7 Quantum computers: physical realization

Exercises: 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9, 7.10, 7.11, 7.12, 7.13, 7.14, 7.15, 7.16, 7.17, 7.18, 7.19, 7.20, 7.21, 7.22, 7.23, 7.24, 7.25, 7.26, 7.27, 7.28, 7.29, 7.30, 7.31, 7.32, 7.33, 7.34, 7.35, 7.36, 7.37, 7.38, 7.39, 7.40, 7.41, 7.42, 7.43, 7.44, 7.45, 7.46, 7.47, 7.48, 7.49, 7.50, 7.51, 7.52.

7.1

$$\begin{aligned}
a^\dagger a &= \frac{1}{2m\hbar\omega}\left(m\omega x - ip\right)\left(m\omega x + ip\right) \\
&= \frac{1}{2m\hbar\omega}\left(p^2 + m^2\omega^2x^2 + im\omega[x,p]\right) \\
&= \frac{1}{2m\hbar\omega}\left(p^2 + m^2\omega^2x^2 - \hbar m\omega\right) \\
&= \frac{1}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\right) - \frac{1}{2} \\
&= \frac{H}{\hbar\omega} - \frac{1}{2}.
\end{aligned}$$

7.2

$$\begin{aligned}
\left[a, a^\dagger\right] &= aa^\dagger - a^\dagger a \\
&= \frac{H}{\hbar\omega} + \frac{1}{2} - \left(\frac{H}{\hbar\omega} - \frac{1}{2}\right) = 1.
\end{aligned}$$

7.3

$$\begin{aligned}
[H, a] &= \left(\hbar\omega a^\dagger a + \frac{1}{2}\right)a - a\left(\hbar\omega a^\dagger a + \frac{1}{2}\right) \\
&= \hbar\omega a^\dagger a a - \hbar\omega a a^\dagger a \\
&= \hbar\omega\left[a^\dagger, a\right]a = -\hbar\omega a.
\end{aligned}$$

From this result we see that $[[H,a],a] = 0$. So considering that $H|\psi\rangle = E|\psi\rangle$ with the energy satisfying $E \geq n\hbar\omega$ we may write

$$\begin{aligned}
Ha^n|\psi\rangle &= \left(aH + [H,a]\right)a^{n-1}|\psi\rangle \\
&= \left(a^2H + 2[H,a]a\right)a^{n-2}|\psi\rangle \\
&\vdots \\
&= \left(a^nH + n[H,a]a^{n-1}\right)|\psi\rangle \\
&= \left(E - n\hbar\omega\right)a^n|\psi\rangle.
\end{aligned}$$

So the state $a^n|\psi\rangle$ is an eigenstate of H with eigenvalue $E - n\hbar\omega$.

7.4

$$
\begin{aligned}
|n\rangle &= \frac{a^\dagger}{\sqrt{n}} |n-1\rangle \\
&= \frac{\left(a^\dagger\right)^2}{\sqrt{n(n-1)}} |n-2\rangle \\
&\vdots \\
&= \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} |0\rangle .
\end{aligned}
$$

7.5

From Equation (7.12) we have

$$
a|n\rangle = \sqrt{n}\,|n-1\rangle .
$$

Then using Equation (7.11) we obtain

$$
\begin{aligned}
a^\dagger a|n\rangle &= \sqrt{n} a^\dagger |n-1\rangle \\
&= \sqrt{n}\sqrt{(n-1)+1}\,|(n-1)+1\rangle \\
&= n|n\rangle ,
\end{aligned}
$$

which is consistent with Equation (7.10). Besides, since $\langle n|n\rangle = 1$ we expect that $\langle n|H|n\rangle$ will result in the energy E_n associated with the state $|n\rangle$, and using Equation (7.10) we verify that

$$
\begin{aligned}
\langle n|H|n\rangle &= \hbar\omega \langle n|a^\dagger a|n\rangle + \frac{1}{2}\langle n|n\rangle \\
&= n\hbar\omega + \frac{1}{2} = E_n .
\end{aligned}
$$

7.6

$$
\begin{aligned}
a|\alpha\rangle &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}} a|n\rangle \\
&= e^{-|\alpha|^2/2}\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle .
\end{aligned}
$$

Now we make the variable substitution $n-1 \to m$, yielding

$$
\begin{aligned}
a|\alpha\rangle &= e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\alpha^{m+1}}{\sqrt{(m+1)!}}\sqrt{m+1}\,|m\rangle \\
&= \alpha e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle = \alpha|\alpha\rangle .
\end{aligned}
$$

So the coherent state is an eigenstate of the photon annihilation operator with eigenvalue α.

7.7

Let us consider a generic state $|\psi_{\text{in}}\rangle = c_0|01\rangle + c_1|10\rangle$. The circuit applies a phase shift of π on the $|01\rangle$ component meaning we obtain $|\psi_{\text{out}}\rangle = e^{i\pi}c_0|01\rangle + c_1|10\rangle$. In column vector representation we have Equation (7.23), that is

$$|\psi_{\text{out}}\rangle = \begin{bmatrix} e^{i\pi}c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} e^{i\pi} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} e^{i\pi} & 0 \\ 0 & 1 \end{bmatrix}|\psi_{\text{in}}\rangle$$

7.8

$$\begin{aligned} P|\alpha\rangle &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}P|n\rangle \\ &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{in\Delta}|n\rangle \\ &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha e^{i\Delta})^n}{\sqrt{n!}}|n\rangle \\ &= |\alpha e^{i\Delta}\rangle . \end{aligned}$$

7.9

*It seems there are some inconsistencies in this subsection. Considering B as given by Equation (7.25), $|01\rangle = a^\dagger|00\rangle$ and $|10\rangle = b^\dagger|00\rangle$ we should have $B|01\rangle = \cos\theta|01\rangle - \sin\theta|10\rangle$ and $B|10\rangle = \sin\theta|01\rangle + \cos\theta|10\rangle$.

Let us consider a generic state $|\psi\rangle = c_0|01\rangle + c_1|10\rangle$, the action of the circuit is

$$\begin{aligned} |\psi\rangle &\xrightarrow{B} c_0\frac{|01\rangle - |10\rangle}{\sqrt{2}} + c_1\frac{|01\rangle + |10\rangle}{\sqrt{2}} \\ &\xrightarrow{P_\pi|01\rangle} c_0\frac{-|01\rangle - |10\rangle}{\sqrt{2}} + c_1\frac{-|01\rangle + |10\rangle}{\sqrt{2}} \\ &= -c_0\frac{|01\rangle + |10\rangle}{\sqrt{2}} - c_1\frac{|01\rangle - |10\rangle}{\sqrt{2}}. \end{aligned}$$

This is indeed a Hadamard gate with an overall phase of π.

7.10

The action of the first circuit is basically $B^\dagger B$, and since B is unitary this is obviously the identity operation. For the second circuit we may consider a generic state $|\psi\rangle = c_0|01\rangle + c_1|10\rangle$

$$\begin{aligned} |\psi\rangle &\xrightarrow{B} c_0\frac{|01\rangle - |10\rangle}{\sqrt{2}} + c_1\frac{|01\rangle + |10\rangle}{\sqrt{2}} \\ &\xrightarrow{P_\varphi|01\rangle} c_0\frac{e^{i\varphi}|01\rangle - |10\rangle}{\sqrt{2}} + c_1\frac{e^{i\varphi}|01\rangle + |10\rangle}{\sqrt{2}} \\ &= e^{i\varphi}\frac{c_0 + c_1}{\sqrt{2}}|01\rangle + \frac{-c_0 + c_1}{\sqrt{2}}|10\rangle \end{aligned}$$

$$
\begin{aligned}
&\xrightarrow{B^\dagger} e^{i\varphi}\frac{c_0+c_1}{\sqrt{2}}\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right)+\frac{-c_0+c_1}{\sqrt{2}}\left(\frac{-|01\rangle+|10\rangle}{\sqrt{2}}\right)\\
&\quad=\frac{c_0(e^{i\varphi}+1)+c_1(e^{i\varphi}-1)}{2}|01\rangle+\frac{c_0(e^{i\varphi}-1)+c_1(e^{i\varphi}+1)}{2}|10\rangle\,.
\end{aligned}
$$

So in terms of rotations, the circuit performs

$$
\frac{1}{2}\begin{bmatrix}e^{i\varphi}+1 & e^{i\varphi}-1\\ e^{i\varphi}-1 & e^{i\varphi}+1\end{bmatrix}\begin{bmatrix}c_0\\ c_1\end{bmatrix}=e^{i\varphi/2}\begin{bmatrix}\cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2}\\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{bmatrix}\begin{bmatrix}c_0\\ c_1\end{bmatrix}=e^{i\varphi/2}R_x(\varphi)|\psi\rangle\,.
$$

