langle i \,|\, AB \,|\, i \rangle \\
&= \sum_i \sum_j \langle i \,|\, A \,|\, j \rangle \langle j \,|\, B \,|\, i \rangle \\
&= \sum_i \sum_j \langle j \,|\, B \,|\, i \rangle \langle i \,|\, A \,|\, j \rangle \\
&= \sum_j \langle j \,|\, BA \,|\, j \rangle \\
&= \operatorname{tr}(BA).
\end{aligned}$$

2.38

$$\begin{aligned}
\operatorname{tr}(A+B) &= \sum_i \langle i \,|\, (A+B) \,|\, i \rangle \\
&= \sum_i \langle i \,|\, A \,|\, i \rangle + \sum_i \langle i \,|\, B \,|\, i \rangle \\
&= \operatorname{tr}(A) + \operatorname{tr}(B),
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr}(zA) &= \sum_i \langle i \,|\, (zA) \,|\, i \rangle \\
&= z \sum_i \langle i \,|\, A \,|\, i \rangle \\
&= z \operatorname{tr}(A).
\end{aligned}$$

2.39

For linear operators A and B_i acting on V, and $z_i \in \mathbb{C}$, we have property (1):

$$\begin{aligned}
\left(A, \sum_i z_i B_i\right) = \operatorname{tr}\left(\sum_i z_i A^\dagger B_i\right) &= \sum_i z_i \operatorname{tr}\big(A^\dagger B_i\big) \\
&= \sum_i z_i \left(A, B_i\right),
\end{aligned}$$

property (2):

$$\begin{aligned}
(A, B_i) = \operatorname{tr}\big(A^\dagger B_i\big) &= \sum_j \left\langle j \,\middle|\, A^\dagger B_i \,\middle|\, j \right\rangle \\
&= \left(\sum_j \left\langle j \,\middle|\, B_i^\dagger A \,\middle|\, j \right\rangle\right)^* \\
&= \left(\operatorname{tr}\left(B_i^\dagger A\right)\right)^* \\
&= \left(B_i, A\right)^*,
\end{aligned}$$

and property (3):

$$
\begin{aligned}
(A, A) = \operatorname{tr}\bigl(A^\dagger A\bigr) &= \sum_i \langle i \,|\, A^\dagger A \,|\, i \rangle \\
&= \sum_i \| A \,| i \rangle \|^2 \geq 0.
\end{aligned}
$$

If $\dim(V) = d$ the transformations $V \to V$ can be represented by $d \times d$ matrices $M \in L_V$, meaning there are d^2 independent parameters necessary to write a transformation matrix, thus $\dim(L_V) = d^2$.

If $|i\rangle$ for $i \in \{1, \cdots, d\}$ is an orthonormal basis for V then the set of d^2 operators $\{|i\rangle\langle j|\}$ for $i, j \in \{1, \cdots, d\}$ forms an orthonormal basis for L_V, because for any i, j, k and l we have

$$
\begin{aligned}
(|i\rangle\langle j| \,, |k\rangle\langle l|) = \operatorname{tr}\Bigl((|i\rangle\langle j|)^\dagger \, |k\rangle\langle l| \Bigr) &= \sum_{m=1}^{d} \langle m \,|\, j \rangle \, \langle i \,|\, k \rangle \, \langle l \,|\, m \rangle \\
&= \delta_{ik} \sum_{m=1}^{d} \langle l \,|\, m \rangle \, \langle m \,|\, j \rangle \\
&= \delta_{ik} \delta_{jl}.
\end{aligned}
$$

2.40

$$
\begin{aligned}
[X, Y] &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = 2iZ, \\
[Y, Z] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2iX, \\
[Z, X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = 2iY.
\end{aligned}
$$

2.41

$$
\begin{aligned}
\{Y, X\} = \{X, Y\} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0, \\
\{Z, Y\} = \{Y, Z\} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = 0, \\
\{X, Z\} = \{Z, X\} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0,
\end{aligned}
$$

$$
X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,
$$

$$
\begin{aligned}
Y^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \\
Z^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.
\end{aligned}
$$

2.42

$$
\begin{aligned}
AB &= \frac{1}{2}\left(2AB + BA - BA\right) \\
&= \frac{1}{2}\left(AB - BA + AB + BA\right) \\
&= \frac{[A, B] + \{A, B\}}{2}.
\end{aligned}
$$

2.43

$$
\begin{aligned}
\sigma_j \sigma_k = \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2} &= \frac{2i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l + 2\delta_{jk} I}{2} \\
&= \delta_{jk} I + i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l.
\end{aligned}
$$

2.44

If $[A, B] = \{A, B\} = 0$ and A is invertible, then

$$
[A, B] + \{A, B\} = 2AB = 0 \quad \Longrightarrow \quad 2A^{-1}AB = 0 \quad \Longrightarrow \quad B = 0.
$$

2.45

$$
\begin{aligned}
[A, B]^\dagger &= (AB - BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= \left[B^\dagger, A^\dagger\right].
\end{aligned}
$$

2.46

$$
\begin{aligned}
[A, B] &= AB - BA \\
&= -\left(BA - AB\right) \\
&= -[B, A].
\end{aligned}
$$

2.47

$$
i[A, B] = iAB - iBA
$$

$$
\begin{aligned}
&= \left(-iB^\dagger A^\dagger + iA^\dagger B^\dagger\right)^\dagger \\
&= (iAB - iBA)^\dagger \\
&= (i[A,B])^\dagger .
\end{aligned}
$$

2.48

$$
\begin{aligned}
P &= IP = PI, \\
U &= UI = IU.
\end{aligned}
$$

By the spectral theorem H can be written as

$$
H = \sum_i \lambda_i \left|i\right\rangle\!\left\langle i\right| ,
$$

thus

$$
\begin{aligned}
\sqrt{H^\dagger H} = \sqrt{HH^\dagger} = \sqrt{H^2} &= \sqrt{\sum_i \sum_j \lambda_i \lambda_j \left|i\right\rangle \left\langle i \,|\, j\right\rangle \left\langle j\right|} \\
&= \sqrt{\sum_i \lambda_i^2 \left|i\right\rangle\!\left\langle i\right|} \\
&= \sum_i \left|\lambda_i\right| \left|i\right\rangle\!\left\langle i\right| \\
\implies \quad H &= U \sum_i \left|\lambda_i\right| \left|i\right\rangle\!\left\langle i\right| = \sum_i \left|\lambda_i\right| \left|i\right\rangle\!\left\langle i\right| U,
\end{aligned}
$$

where $U = H\left(\sqrt{H^2}\right)^{-1}$.

