

2 Introduction to quantum mechanics

Exercises: 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21, 2.22, 2.23, 2.24, 2.25, 2.26, 2.27, 2.28, 2.29, 2.30, 2.31, 2.32, 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, 2.39, 2.40, 2.41, 2.42, 2.43, 2.44, 2.45, 2.46, 2.47, 2.48, 2.49, 2.50, 2.51, 2.52, 2.53, 2.54, 2.55, 2.56, 2.57, 2.58, 2.59, 2.60, 2.61, 2.62, 2.63, 2.64, 2.65, 2.66, 2.67, 2.68, 2.69, 2.70, 2.71, 2.72, 2.73, 2.74, 2.75, 2.76, 2.77, 2.78, 2.79, 2.80, 2.81, 2.82.

2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0.$$

2.2

$$\begin{aligned} A|0\rangle &= |1\rangle = 0|0\rangle + 1|1\rangle, \\ A|1\rangle &= |0\rangle = 1|0\rangle + 0|1\rangle. \end{aligned}$$

So writing the basis vectors as

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we have

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

2.3

$$\begin{aligned} BA|v_i\rangle &= B\left(\sum_j A_{ji}|w_j\rangle\right) \\ &= \sum_j A_{ji}B|w_j\rangle \\ &= \sum_j A_{ji} \sum_k B_{kj}|x_k\rangle \\ &= \sum_j \sum_k B_{kj}A_{ji}|x_k\rangle, \end{aligned}$$

but $\sum_j B_{kj}A_{ji}$ is precisely B's k -th row multiplied by A's i -th column (matrix multiplication), thus we have

$$BA|v_i\rangle = \sum_k (BA)_{ki}|x_k\rangle.$$

2.4

$$I |v_i\rangle = \sum_j I_{ji} |v_j\rangle = |v_i\rangle,$$

that is, I is such that $I_{ji} = 0$ for $j \neq i$ and $I_{ji} = 1$ for $j = i$, meaning I is represented by the identity matrix.

2.5

For $|v\rangle$ and $|w_i\rangle \in \mathbb{C}^n$, where

$$|v\rangle = (v_1, \dots, v_n), |w_i\rangle = (w_{i1}, \dots, w_{in}),$$

we have property (1):

$$\begin{aligned} \left(|v\rangle, \sum_i \lambda_i |w_i\rangle \right) &= \left((v_1, \dots, v_n), \sum_i \lambda_i (w_{i1}, \dots, w_{in}) \right) = \sum_j v_j^* \sum_i \lambda_i w_{ij} \\ &= \sum_i \lambda_i \sum_j v_j^* w_{ij} \\ &= \sum_i \lambda_i ((v_1, \dots, v_n), (w_{i1}, \dots, w_{in})) \\ &= \sum_i \lambda_i (|v\rangle, |w_i\rangle), \end{aligned}$$

property (2):

$$\begin{aligned} (|v\rangle, |w_i\rangle) &= \sum_j v_j^* w_{ij} \\ &= \left(\sum_j w_{ij}^* v_j \right)^* \\ &= (|w_i\rangle, |v\rangle)^*, \end{aligned}$$

and property (3):

$$\begin{aligned} (|v\rangle, |v\rangle) &= \sum_i v_i^* v_i \\ &= \sum_i |v_i|^2 \geq 0. \end{aligned}$$

2.6

Using property (2) we have

$$\left(\sum_i \lambda_i |w_i\rangle, |v\rangle \right) = \left(|v\rangle, \sum_i \lambda_i |w_i\rangle \right)^*,$$

and using properties (1) and (2) yields

$$\begin{aligned} \left(|v\rangle, \sum_i \lambda_i |w_i\rangle \right)^* &= \left(\sum_i \lambda_i (|v\rangle, |w_i\rangle) \right)^* \\ &= \sum_i \lambda_i^* (|w_i\rangle, |v\rangle). \end{aligned}$$

2.7

$$\langle w | v \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0.$$

$$\| |w\rangle \| = \| |v\rangle \| = \sqrt{2},$$

so the normalized forms are given by

$$\frac{|w\rangle}{\| |w\rangle \|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{|v\rangle}{\| |v\rangle \|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

2.8

For $|v_1\rangle$ and $|v_2\rangle$ we have

$$\begin{aligned} \langle v_1 | v_2 \rangle &= \left(\frac{\langle w_1 |}{\| |w_1\rangle \|} \right) \left(\frac{|w_2\rangle \langle v_1 | w_2 \rangle |v_1\rangle}{\| |w_2\rangle \|} \right) \\ &= \frac{\langle w_1 | w_2 \rangle - \langle v_1 | w_2 \rangle \langle w_1 | v_1 \rangle}{\| |w_1\rangle \| \| |w_2\rangle \|} \\ &= \frac{\langle w_1 | w_2 \rangle - \frac{\langle w_1 | w_2 \rangle}{\| |w_1\rangle \|} \| w_1 \|}{\| |w_1\rangle \| \| |w_2\rangle \|} \\ &= 0. \end{aligned}$$

Because it is a structure constructed inductively the same should hold for any $|v_i\rangle$. So by the end of the Gram–Schmidt process we have $\dim(V)$ vectors $|v_i\rangle$ satisfying $\langle v_i | v_j \rangle = \delta_{ij}$, thus the set $\{|v_i\rangle\}$ is an orthonormal basis for V .

2.9

$$\begin{aligned} I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|, \\ X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|, \\ Y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i |0\rangle\langle 1| + i |1\rangle\langle 0|. \end{aligned}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|.$$

2.10

All matrix elements can be calculated as

$$\langle v_l | v_j \rangle \langle v_k | v_m \rangle = \delta_{lj}\delta_{km},$$

thus it is an operator whose matrix representation has value 1 at the j 'th row and k 'th column and 0 everywhere else.

2.11

$$X : \det(X - \lambda I) = \lambda^2 - 1 = 0 \implies \text{eigenvalues} = \{-1, 1\}.$$

For eigenvalue 1 :

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle).$$

For eigenvalue -1 :

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle).$$

$$Y : \det(Y - \lambda I) = \lambda^2 - 1 = 0 \implies \text{eigenvalues} = \{-1, 1\}.$$

For eigenvalue 1 :

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle).$$

For eigenvalue -1 :

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - i|1\rangle).$$

$$Z : \text{is already diagonal in the computational basis} \implies \text{eigenvalues} = \{-1, 1\}.$$

For eigenvalue 1 :

$$\text{eigenvector} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle.$$

For eigenvalue -1 :

$$\text{eigenvector} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle.$$

2.12

$$\det \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = 0 \implies \text{eigenvalue} = 1 \text{ (degenerate).}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \implies a+b=b \implies \text{eigenvector} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle.$$

And there is no other eigenvector for the degenerate eigenvalue, thus this matrix is not diagonalizable.

2.13

$$\begin{aligned} (|a\rangle, (|w\rangle\langle v|)^\dagger |b\rangle) &= (\langle v | a\rangle |w\rangle, |b\rangle) \\ &= \langle v | a\rangle^* (|w\rangle, |b\rangle) \\ &= \langle a | v\rangle \langle w | b\rangle \\ &= (|a\rangle, (|v\rangle\langle w|) |b\rangle). \end{aligned}$$

2.14

$$\begin{aligned} \left(\left(\sum_i a_i A_i \right)^\dagger |v\rangle, |w\rangle \right) &= \left(|v\rangle, \sum_i a_i A_i |w\rangle \right) \\ &= \sum_i a_i (|v\rangle, A_i |w\rangle) \\ &= \sum_i a_i (A_i^\dagger |v\rangle, |w\rangle) \\ &= \left(\sum_i a_i^* A_i^\dagger |v\rangle, |w\rangle \right). \end{aligned}$$

2.15

$$\begin{aligned} ((A^\dagger)^\dagger |v\rangle, |w\rangle) &= (|v\rangle, A^\dagger |w\rangle) \\ &= (A^\dagger |w\rangle, |v\rangle)^* \\ &= (|w\rangle, A |v\rangle)^* \\ &= (A |v\rangle, |w\rangle). \end{aligned}$$

2.16

$$P^2 = \left(\sum_{i=1}^k |i\rangle\langle i| \right) \left(\sum_{j=1}^k |j\rangle\langle j| \right)$$

$$\begin{aligned}
&= \sum_{i=1}^k \sum_{j=1}^k |i\rangle \langle i| j\rangle |j\rangle \\
&= \sum_{i=1}^k \sum_{j=1}^k |i\rangle \langle j| \delta_{ij} \\
&= \sum_{i=1}^k |i\rangle \langle i| = P.
\end{aligned}$$

2.17

Let H be a normal operator. Then there exists an operator M that diagonalizes H to H_d , that is

$$\begin{aligned}
H &= M^\dagger H_d M, \\
H^\dagger &= M^\dagger H_d^\dagger M.
\end{aligned}$$

If H has real eigenvalues then $H_d^\dagger = H_d$, thus

$$H^\dagger = M^\dagger H_d^\dagger M = M^\dagger H_d M = H.$$

Conversely, if H is Hermitian then $H_d = H_d^\dagger$, which means all eigenvalues are real.