7.11

$$
\begin{aligned}
B|2,0\rangle&=B\frac{b^\dagger}{\sqrt{2}}|1,0\rangle\\
&=\frac{1}{\sqrt{2}}Bb^\dagger B^\dagger B|1,0\rangle\\
&=\frac{1}{\sqrt{2}}\left(\frac{b^\dagger+a^\dagger}{\sqrt{2}}\right)\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right)\\
&=\frac{1}{\sqrt{2}}|1,1\rangle+\frac{1}{2}\Big(|0,2\rangle+|2,0\rangle\Big)
\end{aligned}
$$

7.12

$$
\begin{aligned}
B|\alpha\rangle|\beta\rangle&=e^{-\left(|\alpha|^2+|\beta|^2\right)/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^m\beta^n}{\sqrt{m!n!}}B|n,m\rangle\\
&=e^{-\left(|\alpha|^2+|\beta|^2\right)/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^m\beta^n}{\sqrt{m!n!}}B\frac{(a^\dagger)^m}{\sqrt{m!}}\frac{(b^\dagger)^n}{\sqrt{n!}}|0,0\rangle\\
&=e^{-\left(|\alpha|^2+|\beta|^2\right)/2}B\exp\big(\alpha a^\dagger+\beta b^\dagger\big)|0,0\rangle
\end{aligned}
$$

We have that $Ba^\dagger B^\dagger=a^\dagger\cos\theta-b^\dagger\sin\theta$ and $Bb^\dagger B^\dagger=b^\dagger\cos\theta+a^\dagger\sin\theta$, therefore

$$
\begin{aligned}
B|\alpha\rangle|\beta\rangle&=e^{-\left(|\alpha|^2+|\beta|^2\right)/2}B\exp\big(\alpha a^\dagger+\beta b^\dagger\big)B^\dagger B|0,0\rangle\\
&=e^{-\left(|\alpha|^2+|\beta|^2\right)/2}\exp\big(\alpha(a^\dagger\cos\theta-b^\dagger\sin\theta)+\beta(b^\dagger\cos\theta+a^\dagger\sin\theta)\big)|0,0\rangle\\
&=e^{-\left(|\alpha|^2+|\beta|^2\right)/2}\exp\big((\alpha\cos\theta+\beta\sin\theta)a^\dagger+(\beta\cos\theta-\alpha\sin\theta)b^\dagger\big)|0,0\rangle\\
&=e^{-\left(|\alpha|^2+|\beta|^2\right)/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha\cos\theta+\beta\sin\theta)^m(\beta\cos\theta-\alpha\sin\theta)^n}{\sqrt{m!n!}}\frac{(a^\dagger)^m}{\sqrt{m!}}\frac{(b^\dagger)^n}{\sqrt{n!}}|0,0\rangle\\
&=e^{-\left(|\alpha|^2+|\beta|^2\right)/2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha\cos\theta+\beta\sin\theta)^m(\beta\cos\theta-\alpha\sin\theta)^n}{\sqrt{m!n!}}|n,m\rangle\\
&=|\alpha\cos\theta+\beta\sin\theta\rangle|\beta\cos\theta-\alpha\sin\theta\rangle\,.
\end{aligned}
$$

7.13

-

7.14

$$
\begin{aligned}
K\left|\alpha\right\rangle\left|n\right\rangle &= e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}e^{i\chi La^\dagger ab^\dagger b}\left|m\right\rangle\left|n\right\rangle \\
&= e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\sum_{j=0}^{\infty}\frac{(i\chi L)^j}{j!}(a^\dagger a)^j\left|m\right\rangle(b^\dagger b)\left|n\right\rangle \\
&= e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\sum_{j=0}^{\infty}\frac{(i\chi Lnm)^j}{j!}\left|m\right\rangle\left|n\right\rangle \\
&= e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\left(\alpha e^{i\chi Ln}\right)^m}{\sqrt{m!}}\left|m\right\rangle\left|n\right\rangle \\
&= \left|\alpha e^{i\chi Ln}\right\rangle\left|n\right\rangle .
\end{aligned}
$$

Now this result implies that for two coherent states $\left|\alpha\right\rangle$ and $\left|\beta\right\rangle$ we have

$$
\begin{aligned}
K\left|\alpha\right\rangle\left|\beta\right\rangle &= e^{-|\beta|^2/2}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}K\left|\alpha\right\rangle\left|m\right\rangle \\
&= e^{-|\beta|^2/2}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}\left|\alpha e^{i\chi Lm}\right\rangle\left|m\right\rangle .
\end{aligned}
$$

Therefore ρ_a can be calculated as

$$
\begin{aligned}
\rho_a &= \mathrm{Tr}_b\left[e^{-|\beta|^2}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\frac{\beta^m\beta^{*m'}}{\sqrt{m!m'!}}\left|\alpha e^{i\chi Lm}\middle\rangle\middle\langle\alpha e^{i\chi Lm'}\right|\otimes\left|m\middle\rangle\middle\langle m'\right|\right] \\
&= e^{-|\beta|^2}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\frac{\beta^m\beta^{*m'}}{\sqrt{m!m'!}}\left|\alpha e^{i\chi Lm}\middle\rangle\middle\langle\alpha e^{i\chi Lm'}\right|\sum_{j=0}^{\infty}\left\langle j|m\right\rangle\left\langle m'|j\right\rangle \\
&= e^{-|\beta|^2}\sum_{m=0}^{\infty}\frac{|\beta|^{2m}}{m!}\left|\alpha e^{i\chi Lm}\middle\rangle\middle\langle\alpha e^{i\chi Lm}\right| .
\end{aligned}
$$

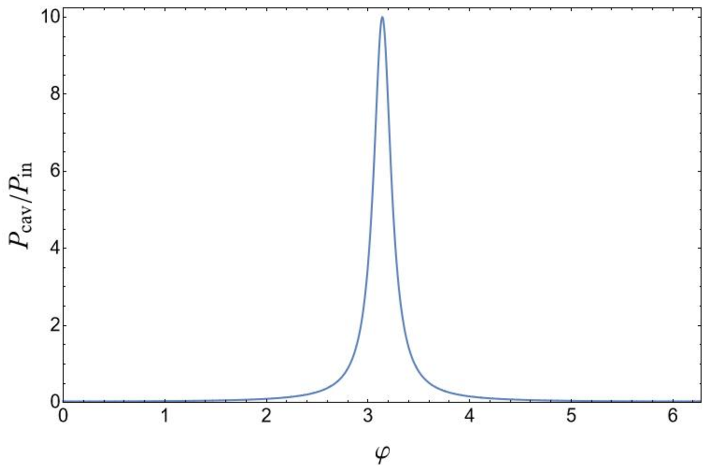
The terms for $m=|\beta|^2-2$, $m=|\beta|^2-1$, $m=|\beta|^2$ and $m=|\beta|^2+1$ have, respectively, amplitudes

$$
\begin{aligned}
A_{-2} &= e^{-|\beta|^2}\frac{|\beta|^{2|\beta|^2-4}}{(|\beta|^2-2)!} = e^{-|\beta^2|}\frac{|\beta|^{2|\beta|^2}}{|\beta|^2!}\frac{|\beta|^4-|\beta|^2}{|\beta|^4}, \\
A_{-1} &= e^{-|\beta|^2}\frac{|\beta|^{2|\beta|^2-2}}{(|\beta|^2-1)!} = e^{-|\beta^2|}\frac{|\beta|^{2|\beta|^2}}{|\beta|^2!}\frac{|\beta|^2}{|\beta|^2}, \\
A_0 &= e^{-|\beta^2|}\frac{|\beta|^{2|\beta|^2}}{|\beta|^2!}, \\
A_{+1} &= e^{-|\beta^2|}\frac{|\beta|^{2|\beta|^2+2}}{(|\beta|^2+1)!} = e^{-|\beta^2|}\frac{|\beta|^{2|\beta|^2}}{|\beta|^2!}\frac{|\beta|^2}{|\beta|^2+1}.
\end{aligned}
$$

We see that $A_{-1}=A_0$ but clearly $A_0>A_{-2}$ and $A_0>A_{+1}$. By induction, it is clear that these relations are still valid for all $0\leq m\leq|\beta|^2-2$ and $m\geq|\beta|^2+1$ since the amplitude for $m=0$ is $e^{-|\beta|^2}<A_0$ and the amplitude vanishes for $m\to\infty$. Therefore the main contribution to the sum is

for $m = |\beta|^2$.

7.15



The peak occurs for $\varphi = \pi$, which is the same as considering $d = \pi c/\omega$.