2.49

Let M be a normal matrix, then by the spectral theorem we can write

$$
M = \sum_i \lambda_i \left|i\right\rangle\!\left\langle i\right| ,
$$

thus

$$
\begin{aligned}
\sqrt{M^\dagger M} = \sqrt{MM^\dagger} &= \sqrt{\sum_i \sum_j \lambda_i \lambda_j^* \left|i\right\rangle \left\langle i \,|\, j\right\rangle \left\langle j\right|} \\
&= \sqrt{\sum_i \left|\lambda_i\right|^2 \left|i\right\rangle\!\left\langle i\right|} \\
&= \sum \left|\lambda_i\right| \left|i\right\rangle\!\left\langle i\right| \\
\implies \quad M &= U \sum_i \left|\lambda_i\right| \left|i\right\rangle\!\left\langle i\right| = \sum_i \left|\lambda_i\right| \left|i\right\rangle\!\left\langle i\right| U,
\end{aligned}
$$

where $U = M\left(\sqrt{M^\dagger M}\right)^{-1}$.

2.50

Labeling the matrix as M we have

$$\begin{aligned} J^2 = M^\dagger M &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \\ K^2 = MM^\dagger &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} &\det\big(J^2 - \lambda I\big) = \lambda^2 - 3\lambda + 1 = 0 &&\implies \text{eigenvalues} = \frac{3 \pm \sqrt{5}}{2}. \\ &\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{3 \pm \sqrt{5}}{2}\begin{bmatrix} a \\ b \end{bmatrix} &&\implies \text{eigenvectors} = |j_\pm\rangle = \frac{1}{\sqrt{10 \mp 2\sqrt{5}}}\begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}. \\ &\det\big(K^2 - \lambda I\big) = \lambda^2 - 3\lambda + 1 = 0 &&\implies \text{eigenvalues} = \frac{3 \pm \sqrt{5}}{2}. \\ &\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{3 \pm \sqrt{5}}{2}\begin{bmatrix} a \\ b \end{bmatrix} &&\implies \text{eigenvectors} = |k_\pm\rangle = \frac{1}{\sqrt{10 \pm 2\sqrt{5}}}\begin{bmatrix} 2 \\ 1 \pm \sqrt{5} \end{bmatrix}. \end{aligned}$$

So the left and right positive operators are given by

$$\begin{aligned} J &= \sqrt{\frac{3+\sqrt{5}}{2}}\,|j_+\rangle\langle j_+| + \sqrt{\frac{3-\sqrt{5}}{2}}\,|j_-\rangle\langle j_-| = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \\ K &= \sqrt{\frac{3+\sqrt{5}}{2}}\,|k_+\rangle\langle k_+| + \sqrt{\frac{3-\sqrt{5}}{2}}\,|k_-\rangle\langle k_-| = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \end{aligned}$$

and the unitary is given by

$$U = MJ^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

2.51

$$H^\dagger H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

2.52

$$H^2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

2.53

$$\det(H-\lambda I)=\lambda^2-1=0 \quad\Longrightarrow\quad \text{eigenvalues}=\{-1,1\}.$$

For eigenvalue 1 :

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}a\\ b\end{bmatrix} \quad\Longrightarrow\quad \text{eigenvector}=\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix}1\\ -1+\sqrt{2}\end{bmatrix}$$

For eigenvalue -1 :

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=-\begin{bmatrix}a\\ b\end{bmatrix} \quad\Longrightarrow\quad \text{eigenvector}=\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1\\ -1-\sqrt{2}\end{bmatrix}.$$

2.54

If $[A,B]=0$ then they share a common eigenbasis $\{|i\rangle\}$, that is

$$\begin{aligned}A&=\sum_i \alpha_i\,|i\rangle\langle i|\,,\\ B&=\sum_i \beta_i\,|i\rangle\langle i|\,.\end{aligned}$$

$$\begin{aligned}\Longrightarrow\quad \exp(A)\exp(B)&=\sum_i \exp(\alpha_i)\,|i\rangle\langle i|\sum_j \exp(\beta_j)\,|j\rangle\langle j|\\ &=\sum_i\sum_j \exp(\alpha_i+\beta_j)\,|i\rangle\,\langle i\,|\,j\rangle\,\langle j|\\ &=\sum_i \exp(\alpha_i+\beta_i)\,|i\rangle\langle i|\\ &=\exp(A+B).\end{aligned}$$

2.55

Considering that the Hamiltonian has a spectral decomposition $\sum_j E_j(t_1,t_2)\,|j\rangle\langle j|$ yields

$$\begin{aligned}U^\dagger(t_1,t_2)U(t_1,t_2)&=\exp\left(\frac{iH(t_1,t_2)}{\hbar}\right)\exp\left(\frac{-iH(t_1,t_2)}{\hbar}\right)\\ &=\sum_j\sum_k \exp\left(\frac{iE_j(t_1,t_2)}{\hbar}\right)\exp\left(\frac{-iE_k(t_1,t_2)}{\hbar}\right)|j\rangle\,\langle j\,|\,k\rangle\,\langle k|\\ &=\sum_j \exp\left(\frac{iE_j(t_1,t_2)-iE_j(t_1,t_2)}{\hbar}\right)|j\rangle\langle j|\\ &=\exp(0)\sum_j |j\rangle\langle j|\\ &=I.\end{aligned}$$

2.56

If U is unitary then it can be written as $\sum_j \exp(i\theta_j) |j\rangle\langle j|$, where $\theta_j \in \mathbb{R}$, thus

$$\begin{aligned}
K = -i\log(U) &= -i\sum_j \log[\exp(i\theta_j)] |j\rangle\langle j| \\
&= -i\sum_j i\theta_j |j\rangle\langle j| \\
&= \left(-i\sum_j i\theta_j |j\rangle\langle j|\right)^\dagger \\
&= K^\dagger.
\end{aligned}$$

2.57

After a measurement is performed over an initial state $|\psi\rangle$, using the operators set $\{L_l\}$, the state is transformed to the one associated with the measurement outcome l, given by

$$|\psi_l\rangle = \frac{L_l |\psi\rangle}{\sqrt{\left\langle \psi \,\middle|\, L_l^\dagger L_l \,\middle|\, \psi \right\rangle}}.$$