2.18

Let $|v\rangle$ be an eigenvector of U . Then

$$\begin{aligned}
U|v\rangle &= \lambda|v\rangle, \\
\langle v|U^\dagger &= \lambda^*\langle v|.
\end{aligned}$$

But we must have

$$\langle v|U^\dagger U|v\rangle = \lambda^* \lambda \langle v|v\rangle = \langle v|v\rangle \implies \lambda^* \lambda = 1 \implies \lambda = e^{i\theta}.$$

2.19

$$\begin{aligned}
X : X^\dagger &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X \\
X^\dagger X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\
Y : Y^\dagger &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^\dagger = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y \\
Y^\dagger Y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
\end{aligned}$$

$$Z : Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

$$Z^\dagger Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

2.20

$$I = \sum_i |v_i\rangle\langle v_i| = \sum_i |w_i\rangle\langle w_i|.$$

Then A'_{ij} can be written as

$$\begin{aligned} A'_{ij} &= \langle v_i | A | v_j \rangle = \sum_k \sum_l \langle v_i | w_k \rangle \langle w_k | A | w_l \rangle \langle w_l | v_j \rangle \\ &= \sum_k \sum_l \langle v_i | w_k \rangle A''_{kl} \langle w_l | v_j \rangle. \end{aligned}$$

If $|v_i\rangle$ and $|w_i\rangle$ are both orthonormal bases then there exists a unitary U such that $|w_i\rangle = U|v_i\rangle$, thus

$$A'_{ij} = \sum_k \sum_l U_{ik} A''_{kl} U_{lj}^\dagger.$$

2.21

Let $|v\rangle \in V(\dim = 1)$ be an eigenvector of $M = M^\dagger$ with eigenvalue λ and also element of subspace P . Then

$$\begin{aligned} M &= (P + Q) M (P + Q) \\ &= PMP + PMQ + QMP + QMQ. \end{aligned}$$

Because M is Hermitian we have that $PMP = \lambda P$ and QMQ are normal and diagonal with respect to an orthonormal basis for the subspaces P and Q respectively. Also

$$\begin{aligned} QMP &= 0, \\ PMQ &= (QMP)^\dagger = 0. \end{aligned}$$

Thus $M = PMP + QMQ$ is diagonal with respect to some orthonormal basis for space V . And by induction, this must be true for higher dimensional Hilbert spaces.

2.22

Let $|v\rangle$ and $|w\rangle$ be two eigenvectors of a Hermitian operator H with different eigenvalues. Then

$$\begin{aligned} H|v\rangle &= \alpha|v\rangle, \\ H|w\rangle &= \beta|w\rangle. \end{aligned}$$

But we must have

$$\langle w | H | v \rangle = \alpha \langle w | v \rangle = \beta \langle w | v \rangle \implies (\alpha - \beta) \langle w | v \rangle = 0.$$

Since $\alpha \neq \beta$, $|v\rangle$ and $|w\rangle$ must be orthogonal.

2.23

Let $|v\rangle$ be an eigenvector of the projector P . Then

$$\begin{aligned} P|v\rangle &= \lambda|v\rangle, \\ P^2|v\rangle &= \lambda P|v\rangle = \lambda^2|v\rangle. \end{aligned}$$

Since P is a projector, $P = P^2$, thus

$$\lambda = \lambda^2 \implies \lambda = 0 \text{ or } 1.$$

2.24

Defining

$$B := \frac{A + A^\dagger}{2}, C := \frac{A - A^\dagger}{2i},$$

we can write operator A as $A = B + iC$. If A is positive then

$$(\langle v |, A |v\rangle) = (\langle v |, B |v\rangle) + i (\langle v |, C |v\rangle) \geq 0.$$

But that is only possible if $C = 0 \Rightarrow A = A^\dagger$.

2.25

$$\begin{aligned} A|v\rangle &= |w\rangle, \\ \langle v | A^\dagger &= \langle w |, \\ \implies \langle v | A^\dagger A | v \rangle &= \langle w | w \rangle \geq 0. \end{aligned}$$

2.26

$$|\psi\rangle^{\otimes 2} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) = \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle}{2}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{aligned} |\psi\rangle^{\otimes 3} &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \\ &= \frac{|0\rangle|0\rangle|0\rangle + |0\rangle|0\rangle|1\rangle + |0\rangle|1\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |1\rangle|0\rangle|0\rangle + |1\rangle|0\rangle|1\rangle + |1\rangle|1\rangle|0\rangle + |1\rangle|1\rangle|1\rangle}{2\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \end{aligned}$$

2.27

$$\begin{aligned} X \otimes Z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\ I \otimes X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ X \otimes I &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The last two examples show that the tensor product is non-commutative.

2.28

Let A be represented by an $m \times n$ matrix. Then

$$(A \otimes B)^* = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* = \begin{bmatrix} A_{11}^*B^* & \cdots & A_{1n}^*B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} = A^* \otimes B^*,$$

$$(A \otimes B)^T = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T = \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} = A^T \otimes B^T,$$

$$(A \otimes B)^\dagger = ((A \otimes B)^*)^T = (A^* \otimes B^*)^T = (A^*)^T \otimes (B^*)^T = A^\dagger \otimes B^\dagger.$$

2.29

$$\begin{aligned} (U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) &= (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) \\ &= U_1^\dagger U_1 \otimes U_2^\dagger U_2 \\ &= I_1 \otimes I_2 \end{aligned}$$

2.30

$$\begin{aligned} H_1 \otimes H_2 &= H_1^\dagger \otimes H_2^\dagger \\ &= (H_1 \otimes H_2)^\dagger \end{aligned}$$

2.31

$$\begin{aligned} (\langle v | \otimes |w \rangle, (A \otimes B) |v\rangle \otimes |w\rangle) &= (\langle v | \otimes |w \rangle, A |v\rangle \otimes B |w\rangle) \\ &= \langle v | A | v \rangle \langle w | B | w \rangle. \end{aligned}$$

Since $\langle v | A | v \rangle \geq 0$ and $\langle w | B | w \rangle \geq 0$ it follows that

$$(\langle v | \otimes |w \rangle, (A \otimes B) |v\rangle \otimes |w\rangle) \geq 0.$$

2.32

$$\begin{aligned} (P_1 \otimes P_2)^2 &= (P_1 \otimes P_2)(P_1 \otimes P_2) \\ &= P_1^2 \otimes P_2^2 \\ &= P_1 \otimes P_2. \end{aligned}$$

2.33

$$\begin{aligned} H &= \frac{1}{\sqrt{2}} [(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|] \\ &= \frac{1}{\sqrt{2}} (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\ &= \frac{1}{\sqrt{2}} \sum_{x,y=0}^1 (-1)^{x \cdot y} |x\rangle\langle y|. \end{aligned}$$

Thus

$$\begin{aligned} H^{\otimes n} &= \frac{1}{\sqrt{2}} \sum_{x_1, y_1=0}^1 (-1)^{x_1 \cdot y_1} |x_1\rangle\langle y_1| \otimes \cdots \otimes \frac{1}{\sqrt{2}} \sum_{x_n, y_n=0}^1 (-1)^{x_n \cdot y_n} |x_n\rangle\langle y_n| \\ &= \frac{1}{\sqrt{2^n}} \sum_{x_1, y_1=0}^1 \cdots \sum_{x_n, y_n=0}^1 (-1)^{x_1 \cdot y_1} \cdots (-1)^{x_n \cdot y_n} |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_n\rangle\langle y_n|. \end{aligned}$$

