Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \begin{cases} |x| < 1 & \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \\ x \neq 1 & \sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x} \end{cases}$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \ln n + O(1) \qquad \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Approximations & Formulas

- $e^x > 1 + x, e^{-x} > 1 - x, e^{-x} \approx 1 - x$
- $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$
- $k! \approx k^k e^{-k} \sqrt{2\pi k + o(k)}$
- if $k = o(\sqrt{n})$ then $\binom{n}{k} \approx \frac{n^k}{k!}$
- **Binomial coeff**: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Binomail Thm**: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-1} b^i$
- **Jensen's Inequality**: $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$, if f is convex. Convex if $\forall x: \ f''(x) \geq 0$

Asymptotic Notation

Symbol	$L = \lim_{n\to\infty} \frac{f(n)}{g(n)}$	Relation
$f(n) = \mathcal{O}(g(n))$	$L < \infty$	$f \preceq g$
$f(n) = \Omega(g(n))$	$L > 0$	$f \succeq g$
$f(n) = \Theta(g(n))$	$0 < L < \infty$	$f \asymp g$
$f(n) = o(g(n))$	$L = 0$	$f \prec g$
$f(n) = \omega(g(n))$	$L = \infty$	$f \succ g$
$f(n) = g(n) + o(g(n))$	$L = 1$	$f(n) \sim g(n)$

Table 1: Notations and their relationships

Combinatorics

- Ordered, w/ repetition: n^k
- Ordered, w/o repetition: $\frac{n!}{(n-k)!}$
- Unordered, w/ repetition: $\binom{n+k-1}{k}$
- Unordered, w/o repetition: $\binom{n}{k}$

Basic-Probability

Axioms probability distribution:

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $\mathbb{P}[\bigcup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$ if $A_i \cap A_j = \emptyset$ for $i \neq j$

Probability of an event A: $\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$

- $\mathbb{P}[\overline{A}] = 1 - \mathbb{P}[A]$
- If $A \subseteq B$ then $\mathbb{P}[B] = \mathbb{P}[A] + \mathbb{P}[B \setminus A] \geq \mathbb{P}[A]$
- $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$
- $\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] - \mathbb{P}[A \cap B] - \mathbb{P}[A \cap C] - \mathbb{P}[B \cap C] + \mathbb{P}[A \cap B \cap C]$
- A, B independent iff $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$
- cond. independent iff $\mathbb{P}[A \cap B \mid C] = \mathbb{P}[A \mid C]\mathbb{P}[B \mid C]$
- if mutually indep: $\mathbb{P}[\bigcap_{i=1}^{n} A_i] = \prod_{i=1}^{n} \mathbb{P}[A_i]$

Union Bound
$\mathbb{P}[\bigcup_i A_i] \leq \sum_i \mathbb{P}[A_i]$

With high probability (WHP) probability $\geq 1 - \frac{1}{f(n)}$ for $f(n) = \Omega(n^c)$, for $c > 0$. $\iff$ probability goes to 1 as $n \to \infty$.

Conditional Probability $\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$

A and B are independent iff $\mathbb{P}[A \mid B] = \mathbb{P}[A]$

Bayes Rule

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A]\mathbb{P}[A]}{\mathbb{P}[B]}$$

Law of Total Probability

$$\mathbb{P}[A] = \sum_i \mathbb{P}[A \mid B_i]\mathbb{P}[B_i], \ B_i \text{ partition of } \Omega$$

Random Variables

Expectation

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}[X = x] \stackrel{X \geq 0}{=} \sum_x \mathbb{P}[X > x]$$

- Linearity: $\mathbb{E}[aX + Y] = a\mathbb{E}[X] + \mathbb{E}[Y]$
- X, Y independent: $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$

