Constraining Earth model properties through Backus-Gilbert SOLA inferences

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Motivation and Aims

- Insufficient seismic data leaves many Earth regions poorly constrained
- Inferences can offer advantages over inversions in constraining sparsely covered properties.
- The Backus Gilbert SOLA (BG SOLA) method, a deterministic inference method, can provide bounds on poorly covered model properties without strong regularization or model discretization.
- Historically, BG SOLA has been used to constrain local averages of seismic tomography models[1,2,3,4,5], and some gradient information.
- In this work we delve deeper into what properties can be constrained with BG SOLA with applications to Earth seismic tomography, and more specifically, using normal modes.

Inversion versus Inference

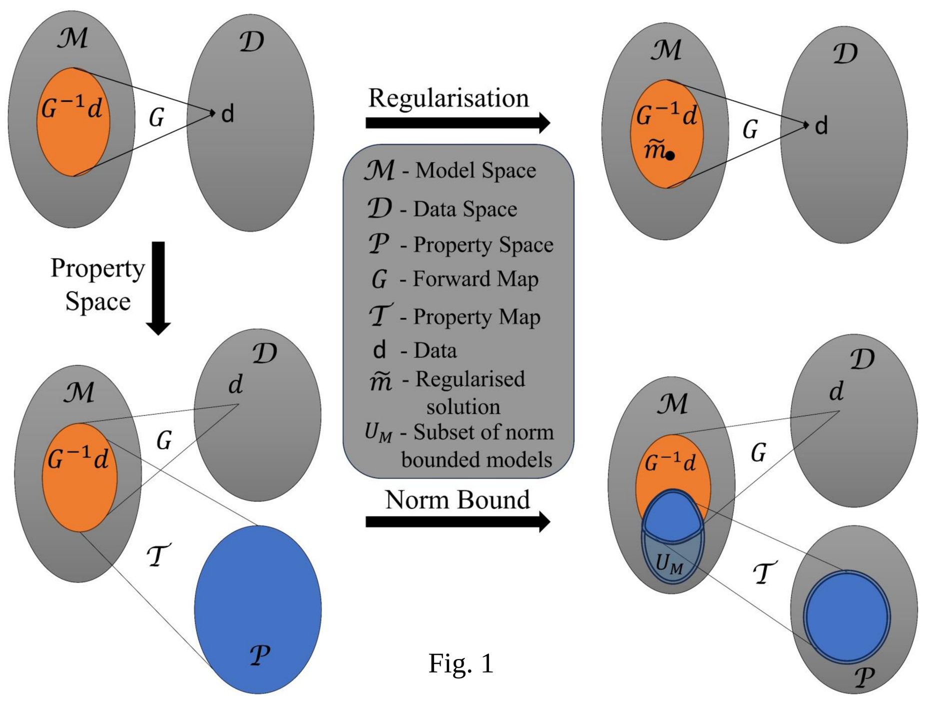


Fig. 1

Inversion

- Constrains the model space to a single point through regularization

BG SOLA

- Constrains the model space only to a bounded subspace
- A certain property of the constrained models is further bounded within a property space

The model m relates to data d through $G(m)=d$, representing data constraints within the model space. Insufficient data yield an underdetermined inverse problem, where a continuous model with finite data may result in infinite or no solutions[6]. Both model discretization and regularisation strongly constrain the model space, transforming the ill-posed problem into an invertible one with a unique and complete solution. However, the solution's accuracy relies on the validity of several assumptions. Sparse data might leave parts of the model influenced solely by these assumptions, potentially leading to "artefact" features. Uncertainties are pivotal in identifying trustworthy model regions amidst these constraints. Deterministic inferences such as Backus-Gilbert SOLA avoid collapsing the entire model space into a single point. They start with a weaker constraint (model norm bound) and focus on specific properties rather than the whole model, providing bounds (uncertainties) on true model solution properties rather than a complete solution.

SOLA

Problem

Given: $d_i = \int K_i m$, $\|m\| \leq M$

Find: $p_j = \int T_j m$

Solution[7]

$$p_j = \tilde{p}_j \pm \epsilon_j \text{ where } \epsilon_j = \sqrt{M^2 - \|\tilde{m}\|^2} H_{jj}$$

d_i - data	T_j - Target kernels	M - Norm bound
K_i - Sensitivity kernels	$\tilde{p}_j$ - Least norm solution property	$\|\tilde{m}\|$ - Least norm
m - Unknown model		H_{jj} - Bounding Hyperellipsoid matrix
p_j - Model properties	ϵ_j - Property bounds	

Given a data constraint and a norm bound (L_2), we find the corresponding property bounds.