Then, if a measurement is performed over $|\psi_l\rangle$, using the operators set $\{M_m\}$, the final state can be written as

$$\begin{aligned}
|\psi_{lm}\rangle = \frac{M_m |\psi_l\rangle}{\sqrt{\left\langle \psi_l \,\middle|\, M_m^\dagger M_m \,\middle|\, \psi_l \right\rangle}} &= \frac{M_m}{\sqrt{\left(\frac{\langle\psi| L_l^\dagger}{\sqrt{\left\langle \psi \,\middle|\, L_l^\dagger L_l \,\middle|\, \psi \right\rangle}}\right) M_m^\dagger M_m \left(\frac{L_l|\psi\rangle}{\sqrt{\left\langle \psi \,\middle|\, L_l^\dagger L_l \,\middle|\, \psi \right\rangle}}\right)}} \left(\frac{L_l |\psi\rangle}{\sqrt{\left\langle \psi \,\middle|\, L_l^\dagger L_l \,\middle|\, \psi \right\rangle}}\right) \\
&= \frac{M_m L_l |\psi\rangle}{\sqrt{\left\langle \psi \,\middle|\, L_l^\dagger M_m^\dagger M_m L_l \,\middle|\, \psi \right\rangle}}.
\end{aligned}$$

Alternatively, if a measurement was to be performed over $|\psi\rangle$, using the operators set $\{N_{lm}\}$, the final state would be

$$\frac{N_{lm} |\psi\rangle}{\sqrt{\left\langle \psi \,\middle|\, N_{lm}^\dagger N_{lm} \,\middle|\, \psi \right\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\left\langle \psi \,\middle|\, L_l^\dagger M_m^\dagger M_m L_l \,\middle|\, \psi \right\rangle}} = |\psi_{lm}\rangle .$$

2.58

$$\langle M\rangle = \langle \psi \,|\, M \,|\, \psi\rangle = m \langle \psi \,|\, \psi\rangle = m,$$

$$\Delta(M) = \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{\langle \psi \,|\, MM \,|\, \psi\rangle - m^2} = \sqrt{m^2 - m^2} = 0.$$

2.59

$$\langle X \rangle = \langle 0 \,|\, X \,|\, 0 \rangle = \langle 0 \,|\, 1 \rangle = 0,$$

$$\Delta(X) = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{\langle 0 \,|\, I \,|\, 0 \rangle - 0} = \sqrt{1 - 0} = 1$$

2.60

$$\begin{aligned}
\vec{v} \cdot \vec{\sigma} &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}.
\end{aligned}$$

$$\det(\vec{v} \cdot \vec{\sigma} - \lambda I) = \lambda^2 - \left(v_1^2 + v_2^2 + v_3^2\right) = 0.$$

Since $\vec{v}$ is a unit vector, $v_1^2 + v_2^2 + v_3^2 = 1$ and the eigenvalues are $\{-1, 1\}$.

For eigenvalue 1:

$$\begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvector} = |e_+\rangle = \frac{1}{\sqrt{2 + 2v_3}} \begin{bmatrix} 1 + v_3 \\ v_1 + iv_2 \end{bmatrix}.$$

For eigenvalue -1:

$$\begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvector} = |e_-\rangle = \frac{1}{\sqrt{2 - 2v_3}} \begin{bmatrix} 1 - v_3 \\ -v_1 - iv_2 \end{bmatrix}.$$

The projector operators are then given by

$$\begin{aligned}
P_+ = |e_+\rangle\langle e_+| &= \frac{1}{\sqrt{2 + 2v_3}} \begin{bmatrix} 1 + v_3 \\ v_1 + iv_2 \end{bmatrix} \frac{1}{\sqrt{2 + 2v_3}} \begin{bmatrix} 1 + v_3 & v_1 - iv_2 \end{bmatrix} \\
&= \frac{1}{2\,(1 + v_3)} \begin{bmatrix} (1 + v_3)^2 & (1 + v_3)\,(v_1 - iv_2) \\ (1 + v_3)\,(v_1 + iv_2) & v_1^2 + v_2^2 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{bmatrix} \\
&= \frac{1}{2}\,(I + \vec{v} \cdot \vec{\sigma}) \\
P_- = |e_-\rangle\langle e_-| &= \frac{1}{\sqrt{2 - 2v_3}} \begin{bmatrix} 1 - v_3 \\ -v_1 - iv_2 \end{bmatrix} \frac{1}{\sqrt{2 - 2v_3}} \begin{bmatrix} 1 - v_3 & -v_1 + iv_2 \end{bmatrix} \\
&= \frac{1}{2\,(1 - v_3)} \begin{bmatrix} (1 - v_3)^2 & (1 - v_3)\,(-v_1 + iv_2) \\ (1 - v_3)\,(-v_1 - iv_2) & v_1^2 + v_2^2 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\begin{bmatrix} 1 - v_3 & -v_1 + iv_2 \\ -v_1 - iv_2 & 1 + v_3 \end{bmatrix} \\
&= \frac{1}{2}\left(I - \vec{v}\cdot\vec{\sigma}\right).
\end{aligned}
$$

2.61

$$
\begin{aligned}
p(1) = \langle 0 \,|\, P_+ \,|\, 0\rangle &= \frac{1}{2}\left(\langle 0 \,|\, I \,|\, 0\rangle + \langle 0 \,|\, \vec{v}\cdot\vec{\sigma} \,|\, 0\rangle\right) \\
&= \frac{1}{2}\left(\langle 0 \,|\, I \,|\, 0\rangle + v_1\langle 0 \,|\, X \,|\, 0\rangle + v_2\langle 0 \,|\, Y \,|\, 0\rangle + v_3\langle 0 \,|\, Z \,|\, 0\rangle\right) \\
&= \frac{1}{2}\left(1 + 0 + 0 + v_3\right) \\
&= \frac{1 + v_3}{2}.
\end{aligned}
$$

$$
\begin{aligned}
|\psi\rangle = \frac{P_+\,|0\rangle}{\sqrt{\langle 0 \,|\, P_+ \,|\, 0\rangle}} &= \frac{1}{2}\left(\frac{I\,|0\rangle + v_1 X\,|0\rangle + v_2 Y\,|0\rangle + v_3 Z\,|0\rangle}{\sqrt{\frac{1+v_3}{2}}}\right) \\
&= \frac{(1+v_3)\,|0\rangle + (v_1 + iv_2)\,|1\rangle}{\sqrt{2+2v_3}} \\
&= |e_+\rangle\,.
\end{aligned}
$$