Variance

$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

- If X, Y are independent: $\mathbb{V}[X + Y] = \mathbb{V}[X] + \mathbb{V}[Y]$
- General: $\mathbb{V}[X + Y] = \mathbb{V}[X] + \mathbb{V}[Y] + 2\text{Cov}[X, Y]$
- Convariance: $\text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- $\mathbb{V}[cX] = c^2\mathbb{V}[X]$
- $\mathbb{V}[X] = 0 \iff X =$ constant, else: $\mathbb{V}[X] > 0$
- $\mathbb{V}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{V}[X_i]$
- **Standard deviation:** $\sigma_X = \sqrt{\mathbb{V}[X]}$

Joint PMF

$$p_{X,Y}(x, y) = \mathbb{P}[X = x \cap Y = y]$$

$$\mathbb{E}[f(X, Y)] = \sum_{x,y} f(x, y) \cdot p_{X,Y}(x, y)$$

Distributions

$q = 1 - p$	PMF $\mathbb{P}[X = k]$	$\mathbb{E}[X]$	Var(X)
Uniform	$\frac{1}{n}$	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$
Binomial	$\binom{n}{k} p^k q^{n-k}$	np	npq
Bernoulli	$p^k q^{1-k}$	p	pq
Geometric	$q^{k-1} p$	$\frac{1}{p}$	$\frac{q}{p^2}$
Poisson	$\frac{\lambda^k}{k!} e^{-\lambda}$	λ	λ
Hypergeom	$\frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$	$\frac{nk}{N}$	$\frac{N-n}{N-1}(n)\frac{k}{N}\frac{N-k}{N}$

- Binomial: # of successes in n trials, p success rate
- Bernoulli: Single trial, p success rate
- Geometric: Trials until first success
- Poisson: Approximates binomial distribution for $n \to \infty$: if $X \in \text{Bin}(n, p)$, $\mu = pn$, then for $n \to \infty$:

$$\mathbb{P}(X = i) \approx \frac{\mu^i e^{-\mu}}{i!}$$

  Use if n large and p small.
- Hypegeom: N total, n sample, x success, k defect

Tail Bounds

Markov's Inequality If X is a non-negative random variable and $a > 0$, then:

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a} \qquad \mathbb{P}\left[X \geq b \cdot \mathbb{E}\left[X\right]\right] \leq \frac{1}{b}$$

Chebyshev's Inequality If X is a random variable with finite expected value μ and finite non-zero variance σ^2, then for any real number $k > 0$,

$$\mathbb{P}[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2} \qquad \mathbb{P}[|X - \mu| \geq b] \leq \frac{\mathbb{V}[X]}{b^2}$$

$$|X - \mu| \geq a\sigma \iff (X \geq a\sigma + \mu) \cup (X \leq -a\sigma + \mu)$$

Chernoff Bounds If $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables, each equal to 1 with probability p, and if $X = X_1 + X_2 + \ldots + X_n$, then for $0 < \delta < 1$:

$$\mathbb{P}[X < (1-\delta)\mathbb{E}[X]] < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}, \text{for } \delta \in (0,1)$$

$$\mathbb{P}[X > (1+\delta)\mathbb{E}[X]] < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}, \text{for } \delta \in (0,\infty)$$

Randomized Algorithms

Monte Carlo Finite time, might be wrong.
Las Vegas Always correct, unbounded time.

Fingerprinting

- if $n \in \mathbb{Z}$ has N-bits, then at most N primes can divide n.
- $p_i \approx i \ln i, \quad p_i = \text{ith prime}$
- **Prime number theorem**: $\pi(n) \approx \frac{n}{\ln n}$, where $\pi(n)$ is the number of primes less than n.
- **Fermats Little**: If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \mod p$.
- $x \mod p = y \mod p \iff p\, divides\, |x - y|$
- Then use that to get something like:

$$\mathbb{P}\left[p \text{ divs } |x-y|\right] = \frac{\#\text{ primes dividing } |x-y|}{\#\text{ primes}}$$

, where m is max possible prime.