Visual Example

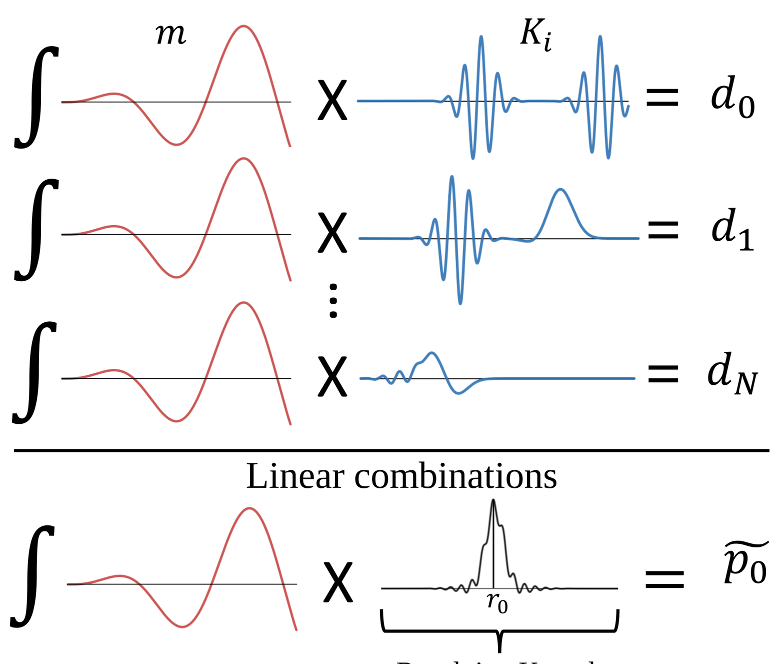
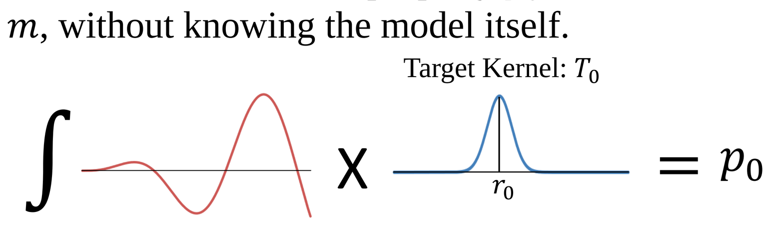


Fig. 2 Resolving Kernel

We want to find a local property p_0 of the unknown model m, without knowing the model itself.



- We obtain an approximation $\widetilde{p_0}$ from linear combinations of the data constraints.
- The true property p_0 is somewhere inside $\widetilde{p_0} \pm \epsilon_0$
- Property error bound ϵ_0 depends on H, which can be computed from the difference between the resolving kernel and the target kernel.

Different Targets

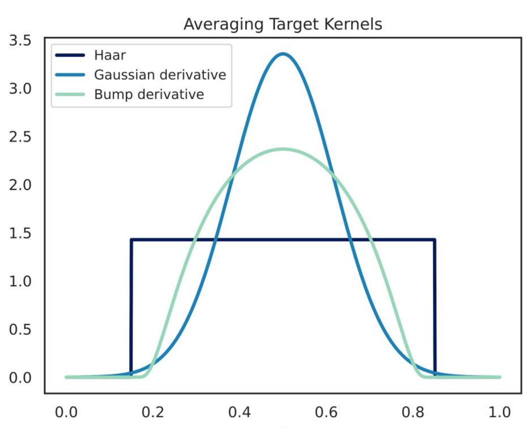


Fig. 3

- Target kernel choice defines the model property that will be constrained
- Various local averages (left) differ in constraint effectiveness.
- Derivatives of averaging target kernels yield gradient information (right)