7.16

We have that

$$\int Y^*_{l_1 m_1} Y_{1m} Y_{l_2 m_2} \,\mathrm{d}\Omega = C \int P_{l_1 m_1}(\cos\theta) P_{1m}(\cos\theta) P_{l_2 m_2}(\cos\theta) e^{i(m_2 - m_1)\varphi} e^{im\varphi} \,\mathrm{d}\Omega,$$

where C is some constant depending on the orbital angular momenta and their components. First let us analyze the condition for $m_2 - m_1$. Since $m = \pm 1$, if $m_2 - m_1 \neq \pm 1$ we will have a term proportional to $e^{\pm in\varphi}$ where n is an integer. Since the φ dependence is in this term exclusively the integral will vanish when integrated over $\varphi \in [0, 2\pi]$, therefore we must have $m_2 - m_1 = \pm 1$. **Incomplete...**

7.17

For $\omega = \delta = 0$ the Hamiltonian is given by $H = g\left(a^\dagger \sigma_- + a\sigma_+\right)$, so

$$\begin{aligned}
H\left|\chi_n\right\rangle &= g\frac{1}{\sqrt{2}}\left[a^\dagger\sigma_-\left|n,1\right\rangle + a\sigma_+\left|n,1\right\rangle + a^\dagger\sigma_-\left|n+1,0\right\rangle + a\sigma_+\left|n+1,0\right\rangle\right] \\
&= g\frac{1}{\sqrt{2}}\left[\sqrt{n+1}\left|n+1,0\right\rangle + \sqrt{n+1}\left|n,1\right\rangle\right] \\
&= g\sqrt{n+1}\frac{1}{\sqrt{2}}\left[\left|n,1\right\rangle + \left|n+1,0\right\rangle\right] = g\sqrt{n+1}\left|\chi_n\right\rangle,
\end{aligned}$$

$$\begin{aligned}
H\left|\overline{\chi}_n\right\rangle &= g\frac{1}{\sqrt{2}}\left[a^\dagger\sigma_-\left|n,1\right\rangle + a\sigma_+\left|n,1\right\rangle - a^\dagger\sigma_-\left|n+1,0\right\rangle - a\sigma_+\left|n+1,0\right\rangle\right] \\
&= g\frac{1}{\sqrt{2}}\left[\sqrt{n+1}\left|n+1,0\right\rangle - \sqrt{n+1}\left|n,1\right\rangle\right]
\end{aligned}$$

$$= -g\sqrt{n+1}\frac{1}{\sqrt{2}}\left[|n,1\rangle + |n+1,0\rangle\right] = -g\sqrt{n+1}\,|\chi_n\rangle\,.$$

7.18

*Sine and cossine are exchanged in Equation (7.78). Apparently the basis state order should be $|00\rangle$, $|10\rangle$ and $|01\rangle$ instead of $|00\rangle$, $|01\rangle$ and $|10\rangle$. And the Hamiltonian should not be multiplied by -1 in Equation (7.76).

Let us divide the Hamiltonian in two parts: one associated with the subspace spanned by $|00\rangle$ and other with the one spanned by $|01\rangle$ and $|10\rangle$. For the subspace spanned by $|00\rangle$ the Hamiltonian is just $H_1 = \delta\,|00\rangle\langle 00|$, so immediately we have

$$\begin{aligned} U_1 &= e^{-iH_1 t} \\ &= e^{-i\delta t}\,|00\rangle\langle 00|\,. \end{aligned}$$

For the second subspace we may write the Hamiltonian as $H_2 = gX + \delta Z$, or equivalently $H_2 = \vec{n}\cdot\vec{\sigma}$ for $\vec{n} = (g, 0, \delta)$, so

$$\begin{aligned} U_2 &= e^{-iH_2 t} \\ &= \cos(|\vec{n}|t)I - i\hat{n}\cdot\vec{\sigma}\sin(|\vec{n}|t) \\ &= \cos\Big(\sqrt{g^2+\delta^2}t\Big)I - i\frac{gX+\delta Z}{\sqrt{g^2+\delta^2}}\sin\Big(\sqrt{g^2+\delta^2}t\Big). \end{aligned}$$

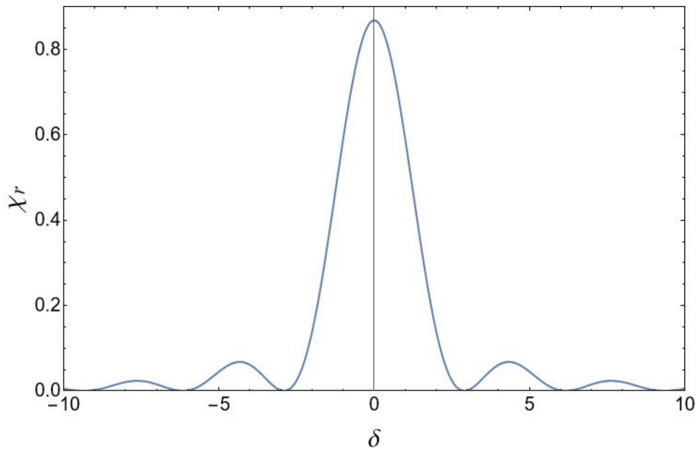
Now we identify the Rabi frequency $\Omega = \sqrt{g^2+\delta^2}$ and use the fact that $I = |10\rangle\langle 10| + |01\rangle\langle 01|$ (I denotes the identity only in this subspace), $X = |10\rangle\langle 01| + |01\rangle\langle 10|$ and $Z = |10\rangle\langle 10| - |01\rangle\langle 01|$ to rewrite

$$\begin{aligned} U_2 &= \cos\Omega t\Big(\,|10\rangle\langle 10| + |01\rangle\langle 01|\,\Big) - i\frac{g}{\Omega}\sin\Omega t\Big(\,|10\rangle\langle 01| + |01\rangle\langle 10|\,\Big) - i\frac{\delta}{\Omega}\sin\Omega t\Big(\,|10\rangle\langle 10| - |01\rangle\langle 01|\,\Big) \\ &= \left(\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t\right)|10\rangle\langle 10| + \left(\cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t\right)|01\rangle\langle 01| - i\frac{g}{\Omega}\sin\Omega t\Big(\,|10\rangle\langle 01| + |01\rangle\langle 10|\,\Big). \end{aligned}$$

Adding the solutions of both subspaces we obtain Equation (7.77)

$$\begin{aligned} U = e^{-i\delta t}\,|00\rangle\langle 00| &+ \left(\cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t\right)|01\rangle\langle 01| + \left(\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t\right)|10\rangle\langle 10| \\ &- i\frac{g}{\Omega}\sin\Omega t\Big(\,|01\rangle\langle 10| + |10\rangle\langle 01|\,\Big). \end{aligned}$$

7.19



The oscillations are due to the dependency over a squared sine of the Rabi frequency, a quantity that also depends on the detuning δ.

7.20

Taking only the matrix elements in which the atom stays in the ground state gives us

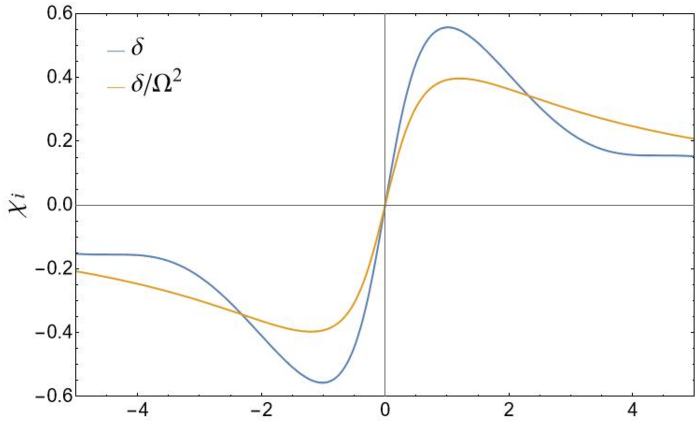
$$U' = e^{-i\delta t} |00\rangle\langle 00| + \left(\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t\right)|10\rangle\langle 10|\,.$$

Tracing over the atom space and multiplying everything by a global phase $e^{i\delta t}$ yields

$$\mathrm{Tr}_{\text{atom}}[U'] = |0\rangle\langle 0| + e^{i\delta t}\left(\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t\right)|1\rangle\langle 1|\,.$$

The phase shift is the difference in the argument of the amplitudes between states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$

$$\chi_i = \arg\left[e^{i\delta t}\left(\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t\right)\right].$$



7.21

Since H is diagonal for H_0, H_1 and H_2 we have that

$$e^{iHt} = \begin{bmatrix} e^{iH_0t} & 0 & 0 \\ 0 & e^{iH_1t} & 0 \\ 0 & 0 & e^{iH_2t}. \end{bmatrix}$$

Now, since H_0 acts only over the subspace panned by $|000\rangle$ we have that $e^{iH_0t} = e^{-i\delta t}\,|000\rangle\langle 000|$. H_1 can be divided in two parts, one that acts over the subspace spanned by $|100\rangle$ and $|001\rangle$ and other over the subspace spanned by $|010\rangle$ and $|002\rangle$. So we may write

$$\begin{aligned} H_1^{(1)} &= g_a\Big(\,|100\rangle\langle 001| + |001\rangle\langle 100|\,\Big) - \delta\Big(\,|100\rangle\langle 100| - |001\rangle\langle 001|\,\Big) \\ &= g_a X - \delta Z, \end{aligned}$$

$$\begin{aligned} H_1^{(2)} &= g_b\Big(\,|010\rangle\langle 002| + |002\rangle\langle 010|\,\Big) - \delta\Big(\,|010\rangle\langle 010| - |002\rangle\langle 002|\,\Big) \\ &= g_b X - \delta Z. \end{aligned}$$