Continuous Master Theorem

Solves recurrences of the form:

$$F_n = t_n + \sum_{0 \leq j \leq n} w_{n,j} F_j, \quad \text{for some } n \geq n_0$$

With $t_n = \Theta(n^a \log^b n)$, $a \geq 0$, $b > -1$

1. Find the shape function: $\omega(z) = \lim_{n\to\infty} n \cdot \omega_{n,z\cdot n}$
2. $\mathcal{H} = 1 - \int_0^1 \omega(z) z^a \, dz$, $\mathcal{H}' = -(b+1)\int_0^1 \omega(z) z^a \ln z \, dz$

Then:

$$F_n = \begin{cases} \frac{t_n}{\mathcal{H}} + o(t_n) & \text{if } \mathcal{H} > 0, \\ \frac{t_n}{\mathcal{H}'} \ln n + o(t_n \log n) & \text{if } \mathcal{H} = 0 \text{ and } \mathcal{H}' \neq 0, \\ \Theta(t_n n^{\alpha}) & \text{if } \mathcal{H} < 0 \end{cases}$$

where α is the solution to: $1 - \int_0^1 \omega(z) z^{\alpha} \, dz = 0$

Algorithms

Random Quicksort:

```
function RAND-QUICKSORT(A)
    if A.size() ≤ 3 then
        INSERTION-SORT(A)
        return A
    end if
    choose a ∈ A uniformly at random
    Put in A⁻ all elements < a and in A⁺ all elements > a
    Rand-Quicksort(A⁻)
    Rand-Quicksort(A⁺)
    A := A⁻ · {a} · A⁺
end function
```

```
function RANDOMIZED MEDIAN(S set of n elements)
    R := Sample ⌈n^(3/4)⌉) elements from S with replacement
    SORT(R)
    d := R[½⌊n^(3/4) − √n⌋]
    u := R[½⌈n^(3/4) + √n⌉]
    C := {x ∈ S | d ≤ x ≤ u}
    l_d := # of elements in S < d
    l_u := # of elements in S > u
    if l_d > n/2 or l_u > n/2 or |C| > 4n^(3/4) then
        return fail
    end if
    SORT(C)
    return C[⌊n/2⌋ − l_d + 1]
end function
```

Randomized Median finds median with probability $\geq 1 - \frac{1}{n^{1/4}}$
Proof:

- $E_1 = d > m \equiv l_d > n/2$
- $E_2 = u < m \equiv l_u > n/2$
- $E_3 = |C| > 4n^{3/4}$

$$\mathbb{P}\left[\text{fail}\right] = \mathbb{P}\left[E_1 \cup E_2 \cup E_3\right]$$

E1:

1. $Y = |\{x \in R \mid x \leq m\}|$, $\mathbb{P}\left[E_1\right] = \mathbb{P}\left[Y < n^{3/4}/2 - \sqrt{n}\right]$
2. $Y_j = 1 \iff$ value in j-th position in R is $\leq m$.
3. $Y = \sum_{j=1}^{n^{3/4}} Y_j$, and Y_j independent
4. There are $\frac{n-1}{2} + 1$ Elements $\leq m$ in S
5. $\mathbb{P}\left[Y_j = 1\right] = \frac{1}{2} + \frac{1}{2n}$
6. $Y \sim \text{Bin}(n^{3/4}, \frac{1}{2} + \frac{1}{2n})$
7. $\mathbb{E}$, $\mathbb{V}$, then chebyshev

E3:

- $E_{3,1}$ = At least $2n^{3/4}$ items in C are $> m$
- $E_{3,2}$= At least $2n^{3/4}$ items in C are $< m$

$$\mathbb{P}\left[E_3\right] \leq \mathbb{P}\left[E_{3,1} \cup E_{3,2}\right] \leq \mathbb{P}\left[E_{3,1}\right] + \mathbb{P}\left[E_{3,2}\right]$$

$E_{3,1}$ happens when rank(u) in $\tilde{S}$ is $\geq n/2 + 2n^{3/4}$ etc.