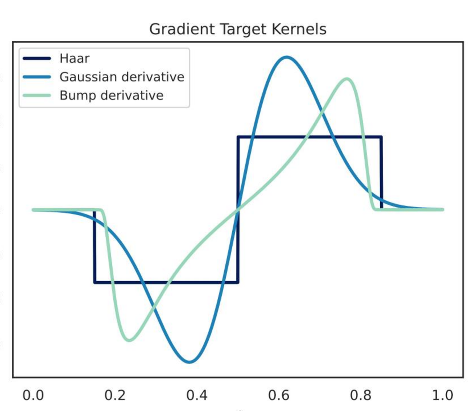


Fig. 5

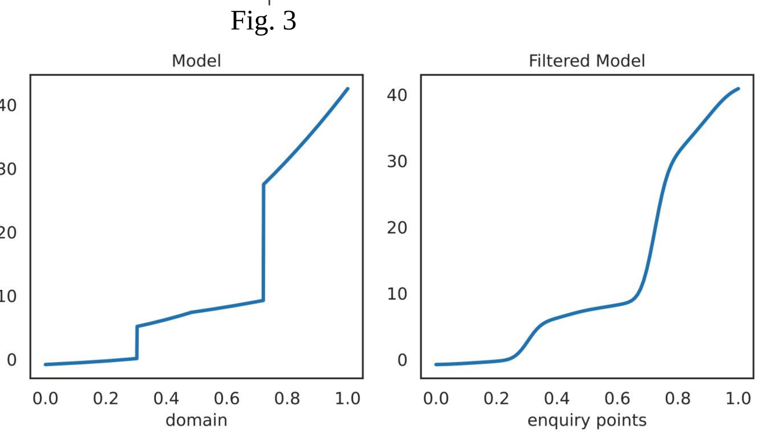


Fig. 4 : Using a Gaussian averaging target kernel we obtain a smoothed version of the true model.

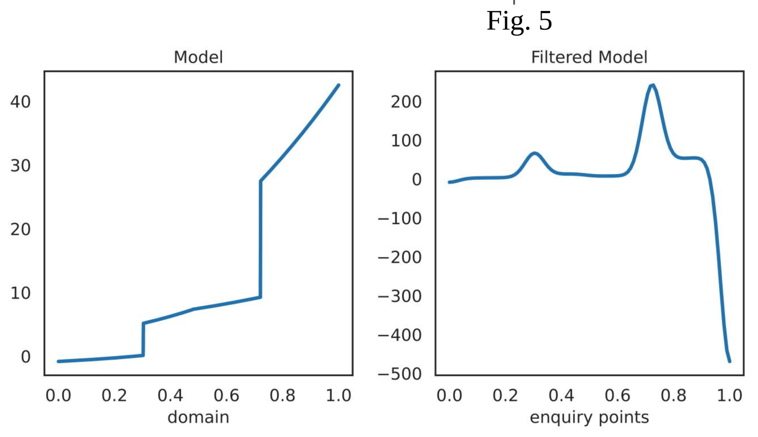


Fig. 6: Using the derivative of a Gaussian averaging target kernel we obtain a smoothed version of the true model's gradient.

Results

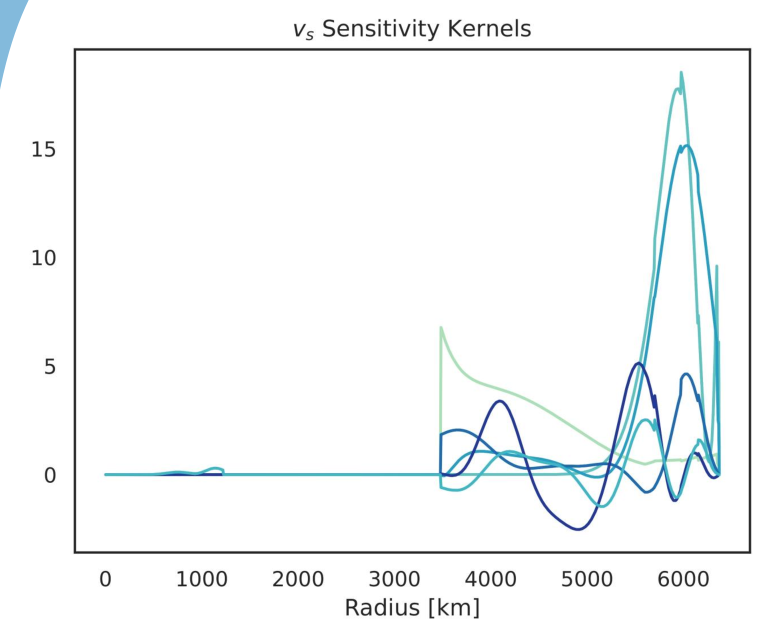


Fig. 7: Examples of 1D normal mode sensitivity kernels to vs[8] computed in the PREM model[9] plotted from the center of the Earth (left) to the surface (right).

