The maximum likelihood estimators of functions of the parameters

$\phi = f(\theta)$ $\quad \hat{\phi}$. maximum likelihood estimator

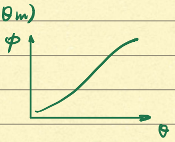
The functions of the maximum likelihood estimators of the parameters.

$\hat{\theta} = \operatorname{argmax} L(\theta) \qquad \phi = f(\hat{\theta})$

Corollary 3.2.1. $\quad \hat{\theta}_1, \cdots, \hat{\theta}_m \qquad \theta_1, \cdots, \theta_m$ 的 MLE

$$\begin{matrix} \phi_1(\hat{\theta}_1, \cdots, \hat{\theta}_m) & \xrightarrow{MLE} & \phi_1(\theta_1, \cdots, \theta_m) \\ \phi_m(\hat{\theta}_1, \cdots, \hat{\theta}_m) & & \phi_m(\theta_1, \cdots, \theta_m) \end{matrix}$$

Lemma.



ϕ a single value function, single value inverse

$f(\theta)$, $\theta \in S$

$\phi \in S^*$

Each $\theta \in S$, there is a unique $\theta^* \in S^*$ corresponding

$g(\theta^*) = f[\phi^{-1}(\theta^*)] \qquad \theta^* = \phi(\theta)$

$\quad = f(\theta) \qquad \theta = \phi^{-1}(\theta^*)$

(1) $f(\theta) \Rightarrow \max$ at $\theta = \theta_0$

$g(\theta^*) \Rightarrow \max$ at $\theta^* = \theta^*_0 = \phi(\theta_0)$

(2) maximum of $f(\theta)$ is unique at θ_0

maximum of $g(\theta^*)$ is also unique at θ_0^*

Proof: $\because f(\theta) \Rightarrow \max$ at $\theta = \theta_0$

$\therefore f(\theta_0) \geq f(\theta) \quad \forall \theta \in S$

$\therefore \forall \theta^* \in S^*$

$$\begin{aligned} g(\theta^*) &= f[\phi^{-1}(\theta^*)] \\ &= f(\theta) \leq f(\theta_0) \\ &= f(\phi^{-1}(\theta_0^*)) \\ &= g(\theta_0^*) \end{aligned}$$

$\therefore g(\theta_0^*)$ reaches maximum

Theorem 3

$\hat{\mu} = \bar{x} = \frac{1}{N}\sum_{\alpha=1}^{N} x_\alpha$, $\qquad$ MLE

$\hat{\Sigma} = \frac{1}{N}\sum_{\alpha=1}^{N}(x_\alpha - \bar{x})(x_\alpha - \bar{x})^T$

$\hat{\sigma}_i^2 = \frac{1}{N}\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)^2 \qquad x_{i\alpha}$ 是一个 random variable

是 X 中第 i 个 element 的第 α 次的 observation

$$\hat{\rho}_{ij} = \frac{\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)}{\sqrt{\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)^2}\sqrt{\sum_{\alpha=1}^{N}(x_{j\alpha} - \bar{x}_j)^2}}$$

Lemma 8

orthogonal $C^TC = CC^T = I_N$

$C = (c_{\alpha\beta})$ is orthogonal, $y_\alpha = \sum_{\beta=1}^{N} c_{\alpha\beta} x_\beta$

then $\sum_{\alpha=1}^{N} x_\alpha x_\alpha^T = \sum_{\alpha=1}^{N} y_\alpha y_\alpha^T$

$\because \quad y_\alpha = \sum_{\beta=1}^{N} c_{\alpha\beta} x_\beta \qquad X = [x_1, \cdots, x_N]$

$Y = [Y_1, \cdots, Y_N]$

$$\begin{aligned} y_\alpha &= c_{\alpha 1}x_1 + c_{\alpha 2}x_2 + \cdots + c_{\alpha N}x_N \\ &= [x_1, x_2, \cdots, x_N][c_{\alpha 1}, c_{\alpha 2}, \cdots, c_{\alpha N}]^T \end{aligned}$$

$Y = X \cdot C^T$

$\therefore \quad \sum_{\alpha=1}^{N} y_\alpha y_\alpha^T = YY^T = XC^T(XC^T)^T = XC^TCX^T = XX^T = \sum_{\alpha=1}^{N} x_\alpha x_\alpha^T$

4. Unbias

An estimator of a parameter vector θ is unbias iff $E_\theta(t) = \theta$

$E(\bar{x}) = \frac{1}{N} E\left(\sum_{\alpha=1}^{N} x_\alpha\right) = \mu \Rightarrow$ unbias estimator

$E(\hat{\Sigma}) = \frac{1}{N} E\left(\sum_{\alpha=1}^{N-1} z_\alpha z_\alpha^T\right) = \frac{N-1}{N} \sum_{\alpha=1}^{N} (x_\alpha - \bar{x})(x_\alpha - \bar{x})^T$

5. Sufficiency

A statistic $t(y)$ is sufficient for θ iff density $f(y|\theta)$ can be factored as

$$f(y|\theta) = \underbrace{g(t(y), \theta)}_{\text{non negative}} \underbrace{h(y)}_{\text{does not depend on } \theta}$$

Theorem 3.4.1

1. $\bar{x}$, S are sufficient for μ, Σ, μ is not given.
2. μ is given, $\sum_{\alpha=1}^{N} (x_\alpha - \mu)(x_\alpha - \mu)^T$ is sufficient for Σ
3. Σ is given, $\bar{x}$ is sufficient for μ

$\because$ $x_1, \cdots x_n$ is normal

$\prod_{\alpha=1}^{N} \text{normal}(x_\alpha | \mu, \Sigma)$

$= (2\pi)^{-\frac{1}{2}Np} |\Sigma|^{-\frac{1}{2}N} \exp\left[-\frac{1}{2} \operatorname{tr} \Sigma^{-1} \sum_{\alpha=1}^{N} (x_\alpha - \mu)(x_\alpha - \mu)^T\right]$

$= (2\pi)^{-\frac{1}{2}Np} |\Sigma|^{-\frac{1}{2}N} \exp\left\{-\frac{1}{2}\left[N(\bar{x}-\mu)^T \Sigma^{-1} (\bar{x}-\mu) + (N-1) \operatorname{tr} \Sigma^{-1} S\right]\right\}$

$h(x_1, \cdots, x_N) = \exp\left\{-\frac{1}{2}(N-1) \operatorname{tr} \Sigma^{-1} S\right\}$