If we define $|x\rangle$ and $|y\rangle$ as the bit sequence states

$$\begin{aligned} |x\rangle &:= \bigotimes_{i=1}^n |x_i\rangle, \\ |y\rangle &:= \bigotimes_{i=1}^n |y_i\rangle, \end{aligned}$$

and $x \cdot y$ as the bitwise product

$$x \cdot y := \bigoplus_{i=1}^n x_i \cdot y_i,$$

where \oplus denotes sum modulo-2 (or the XOR logic operation), then we can write

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle\langle y|.$$

$$\begin{aligned} H^{\otimes 2} &= \frac{1}{\sqrt{2^2}} [(-1)^{0 \oplus 0} |00\rangle\langle 00| + (-1)^{0 \oplus 0} |01\rangle\langle 00| + (-1)^{0 \oplus 0} |10\rangle\langle 00| + (-1)^{0 \oplus 0} |11\rangle\langle 00| \\ &\quad + (-1)^{0 \oplus 0} |00\rangle\langle 01| + (-1)^{0 \oplus 1} |01\rangle\langle 01| + (-1)^{0 \oplus 0} |10\rangle\langle 01| + (-1)^{0 \oplus 1} |11\rangle\langle 01| \\ &\quad + (-1)^{0 \oplus 0} |00\rangle\langle 10| + (-1)^{0 \oplus 0} |01\rangle\langle 10| + (-1)^{1 \oplus 0} |10\rangle\langle 10| + (-1)^{1 \oplus 0} |11\rangle\langle 10| \\ &\quad + (-1)^{0 \oplus 0} |00\rangle\langle 11| + (-1)^{0 \oplus 1} |01\rangle\langle 11| + (-1)^{1 \oplus 0} |10\rangle\langle 11| + (-1)^{1 \oplus 1} |11\rangle\langle 11|] \end{aligned}$$

$$\implies H^{\otimes 2} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

2.34

$$\det \begin{bmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{bmatrix} = \lambda^2 - 8\lambda + 7 = 0 \implies \text{eigenvalues} = \{1, 7\}.$$

For eigenvalue 1 :

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = |v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For eigenvalue 7 :

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 7 \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = |v_7\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So labeling the matrix as M we have

$$\sqrt{M} = \sqrt{1}|v_1\rangle\langle v_1| + \sqrt{7}|v_7\rangle\langle v_7| = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\sqrt{7}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7} \end{bmatrix},$$

$$\log(M) = \log(1)|v_1\rangle\langle v_1| + \log(7)|v_7\rangle\langle v_7| = \frac{\log(7)}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

2.35

$$\begin{aligned} \vec{v} \cdot \vec{\sigma} &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}. \end{aligned}$$

$$\det(\vec{v} \cdot \vec{\sigma} - \lambda I) = \lambda^2 - (v_1^2 + v_2^2 + v_3^2) = 0.$$

Since \vec{v} is a unit vector, $v_1^2 + v_2^2 + v_3^2 = 1$ and the eigenvalues are $\{-1, 1\}$. Labeling the respective eigenvectors as $|e_-\rangle$ and $|e_+\rangle$ we have

$$\begin{aligned} I &= |e_+\rangle\langle e_+| + |e_-\rangle\langle e_-|, \\ \vec{v} \cdot \vec{\sigma} &= |e_+\rangle\langle e_+| - |e_-\rangle\langle e_-|. \end{aligned}$$

$$\begin{aligned} \exp(i\theta \vec{v} \cdot \vec{\sigma}) &= \exp(i\theta)|e_+\rangle\langle e_+| + \exp(-i\theta)|e_-\rangle\langle e_-| \\ &= \cos(\theta)|e_+\rangle\langle e_+| + i\sin(\theta)|e_+\rangle\langle e_+| + \cos(\theta)|e_-\rangle\langle e_-| - i\sin(\theta)|e_-\rangle\langle e_-| \\ &= \cos(\theta)(|e_+\rangle\langle e_+| + |e_-\rangle\langle e_-|) + i\sin(\theta)(|e_+\rangle\langle e_+| - |e_-\rangle\langle e_-|) \\ &= \cos(\theta)I + i\sin(\theta)\vec{v} \cdot \vec{\sigma}. \end{aligned}$$

2.36

$$\begin{aligned} \text{tr}(X) &= 0 + 0 = 0, \\ \text{tr}(Y) &= 0 + 0 = 0, \\ \text{tr}(Z) &= 1 + (-1) = 0. \end{aligned}$$

2.37

$$\begin{aligned}
\text{tr}(AB) &= \sum_i \langle i | AB | i \rangle \\
&= \sum_i \sum_j \langle i | A | j \rangle \langle j | B | i \rangle \\
&= \sum_i \sum_j \langle j | B | i \rangle \langle i | A | j \rangle \\
&= \sum_j \langle j | BA | j \rangle \\
&= \text{tr}(BA).
\end{aligned}$$

2.38

$$\begin{aligned}
\text{tr}(A + B) &= \sum_i \langle i | (A + B) | i \rangle \\
&= \sum_i \langle i | A | i \rangle + \sum_i \langle i | B | i \rangle \\
&= \text{tr}(A) + \text{tr}(B),
\end{aligned}$$

$$\begin{aligned}
\text{tr}(zA) &= \sum_i \langle i | (zA) | i \rangle \\
&= z \sum_i \langle i | A | i \rangle \\
&= z \text{tr}(A).
\end{aligned}$$

2.39

For linear operators A and B_i acting on V , and $z_i \in \mathbb{C}$, we have property (1):

$$\begin{aligned}
\left(A, \sum_i z_i B_i \right) &= \text{tr} \left(\sum_i z_i A^\dagger B_i \right) = \sum_i z_i \text{tr}(A^\dagger B_i) \\
&= \sum_i z_i (A, B_i),
\end{aligned}$$

property (2):

$$\begin{aligned}
(A, B_i) &= \text{tr}(A^\dagger B_i) = \sum_j \langle j | A^\dagger B_i | j \rangle \\
&= \left(\sum_j \langle j | B_i^\dagger A | j \rangle \right)^* \\
&= \left(\text{tr}(B_i^\dagger A) \right)^* \\
&= (B_i, A)^*,
\end{aligned}$$

and property (3):

$$\begin{aligned}(A, A) &= \text{tr}(A^\dagger A) = \sum_i \langle i | A^\dagger A | i \rangle \\ &= \sum_i \|A | i \rangle\|^2 \geq 0.\end{aligned}$$

If $\dim(V) = d$ the transformations $V \rightarrow V$ can be represented by $d \times d$ matrices $M \in L_V$, meaning there are d^2 independent parameters necessary to write a transformation matrix, thus $\dim(L_V) = d^2$.

If $|i\rangle$ for $i \in \{1, \dots, d\}$ is an orthonormal basis for V then the set of d^2 operators $\{|i\rangle\langle j|\}$ for $i, j \in \{1, \dots, d\}$ forms an orthonormal basis for L_V , because for any i, j, k and l we have

$$\begin{aligned}(|i\rangle\langle j|, |k\rangle\langle l|) &= \text{tr}\left((|i\rangle\langle j|)^\dagger |k\rangle\langle l|\right) = \sum_{m=1}^d \langle m | j \rangle \langle i | k \rangle \langle l | m \rangle \\ &= \delta_{ik} \sum_{m=1}^d \langle l | m \rangle \langle m | j \rangle \\ &= \delta_{ik} \delta_{jl}.\end{aligned}$$

2.40

$$\begin{aligned}[X, Y] &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = 2iZ, \\ [Y, Z] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2iX, \\ [Z, X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = 2iY.\end{aligned}$$

2.41

$$\begin{aligned}\{Y, X\} &= \{X, Y\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0, \\ \{Z, Y\} &= \{Y, Z\} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = 0, \\ \{X, Z\} &= \{Z, X\} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0,\end{aligned}$$

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$Y^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

$$Z^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

2.42

$$\begin{aligned} AB &= \frac{1}{2} (2AB + BA - BA) \\ &= \frac{1}{2} (AB - BA + AB + BA) \\ &= \frac{[A, B] + \{A, B\}}{2}. \end{aligned}$$

2.43

$$\begin{aligned} \sigma_j \sigma_k &= \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2} = \frac{2i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l + 2\delta_{jk} I}{2} \\ &= \delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l. \end{aligned}$$