- Toy problem: use different targets to constrain properties of an unknown 1D model in the error-free data case:
- The v_s sensitivity kernels shown in (Fig. 7) relate the (synthetic) data to the model.
- The model norm bound is assumed to be 4 times higher than the real model norm.
- We compute and plot (Fig. 8) the relative error bounds of the properties:

$$\textit{relative error bound} = \frac{\epsilon}{\max(\tilde{p}) - \min(\tilde{p})}$$

where $\tilde{p}$ is the least norm property (see Fig. 9 for example).

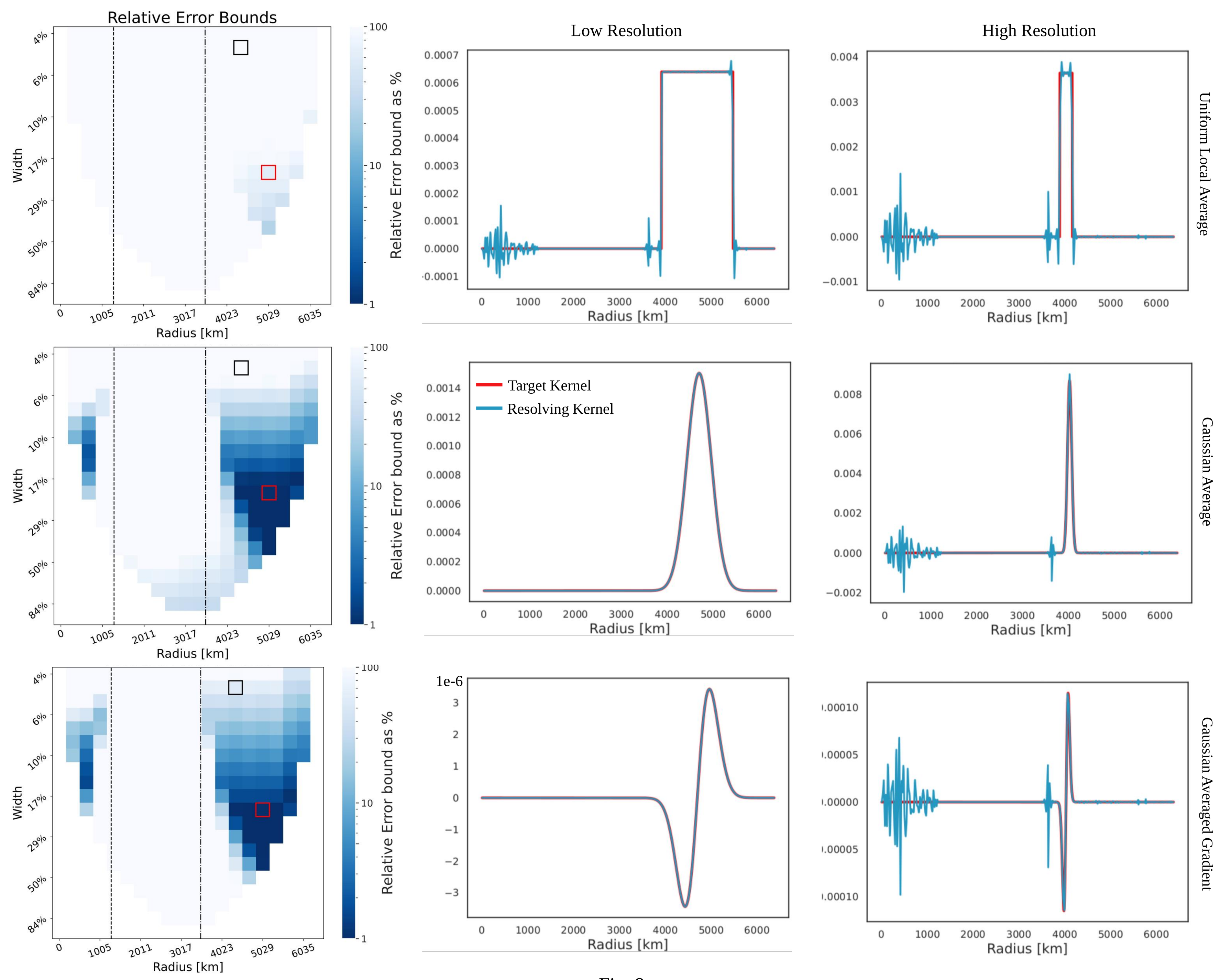


Fig. 8

Local averages are poorly constrained at all depths and all resolutions. Local Gaussian averages are well constrained in the mantle and outer inner core. The outer core is not well constrained due to the lack of sensitivity to v_s. The Gaussian averaged gradient shows similar relative error bounds pattern to local Gaussian averages but is slightly better constrained at higher resolutions. The poor constraint of local uniform averages is because our sensitivity kernels are very dissimilar to the boxcar kernels.

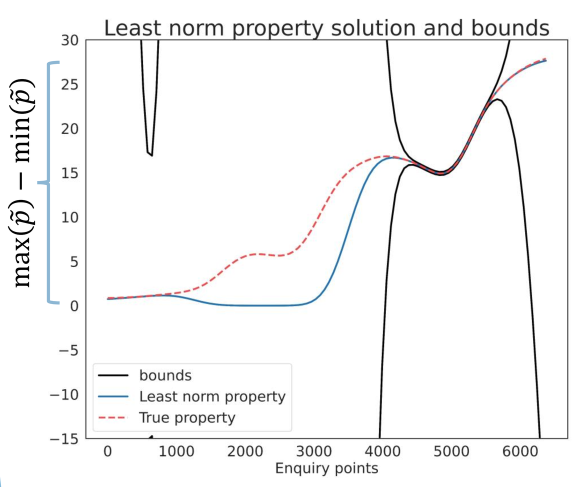


Fig. 9: Property bounds for a norm bound 4 times bigger than the true model norm Black: $\tilde{p} \pm \epsilon$, Blue: $\tilde{p}$

- Here we choose as target kernel the Gaussian average with width 25% of the domain length.