2.62

If the measurement operators P_m and the POVM elements E_m coincide then

$$
E_m = P_m^\dagger P_m = P_m \quad\Longrightarrow\quad P_m \text{ are positive operators} \quad\Longrightarrow\quad P_m^\dagger = P_m,
$$

thus

$$
P_m^\dagger P_m = P_m^2 \quad\Longrightarrow\quad P_m^2 = P_m.
$$

2.63

Unitaries U_m rise naturally from the left polar decomposition of operators M_m, given by

$$
M_m = U_m\sqrt{M_m^\dagger M_m} = U_m\sqrt{E_m}.
$$

2.64

We can use an idea analogous to the Gram-Schmidt process to produce the states

$$
|\phi_i\rangle = |\psi_i\rangle - \sum_{j=1\,(j\neq i)}^{m} \frac{\langle \psi_j \,|\, \psi_i\rangle\,|\psi_j\rangle}{\left\||\psi_j\rangle\right\|^2},
$$

orthogonal to all states $|\psi_j\rangle$ with $j \neq i$. So by choosing

$$E_i = \alpha_i \left|\phi_i\middle\rangle\!\middle\langle\phi_i\right|, \text{ for } i \in \{1, \cdots, m\}, \text{ and } E_{m+1} = I - \sum_{i=1}^{m} E_i,$$

where α_i are constants such that E_{m+1} is a positive operator, then we satisfy the condition $\sum_i E_i = I$, and for any measurement outcome E_i, for $i \in \{1, \cdots, m\}$, Bob knows with certainty that the received state is $|\psi_i\rangle$, because the probability of obtaining E_i upon receiving any other state $|\psi_j\rangle$ is

$$\langle \psi_j \,|\, E_i \,|\, \psi_j \rangle = \alpha_i \langle \psi_j \,|\, \phi_i \rangle \langle \phi_i | \psi_j \rangle = 0.$$

2.65

In the Hadamard basis $\{|+\rangle, |-\rangle\}$ the states are written as

$$\begin{aligned} \frac{|0\rangle + |1\rangle}{\sqrt{2}} &= |+\rangle, \\ \frac{|0\rangle - |1\rangle}{\sqrt{2}} &= |-\rangle. \end{aligned}$$

2.66

$$\begin{aligned} \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) &= \frac{1}{2}\big(\langle 0 \,|\, X_1 \,|\, 0 \rangle \langle 0 \,|\, Z_2 \,|\, 0 \rangle + \langle 0 \,|\, X_1 \,|\, 1 \rangle \langle 0 \,|\, Z_2 \,|\, 1 \rangle \\ &\quad + \langle 1 \,|\, X_1 \,|\, 0 \rangle \langle 1 \,|\, Z_2 \,|\, 0 \rangle + \langle 1 \,|\, X_1 \,|\, 1 \rangle \langle 1 \,|\, Z_2 \,|\, 1 \rangle \big) \\ &= \frac{1}{2}\big(\langle 0 \,|\, 1 \rangle \langle 0 \,|\, 0 \rangle + \langle 0 \,|\, 0 \rangle \langle 0 \,|\, 1 \rangle + \langle 1 \,|\, 1 \rangle \langle 1 \,|\, 0 \rangle + \langle 1 \,|\, 0 \rangle \langle 1 \,|\, 1 \rangle \big) \\ &= 0. \end{aligned}$$

2.67

Let $W^\perp$ be the orthogonal complement of the subspace W, then we have $V = W \oplus W^\perp$. Also, consider that $\{|w_i\rangle\}$ and $\left\{|w_i^\perp\rangle\right\}$ are orthonormal basis for the subspaces W and $W^\perp$ respectively. We can define an operator U' given by

$$U' \coloneqq \sum_{i=1}^{\dim(W)} U \left|w_i\middle\rangle\!\middle\langle w_i\right| + \sum_{i=1}^{\dim(W^\perp)} \left|w_i^\perp\middle\rangle\!\middle\langle w_i^\perp\right|.$$

Any vector $|v\rangle \in V$ can be written as

$$|v\rangle = \sum_{i=1}^{\dim(W)} a_i \,|w_i\rangle + \sum_{i=1}^{\dim(W^\perp)} b_i \,|w_i^\perp\rangle,$$

thus

$$
\begin{aligned}
U' \left|v\right\rangle &= \sum_{i=1}^{\dim(W)} \sum_{j=1}^{\dim(W)} a_j U \left|w_i\right\rangle \left\langle w_i \,\middle|\, w_j\right\rangle + \sum_{i=1}^{\dim(W^\perp)} \sum_{i=j}^{\dim(W^\perp)} b_j \left|w_i^\perp\right\rangle \left\langle w_i^\perp \,\middle|\, w_j^\perp\right\rangle \\
&= \sum_{i=1}^{\dim(W)} a_i U \left|w_i\right\rangle + \sum_{i=1}^{\dim(W^\perp)} b_i \left|w_i^\perp\right\rangle \quad \in V.
\end{aligned}
$$

$$
\begin{aligned}
U'^\dagger U' &= \sum_{i=1}^{\dim(W)} \sum_{j=1}^{\dim(W)} \left|w_i\right\rangle \left\langle w_i \,\middle|\, U^\dagger U \,\middle|\, w_j\right\rangle \left\langle w_j\right| + \sum_{i=1}^{\dim(W^\perp)} \sum_{j=1}^{\dim(W^\perp)} \left|w_i^\perp\right\rangle \left\langle w_i^\perp \,\middle|\, w_j^\perp\right\rangle \left\langle w_j^\perp\right| \\
&= \sum_{i=1}^{\dim(W)} \left|w_i\middle\rangle\middle\langle w_i\right| + \sum_{i=1}^{\dim(W^\perp)} \left|w_i^\perp\middle\rangle\middle\langle w_i^\perp\right| \\
&= I.
\end{aligned}
$$

So we clearly have a unitary operator $U' : V \to V$. Any vector $\left|w\right\rangle \in W$ can be written as

$$
\left|w\right\rangle = \sum_{i=1}^{\dim(W)} c_i \left|w_i\right\rangle,
$$

thus

$$
\begin{aligned}
U' \left|w\right\rangle &= \sum_{i=1}^{\dim(W)} \sum_{j=1}^{\dim(W)} c_j U \left|w_i\right\rangle \left\langle w_i \,\middle|\, w_j\right\rangle + \sum_{i=1}^{\dim(W^\perp)} \sum_{j=1}^{\dim(W)} c_j \left|w_i^\perp\right\rangle \left\langle w_i^\perp \,\middle|\, w_j\right\rangle \\
&= \sum_{i=1}^{\dim(W)} c_i U \left|w_i\right\rangle \\
&= U \left|w\right\rangle.
\end{aligned}
$$

Therefore, there exists a unitary operator $U' : V \to V$ which extends U.