2.44

If $[A, B] = \{A, B\} = 0$ and A is invertible, then

$$[A, B] + \{A, B\} = 2AB = 0 \implies 2A^{-1}AB = 0 \implies B = 0.$$

2.45

$$\begin{aligned} [A, B]^\dagger &= (AB - BA)^\dagger \\ &= B^\dagger A^\dagger - A^\dagger B^\dagger \\ &= [B^\dagger, A^\dagger]. \end{aligned}$$

2.46

$$\begin{aligned} [A, B] &= AB - BA \\ &= -(BA - AB) \\ &= -[B, A]. \end{aligned}$$

2.47

$$i[A, B] = iAB - iBA$$

$$\begin{aligned}
&= (-iB^\dagger A^\dagger + iA^\dagger B^\dagger)^\dagger \\
&= (iAB - iBA)^\dagger \\
&= (i[A, B])^\dagger.
\end{aligned}$$

2.48

$$\begin{aligned}
P &= IP = PI, \\
U &= UI = IU.
\end{aligned}$$

By the spectral theorem H can be written as

$$H = \sum_i \lambda_i |i\rangle\langle i|,$$

thus

$$\begin{aligned}
\sqrt{H^\dagger H} &= \sqrt{HH^\dagger} = \sqrt{H^2} = \sqrt{\sum_i \sum_j \lambda_i \lambda_j |i\rangle\langle i| |j\rangle\langle j|} \\
&= \sqrt{\sum_i \lambda_i^2 |i\rangle\langle i|} \\
&= \sum_i |\lambda_i| |i\rangle\langle i| \\
\implies H &= U \sum_i |\lambda_i| |i\rangle\langle i| = \sum_i |\lambda_i| |i\rangle\langle i| U,
\end{aligned}$$

where $U = H \left(\sqrt{H^2} \right)^{-1}$.

2.49

Let M be a normal matrix, then by the spectral theorem we can write

$$M = \sum_i \lambda_i |i\rangle\langle i|,$$

thus

$$\begin{aligned}
\sqrt{M^\dagger M} &= \sqrt{MM^\dagger} = \sqrt{\sum_i \sum_j \lambda_i \lambda_j^* |i\rangle\langle i| |j\rangle\langle j|} \\
&= \sqrt{\sum_i |\lambda_i|^2 |i\rangle\langle i|} \\
&= \sum_i |\lambda_i| |i\rangle\langle i| \\
\implies M &= U \sum_i |\lambda_i| |i\rangle\langle i| = \sum_i |\lambda_i| |i\rangle\langle i| U,
\end{aligned}$$

where $U = M \left(\sqrt{M^\dagger M} \right)^{-1}$.

2.50

Labeling the matrix as M we have

$$J^2 = M^\dagger M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

$$K^2 = MM^\dagger = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$\det(J^2 - \lambda I) = \lambda^2 - 3\lambda + 1 = 0 \implies \text{eigenvalues} = \frac{3 \pm \sqrt{5}}{2}.$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{3 \pm \sqrt{5}}{2} \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvectors} = |j_\pm\rangle = \frac{1}{\sqrt{10 \mp 2\sqrt{5}}} \begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}.$$

$$\det(K^2 - \lambda I) = \lambda^2 - 3\lambda + 1 = 0 \implies \text{eigenvalues} = \frac{3 \pm \sqrt{5}}{2}.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{3 \pm \sqrt{5}}{2} \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvectors} = |k_\pm\rangle = \frac{1}{\sqrt{10 \pm 2\sqrt{5}}} \begin{bmatrix} 2 \\ 1 \pm \sqrt{5} \end{bmatrix}.$$

So the left and right positive operators are given by

$$J = \sqrt{\frac{3 + \sqrt{5}}{2}} |j_+\rangle\langle j_+| + \sqrt{\frac{3 - \sqrt{5}}{2}} |j_-\rangle\langle j_-| = \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

$$K = \sqrt{\frac{3 + \sqrt{5}}{2}} |k_+\rangle\langle k_+| + \sqrt{\frac{3 - \sqrt{5}}{2}} |k_-\rangle\langle k_-| = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix},$$

and the unitary is given by

$$U = MJ^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

2.51

$$H^\dagger H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

2.52

$$H^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

2.53

$$\det(H - \lambda I) = \lambda^2 - 1 = 0 \implies \text{eigenvalues} = \{-1, 1\}.$$

For eigenvalue 1 :

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = \frac{1}{\sqrt{4 - 2\sqrt{2}}} \begin{bmatrix} 1 \\ -1 + \sqrt{2} \end{bmatrix}$$

For eigenvalue -1 :

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = \frac{1}{\sqrt{4 + 2\sqrt{2}}} \begin{bmatrix} 1 \\ -1 - \sqrt{2} \end{bmatrix}.$$

2.54

If $[A, B] = 0$ then they share a common eigenbasis $\{|i\rangle\}$, that is

$$A = \sum_i \alpha_i |i\rangle\langle i|,$$

$$B = \sum_i \beta_i |i\rangle\langle i|.$$

$$\implies \exp(A) \exp(B) = \sum_i \exp(\alpha_i) |i\rangle\langle i| \sum_j \exp(\beta_j) |j\rangle\langle j|$$

$$= \sum_i \sum_j \exp(\alpha_i + \beta_j) |i\rangle\langle i| |j\rangle\langle j|$$

$$= \sum_i \exp(\alpha_i + \beta_i) |i\rangle\langle i|$$

$$= \exp(A + B).$$

2.55

Considering that the Hamiltonian has a spectral decomposition $\sum_j E_j(t_1, t_2) |j\rangle\langle j|$ yields

$$\begin{aligned} U^\dagger(t_1, t_2) U(t_1, t_2) &= \exp\left(\frac{iH(t_1, t_2)}{\hbar}\right) \exp\left(\frac{-iH(t_1, t_2)}{\hbar}\right) \\ &= \sum_j \sum_k \exp\left(\frac{iE_j(t_1, t_2)}{\hbar}\right) \exp\left(\frac{-iE_k(t_1, t_2)}{\hbar}\right) |j\rangle\langle j| |k\rangle\langle k| \\ &= \sum_j \exp\left(\frac{iE_j(t_1, t_2) - iE_j(t_1, t_2)}{\hbar}\right) |j\rangle\langle j| \\ &= \exp(0) \sum_j |j\rangle\langle j| \\ &= I. \end{aligned}$$

2.56

If U is unitary then it can be written as $\sum_j \exp(i\theta_j) |j\rangle\langle j|$, where $\theta_j \in \mathbb{R}$, thus

$$\begin{aligned} K &= -i \log(U) = -i \sum_j \log[\exp(i\theta_j)] |j\rangle\langle j| \\ &= -i \sum_j i\theta_j |j\rangle\langle j| \\ &= \left(-i \sum_j i\theta_j |j\rangle\langle j| \right)^\dagger \\ &= K^\dagger. \end{aligned}$$

2.57

After a measurement is performed over an initial state $|\psi\rangle$, using the operators set $\{L_l\}$, the state is transformed to the one associated with the measurement outcome l , given by

$$|\psi_l\rangle = \frac{L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}}.$$

Then, if a measurement is performed over $|\psi_l\rangle$, using the operators set $\{M_m\}$, the final state can be written as

$$\begin{aligned} |\psi_{lm}\rangle &= \frac{M_m |\psi_l\rangle}{\sqrt{\langle\psi_l| M_m^\dagger M_m |\psi_l\rangle}} = \frac{M_m}{\sqrt{\left(\frac{\langle\psi| L_l^\dagger}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}}\right) M_m^\dagger M_m \left(\frac{L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}}\right)}} \left(\frac{L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}} \right) \\ &= \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger M_m^\dagger M_m L_l |\psi\rangle}}. \end{aligned}$$

Alternatively, if a measurement was to be performed over $|\psi\rangle$, using the operators set $\{N_{lm}\}$, the final state would be

$$\frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi| N_{lm}^\dagger N_{lm} |\psi\rangle}} = \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger M_m^\dagger M_m L_l |\psi\rangle}} = |\psi_{lm}\rangle.$$

2.58

$$\langle M \rangle = \langle \psi | M | \psi \rangle = m \langle \psi | \psi \rangle = m,$$

$$\Delta(M) = \sqrt{\langle M^2 \rangle - \langle M \rangle^2} = \sqrt{\langle \psi | MM | \psi \rangle - m^2} = \sqrt{m^2 - m^2} = 0.$$

2.59

$$\langle X \rangle = \langle 0 | X | 0 \rangle = \langle 0 | 1 \rangle = 0,$$

$$\Delta(X) = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{\langle 0 | I | 0 \rangle - 0} = \sqrt{1 - 0} = 1$$

2.60

$$\begin{aligned}\vec{v} \cdot \vec{\sigma} &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}.\end{aligned}$$

$$\det(\vec{v} \cdot \vec{\sigma} - \lambda I) = \lambda^2 - (v_1^2 + v_2^2 + v_3^2) = 0.$$

Since \vec{v} is a unit vector, $v_1^2 + v_2^2 + v_3^2 = 1$ and the eigenvalues are $\{-1, 1\}$.