- The generalized BG SOLA method produces bounds (black lines) instead of an "exact answer".
- Regions where the property can be constrained will produce tight bounds.
- Knowing the true model norm results in much tighter bounds (Fig. 10).

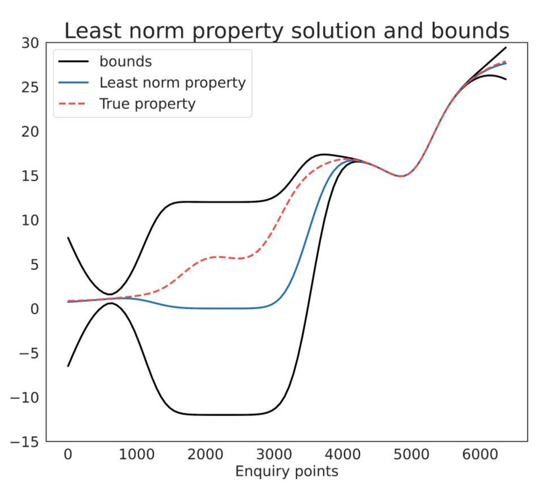


Fig. 10: Property bounds for a norm bound equal to the true model norm. Black: $\tilde{p} \pm \epsilon$, Blue: $\tilde{p}$

Summary

- Deterministic linear inferences rely only on information from the data, and a prior norm bound on the model space.
- Constraining the properties of the model instead of the model itself.
- BG SOLA offers a computationally tractable and mathematically simple method for inferring properties of an unknown model if the forward relation is linear.
- Using different types of target kernels, we can obtain different properties of the unknown model.
- Some properties are better constrained than others depending on the data used.

Future Work

- Implement measurement errors in the method.
- Expand the current method to account for relations where the data depends on multiple model parameters (such as v_s, v_p, ρ and internal discontinuities).
- Apply the method with measurement errors to real 1D normal mode data.
- Expand the theory to various types of 3D target kernels and apply it to 3D data.

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