Numerical Analysis 1 – Class 8

Thursday, March 11th 2021

Subjects covered

- Mini-project presentations

Readings

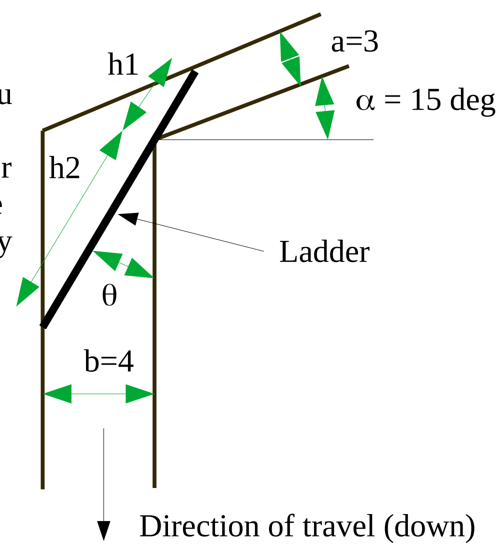
- "An Underdetermined Linear System for GPS", D. Kalman (lined on Canvas).

Problems

Most of the following problems require you to write a program. For each program you write, please make sure you also write a test which validates your program. Please use Canvas to upload your submissions under the "Assignments" link for this problem set.

Problem 1

The goal of this problem is to show that seemingly simple problems can often only be solved using numerical methods. Imagine that you are working in the maintenance crew of a medieval castle in France. To fix a problem on the roof you need to carry an 11 foot long ladder down a narrow hallway and around a corner with an odd angle. The geometry of the hallway is shown in the figure at right. One hallway has width 3 feet, the other's width is 4 feet and the turn angle is 75 degrees. The question is, can you do it? Will the ladder fit around the corner?



To solve the problem please write down an expression which gives the length of the ladder L assuming it touches the walls as shown in the figure. Then find the maximum allowed length as a function of turn angle θ. Knowing that angle, you can compute the ladder length at the tightest spot in the hallway.

The point of the problem is that solving for θ involves a nasty expression involving trig functions. The expression may not be reducible to closed form easily (if at all). However, the problem is ideally suited for solution using 1-dimensional Newton's Method.

Please write a program which computes the maximum ladder length which will fit around the corner using Newton's Method. Feel free to use my implementation of 1D Newton's Method on Canvas as part of your program. As a paper-and-pencil exercise, please also turn in your derivations of the equations to solve as well as the expressions used in Newton's Method.

Regarding testing, I suggest you try the following:

- Make a plot of your expression for the max ladder length vs. θ and verify it is sane based on the geometry of the problem (i.e. consider what is the allowed ladder length as θ sweeps from 0 to

75 degrees.

- Consider an easy geometry where a=b and α=0 degrees. In this case the ladder length L may be obtained by inspection. Configure your program to run this case and verify you get the correct answer.

Problem 2

Consider the 4th order polynomial equation

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \quad (1)$$

and recall that the Fundamental Theorem of Algebra states that this equation has 4 (possibly non-unique) solutions over the complex numbers, $x_i \in \mathbb{C}$. These solutions are called the “roots” of the equation.

The 16th century French mathematician Francois Vieta discovered a set of equations relating the roots of a polynomial of degree n to the coefficients a_i . They are:

$$x_1 + x_2 + x_3 + \cdots + x_n = -\frac{a_{n-1}}{a_n}$$

$$x_1(x_2 + x_3 + \cdots + x_n) + x_2(x_3 + \cdots) + \cdots + x_{n-1} x_n = \frac{a_{n-2}}{a_n}$$

$$\vdots$$

$$x_1 x_2 x_3 \cdots x_n = (-1)^n \frac{a_0}{a_n}$$

Wikipedia has an excellent article about this theorem if you want more details. Notice that Vieta's equations provide a system of *n* nonlinear equations in *n* unknowns. This suggests that one method to find the solutions of equation (1) is to use a rootfinder like Newton's Method.

Please do the following:

- Consider the 4th degree polynomial equation

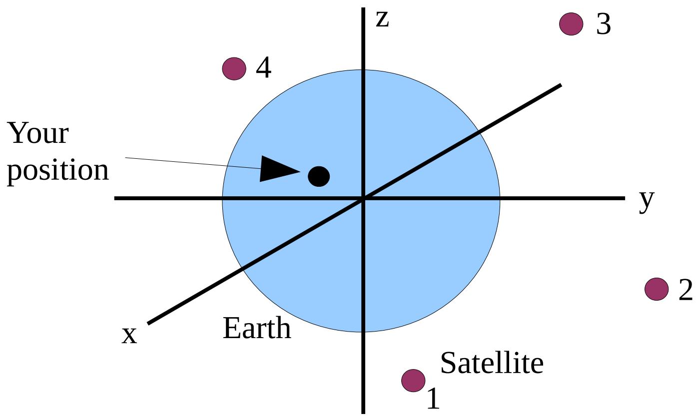
$$x^4 + 0.4\, x^3 - 0.5\, x^2 + 1.2\, x - 1.5 = 0 \quad (2)$$

- Using Vieta's formulas, write down the system of equations obeyed by the roots x_i .
- From the system, derive and write down the Jacobian matrix.
- Now use Newton's Method to compute the roots of equation (2). Feel free to adapt and modify the N-dimensional Newton's Method code on Canvas to solve this problem. Your program should set the coefficients of the polynomial, guess an initial starting point, call Newton's Method, and then print out the roots it finds. Note that some roots may be complex.
- Regarding testing, your test program should verify that when you put each of the roots into equation (2), the result is zero.

Problem 3

Here is an example of using GPS (global positioning system) to determine your position on earth. This example is explained in more detail in the paper "An underdetermined linear system for GPS", by Dan Kalman (on Canvas).

Consider a GPS system consisting of four satellites orbiting earth.



Each satellite broadcasts the time on its internal clock, and its position. Some time after broadcast you receive the signal from each of the four satellites. The delay time is determined by the distance between the satellite and you, as well as the speed of radio waves (i.e. the speed of light). You don't have your own clock, so you have only the information sent by the satellites to find your position.

The distance from your position to satellite i is

$$d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$$

where (x, y, z) represents your position, and (xi, yi, zi) the position of satellite i. Each satellite broadcasts its position, so you know the triplet (xi, yi, zi) for each satellite. This distance is also

$$d_i = c\left(t_{local} - t_i(0)\right)$$

where c is the velocity of light (c = 0.047 in our units), tlocal is your local time, and ti(0) is the time at which satellite i broadcast the message you are receiving at time t. (This parameter is contained in the position message broadcast by the satellite.)

Your assignment: Set up a system of 4 simultaneous, nonlinear equations which govern the signal you receive, and solve it using Newton's method to get your position (x, y, z) on the earth's surface, as well as your local time.

Feel free to adapt and modify the 2D Newton's Method code on Canvas to solve this problem. Use the data below, which are the position and time measurements you receive from each satellite.

Satellite	**x**	**y**	**z**	**t(0)**
1	1.2000	2.3000	0.2000	9.9999

2	-0.5000	1.5000	1.8000	13.0681
3	-1.7000	0.8000	1.3000	2.0251
4	1.7000	1.4000	-0.5000	10.5317

Note that the units are normalized so that the earth's radius is 1. (That's why c = 0.047.) Check your result – make sure the position you find is on the earth's surface. As an additional check, you may use the method described in Kalman's paper to create a set of linear equations which are easier to solve than those above. The result you get from Newton's method should match the result you get from the linear equations.