For eigenvalue 1:

$$\begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = |e_+\rangle = \frac{1}{\sqrt{2+2v_3}} \begin{bmatrix} 1+v_3 \\ v_1+iv_2 \end{bmatrix}.$$

For eigenvalue -1:

$$\begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvector} = |e_-\rangle = \frac{1}{\sqrt{2-2v_3}} \begin{bmatrix} 1-v_3 \\ -v_1-iv_2 \end{bmatrix}.$$

The projector operators are then given by

$$\begin{aligned}P_+ &= |e_+\rangle\langle e_+| = \frac{1}{\sqrt{2+2v_3}} \begin{bmatrix} 1+v_3 \\ v_1+iv_2 \end{bmatrix} \frac{1}{\sqrt{2+2v_3}} \begin{bmatrix} 1+v_3 & v_1-iv_2 \end{bmatrix} \\ &= \frac{1}{2(1+v_3)} \begin{bmatrix} (1+v_3)^2 & (1+v_3)(v_1-iv_2) \\ (1+v_3)(v_1+iv_2) & v_1^2+v_2^2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \\ &= \frac{1}{2} (I + \vec{v} \cdot \vec{\sigma})\end{aligned}$$

$$\begin{aligned}P_- &= |e_-\rangle\langle e_-| = \frac{1}{\sqrt{2-2v_3}} \begin{bmatrix} 1-v_3 \\ -v_1-iv_2 \end{bmatrix} \frac{1}{\sqrt{2-2v_3}} \begin{bmatrix} 1-v_3 & -v_1+iv_2 \end{bmatrix} \\ &= \frac{1}{2(1-v_3)} \begin{bmatrix} (1-v_3)^2 & (1-v_3)(-v_1+iv_2) \\ (1-v_3)(-v_1-iv_2) & v_1^2+v_2^2 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \begin{bmatrix} 1 - v_3 & -v_1 + iv_2 \\ -v_1 - iv_2 & 1 + v_3 \end{bmatrix} \\
&= \frac{1}{2} (I - \vec{v} \cdot \vec{\sigma}).
\end{aligned}$$

2.61

$$\begin{aligned}
p(1) = \langle 0 | P_+ | 0 \rangle &= \frac{1}{2} (\langle 0 | I | 0 \rangle + \langle 0 | \vec{v} \cdot \vec{\sigma} | 0 \rangle) \\
&= \frac{1}{2} (\langle 0 | I | 0 \rangle + v_1 \langle 0 | X | 0 \rangle + v_2 \langle 0 | Y | 0 \rangle + v_3 \langle 0 | Z | 0 \rangle) \\
&= \frac{1}{2} (1 + 0 + 0 + v_3) \\
&= \frac{1 + v_3}{2}.
\end{aligned}$$

$$\begin{aligned}
|\psi\rangle &= \frac{P_+ |0\rangle}{\sqrt{\langle 0 | P_+ | 0 \rangle}} = \frac{1}{2} \left(\frac{I |0\rangle + v_1 X |0\rangle + v_2 Y |0\rangle + v_3 Z |0\rangle}{\sqrt{\frac{1+v_3}{2}}} \right) \\
&= \frac{(1 + v_3) |0\rangle + (v_1 + iv_2) |1\rangle}{\sqrt{2 + 2v_3}} \\
&= |e_+\rangle.
\end{aligned}$$

2.62

If the measurement operators P_m and the POVM elements E_m coincide then

$$E_m = P_m^\dagger P_m = P_m \implies P_m \text{ are positive operators} \implies P_m^\dagger = P_m,$$

thus

$$P_m^\dagger P_m = P_m^2 \implies P_m^2 = P_m.$$

2.63

Unitaries U_m rise naturally from the left polar decomposition of operators M_m , given by

$$M_m = U_m \sqrt{M_m^\dagger M_m} = U_m \sqrt{E_m}.$$

2.64

We can use an idea analogous to the Gram-Schmidt process to produce the states

$$|\phi_i\rangle = |\psi_i\rangle - \sum_{j=1 (j \neq i)}^m \frac{\langle \psi_j | \psi_i \rangle |\psi_j\rangle}{\| |\psi_j\rangle \|^2},$$

orthogonal to all states $|\psi_j\rangle$ with $j \neq i$. So by choosing

$$E_i = \alpha_i |\phi_i\rangle\langle\phi_i|, \text{ for } i \in \{1, \dots, m\}, \text{ and } E_{m+1} = I - \sum_{i=1}^m E_i,$$

where α_i are constants such that E_{m+1} is a positive operator, then we satisfy the condition $\sum_i E_i = I$, and for any measurement outcome E_i , for $i \in \{1, \dots, m\}$, Bob knows with certainty that the received state is $|\psi_i\rangle$, because the probability of obtaining E_i upon receiving any other state $|\psi_j\rangle$ is

$$\langle\psi_j|E_i|\psi_j\rangle = \alpha_i \langle\psi_j|\phi_i\rangle\langle\phi_i|\psi_j\rangle = 0.$$

2.65

In the Hadamard basis $\{|+\rangle, |-\rangle\}$ the states are written as

$$\begin{aligned} \frac{|0\rangle + |1\rangle}{\sqrt{2}} &= |+\rangle, \\ \frac{|0\rangle - |1\rangle}{\sqrt{2}} &= |-\rangle. \end{aligned}$$

2.66

$$\begin{aligned} \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) &= \frac{1}{2} \left(\langle 0|X_1|0\rangle\langle 0|Z_2|0\rangle + \langle 0|X_1|1\rangle\langle 0|Z_2|1\rangle \right. \\ &\quad \left. + \langle 1|X_1|0\rangle\langle 1|Z_2|0\rangle + \langle 1|X_1|1\rangle\langle 1|Z_2|1\rangle \right) \\ &= \frac{1}{2} \left(\langle 0|1\rangle\langle 0|0\rangle + \langle 0|0\rangle\langle 0|1\rangle + \langle 1|1\rangle\langle 1|0\rangle + \langle 1|0\rangle\langle 1|1\rangle \right) \\ &= 0. \end{aligned}$$

2.67

Let W^\perp be the orthogonal complement of the subspace W , then we have $V = W \oplus W^\perp$. Also, consider that $\{|w_i\rangle\}$ and $\{|w_i^\perp\rangle\}$ are orthonormal basis for the subspaces W and W^\perp respectively. We can define an operator U' given by

$$U' := \sum_{i=1}^{\dim(W)} U |w_i\rangle\langle w_i| + \sum_{i=1}^{\dim(W^\perp)} |w_i^\perp\rangle\langle w_i^\perp|.$$

Any vector $|v\rangle \in V$ can be written as

$$|v\rangle = \sum_{i=1}^{\dim(W)} a_i |w_i\rangle + \sum_{i=1}^{\dim(W^\perp)} b_i |w_i^\perp\rangle,$$

thus

$$\begin{aligned} U' |v\rangle &= \sum_{i=1}^{\dim(W)} \sum_{j=1}^{\dim(W)} a_j U |w_i\rangle \langle w_i| w_j\rangle + \sum_{i=1}^{\dim(W^\perp)} \sum_{i=j}^{\dim(W^\perp)} b_j |w_i^\perp\rangle \langle w_i^\perp| w_j^\perp\rangle \\ &= \sum_{i=1}^{\dim(W)} a_i U |w_i\rangle + \sum_{i=1}^{\dim(W^\perp)} b_i |w_i^\perp\rangle \in V. \end{aligned}$$

$$\begin{aligned} U'^\dagger U' &= \sum_{i=1}^{\dim(W)} \sum_{j=1}^{\dim(W)} |w_i\rangle \langle w_i| U^\dagger U |w_j\rangle \langle w_j| + \sum_{i=1}^{\dim(W^\perp)} \sum_{j=1}^{\dim(W^\perp)} |w_i^\perp\rangle \langle w_i^\perp| w_j^\perp\rangle \langle w_j^\perp| \\ &= \sum_{i=1}^{\dim(W)} |w_i\rangle \langle w_i| + \sum_{i=1}^{\dim(W^\perp)} |w_i^\perp\rangle \langle w_i^\perp| \\ &= I. \end{aligned}$$