2.68

Suppose there are states $\left|a\right\rangle = \alpha_0 \left|0\right\rangle + \alpha_1 \left|1\right\rangle$ and $\left|b\right\rangle = \beta_0 \left|0\right\rangle + \beta_1 \left|1\right\rangle$ such that $\left|\psi\right\rangle$ can be written as $\left|\psi\right\rangle = \left|a\right\rangle \left|b\right\rangle$. Then explicitly we have

$$
\begin{aligned}
\left|\psi\right\rangle = \left(\alpha_0 \left|0\right\rangle + \alpha_1 \left|1\right\rangle\right)\left(\beta_0 \left|0\right\rangle + \beta_1 \left|1\right\rangle\right) &= \alpha_0\beta_0 \left|00\right\rangle + \alpha_0\beta_1 \left|01\right\rangle + \alpha_1\beta_0 \left|10\right\rangle + \alpha_1\beta_1 \left|11\right\rangle \\
&= \frac{1}{\sqrt{2}} \left|00\right\rangle + \frac{1}{\sqrt{2}} \left|11\right\rangle,
\end{aligned}
$$

but there are no possible combination of values for $\alpha_0, \alpha_1, \beta_0$ and β_1 that satisfies this equality.

2.69

Labeling the Bell states as

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \\
|\Phi^-\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \\
|\Psi^+\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \\
|\Psi^-\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}},
\end{aligned}
$$

we have the following relations:

$$
\begin{aligned}
\langle \Phi^\pm | \Phi^\pm \rangle &= \frac{\langle 00| \pm \langle 11|}{\sqrt{2}} \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle \langle 0|0\rangle \pm \langle 0|1\rangle \langle 0|1\rangle \pm \langle 1|0\rangle \langle 1|0\rangle + \langle 1|1\rangle \langle 1|1\rangle}{2} = 1, \\
\langle \Psi^\pm | \Psi^\pm \rangle &= \frac{\langle 01| \pm \langle 10|}{\sqrt{2}} \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle \langle 1|1\rangle \pm \langle 0|1\rangle \langle 1|0\rangle \pm \langle 1|0\rangle \langle 0|1\rangle + \langle 1|1\rangle \langle 0|0\rangle}{2} = 1, \\
\langle \Phi^\pm | \Phi^\mp \rangle &= \frac{\langle 00| \pm \langle 11|}{\sqrt{2}} \frac{|00\rangle \mp |11\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle \langle 0|0\rangle \mp \langle 0|1\rangle \langle 0|1\rangle \pm \langle 1|0\rangle \langle 1|0\rangle - \langle 1|1\rangle \langle 1|1\rangle}{2} = 0, \\
\langle \Psi^\pm | \Psi^\mp \rangle &= \frac{\langle 01| \pm \langle 10|}{\sqrt{2}} \frac{|01\rangle \mp |10\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle \langle 1|1\rangle \mp \langle 0|1\rangle \langle 1|0\rangle \pm \langle 1|0\rangle \langle 0|1\rangle - \langle 1|1\rangle \langle 0|0\rangle}{2} = 0, \\
\langle \Phi^\pm | \Psi^\pm \rangle &= \frac{\langle 00| \pm \langle 11|}{\sqrt{2}} \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle \langle 0|1\rangle \pm \langle 0|1\rangle \langle 0|0\rangle \pm \langle 1|0\rangle \langle 1|1\rangle + \langle 1|1\rangle \langle 1|0\rangle}{2} = 0, \\
\langle \Phi^\pm | \Psi^\mp \rangle &= \frac{\langle 00| \pm \langle 11|}{\sqrt{2}} \frac{|01\rangle \mp |10\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle \langle 0|1\rangle \mp \langle 0|1\rangle \langle 0|0\rangle \pm \langle 1|0\rangle \langle 1|1\rangle - \langle 1|1\rangle \langle 1|0\rangle}{2} = 0.
\end{aligned}
$$

2.70

$$
\begin{aligned}
\langle \Phi^\pm | E \otimes I | \Phi^\pm \rangle &= \frac{\langle 0|E|0\rangle \langle 0|0\rangle \pm \langle 0|E|1\rangle \langle 0|1\rangle \pm \langle 1|E|0\rangle \langle 1|0\rangle + \langle 1|E|1\rangle \langle 1|1\rangle}{2} \\
&= \frac{\langle 0|E|0\rangle + \langle 1|E|1\rangle}{2}, \\
\langle \Psi^\pm | E \otimes I | \Psi^\pm \rangle &= \frac{\langle 0|E|0\rangle \langle 1|1\rangle \pm \langle 0|E|1\rangle \langle 1|0\rangle \pm \langle 1|E|0\rangle \langle 0|1\rangle + \langle 1|E|1\rangle \langle 0|0\rangle}{2} \\
&= \frac{\langle 0|E|0\rangle + \langle 1|E|1\rangle}{2}.
\end{aligned}
$$

So, if Eve intercepts Alice's qubit and performs a measurement using measurement operators $\{M_m\}$ the probability of obtaining outcome m is

$$
\langle \psi | M_m^\dagger M_m | \psi \rangle = \frac{\langle 0 | M_m^\dagger M_m | 0 \rangle + \langle 1 | M_m^\dagger M_m | 1 \rangle}{2}
$$

for all m, independently of the four possible states $|\psi\rangle$. Therefore Eve could not infer anything about the qubit sent by Alice.

2.71

Let $\{|\psi_i\rangle\}$ be an orthonormal basis for which the density operator is diagonal. Then

$$\begin{aligned} \rho^2 &= \sum_i \sum_j p_i p_j \left|\psi_i\right\rangle \left\langle \psi_i \,\middle|\, \psi_j \right\rangle \left\langle \psi_j \right| \\ &= \sum_i p_i^2 \left|\psi_i\right\rangle\!\left\langle \psi_i \right| \\ \implies \quad \operatorname{tr}\!\left(\rho^2\right) &= \sum_j \left\langle \psi_j \,\middle|\, \left(\sum_i p_i^2 \left|\psi_i\right\rangle\!\left\langle \psi_i \right|\right) \,\middle|\, \psi_j \right\rangle \\ &= \sum_i \sum_j p_i^2 \left\langle \psi_j \,\middle|\, \psi_i \right\rangle \left\langle \psi_i \,\middle|\, \psi_j \right\rangle \\ &= \sum_j p_j^2 \leq 1, \end{aligned}$$

because $\sum_i p_i = 1$ and $p_i \leq 1$ for all i. Equality would only occur if $p_i = 1$ for some i and $p_j = 0$ for all $j \neq i$, that is, a pure state.