Exponentiation of these two "sub-Hamiltonians" yields respectively (see Exercise 7.18)

$$\begin{aligned} U_1^{(1)} = &\left(\cos\Omega_a t - i\frac{\delta}{\Omega_a}\sin\Omega_a t\right)|100\rangle\langle 100| + \left(\cos\Omega_a t + i\frac{\delta}{\Omega_a}\sin\Omega_a t\right)|001\rangle\langle 001| \\ &+ i\frac{g_a}{\Omega_a}\Big(\,|001\rangle\langle 100| + |100\rangle\langle 001|\,\Big), \end{aligned}$$

$$\begin{aligned} U_1^{(2)} = &\left(\cos\Omega_b t - i\frac{\delta}{\Omega_b}\sin\Omega_b t\right)|010\rangle\langle 010| + \left(\cos\Omega_a t + i\frac{\delta}{\Omega_a}\sin\Omega_a t\right)|002\rangle\langle 002| \\ &+ i\frac{g_a}{\Omega_a}\Big(\,|010\rangle\langle 002| + |002\rangle\langle 010|\,\Big), \end{aligned}$$

where we have defined $\Omega_a \coloneqq \sqrt{g_a^2 + \delta^2}$ and $\Omega_b \coloneqq \sqrt{g_b^2 + \delta^2}$. From these results we can calculate φ_a and φ_b as

$$\begin{aligned} \varphi_a &= \arg\left[\cos\Omega_a t - i\frac{\delta}{\Omega_a}\sin\Omega_a t\right] - \arg\left[e^{-i\delta t}\right] \\ &= \arg\left[e^{i\delta t}\left(\cos\Omega_a t - i\frac{\delta}{\Omega_a}\sin\Omega_a t\right)\right], \end{aligned}$$

$$\begin{aligned} \varphi_b &= \arg\left[\cos\Omega_b t - i\frac{\delta}{\Omega_b}\sin\Omega_b t\right] - \arg\left[e^{-i\delta t}\right] \\ &= \arg\left[e^{i\delta t}\left(\cos\Omega_b t - i\frac{\delta}{\Omega_b}\sin\Omega_b t\right)\right]. \end{aligned}$$

Finally, for H_2 we are only interested in the terms proportional to $|110\rangle\langle 110|$, which corresponds to the topmost and leftmost entry of H_2. We can verify that

$$H_2^2 = \begin{bmatrix} \Omega'^2 & 0 & 0 \\ 0 & \Omega_a^2 & g_a g_b \\ 0 & g_a g_b & \Omega_b^2 \end{bmatrix} \quad \Longrightarrow \quad H_2^3 = \begin{bmatrix} -\delta\Omega'^2 & \cdots \\ \vdots & \ddots \end{bmatrix}.$$

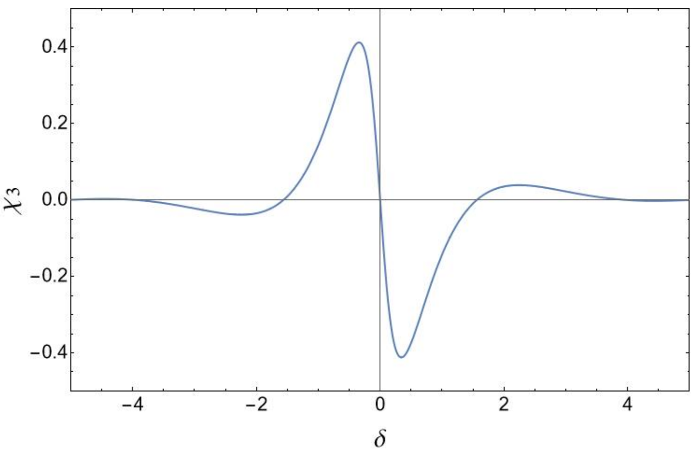
So by induction, caring only for the subspace spanned by $|110\rangle$, we conclude that

$$\begin{aligned} U_2 &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} H_2^j \\ &= \left(\sum_{k=0}^{\infty} \frac{(it)^{2k}}{(2k)!} \Omega'^{2k} - \delta \sum_{k=0}^{\infty} \frac{(it)^{2k+1}}{(2k+1)!} \Omega'^{2k} \right) |110\rangle\langle 110| + \cdots \\ &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (\Omega' t)^{2k} - i \frac{\delta}{\Omega'} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\Omega' t)^{2k+1} \right) |110\rangle\langle 110| + \cdots \\ &= \left(\cos \Omega' t - i \frac{\delta}{\Omega'} \sin \Omega' t \right) |110\rangle\langle 110| + \cdots . \end{aligned}$$

From this result we finally obtain

$$\begin{aligned} \varphi_{ab} &= \arg\left[\cos \Omega' t - i \frac{\delta}{\Omega'} \sin \Omega' t \right] - \arg\left[e^{-i\delta t} \right] \\ &= \arg\left[e^{i\delta t} \left(\cos \Omega' t - i \frac{\delta}{\Omega'} \sin \Omega' t \right) \right]. \end{aligned}$$

We can compute the Kerr phase shift as $\chi_3 = \varphi_{ab} - \varphi_a - \varphi_b$. The graph shows the Kerr phase shift as a function of the detuning for $g_a = g_b = 1$ and $t = 0.98$.



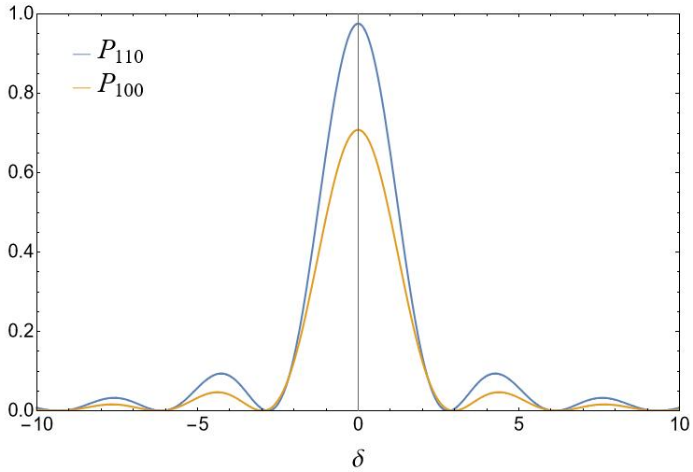
7.22

$$\begin{aligned} P_{110} := 1 - \langle 110|U|110\rangle = 1 - &\left| \cos \Omega' t - i \frac{\delta}{\Omega'} \sin \Omega' t \right|^2 \\ &= 1 - \cos^2(\Omega' t) - \frac{\delta^2}{\Omega'^2} \sin^2(\Omega' t) \end{aligned}$$

$$= \left(1 - \frac{\delta^2}{\delta^2 + g_a^2 + g_b^2}\right) \sin^2\left(\sqrt{\delta^2 + g_a^2 + g_b^2}\,t\right),$$

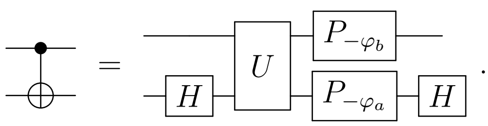
$$\begin{aligned} P_{100} \coloneqq 1 - \langle 100|U|100\rangle &= 1 - \left|\cos\Omega_a t - i\frac{\delta}{\Omega_a}\sin\Omega_a t\right|^2 \\ &= 1 - \cos^2(\Omega_a t) - \frac{\delta^2}{\Omega_a^2}\sin^2(\Omega_a t) \\ &= \left(1 - \frac{\delta^2}{\delta^2 + g_a^2}\right)\sin^2\left(\sqrt{\delta^2 + g_a^2}\,t\right). \end{aligned}$$

The graph shows P_{110} and P_{100} as function of the detuning for $g_a = g_b = 1$ and $t = 0.98$.