So we clearly have a unitary operator $U' : V \rightarrow V$. Any vector $|w\rangle \in W$ can be written as

$$|w\rangle = \sum_{i=1}^{\dim(W)} c_i |w_i\rangle,$$

thus

$$\begin{aligned} U' |w\rangle &= \sum_{i=1}^{\dim(W)} \sum_{j=1}^{\dim(W)} c_j U |w_i\rangle \langle w_i| w_j\rangle + \sum_{i=1}^{\dim(W^\perp)} \sum_{j=1}^{\dim(W)} c_j |w_i^\perp\rangle \langle w_i^\perp| w_j\rangle \\ &= \sum_{i=1}^{\dim(W)} c_i U |w_i\rangle \\ &= U |w\rangle. \end{aligned}$$

Therefore, there exists a unitary operator $U' : V \rightarrow V$ which extends U .

2.68

Suppose there are states $|a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ and $|b\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$ such that $|\psi\rangle$ can be written as $|\psi\rangle = |a\rangle |b\rangle$. Then explicitly we have

$$\begin{aligned} |\psi\rangle &= (\alpha_0 |0\rangle + \alpha_1 |1\rangle) (\beta_0 |0\rangle + \beta_1 |1\rangle) = \alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle \\ &= \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle, \end{aligned}$$

but there are no possible combination of values for $\alpha_0, \alpha_1, \beta_0$ and β_1 that satisfies this equality.

2.69

Labeling the Bell states as

$$\begin{aligned} |\Phi^+\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \\ |\Phi^-\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \\ |\Psi^+\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \\ |\Psi^-\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \end{aligned}$$

we have the following relations:

$$\begin{aligned} \langle \Phi^\pm | \Phi^\pm \rangle &= \frac{\langle 00 | \pm \langle 11 |}{\sqrt{2}} \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} = \frac{\langle 0 | 0 \rangle \langle 0 | 0 \rangle \pm \langle 0 | 1 \rangle \langle 0 | 1 \rangle \pm \langle 1 | 0 \rangle \langle 1 | 0 \rangle + \langle 1 | 1 \rangle \langle 1 | 1 \rangle}{2} = 1, \\ \langle \Psi^\pm | \Psi^\pm \rangle &= \frac{\langle 01 | \pm \langle 10 |}{\sqrt{2}} \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} = \frac{\langle 0 | 0 \rangle \langle 1 | 1 \rangle \pm \langle 0 | 1 \rangle \langle 1 | 0 \rangle \pm \langle 1 | 0 \rangle \langle 0 | 1 \rangle + \langle 1 | 1 \rangle \langle 0 | 0 \rangle}{2} = 1, \\ \langle \Phi^\pm | \Phi^\mp \rangle &= \frac{\langle 00 | \pm \langle 11 |}{\sqrt{2}} \frac{|00\rangle \mp |11\rangle}{\sqrt{2}} = \frac{\langle 0 | 0 \rangle \langle 0 | 0 \rangle \mp \langle 0 | 1 \rangle \langle 0 | 1 \rangle \pm \langle 1 | 0 \rangle \langle 1 | 0 \rangle - \langle 1 | 1 \rangle \langle 1 | 1 \rangle}{2} = 0, \\ \langle \Psi^\pm | \Psi^\mp \rangle &= \frac{\langle 01 | \pm \langle 10 |}{\sqrt{2}} \frac{|01\rangle \mp |10\rangle}{\sqrt{2}} = \frac{\langle 0 | 0 \rangle \langle 1 | 1 \rangle \mp \langle 0 | 1 \rangle \langle 1 | 0 \rangle \pm \langle 1 | 0 \rangle \langle 0 | 1 \rangle - \langle 1 | 1 \rangle \langle 0 | 0 \rangle}{2} = 0, \\ \langle \Phi^\pm | \Psi^\pm \rangle &= \frac{\langle 00 | \pm \langle 11 |}{\sqrt{2}} \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} = \frac{\langle 0 | 0 \rangle \langle 0 | 1 \rangle \pm \langle 0 | 1 \rangle \langle 0 | 0 \rangle \pm \langle 1 | 0 \rangle \langle 1 | 1 \rangle + \langle 1 | 1 \rangle \langle 1 | 0 \rangle}{2} = 0, \\ \langle \Phi^\pm | \Psi^\mp \rangle &= \frac{\langle 00 | \pm \langle 11 |}{\sqrt{2}} \frac{|01\rangle \mp |10\rangle}{\sqrt{2}} = \frac{\langle 0 | 0 \rangle \langle 0 | 1 \rangle \mp \langle 0 | 1 \rangle \langle 0 | 0 \rangle \pm \langle 1 | 0 \rangle \langle 1 | 1 \rangle - \langle 1 | 1 \rangle \langle 1 | 0 \rangle}{2} = 0. \end{aligned}$$

2.70

$$\begin{aligned} \langle \Phi^\pm | E \otimes I | \Phi^\pm \rangle &= \frac{\langle 0 | E | 0 \rangle \langle 0 | 0 \rangle \pm \langle 0 | E | 1 \rangle \langle 0 | 1 \rangle \pm \langle 1 | E | 0 \rangle \langle 1 | 0 \rangle + \langle 1 | E | 1 \rangle \langle 1 | 1 \rangle}{2} \\ &= \frac{\langle 0 | E | 0 \rangle + \langle 1 | E | 1 \rangle}{2}, \\ \langle \Psi^\pm | E \otimes I | \Psi^\pm \rangle &= \frac{\langle 0 | E | 0 \rangle \langle 1 | 1 \rangle \pm \langle 0 | E | 1 \rangle \langle 1 | 0 \rangle \pm \langle 1 | E | 0 \rangle \langle 0 | 1 \rangle + \langle 1 | E | 1 \rangle \langle 0 | 0 \rangle}{2} \\ &= \frac{\langle 0 | E | 0 \rangle + \langle 1 | E | 1 \rangle}{2}. \end{aligned}$$

So, if Eve intercepts Alice's qubit and performs a measurement using measurement operators $\{M_m\}$ the probability of obtaining outcome m is

$$\langle \psi | M_m^\dagger M_m | \psi \rangle = \frac{\langle 0 | M_m^\dagger M_m | 0 \rangle + \langle 1 | M_m^\dagger M_m | 1 \rangle}{2}$$

for all m , independently of the four possible states $|\psi\rangle$. Therefore Eve could not infer anything about the qubit sent by Alice.

2.71

Let $\{|\psi_i\rangle\}$ be an orthonormal basis for which the density operator is diagonal. Then

$$\begin{aligned}\rho^2 &= \sum_i \sum_j p_i p_j |\psi_i\rangle \langle \psi_i| |\psi_j\rangle \langle \psi_j| \\ &= \sum_i p_i^2 |\psi_i\rangle \langle \psi_i| \\ \implies \text{tr}(\rho^2) &= \sum_j \left\langle \psi_j \left| \left(\sum_i p_i^2 |\psi_i\rangle \langle \psi_i| \right) \right| \psi_j \right\rangle \\ &= \sum_i \sum_j p_i^2 \langle \psi_j | \psi_i \rangle \langle \psi_i | \psi_j \rangle \\ &= \sum_j p_j^2 \leq 1,\end{aligned}$$

because $\sum_i p_i = 1$ and $p_i \leq 1$ for all i . Equality would only occur if $p_i = 1$ for some i and $p_j = 0$ for all $j \neq i$, that is, a pure state.