2.72

A density operator ρ is positive (hence Hermitian) and has trace equal to one. So for $a, b, c \in \mathbb{R}$, such that $-1 \leq c \leq 1$, we can write ρ as

$$\begin{aligned} \rho &= \frac{1}{2} \begin{bmatrix} 1+c & a-ib \\ a+ib & 1-c \end{bmatrix} \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & -ib \\ ib & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} \right) \\ &= \frac{I + aX + bY + cZ}{2}. \end{aligned}$$

We must have $\operatorname{tr}(\rho^2) \leq 1$, thus

$$\begin{aligned} \operatorname{tr}\!\left(\rho^2\right) &= \frac{1}{4} \operatorname{tr} \begin{bmatrix} a^2 + b^2 + (1+c)^2 & 2\,(a-ib) \\ 2\,(a+ib) & a^2 + b^2 + (1-c)^2 \end{bmatrix} \\ &= \frac{1}{2}\left(1 + a^2 + b^2 + c^2\right) \quad \implies \quad a^2 + b^2 + c^2 \leq 1. \end{aligned}$$

So defining the vector $\vec{r} \coloneqq (a, b, c)$, we have $\|\vec{r}\| \leq 1$ and

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}.$$

For $\rho = I/2$ we have $\vec{r} = 0$, meaning the null vector. So the state would be represented by the origin in the Bloch sphere.

If ρ is pure then

$$\operatorname{tr}\!\left(\rho^2\right) = 1 \quad \iff \quad a^2 + b^2 + c^2 = 1 \quad \iff \quad \|\vec{r}\| = 1.$$

For pure states we have $\rho = |\psi\rangle\langle\psi|$, thus we may write

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\,,$$

where θ and φ are polar angles of the unit Bloch vector $\vec{r}$, as described in Section 1.2.

2.73

$$\begin{aligned}|\psi\rangle &= \rho\rho^{-1}|\psi\rangle\\ &= \sum_i p_i |\psi_i\rangle \langle\psi_i|\,\rho^{-1}\,|\psi\rangle\,.\end{aligned}$$

Because $|\psi\rangle$ is a state in the support of ρ we may write

$$\begin{aligned}&|\psi\rangle = \sum_j a_j |\psi_j\rangle\\ \Longrightarrow\quad &|\psi\rangle = \sum_i\sum_j p_i a_j |\psi_i\rangle \langle\psi_i|\,\rho^{-1}\,|\psi_j\rangle\\ & = \sum_i p_i \langle\psi_i|\,\rho^{-1}\,|\psi_i\rangle\, a_i |\psi_i\rangle\,,\end{aligned}$$

but this equality holds only if

$$p_i = \frac{1}{\langle\psi_i|\,\rho^{-1}\,|\psi_i\rangle}.$$

2.74

$$\begin{aligned}&\rho^{AB} = (|a\rangle|b\rangle)(\langle a|\langle b|) = |a\rangle\langle a|\otimes|b\rangle\langle b|\\ \Longrightarrow\quad &\rho^{A} = \mathrm{tr}_B\big(\rho^{AB}\big) = |a\rangle\langle a|\,\mathrm{tr}(|b\rangle\langle b|)\\ &\phantom{\rho^{A} = \mathrm{tr}_B\big(\rho^{AB}\big)} = |a\rangle\langle a|\,.\end{aligned}$$

2.75

Labeling the density operators as

$$\begin{aligned}\rho^{\Phi^\pm} &= \big|\Phi^\pm\big\rangle\big\langle\Phi^\pm\big| = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}\frac{\langle 00| \pm \langle 11|}{\sqrt{2}} = \frac{|00\rangle\langle 00| \pm |00\rangle\langle 11| \pm |11\rangle\langle 00| + |11\rangle\langle 11|}{2},\\ \rho^{\Psi^\pm} &= \big|\Psi^\pm\big\rangle\big\langle\Psi^\pm\big| = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}\frac{\langle 01| \pm \langle 10|}{\sqrt{2}} = \frac{|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|}{2},\end{aligned}$$

we have the following relations:

$$\begin{aligned}\rho_1^{\Phi^\pm} = \mathrm{tr}_2\Big(\rho^{\Phi^\pm}\Big) &= \frac{|0\rangle\langle 0|\,\mathrm{tr}(|0\rangle\langle 0|) \pm |0\rangle\langle 1|\,\mathrm{tr}(|0\rangle\langle 1|) \pm |1\rangle\langle 0|\,\mathrm{tr}(|1\rangle\langle 0|) + |1\rangle\langle 1|\,\mathrm{tr}(|1\rangle\langle 1|)}{2}\\ &= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2},\end{aligned}$$

$$
\begin{aligned}
\rho_2^{\Phi^\pm} = \mathrm{tr}_1\left(\rho^{\Phi^\pm}\right) &= \frac{\mathrm{tr}(|0\rangle\langle 0|)\,|0\rangle\langle 0| \pm \mathrm{tr}(|0\rangle\langle 1|)\,|0\rangle\langle 1| \pm \mathrm{tr}(|1\rangle\langle 0|)\,|1\rangle\langle 0| + \mathrm{tr}(|1\rangle\langle 1|)\,|1\rangle\langle 1|}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}, \\
\rho_1^{\Psi^\pm} = \mathrm{tr}_2\left(\rho^{\Psi^\pm}\right) &= \frac{|0\rangle\langle 0|\,\mathrm{tr}(|1\rangle\langle 1|) \pm |0\rangle\langle 1|\,\mathrm{tr}(|1\rangle\langle 0|) \pm |1\rangle\langle 0|\,\mathrm{tr}(|0\rangle\langle 1|) + |1\rangle\langle 1|\,\mathrm{tr}(|0\rangle\langle 0|)}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}, \\
\rho_2^{\Psi^\pm} = \mathrm{tr}_1\left(\rho^{\Psi^\pm}\right) &= \frac{\mathrm{tr}(|0\rangle\langle 0|)\,|1\rangle\langle 1| \pm \mathrm{tr}(|0\rangle\langle 1|)\,|1\rangle\langle 0| \pm \mathrm{tr}(|1\rangle\langle 0|)\,|0\rangle\langle 1| + \mathrm{tr}(|1\rangle\langle 1|)\,|0\rangle\langle 0|}{2} \\
&= \frac{|1\rangle\langle 1| + |0\rangle\langle 0|}{2} = \frac{I}{2}.
\end{aligned}
$$