7.23

Notice that for $\varphi_a = \varphi_b = 0$ and $\Delta = \pi$, Equation (7.87) becomes precisely a controlled-Z gate. And since $HZH = X$ it is possible to implement a CNOT gate. If φ_a and φ_b are arbitrary we must only apply inverse phase shifts to the individual qubits. In terms of quantum circuit, if we denote the two-qubit unitary operation in Equation (7.87) as U we would have



We can verify that for a generic initial state $|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ and $\Delta = \pi$ the action of the circuit is

$$\begin{aligned} |\psi\rangle &\xrightarrow{I\otimes H} c_{00}\frac{|00\rangle + |01\rangle}{\sqrt{2}} + c_{01}\frac{|00\rangle - |01\rangle}{\sqrt{2}} + c_{10}\frac{|10\rangle + |11\rangle}{\sqrt{2}} + c_{11}\frac{|10\rangle - |11\rangle}{\sqrt{2}} \\ &\xrightarrow{U} c_{00}\frac{|00\rangle + e^{i\varphi_a}|01\rangle}{\sqrt{2}} + c_{01}\frac{|00\rangle - e^{i\varphi_a}|01\rangle}{\sqrt{2}} + c_{10}\frac{e^{i\varphi_b}|10\rangle + e^{i(\varphi_a+\varphi_b+\pi)}|11\rangle}{\sqrt{2}} \\ &\qquad + c_{11}\frac{e^{i\varphi_b}|10\rangle - e^{i(\varphi_a+\varphi_b+\pi)}|11\rangle}{\sqrt{2}} \\ &\xrightarrow{P_{-\varphi_b}\otimes P_{-\varphi_a}} c_{00}\frac{|00\rangle + |01\rangle}{\sqrt{2}} + c_{01}\frac{|00\rangle - |01\rangle}{\sqrt{2}} + c_{10}\frac{|10\rangle + e^{i\pi}|11\rangle}{\sqrt{2}} + c_{11}\frac{|10\rangle - e^{i\pi}|11\rangle}{\sqrt{2}} \end{aligned}$$

$$
\begin{aligned}
&= c_{00}\frac{|00\rangle + |01\rangle}{\sqrt{2}} + c_{01}\frac{|00\rangle - |01\rangle}{\sqrt{2}} + c_{10}\frac{|10\rangle - |11\rangle}{\sqrt{2}} + c_{11}\frac{|10\rangle + |11\rangle}{\sqrt{2}} \\
\xrightarrow{I\otimes H} & c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|11\rangle + c_{11}|10\rangle \\
&= CX|\psi\rangle .
\end{aligned}
$$

7.24

$$
\begin{aligned}
\mu_N B &\approx 5\times 10^{-27}\ [\mathrm{J/T}] \times 10\ [\mathrm{T}] = 5\times 10^{-26}\ \mathrm{J}, \\
k_{\mathrm{B}}T &\approx 1.4\times 10^{-23}\ [\mathrm{J/K}] \times 300\ [\mathrm{K}] = 4.2\times 10^{-21}\ \mathrm{J}.
\end{aligned}
$$

The thermal energy is around 10^5 times bigger than the nuclear spin energy.

7.25

Using σ_i to denote X, Y or Z for $i = 1, 2$ or 3 respectively we have

$$
\begin{aligned}
[j_i, j_k] &= \frac{1}{4}[\sigma_{i,1} + \sigma_{i,2}, \sigma_{k,1} + \sigma_{k,2}] \\
&= \frac{1}{4}[\sigma_{i,1}, \sigma_{k,1}] + \frac{1}{4}[\sigma_{i,2}, \sigma_{k,2}].
\end{aligned}
$$

Now we use the fact that $[\sigma_i, \sigma_k] = 2i\epsilon_{ikl}\sigma_l$ (see Exercise 2.40) to obtain

$$
\begin{aligned}
[j_i, j_k] &= \frac{2i}{4}\epsilon_{ikl}(\sigma_{l,1} + \sigma_{l,2}) \\
&= i\epsilon_{ikl} j_l.
\end{aligned}
$$

7.26

We should see the operators as $\sigma_{i,1} = \sigma_i \otimes I$ and $\sigma_{i,2} = I \otimes \sigma_i$, where I is the 2×2 identity matrix. Technically, the eigenvalue associated with j_z is $-m_j$. If we want that to be m_j then we need to use a convention where Z is such that $Z|0\rangle = -|0\rangle$ and $Z|1\rangle = |1\rangle$, or in other words, we should multiply Z by -1. So in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis we have

$$
X_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

$$
Y_1 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}
$$

$$Z_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

and hence obtain

$$\begin{aligned} J^2 &= \frac{1}{4}\left((X_1+X_2)^2 + (Y_1+Y_2)^2 + (Z_1+Z_2)^2\right) \\ &= \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} J_z &= \frac{1}{2}(Z_1+Z_2) \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Now to rewrite these results in the $\{|0,0\rangle_J, |1,-1\rangle_J, |1,0\rangle_J, |1,1\rangle_J\}$ basis we need the change of basis matrix. It can be obtained directly from the relations

$$|00\rangle = |1,-1\rangle_J, \quad |01\rangle = \frac{|0,0\rangle_J + |1,0\rangle_J}{\sqrt{2}}, \quad |10\rangle = \frac{-|0,0\rangle_J + |1,0\rangle_J}{\sqrt{2}}, \quad |11\rangle = |1,1\rangle_J,$$

which gives us

$$M = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So finally, the operators J^2 and j_z written in the $\{|0,0\rangle_J, |1,-1\rangle_J, |1,0\rangle_J, |1,1\rangle_J\}$ are

$$MJ^2M^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad Mj_zM^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

7.27

*It seems the states $|1/2,1/2\rangle_k$ and $|1/2,-1/2\rangle_k$, for $k=1$ and 2, are exchanged.

Treating the operators as $\sigma_{i,1} = \sigma_i \otimes I \otimes I$, $\sigma_{i,2} = I \otimes \sigma_i \otimes I$ and $\sigma_{i,3} = I \otimes I \otimes \sigma_i$, where I is the

2×2 identity matrix, and using the same convention for Z as in Ex. 7.26 we have

$$J^2 = \frac{1}{4}\left((X_1+X_2+X_3)^2 + (Y_1+Y_2+Y_3)^2 + (Z_1+Z_2+Z_3)^2\right) = \frac{1}{4}\begin{bmatrix} 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 4 & 7 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 7 & 4 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 15 \end{bmatrix},$$

$$j_z = \frac{1}{2}(Z_1+Z_2+Z_3) = \frac{1}{2}\begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

These results are written in the $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$ basis. From Equations (7.98) to (7.105) we see that the inverse change of basis matrix is given by

$$M^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The order of states we are using in the coupled angular momenta basis is $\{|3/2, 3/2\rangle$, $|3/2, 1/2\rangle$, $|3/2, -1/2\rangle$, $|3/2, -3/2\rangle$, $|1/2, 1/2\rangle_1$, $|1/2, -1/2\rangle_1$, $|1/2, 1/2\rangle_2$, $|1/2, -1/2\rangle_2\}$, which is the order the states are written from Equations (7.98) to (7.105). So finally, the J^2 and j_z operators written in the coupled basis are given by

$$MJ^2M^\dagger = \begin{bmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{15}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{bmatrix}, \quad Mj_zM^\dagger = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

7.28

$$[i_x, i_y] = \begin{bmatrix} -\frac{3i}{2} & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & \frac{3i}{2} \end{bmatrix} = ii_z,$$

$$[i_y, i_z] = \begin{bmatrix} 0 & i\frac{\sqrt{3}}{2} & 0 & 0 \\ i\frac{\sqrt{3}}{2} & 0 & i & 0 \\ 0 & i & 0 & i\frac{\sqrt{3}}{2} \\ 0 & 0 & i\frac{\sqrt{3}}{2} & 0 \end{bmatrix} = ii_x,$$

$$[i_z, i_x] = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} = ii_y.$$

That is, we can write the commutation rules in the compact form $[i_j, i_k] = i\epsilon_{jkl}i_l$, which corresponds to the $SU(2)$ commutation rules.

$$F^2 = \begin{bmatrix} 3 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 2 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 2 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 3 \end{bmatrix}, \quad f_z = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Diagonalizing F^2 and rearranging the diagonal terms in f_z for it to be consistent with F^2 yields

$$F^2 = \begin{bmatrix} 6&0&0&0&0&0&0&0\\ 0&6&0&0&0&0&0&0\\ 0&0&6&0&0&0&0&0\\ 0&0&0&6&0&0&0&0\\ 0&0&0&0&6&0&0&0\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&2 \end{bmatrix}, \quad f_z = \begin{bmatrix} 2&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&-2&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1 \end{bmatrix}.$$

7.29

*The squared sine in p_{decay} should contain an $(\omega-\omega_0)$ instead of $(\omega-\omega_0)^2$.

$$\begin{aligned}\frac{1}{(2\pi c)^3}\frac{8\pi}{3}\int_0^\infty \omega^2 p_{\text{decay}}\,\mathrm{d}\omega &= \frac{1}{(2\pi c)^3}\frac{8\pi}{3}\int_0^\infty \omega^2\frac{\omega_0^2}{2\hbar\omega\epsilon_0 c^2}|\langle 0|\vec{\mu}|1\rangle|^2\frac{4\sin^2\left[\frac{t}{2}(\omega-\omega_0)\right]}{(\omega-\omega_0)^2}\,\mathrm{d}\omega\\ &= \frac{2\omega_0^2|\langle 0|\vec{\mu}|1\rangle|^2}{3\pi^2\hbar\epsilon_0 c^5}\int_0^\infty \frac{\omega}{(\omega-\omega_0)^2}\sin^2\left[\frac{t}{2}(\omega-\omega_0)\right]\mathrm{d}\omega.\end{aligned}$$