2.72

A density operator ρ is positive (hence Hermitian) and has trace equal to one. So for $a, b, c \in \mathbb{R}$, such that $-1 \leq c \leq 1$, we can write ρ as

$$\begin{aligned}\rho &= \frac{1}{2} \begin{bmatrix} 1+c & a-ib \\ a+ib & 1-c \end{bmatrix} \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & -ib \\ ib & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} \right) \\ &= \frac{I + aX + bY + cZ}{2}.\end{aligned}$$

We must have $\text{tr}(\rho^2) \leq 1$, thus

$$\begin{aligned}\text{tr}(\rho^2) &= \frac{1}{4} \text{tr} \begin{bmatrix} a^2 + b^2 + (1+c)^2 & 2(a-ib) \\ 2(a+ib) & a^2 + b^2 + (1-c)^2 \end{bmatrix} \\ &= \frac{1}{2} (1 + a^2 + b^2 + c^2) \implies a^2 + b^2 + c^2 \leq 1.\end{aligned}$$

So defining the vector $\vec{r} := (a, b, c)$, we have $\|\vec{r}\| \leq 1$ and

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}.$$

For $\rho = I/2$ we have $\vec{r} = 0$, meaning the null vector. So the state would be represented by the origin in the Bloch sphere.

If ρ is pure then

$$\text{tr}(\rho^2) = 1 \iff a^2 + b^2 + c^2 = 1 \iff \|\vec{r}\| = 1.$$

For pure states we have $\rho = |\psi\rangle\langle\psi|$, thus we may write

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle,$$

where θ and φ are polar angles of the unit Bloch vector \vec{r} , as described in Section 1.2.

2.73

$$\begin{aligned} |\psi\rangle &= \rho\rho^{-1}|\psi\rangle \\ &= \sum_i p_i |\psi_i\rangle \langle\psi_i| \rho^{-1} |\psi\rangle. \end{aligned}$$

Because $|\psi\rangle$ is a state in the support of ρ we may write

$$\begin{aligned} |\psi\rangle &= \sum_j a_j |\psi_j\rangle \\ \implies |\psi\rangle &= \sum_i \sum_j p_i a_j |\psi_i\rangle \langle\psi_i| \rho^{-1} |\psi_j\rangle \\ &= \sum_i p_i \langle\psi_i| \rho^{-1} |\psi_i\rangle a_i |\psi_i\rangle, \end{aligned}$$

but this equality holds only if

$$p_i = \frac{1}{\langle\psi_i| \rho^{-1} |\psi_i\rangle}.$$

2.74

$$\begin{aligned} \rho^{AB} &= (|a\rangle|b\rangle)(\langle a|\langle b|) = |a\rangle\langle a| \otimes |b\rangle\langle b| \\ \implies \rho^A &= \text{tr}_B(\rho^{AB}) = |a\rangle\langle a| \text{tr}(|b\rangle\langle b|) \\ &= |a\rangle\langle a|. \end{aligned}$$

2.75

Labeling the density operators as

$$\begin{aligned} \rho^{\Phi^\pm} &= |\Phi^\pm\rangle\langle\Phi^\pm| = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \frac{\langle 00| \pm \langle 11|}{\sqrt{2}} = \frac{|00\rangle\langle 00| \pm |00\rangle\langle 11| \pm |11\rangle\langle 00| + |11\rangle\langle 11|}{2}, \\ \rho^{\Psi^\pm} &= |\Psi^\pm\rangle\langle\Psi^\pm| = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \frac{\langle 01| \pm \langle 10|}{\sqrt{2}} = \frac{|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|}{2}, \end{aligned}$$

we have the following relations:

$$\begin{aligned} \rho_1^{\Phi^\pm} &= \text{tr}_2(\rho^{\Phi^\pm}) = \frac{|0\rangle\langle 0| \text{tr}(|0\rangle\langle 0|) \pm |0\rangle\langle 1| \text{tr}(|0\rangle\langle 1|) \pm |1\rangle\langle 0| \text{tr}(|1\rangle\langle 0|) + |1\rangle\langle 1| \text{tr}(|1\rangle\langle 1|)}{2} \\ &= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}, \end{aligned}$$

$$\begin{aligned}
\rho_2^{\Phi^\pm} &= \text{tr}_1(\rho^{\Phi^\pm}) = \frac{\text{tr}(|0\rangle\langle 0|)|0\rangle\langle 0| + \text{tr}(|0\rangle\langle 1|)|0\rangle\langle 1| + \text{tr}(|1\rangle\langle 0|)|1\rangle\langle 0| + \text{tr}(|1\rangle\langle 1|)|1\rangle\langle 1|)}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}, \\
\rho_1^{\Psi^\pm} &= \text{tr}_2(\rho^{\Psi^\pm}) = \frac{|0\rangle\langle 0|\text{tr}(|1\rangle\langle 1|) \pm |0\rangle\langle 1|\text{tr}(|1\rangle\langle 0|) \pm |1\rangle\langle 0|\text{tr}(|0\rangle\langle 1|) + |1\rangle\langle 1|\text{tr}(|0\rangle\langle 0|)}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}, \\
\rho_2^{\Psi^\pm} &= \text{tr}_1(\rho^{\Psi^\pm}) = \frac{\text{tr}(|0\rangle\langle 0|)|1\rangle\langle 1| + \text{tr}(|0\rangle\langle 1|)|1\rangle\langle 0| + \text{tr}(|1\rangle\langle 0|)|0\rangle\langle 1| + \text{tr}(|1\rangle\langle 1|)|0\rangle\langle 0|)}{2} \\
&= \frac{|1\rangle\langle 1| + |0\rangle\langle 0|}{2} = \frac{I}{2}.
\end{aligned}$$

2.76

Let $\dim(A) = m$ and $\dim(B) = n$, and consider that $\{|j\rangle\}$ and $\{|k\rangle\}$ are orthonormal basis for spaces A and B respectively. Then any state $|\psi\rangle \in A \otimes B$ can be written as

$$|\psi\rangle = \sum_{j=1}^m \sum_{k=1}^n a_{jk} |j\rangle |k\rangle.$$

By the singular value decomposition we have

$$a = u \begin{bmatrix} d \\ 0 \end{bmatrix} v \quad \text{if } m > n \quad \text{and} \quad a = u \begin{bmatrix} d & 0 \end{bmatrix} v \quad \text{if } m < n,$$

where u is a unitary $m \times m$ matrix, v is a unitary $n \times n$ matrix and d is a diagonal $\min\{m, n\} \times \min\{m, n\}$ matrix. The 0 just indicates that there are $m - n$ rows with null entries in the case where $m > n$ or $n - m$ columns in the case where $m < n$. If $m > n$ then we can write $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ where u_1 is $m \times n$ and u_2 is $m \times (m - n)$, thus

$$a = u_1 d v \implies a_{jk} = (u_1)_{ji} d_{ii} v_{ik}.$$

Labeling the n column vectors of u_1 as $|i_A\rangle = (u_1)_{ji} |j\rangle$, the n row vectors of v as $|i_B\rangle = v_{ik} |k\rangle$, and $\lambda_i := d_{ii}$ then we may write

$$|\psi\rangle = \sum_{i=1}^n \lambda_i |i_A\rangle |i_B\rangle.$$

Equivalently, if $m < n$ then we can write $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ where v_1 is $m \times n$ and v_2 is $(n - m) \times n$, thus

$$a = u d v_1 \implies a_{jk} = u_{ji} d_{ii} (v_1)_{ik}.$$

Labeling the m column vectors of u as $|i_A\rangle = u_{ji}|j\rangle$, the m row vectors of v_1 as $|i_B\rangle = (v_1)_{ik}$, and $\lambda_i := d_{ii}$ we also may write

$$|\psi\rangle = \sum_{i=1}^m \lambda_i |i_A\rangle |i_B\rangle .$$

2.77

First, notice that if the Schmidt coefficients are non-degenerate then the Schmidt decomposition is unique up to phase. To see that, consider that the Schmidt decomposition of the pure state $|\psi\rangle$ of a composite system $A \otimes B$ is given by

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle \implies \rho = |\psi\rangle\langle\psi| = \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \otimes |i_B\rangle\langle i_B| .$$

So, the eigenvalues of the density operator are given by λ_i^2 , and because the Schmidt coefficients are non-negative, all $\sqrt{\lambda_i^2}$ are uniquely defined. Furthermore, if they are all non-degenerate then the states $|i_A\rangle |i_B\rangle$ associated with each coefficient are also uniquely defined up to phase. Therefore the Schmidt decomposition is unique up to phase.