2.76

Let $\dim(A) = m$ and $\dim(B) = n$, and consider that $\{|j\rangle\}$ and $\{|k\rangle\}$ are orthonormal basis for spaces A and B respectively. Then any state $|\psi\rangle \in A \otimes B$ can be written as

$$
|\psi\rangle = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\,|j\rangle\,|k\rangle\,.
$$

By the singular value decomposition we have

$$
a = u\begin{bmatrix} d \\ 0 \end{bmatrix} v \quad \text{if} \;\; m > n \;\; \text{and} \;\; a = u\begin{bmatrix} d & 0 \end{bmatrix} v \quad \text{if} \;\; m < n,
$$

where u is a unitary $m \times m$ matrix, v is a unitary $n \times n$ matrix and d is a diagonal $\min\{m,n\} \times \min\{m,n\}$ matrix. The 0 just indicates that there are $m-n$ rows with null entries in the case where $m > n$ or $n - m$ columns in the case where $m < n$. If $m > n$ then we can write $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ where u_1 is $m \times n$ and u_2 is $m \times (m-n)$, thus

$$
a = u_1 d v \quad \Longrightarrow \quad a_{jk} = (u_1)_{ji}\, d_{ii} v_{ik}.
$$

Labeling the n column vectors of u_1 as $|i_A\rangle = (u_1)_{ji}\,|j\rangle$, the n row vectors of v as $|i_B\rangle = v_{ik}\,|k\rangle$, and $\lambda_i := d_{ii}$ then we may write

$$
|\psi\rangle = \sum_{i=1}^{n} \lambda_i\,|i_A\rangle\,|i_B\rangle\,.
$$

Equivalently, if $m < n$ then we can write $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ where v_1 is $m \times n$ and v_2 is $(n-m) \times n$, thus

$$
a = u d v_1 \quad \Longrightarrow \quad a_{jk} = u_{ji} d_{ii}\,(v_1)_{ik}\,.
$$

Labeling the m column vectors of u as $|i_A\rangle = u_{ji}|j\rangle$, the m row vectors of v_1 as $|i_B\rangle = (v_1)_{ik}$, and $\lambda_i := d_{ii}$ we also may write

$$|\psi\rangle = \sum_{i=1}^{m} \lambda_i |i_A\rangle |i_B\rangle .$$

2.77

First, notice that if the Schmidt coefficients are non-degenerate then the Schmidt decomposition is unique up to phase. To see that, consider that the Schmidt decomposition of the pure state $|\psi\rangle$ of a composite system $A \otimes B$ is given by

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle \quad \Longrightarrow \quad \rho = |\psi\rangle\langle\psi| = \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \otimes |i_B\rangle\langle i_B| .$$

So, the eigenvalues of the density operator are given by λ_i^2, and because the Schmidt coefficients are non-negative, all $\sqrt{\lambda_i^2}$ are uniquely defined. Furthermore, if they are all non-degenerate then the states $|i_A\rangle |i_B\rangle$ associated with each coefficient are also uniquely defined up to phase. Therefore the Schmidt decomposition is unique up to phase.

Now consider the pure state $|\psi\rangle \in A \otimes B \otimes C$, where A, B and C are one qubit spaces, given by

$$|\psi\rangle = \frac{1}{\sqrt{10}} \big(2|0\rangle|0\rangle|0\rangle + 2|1\rangle|1\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |1\rangle|0\rangle|1\rangle \big).$$

This state can be rewritten as

$$\begin{aligned} |\psi\rangle &= \frac{2}{\sqrt{5}} \left(\frac{1}{\sqrt{2}} |0\rangle|0\rangle + \frac{1}{\sqrt{2}} |1\rangle|1\rangle \right) |0\rangle + \frac{1}{\sqrt{5}} \left(\frac{1}{\sqrt{2}} |0\rangle|1\rangle + \frac{1}{\sqrt{2}} |1\rangle|0\rangle \right) |1\rangle \\ &= \frac{2}{\sqrt{5}} |\Phi^+\rangle |0\rangle + \frac{1}{\sqrt{5}} |\Psi^+\rangle |1\rangle . \end{aligned}$$

This is a Schmidt decomposition $|\psi\rangle = \lambda_0 |0_D\rangle |0_C\rangle + \lambda_1 |1_D\rangle |1_C\rangle$ considering $D := A \otimes B$. Since the Schmidt coefficients are different this decomposition is unique, so $|\psi\rangle$ can be written as a tripartite Schmidt decomposition if and only if $|0_D\rangle$ and $|1_D\rangle$ can be written as $|0_A\rangle|0_B\rangle$ and $|1_A\rangle|1_B\rangle$ respectively. Since they are Bell states this is impossible, thus $|\psi\rangle$ cannot be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle .$$

2.78

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle .$$

Since $\sum_i \lambda_i^2 = 1$, if $\lambda_j = 1$ for some j then it has Schmidt number 1 and

$$|\psi\rangle = |j_A\rangle |j_B\rangle .$$

The converse is immediate.

If $|\psi\rangle$ is a product state then

$$\begin{aligned} |\psi\rangle = |\psi_A\rangle\,|\psi_B\rangle \quad &\Longrightarrow \quad \rho = |\psi\rangle\!\langle\psi| = |\psi_A\rangle\!\langle\psi_A| \otimes |\psi_B\rangle\!\langle\psi_B| \\ &\Longrightarrow \quad \rho^A = \mathrm{tr}_B(\rho) = |\psi_A\rangle\!\langle\psi_A|\,, \\ &\qquad\quad\; \rho^B = \mathrm{tr}_A(\rho) = |\psi_B\rangle\!\langle\psi_B|\,. \end{aligned}$$

The converse is immediate.