Let us now focus on the remainder integral, which we will call $\mathcal{I}$. Making the variable substitution $\omega-\omega_0\equiv x$ yields

$$\begin{aligned}\mathcal{I} &= \int_{-\omega_0}^\infty \frac{x+\omega_0}{x^2}\sin^2\left[\frac{t}{2}x\right]\mathrm{d}x\\ &= \int_{-\omega_0}^\infty \frac{1}{x}\sin^2\left[\frac{t}{2}x\right]\mathrm{d}x + \omega_0\int_{-\omega_0}^\infty \frac{1}{x^2}\sin^2\left[\frac{t}{2}x\right]\mathrm{d}x.\end{aligned}$$

Now, since in general we have $\omega_0 \gg 0$ we can make an approximation where we extend the lower limit of $\mathcal{I}$ to $-\infty$. The first integral contains an odd function and thus vanishes when integrated from $-\infty$ to ∞, so the result is

$$\mathcal{I} \approx \omega_0\int_{-\infty}^\infty \frac{1}{x^2}\sin^2\left[\frac{t}{2}x\right]\mathrm{d}x = \omega_0\frac{t}{2}\pi.$$

Now we can obtain γ_{rad} with

$$\gamma_{\text{rad}} = \frac{2\omega_0^2|\langle 0|\vec{\mu}|1\rangle|^2}{3\pi^2\hbar\epsilon_0 c^5}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} = \frac{\omega_0^3|\langle 0|\vec{\mu}|1\rangle|^2}{3\pi\hbar\epsilon_0 c^5}.$$

7.30

$$\gamma_{\text{rad}}^{\text{ed}} \approx \frac{8\pi^3 q^2 a_0^2\times 10^{45}}{3\pi\hbar\epsilon_0 c^3} \approx 7.5\times 10^7\ \text{s}^{-1}.$$

Or in terms of lifetime, we can say the electron remains in the excited state for about 10^{-8} seconds.

7.31

Up to an immaterial global phase we have

$$\begin{aligned}
R_x(\pi)R_y(\pi/2) &= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= -i \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= e^{-i\pi/2} H.
\end{aligned}$$

7.32

Considering a generic state $|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$, where the first entry is the phonon state and the second one the ion's internal state, the circuit performs the transformation

$$\begin{aligned}
|\psi\rangle &\xrightarrow{I\otimes R_y(\pi/2)} c_{00}\frac{|00\rangle + |01\rangle}{\sqrt{2}} + c_{01}\frac{-|00\rangle + |01\rangle}{\sqrt{2}} + c_{10}\frac{|10\rangle + |11\rangle}{\sqrt{2}} + c_{11}\frac{-|10\rangle + |11\rangle}{\sqrt{2}} \\
&\xrightarrow{CZ} c_{00}\frac{|00\rangle + |01\rangle}{\sqrt{2}} + c_{01}\frac{-|00\rangle + |01\rangle}{\sqrt{2}} + c_{10}\frac{|10\rangle - |11\rangle}{\sqrt{2}} + c_{11}\frac{-|10\rangle - |11\rangle}{\sqrt{2}} \\
&\xrightarrow{I\otimes R_y(-\pi/2)} c_{00}|00\rangle + c_{01}|01\rangle - c_{10}|11\rangle - c_{11}|10\rangle,
\end{aligned}$$

which is a CNOT gate with a relative phase of π over the components where the phonon state is $|1\rangle$.

7.33

We can invert the relation, that is, $|\chi(t)\rangle = e^{-i\omega Zt/2}|\varphi(t)\rangle$, and so the Schrödinger equation becomes

$$\begin{aligned}
i\partial_t\left(e^{-i\omega Zt/2}|\varphi(t)\rangle\right) &= He^{-i\omega Zt/2}|\varphi(t)\rangle \\
&\downarrow \\
\frac{\omega}{2}Ze^{-i\omega Zt/2}|\varphi(t)\rangle + ie^{-i\omega Zt/2}\partial_t|\varphi(t)\rangle &= He^{-i\omega Zt/2}|\varphi(t)\rangle.
\end{aligned}$$

Multiplying both sides, through the left, by $e^{i\omega Zt/2}$ yields Equation (7.129)

$$\begin{aligned}
\frac{\omega}{2}Z|\varphi(t)\rangle + i\partial_t|\varphi(t)\rangle &= e^{i\omega Zt/2}He^{-i\omega Zt/2}|\varphi(t)\rangle \\
&\downarrow \\
i\partial_t|\varphi(t)\rangle &= \left[e^{i\omega Zt/2}He^{-i\omega Zt/2} - \frac{\omega}{2}Z\right]|\varphi(t)\rangle.
\end{aligned}$$

The Hamiltonian that gives rise to Equation (7.135) has the form $\widetilde{\omega}Z/2$ for some frequency $\widetilde{\omega}$. Notice that if we have $\omega = \omega_0$ then defining the states $|\psi(t)\rangle$, such that $|\chi(t)\rangle = e^{-i\widetilde{\omega}Zt/2}|\psi(t)\rangle$, we obtain the following Schrödinger equation:

$$i\partial_t|\psi(t)\rangle = \left[e^{i\widetilde{\omega}Zt/2}He^{-i\widetilde{\omega}Zt/2} - \frac{\omega_0}{2}Z\right]|\psi(t)\rangle.$$

Using Equation (7.130), the rotating frame Hamiltonian becomes

$$\begin{aligned}
e^{i\widetilde{\omega}Zt/2}He^{-i\widetilde{\omega}Zt/2}-\frac{\omega_0}{2}Z &= g\left((X\cos\widetilde{\omega}t - Y\sin\widetilde{\omega}t)\cos\omega_0 t + (Y\cos\widetilde{\omega}t + X\sin\widetilde{\omega}t)\sin\omega_0 t\right)\\
&= gX\left(\cos\widetilde{\omega}t\cos\omega_0 t + \sin\widetilde{\omega}t\sin\omega_0 t\right) + gY\left(\cos\widetilde{\omega}t\sin\omega_0 t - \sin\widetilde{\omega}t\cos\omega_0 t\right)\\
&= g\cos[(\omega_0-\widetilde{\omega})\,t]X + g\sin[(\omega_0-\widetilde{\omega})\,t]Y.
\end{aligned}$$

If we define $g_1(t) := g\cos[(\widetilde{\omega}-\omega_0)\,t]$ and $g_2(t) := g\sin[(\widetilde{\omega}-\omega_0)\,t]$ we obtain the rotating frame Hamiltonian of Equation (7.135). Notice that for $\widetilde{\omega}=\omega$ we obtain the rotating frame Hamiltonian of Equation (7.131), which is just gX at the resonance $\omega=\omega_0$.

7.34

The precession frequency is given by $\omega = \mu_N g_p B/\hbar$, where μ_N is the nuclear Bohr magneton and g_p is the proton g-factor. So we have

$$\omega = \frac{\mu_N g_p}{\hbar}B \approx 3.16\times 10^9\ \mathrm{rad}\cdot\mathrm{s}^{-1}.$$

If we want a $\pi/2$ rotation to be accomplished in 10 microseconds we want the period to be $T=40$ microseconds, meaning $\omega = 2\pi/T \approx 1.57\times 10^5\ \mathrm{rad}\cdot\mathrm{s}^{-1}$, hence

$$\begin{aligned}
B \approx \frac{1.57\times 10^5\times\hbar}{\mu_N g_p} &\approx 5.87\times 10^{-4}\ \mathrm{T}\\
&= 5.87\ \mathrm{G}.
\end{aligned}$$

7.35

We may express the unit vector as $\hat{n} = (s_\theta c_\varphi, s_\theta s_\varphi, c_\theta)$, where we are using s and c to denote, respectively, the sine and cosine functions. Let us also denote the pauli vectors as $\vec{\sigma}_j = (X_j, Y_j, Z_j)$. So the spherical average of $H^{\mathrm{D}}_{1,2}$ is proportional to

$$\begin{aligned}
\int H^{\mathrm{D}}_{1,2}\,\mathrm{d}\Omega = \frac{\gamma_1\gamma_2\hbar}{4r^3}\Bigg[&\vec{\sigma}_1\cdot\vec{\sigma}_2\int_0^{2\pi}\int_0^{\pi} s_\theta\,\mathrm{d}\theta\mathrm{d}\varphi\\
&-3\int_0^{2\pi}\int_0^{\pi}\left(X_1 s_\theta c_\varphi + Y_1 s_\theta s_\varphi + Z_1 c_\theta\right)\left(X_2 s_\theta c_\varphi + Y_2 s_\theta s_\varphi + Z_2 c_\theta\right) s_\theta\,\mathrm{d}\theta\mathrm{d}\varphi\Bigg].
\end{aligned}$$

The first integral yields 4π. For the second integral, notice that for all terms, with the exception of the ones proportional to X_1X_2, Y_1Y_2 and Z_1Z_2, the integral on φ will be either s_φ, c_φ or $s_\varphi c_\varphi$. All these integrals vanish for the interval $[0, 2\pi]$. We are left only with the integral

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}\left(X_1X_2 s_\theta^3 c_\varphi^2 + Y_1Y_2 s_\theta^3 s_\varphi^2 + Z_1Z_2 s_\theta c_\theta^2\right)\mathrm{d}\theta\mathrm{d}\varphi &= \frac{4\pi}{3}\left(X_1X_2 + Y_1Y_2 + Z_1Z_2\right)\\
&= \frac{4\pi}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2.
\end{aligned}$$

We this result, we conclude that

$$\int H_{1,2}^{\mathrm{D}}\,\mathrm{d}\Omega = \frac{\gamma_1\gamma_2\hbar}{4r^3}\left[4\pi\vec{\sigma}_1\cdot\vec{\sigma}_2 - 3\frac{4\pi}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\right] = 0.$$

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7.38

$$R_{x1}^2 = \cos\frac{\pi}{2}I - i\sin\frac{\pi}{2}X_1 = -iX_1 \quad \text{and} \quad e^{-iaZ_1t} = \cos(at)I - i\sin(at)Z_1,$$

therefore

$$\begin{aligned} R_{x1}^2 e^{-iaZ_1t} R_{x1}^2 &= (-iX_1)\left(\cos(at)I - i\sin(at)Z_1\right)(-iX_1) \\ &= -\cos(at)I + i\sin(at)X_1Z_1X_1 \\ &= -\cos(at) - i\sin(at)Z_1 \\ &= -e^{iaZ_1t}. \end{aligned}$$

Since the negative sign is just a global phase, it can be ignored.