Now consider the pure state $|\psi\rangle \in A \otimes B \otimes C$, where A, B and C are one qubit spaces, given by

$$|\psi\rangle = \frac{1}{\sqrt{10}} (2|0\rangle|0\rangle|0\rangle + 2|1\rangle|1\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |1\rangle|0\rangle|1\rangle) .$$

This state can be rewritten as

$$\begin{aligned} |\psi\rangle &= \frac{2}{\sqrt{5}} \left(\frac{1}{\sqrt{2}} |0\rangle|0\rangle + \frac{1}{\sqrt{2}} |1\rangle|1\rangle \right) |0\rangle + \frac{1}{\sqrt{5}} \left(\frac{1}{\sqrt{2}} |0\rangle|1\rangle + \frac{1}{\sqrt{2}} |1\rangle|0\rangle \right) |1\rangle \\ &= \frac{2}{\sqrt{5}} |\Phi^+\rangle|0\rangle + \frac{1}{\sqrt{5}} |\Psi^+\rangle|1\rangle . \end{aligned}$$

This is a Schmidt decomposition $|\psi\rangle = \lambda_0 |0_D\rangle |0_C\rangle + \lambda_1 |1_D\rangle |1_C\rangle$ considering $D := A \otimes B$. Since the Schmidt coefficients are different this decomposition is unique, so $|\psi\rangle$ can be written as a tripartite Schmidt decomposition if and only if $|0_D\rangle$ and $|1_D\rangle$ can be written as $|0_A\rangle |0_B\rangle$ and $|1_A\rangle |1_B\rangle$ respectively. Since they are Bell states this is impossible, thus $|\psi\rangle$ cannot be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle .$$

2.78

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle .$$

Since $\sum_i \lambda_i^2 = 1$, if $\lambda_j = 1$ for some j then it has Schmidt number 1 and

$$|\psi\rangle = |j_A\rangle |j_B\rangle .$$

The converse is immediate.

If $|\psi\rangle$ is a product state then

$$\begin{aligned} |\psi\rangle = |\psi_A\rangle |\psi_B\rangle &\implies \rho = |\psi\rangle\langle\psi| = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| \\ &\implies \rho^A = \text{tr}_B(\rho) = |\psi_A\rangle\langle\psi_A|, \\ &\quad \rho^B = \text{tr}_A(\rho) = |\psi_B\rangle\langle\psi_B|. \end{aligned}$$

The converse is immediate.

2.79

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} |0\rangle |0\rangle + \frac{1}{\sqrt{2}} |1\rangle |1\rangle;$$

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle |+\rangle;$$

For the third state we have

$$\begin{aligned} \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} &\implies \rho = \frac{1}{3} [|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |0\rangle\langle 0| \\ &\quad + |0\rangle\langle 0| \otimes |1\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| \\ &\quad + |1\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|] \\ &\implies \rho^A = \rho^B = \text{tr}_A(\rho) = \frac{1}{3} (2|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \\ &= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

$$\det(\rho^B - \lambda I) = \lambda^2 - \lambda + \frac{1}{9} = 0 \implies \text{eigenvalues} = \lambda_{\pm} = \frac{3 \pm \sqrt{5}}{6}.$$

$$\frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{3 \pm \sqrt{5}}{6} \begin{bmatrix} a \\ b \end{bmatrix} \implies \text{eigenvectors} = |e_{\pm}\rangle = \frac{\exp(i\theta_{\pm})}{\sqrt{10 \mp 2\sqrt{5}}} \begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}.$$

$$\begin{aligned} \text{Choosing the correct phases} \\ (\theta_+ = 0; \theta_- = \frac{\pi}{2}) &\implies \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} = \sqrt{\lambda_+} |e_+\rangle |e_+\rangle + \sqrt{\lambda_-} |e_-\rangle |e_-\rangle. \end{aligned}$$

2.80

$$|\psi\rangle = \sum_i \lambda_i |\psi_i^A\rangle |\psi_i^B\rangle,$$

$$|\varphi\rangle = \sum_i \lambda_i |\varphi_i^A\rangle |\varphi_i^B\rangle.$$

Let $U : A \rightarrow A$ and $V : B \rightarrow B$ be unitary transformations such that $U |\varphi_i^A\rangle = |\psi_i^A\rangle$ and $V |\varphi_i^B\rangle = |\psi_i^B\rangle$, then we may write

$$\begin{aligned} |\psi\rangle &= \sum_i \lambda_i U |\varphi_i^A\rangle \otimes V |\varphi_i^B\rangle \\ &= (U \otimes V) \sum_i \lambda_i |\varphi_i^A\rangle |\varphi_i^B\rangle \\ &= (U \otimes V) |\varphi\rangle. \end{aligned}$$

2.81

Consider that the density operator is given by $\rho^A = \sum_i p_i |i_A\rangle\langle i_A|$, then the two purifications can be written as

$$\begin{aligned} |AR_1\rangle &= \sum_i \sqrt{p_i} |i_A\rangle |i_{R_1}\rangle, \\ |AR_2\rangle &= \sum_i \sqrt{p_i} |i_A\rangle |i_{R_2}\rangle. \end{aligned}$$

Let $U_R : R \rightarrow R$ be a unitary transformation such that $U_R |i_{R_2}\rangle = |i_{R_1}\rangle$, then we may write

$$\begin{aligned} |AR_1\rangle &= \sum_i \sqrt{p_i} I_A |i_A\rangle \otimes U_R |i_{R_2}\rangle \\ &= (I_A \otimes U_R) \sum_i \sqrt{p_i} |i_A\rangle |i_{R_2}\rangle \\ &= (I_A \otimes U_R) |AR_2\rangle. \end{aligned}$$

2.82

$$\begin{aligned} \text{tr}_R \left[\left(\sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle \right) \left(\sum_j \sqrt{p_j} \langle \psi_j | \langle j | \right) \right] &= \sum_i \sum_j \sqrt{p_i p_j} |\psi_i\rangle\langle \psi_j| \text{tr}(|i\rangle\langle j|) \\ &= \sum_i p_i |\psi_i\rangle\langle \psi_i| = \rho. \end{aligned}$$

To measure R in the basis $|i\rangle$ means we are using the set of projective measurement operators $\{I \otimes |i\rangle\langle i|\}$. So, the probability of obtaining outcome i is

$$\begin{aligned} \sum_j \sqrt{p_j} \langle \psi_j | \langle j | (I \otimes |i\rangle\langle i|) \sum_k \sqrt{p_k} |\psi_k\rangle |k\rangle &= \sum_j \sum_k \sqrt{p_j p_k} \langle \psi_j | I | \psi_k\rangle \langle j | i \rangle \langle i | k \rangle \\ &= \sqrt{p_i p_i} \langle \psi_i | I | \psi_i\rangle = p_i, \end{aligned}$$

and the corresponding post-measurement state of system A is $|\psi_i\rangle$.

Due to the unitary freedom in the ensemble for density matrices there exists an ensemble $\{q_j, |\phi_j\rangle\}$

that generates the same density matrix ρ , with the condition that

$$\sqrt{q_j} |\phi_j\rangle = \sum_i u_{ji} \sqrt{p_i} |\psi_i\rangle$$

for some unitary matrix u_{ji} . So, considering an orthonormal basis $\{|r_i\rangle\}$ for R , we may write any purification $|AR\rangle$ as

$$\begin{aligned} |AR\rangle &= \sum_j \sqrt{q_j} |\phi_j\rangle |r_j\rangle \\ &= \sum_j \left(\sum_i u_{ji} \sqrt{p_i} |\psi_i\rangle \right) \otimes |r_j\rangle \\ &= \sum_i \sqrt{p_i} |\psi_i\rangle \otimes \left(\sum_j u_{ji} |r_j\rangle \right). \end{aligned}$$

Now, let $U : R \rightarrow R$ be a unitary transformation such that, for all $|r_i\rangle \in R$, we have $U|r_i\rangle = \sum_j u_{ji} |r_j\rangle := |i\rangle$, thus

$$|AR\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle.$$

It is straightforward to see that $\{|i\rangle\}$ is an orthonormal basis for R since

$$\begin{aligned} \langle i | j \rangle &= \langle r_i | U^\dagger U | r_j \rangle \\ &= \langle r_i | r_j \rangle \\ &= \delta_{ij}, \end{aligned}$$

and we have already shown that, for a purification of this form, if we measure R in the basis $\{|i\rangle\}$, we obtain outcome i with probability p_i , meaning we have post-measurement state $|\psi_i\rangle$ for system A with probability p_i .