2.79

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\,|0\rangle\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,|1\rangle\,;$$

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}\,\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle\,|+\rangle\,;$$

For the third state we have

$$\begin{aligned} \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} \quad \Longrightarrow \quad \rho = \frac{1}{3}\big[\, & |0\rangle\!\langle 0| \otimes |0\rangle\!\langle 0| + |0\rangle\!\langle 0| \otimes |0\rangle\!\langle 1| + |0\rangle\!\langle 1| \otimes |0\rangle\!\langle 0| \\ & + |0\rangle\!\langle 0| \otimes |1\rangle\!\langle 0| + |0\rangle\!\langle 0| \otimes |1\rangle\!\langle 1| + |0\rangle\!\langle 1| \otimes |1\rangle\!\langle 0| \\ & + |1\rangle\!\langle 0| \otimes |0\rangle\!\langle 0| + |1\rangle\!\langle 0| \otimes |0\rangle\!\langle 1| + |1\rangle\!\langle 1| \otimes |0\rangle\!\langle 0|\,\big] \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad \rho^A = \rho^B = \mathrm{tr}_A(\rho) &= \frac{1}{3}\big(2\,|0\rangle\!\langle 0| + |0\rangle\!\langle 1| + |1\rangle\!\langle 0| + |1\rangle\!\langle 1|\,\big) \\ &= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

$$\det\!\big(\rho^B - \lambda I\big) = \lambda^2 - \lambda + \frac{1}{9} = 0 \quad \Longrightarrow \quad \text{eigenvalues} = \lambda_\pm = \frac{3 \pm \sqrt{5}}{6}.$$

$$\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{3 \pm \sqrt{5}}{6}\begin{bmatrix} a \\ b \end{bmatrix} \quad \Longrightarrow \quad \text{eigenvectors} = |e_\pm\rangle = \frac{\exp(i\theta_\pm)}{\sqrt{10 \mp 2\sqrt{5}}}\begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}.$$

$$\begin{array}{c}\text{Choosing the correct phases} \\ \big(\theta_+ = 0\,;\;\; \theta_- = \frac{\pi}{2}\big)\end{array} \quad \Longrightarrow \quad \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} = \sqrt{\lambda_+}\,|e_+\rangle\,|e_+\rangle + \sqrt{\lambda_-}\,|e_-\rangle\,|e_-\rangle\,.$$

2.80

$$|\psi\rangle = \sum_i \lambda_i\,|\psi_i^A\rangle\,|\psi_i^B\rangle\,,$$

$$|\varphi\rangle = \sum_i \lambda_i |\varphi_i^A\rangle |\varphi_i^B\rangle .$$

Let $U : A \to A$ and $V : B \to B$ be unitary transformations such that $U|\varphi_i^A\rangle = |\psi_i^A\rangle$ and $V|\varphi_i^B\rangle = |\psi_i^B\rangle$, then we may write

$$\begin{aligned}
|\psi\rangle &= \sum_i \lambda_i U |\varphi_i^A\rangle \otimes V |\varphi_i^B\rangle \\
&= (U \otimes V) \sum_i \lambda_i |\varphi_i^A\rangle |\varphi_i^B\rangle \\
&= (U \otimes V) |\varphi\rangle .
\end{aligned}$$

2.81

Consider that the density operator is given by $\rho^A = \sum_i p_i |i_A\rangle\langle i_A|$, then the two purifications can be written as

$$\begin{aligned}
|AR_1\rangle &= \sum_i \sqrt{p_i} |i_A\rangle |i_{R_1}\rangle , \\
|AR_2\rangle &= \sum_i \sqrt{p_i} |i_A\rangle |i_{R_2}\rangle .
\end{aligned}$$

Let $U_R : R \to R$ be a unitary transformation such that $U_R |i_{R_2}\rangle = |i_{R_1}\rangle$, then we may write

$$\begin{aligned}
|AR_1\rangle &= \sum_i \sqrt{p_i} I_A |i_A\rangle \otimes U_R |i_{R_2}\rangle \\
&= (I_A \otimes U_R) \sum_i \sqrt{p_i} |i_A\rangle |i_{R_2}\rangle \\
&= (I_A \otimes U_R) |AR_2\rangle .
\end{aligned}$$

2.82

$$\begin{aligned}
\mathrm{tr}_R\left[\left(\sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle\right)\left(\sum_j \sqrt{p_j} \langle\psi_j| \langle j|\right)\right] &= \sum_i \sum_j \sqrt{p_i p_j} |\psi_i\rangle\langle\psi_j| \,\mathrm{tr}(|i\rangle\langle j|) \\
&= \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho.
\end{aligned}$$

To measure R in the basis $|i\rangle$ means we are using the set of projective measurement operators $\{I \otimes |i\rangle\langle i|\}$. So, the probability of obtaining outcome i is

$$\begin{aligned}
\sum_j \sqrt{p_j} \langle\psi_j| \langle j| \big(I \otimes |i\rangle\langle i|\big) \sum_k \sqrt{p_k} |\psi_k\rangle |k\rangle &= \sum_j \sum_k \sqrt{p_j p_k} \langle\psi_j | I | \psi_k\rangle \langle j | i\rangle \langle i | k\rangle \\
&= \sqrt{p_i p_i} \langle\psi_i | I | \psi_i\rangle = p_i,
\end{aligned}$$

and the corresponding post-measurement state of system A is $|\psi_i\rangle$.

Due to the unitary freedom in the ensemble for density matrices there exists an ensemble $\{q_j, |\phi_j\rangle\}$

that generates the same density matrix ρ, with the condition that

$$\sqrt{q_j}\,|\phi_j\rangle = \sum_i u_{ji}\sqrt{p_i}\,|\psi_i\rangle$$

for some unitary matrix u_{ji}. So, considering an orthonormal basis $\{|r_i\rangle\}$ for R, we may write any purification $|AR\rangle$ as

$$\begin{aligned}
|AR\rangle &= \sum_j \sqrt{q_j}\,|\phi_j\rangle\,|r_j\rangle \\
&= \sum_j \left(\sum_i u_{ji}\sqrt{p_i}\,|\psi_i\rangle\right) \otimes |r_j\rangle \\
&= \sum_i \sqrt{p_i}\,|\psi_i\rangle \otimes \left(\sum_j u_{ji}\,|r_j\rangle\right).
\end{aligned}$$

Now, let $U : R \to R$ be a unitary transformation such that, for all $|r_i\rangle \in R$, we have $U\,|r_i\rangle = \sum_j u_{ji}\,|r_j\rangle \coloneqq |i\rangle$, thus

$$|AR\rangle = \sum_i \sqrt{p_i}\,|\psi_i\rangle\,|i\rangle\,.$$

It is straightforward to see that $\{|i\rangle\}$ is an orthonormal basis for R since

$$\begin{aligned}
\langle i\,|\,j\rangle &= \langle r_i\,|\,U^\dagger U\,|\,r_j\rangle \\
&= \langle r_i\,|\,r_j\rangle \\
&= \delta_{ij},
\end{aligned}$$

and we have already shown that, for a purification of this form, if we measure R in the basis $\{|i\rangle\}$, we obtain outcome i with probability p_i, meaning we have post-measurement state $|\psi_i\rangle$ for system A with probability p_i.