7.39

Let U be a unitary operator such that

$$UH^{\mathrm{sys}}U^\dagger = Z \quad \Longrightarrow \quad U^\dagger Z U = H^{\mathrm{sys}}.$$

So we may have

$$\begin{aligned} U^\dagger R_x^2 e^{-iZt} R_x^2 U &= U^\dagger R_x^2 U U^\dagger e^{-iZt} U U^\dagger R_x^2 U \\ &= \left(U^\dagger R_x^2 U\right) e^{-iH^{\mathrm{sys}}t}\left(U^\dagger R_x^2 U\right). \end{aligned}$$

But this same quantity equals

$$\begin{aligned} U^\dagger R_x^2 e^{-iZ_1t} R_x^2 U &= U^\dagger e^{iZt} U \\ &= e^{iH_{\mathrm{sys}}t}. \end{aligned}$$

Therefore we have the relation

$$\left(U^\dagger R_x^2 U\right) e^{-iH^{\mathrm{sys}}t}\left(U^\dagger R_x^2 U\right) = e^{iH_{\mathrm{sys}}t},$$

which is a refocus evolution under the general Hamiltonian H^{sys}, meaning pulses of the form $U^\dagger R_x^2 U$ can be used for that.

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7.42

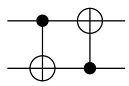
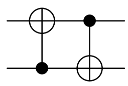
To exchange the positions of c and d we perform

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}a&0&0&0\\0&b&0&0\\0&0&c&0\\0&0&0&d\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}=\begin{bmatrix}a&0&0&0\\0&b&0&0\\0&0&0&d\\0&0&c&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}=\begin{bmatrix}a&0&0&0\\0&b&0&0\\0&0&d&0\\0&0&0&c\end{bmatrix},$$

and then, to exchange the positions of b and c we apply

$$\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}a&0&0&0\\0&b&0&0\\0&0&d&0\\0&0&0&c\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}=\begin{bmatrix}a&0&0&0\\0&0&0&c\\0&0&d&0\\0&b&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}=\begin{bmatrix}a&0&0&0\\0&c&0&0\\0&0&d&0\\0&0&0&b\end{bmatrix},$$

and the resulting matrix is ρ_2. We first used the gate $CX_{(1,2)}$ and then $CX_{(2,1)}$, therefore, on the left we have the composition $P = CX_{(2,1)}CX_{(1,2)}$ and on the right $P^\dagger = CX_{(1,2)}CX_{(2,1)}$, meaning

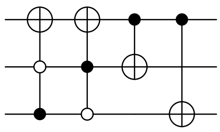
$P =$  and $P^\dagger =$  .

7.43

The initial density operator must be sandwiched with three permutation operators. The three permutations are independent and therefore there is more than one possibility. One of them is, looking at the diagonal, to permutate the 2nd with the 6th entry, the 3rd with the 7th, and the 5th with the 8th.

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

P_1 sends the state $|001\rangle$ to $|101\rangle$ and vice-versa, P_2 sends the state $|010\rangle$ to $|110\rangle$ and vice-versa, and P_3 sends the state $|100\rangle$ to $|111\rangle$. A possible permutation is then given by $P = P_3 P_2 P_1$, which can be identified with the circuit

$P =$  .

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7.47

*The two-qubit operator should be $\exp(iH/8\hbar J)$, or alternatively, the Hamiltonian should be $H = (\pi/4)\hbar J Z_1 Z_2$.

Denoting the two-qubit operator as τ, the action of the circuit is $R_{y2}\tau R_{x2}$. Writing each quantum gate in their matrix representations we have

$$R_{y2} = \exp[-i(\pi/4) I \otimes Y] = \frac{1}{\sqrt{2}} I \otimes I - \frac{i}{\sqrt{2}} I \otimes Y = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\tau = \exp[i(\pi/4)Z \otimes Z] = \frac{1}{\sqrt{2}} I \otimes I + \frac{i}{\sqrt{2}} Z \otimes Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+i & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & 1-i & 0 \\ 0 & 0 & 0 & 1+i \end{bmatrix},$$

$$R_{x2} = \exp[-i(\pi/4)I \otimes X] = \frac{1}{\sqrt{2}} I \otimes I - \frac{i}{\sqrt{2}} I \otimes X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{bmatrix}.$$

Multiplication of the three matrix yields

$$R_{y2}\tau R_{x2} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2+2i & 0 & 0 & 0 \\ 0 & 2-2i & 0 & 0 \\ 0 & 0 & 0 & -2-2i \\ 0 & 0 & 2-2i & 0 \end{bmatrix} = \begin{bmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & 0 & -e^{i\pi/4} \\ 0 & 0 & e^{-i\pi/4} & 0 \end{bmatrix},$$

which is a CNOT gate with extra relative phases between the two components, meaning it correctly executes the CNOT gate only over classical states. In order to correct the relative phases we can apply $R_{z2} = \exp[-i(\pi/4)I \otimes Z]$. The result is

$$R_{z2}R_{y2}\tau R_{x2} = \begin{bmatrix} e^{-i\pi/4} & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & 0 & -e^{i\pi/4} \\ 0 & 0 & e^{-i\pi/4} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

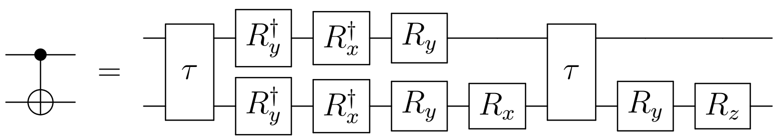
And as it can bee seen, the component $|11\rangle$ still gets a relative phase flip. It is impossible to correct it using only R_z rotations. We need a CZ gate to correct the phase of the $|11\rangle$ right at the beginning. But notice that, up to global phase, CZ corresponds exactly to the operator O shown in Eq. (7.170), which is given by

$$O = R_{y1}R_{x1}^\dagger R_{y1}^\dagger R_{y2}R_{x2}^\dagger R_{y2}^\dagger \tau.$$

So a possible quantum circuit for executing the correct CNOT gate would be

$$R_{z2}R_{y2}\tau R_{x2}R_{y1}R_{x1}^\dagger R_{y1}^\dagger R_{y2}R_{x2}^\dagger R_{y2}^\dagger \tau,$$

meaning



7.48

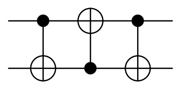
$$R_{y2}\tau R_{x2}R_{x1} = \begin{bmatrix} \frac{1+i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1-i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1+i}{\sqrt{2}} \\ 0 & 0 & \frac{1-i}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1+i}{2} & 0 & \frac{1-i}{2} & 0 \\ 0 & \frac{1-i}{2} & 0 & -\frac{1+i}{2} \\ 0 & \frac{-1+i}{2} & 0 & -\frac{1+i}{2} \\ -\frac{1+i}{2} & 0 & \frac{1-i}{2} & 0 \end{bmatrix}$$

From this result one sees that, if the initial state is given by $|00\rangle$, the final state will be

$$\begin{bmatrix} \frac{1+i}{2} & 0 & \frac{1-i}{2} & 0 \\ 0 & \frac{1-i}{2} & 0 & -\frac{1+i}{2} \\ 0 & \frac{-1+i}{2} & 0 & -\frac{1+i}{2} \\ -\frac{1+i}{2} & 0 & \frac{1-i}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1-i}{2} \\ 0 \\ 0 \\ -\frac{1-i}{2} \end{bmatrix} = e^{-i\pi/4}\frac{|00\rangle - |11\rangle}{\sqrt{2}}.$$

7.49

It is a fact that

SWAP =  .

So let us name the first circuit of Figure 7.19 C_2. And let us define C_1 as the equivalent circuit but with the R_x and R_y rotations applied to the top qubit. Although C_2 yields the wrong relative phases, it is direct to verify that applying $C_1 C_2 C_1$ corrects all relative phases, yielding the SWAP gate.

$$\begin{aligned} C_2 C_1 C_2 &= \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & 0 & -e^{i\frac{\pi}{4}} \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{i\frac{\pi}{4}} \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & 0 & -e^{i\frac{\pi}{4}} \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & 0 & -e^{i\frac{\pi}{4}} \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{2}} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & e^{-i\frac{\pi}{2}} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{i\frac{3\pi}{4}} & 0 & 0 & 0 \\ 0 & 0 & e^{i\frac{3\pi}{4}} & 0 \\ 0 & e^{i\frac{3\pi}{4}} & 0 & 0 \\ 0 & 0 & 0 & e^{i\frac{3\pi}{4}} \end{bmatrix} = e^{i3\pi/4}\text{SWAP}. \end{aligned}$$

7.50

Considering the order $|00\rangle$, $|10\rangle$, $|01\rangle$ and $|11\rangle$ we need, for the $x_0 = 0, 1, 2$ cases, the minus sign to be, respectively, on the first, third, and second element of the diagonal. For the $x_0 = 3$ case we know that the oracle can be written as $O_3 = R_{y1}R_{x1}^{\dagger}R_{y1}^{\dagger}R_{y2}R_{x2}^{\dagger}R_{y2}^{\dagger}\tau$. Breaking this expression into parts we have that

$$R_{y1}R_{x1}^{\dagger}R_{y1}^{\dagger} = \begin{bmatrix} e^{-i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{bmatrix} \quad \text{and} \quad R_{y2}R_{x2}^{\dagger}R_{y2}^{\dagger} = \begin{bmatrix} e^{-i\pi/4} & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{bmatrix}$$

Through direct verification we obtain

$$R_{y1}R_{x1}^{\dagger}R_{y1}^{\dagger}\left(R_{y2}R_{x2}^{\dagger}R_{y2}^{\dagger}\right)^{\dagger}\tau = \begin{bmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{bmatrix} = e^{i\frac{\pi}{4}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (x_0 = 2),$$

$$\left(R_{y1}R_{x1}^{\dagger}R_{y1}^{\dagger}\right)^{\dagger} R_{y2}R_{x2}^{\dagger}R_{y2}^{\dagger}\tau = \begin{bmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 \\ 0 & 0 & e^{-i3\pi/4} & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{bmatrix} = e^{i\frac{\pi}{4}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (x_0 = 1),$$

$$\left(R_{y1}R_{x1}^{\dagger}R_{y1}^{\dagger}\right)^{\dagger}\left(R_{y2}R_{x2}^{\dagger}R_{y2}^{\dagger}\right)^{\dagger}\tau = \begin{bmatrix} e^{i3\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{bmatrix} = e^{-i\frac{\pi}{4}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (x_0 = 0),$$

therefore

$$\begin{aligned} O_2 &= R_{y1}R_{x1}^{\dagger}R_{y1}^{\dagger}R_{y2}R_{x2}R_{y2}^{\dagger}\tau, \\ O_1 &= R_{y1}R_{x1}R_{y1}^{\dagger}R_{y2}R_{x2}^{\dagger}R_{y2}^{\dagger}\tau, \\ O_0 &= R_{y1}R_{x1}R_{y1}^{\dagger}R_{y2}R_{x2}R_{y2}^{\dagger}\tau. \end{aligned}$$

7.51

Using the results of the previous exercise as well as the fact that $H^{\otimes 2} = R_{x1}^2 R_{y1}^{\dagger} R_{x2}^2 R_{y2}^{\dagger}$ and $P = R_{y1}R_{x1}R_{y1}^{\dagger}R_{y2}R_{x2}R_{y2}^{\dagger}\tau$, and that rotations on different qubits commute, we have:

$$\begin{aligned} G &= H^{\otimes 2} P H^{\otimes 2} O_3 \\ &= R_{x1}^2 R_{y1}^{\dagger} R_{x2}^2 R_{y2}^{\dagger} R_{y1}R_{x1}R_{y1}^{\dagger}R_{y2}R_{x2}R_{y2}^{\dagger}\tau R_{x1}^2 R_{y1}^{\dagger} R_{x2}^2 R_{y2}^{\dagger} R_{y1}R_{x1}^{\dagger}R_{y1}^{\dagger}R_{y2}R_{x2}^{\dagger}R_{y2}^{\dagger}\tau \\ &= \left(R_{x1}^2 R_{y1}^{\dagger} R_{y1} R_{x1} R_{y1}^{\dagger}\right)\left(R_{x2}^2 R_{y2}^{\dagger} R_{y2} R_{x2} R_{y2}^{\dagger}\right)\tau\left(R_{x1}^2 R_{y1}^{\dagger} R_{y1} R_{x1}^{\dagger} R_{y1}^{\dagger}\right)\left(R_{x2}^2 R_{y2}^{\dagger} R_{y2} R_{x2}^{\dagger} R_{y2}^{\dagger}\right)\tau \\ &= \left(R_{x1}^3 R_{y1}^{\dagger}\right)\left(R_{x2}^3 R_{y2}^{\dagger}\right)\tau\left(R_{x1} R_{y1}^{\dagger}\right)\left(R_{x2} R_{y2}^{\dagger}\right)\tau \\ &= R_{x1}^{\dagger} R_{y1}^{\dagger} R_{x2}^{\dagger} R_{y2}^{\dagger}\tau R_{x1} R_{y1}^{\dagger} R_{x2} R_{y2}^{\dagger}\tau \quad (x_0 = 3) \end{aligned}$$

$$
\begin{aligned}
G &= H^{\otimes 2} P H^{\otimes 2} O_1 \\
&= H^{\otimes 2} P R_{x1}^2 R_{y1}^\dagger R_{x2}^2 R_{y2}^\dagger R_{y1} R_{x1}^\dagger R_{y1}^\dagger R_{y2} R_{x2} R_{y2}^\dagger \tau \\
&= H^{\otimes 2} P \left(R_{x1}^2 R_{y1}^\dagger R_{y1} R_{x1}^\dagger R_{y1}^\dagger \right) \left(R_{x2}^2 R_{y2}^\dagger R_{y2} R_{x2} R_{y2}^\dagger \right) \tau \\
&= H^{\otimes 2} P \left(R_{x1} R_{y1}^\dagger \right) \left(R_{x2}^3 R_{y2}^\dagger \right) \tau \\
&= R_{x1}^\dagger R_{y1}^\dagger R_{x2}^\dagger R_{y2}^\dagger \tau R_{x1} R_{y1}^\dagger R_{x2}^\dagger R_{y2}^\dagger \tau \quad (x_0 = 2)
\end{aligned}
$$

$$
\begin{aligned}
G &= H^{\otimes 2} P H^{\otimes 2} O_2 \\
&= H^{\otimes 2} P R_{x1}^2 R_{y1}^\dagger R_{x2}^2 R_{y2}^\dagger R_{y1} R_{x1} R_{y1}^\dagger R_{y2} R_{x2}^\dagger R_{y2}^\dagger \tau \\
&= H^{\otimes 2} P \left(R_{x1}^2 R_{y1}^\dagger R_{y1} R_{x1} R_{y1}^\dagger \right) \left(R_{x2}^2 R_{y2}^\dagger R_{y2} R_{x2}^\dagger R_{y2}^\dagger \right) \tau \\
&= H^{\otimes 2} P \left(R_{x1}^3 R_{y1}^\dagger \right) \left(R_{x2} R_{y2}^\dagger \right) \tau \\
&= R_{x1}^\dagger R_{y1}^\dagger R_{x2}^\dagger R_{y2}^\dagger \tau R_{x1}^\dagger R_{y1}^\dagger R_{x2} R_{y2}^\dagger \tau \quad (x_0 = 1)
\end{aligned}
$$

$$
\begin{aligned}
G &= H^{\otimes 2} P H^{\otimes 2} O_0 \\
&= H^{\otimes 2} P R_{x1}^2 R_{y1}^\dagger R_{x2}^2 R_{y2}^\dagger R_{y1} R_{x1} R_{y1}^\dagger R_{y2} R_{x2} R_{y2}^\dagger \tau \\
&= H^{\otimes 2} P \left(R_{x1}^2 R_{y1}^\dagger R_{y1} R_{x1} R_{y1}^\dagger \right) \left(R_{x2}^2 R_{y2}^\dagger R_{y2} R_{x2} R_{y2}^\dagger \right) \tau \\
&= H^{\otimes 2} P \left(R_{x1}^3 R_{y1}^\dagger \right) \left(R_{x2}^3 R_{y2}^\dagger \right) \tau \\
&= R_{x1}^\dagger R_{y1}^\dagger R_{x2}^\dagger R_{y2}^\dagger \tau R_{x1}^\dagger R_{y1}^\dagger R_{x2}^\dagger R_{y2}^\dagger \tau \quad (x_0 = 0)
\end{aligned}
$$

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