# Constraining Earth model properties through Backus-Gilbert SOLA inferences

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## Introduction and Motivation

- Seismic tomography uses inversions to obtain Earth models. However, insufficient and inadequate data leads to solution non-uniqueness[1]. Regularizations are used to break this non-uniqueness, at the cost of introducing prior information. Our prior assumptions have a strong effect on the solutions in regions with bad data coverage. Quantifying uncertainty and resolution is also often difficult.
- Deterministic Linear Inferences, such as Backus-Gilbert SOLA, constrain properties of the true model, instead of the model. This can be done locally, and independently of other locations. They require weaker prior constraints and offer robust uncertainty quantification and resolution analysis.
- BG SOLA has been used in helioseismology[2], signal processing[3], and recently seismic tomography[4,5,6] to obtain local averages estimates of the unknown true model.

## Summary

We show that:

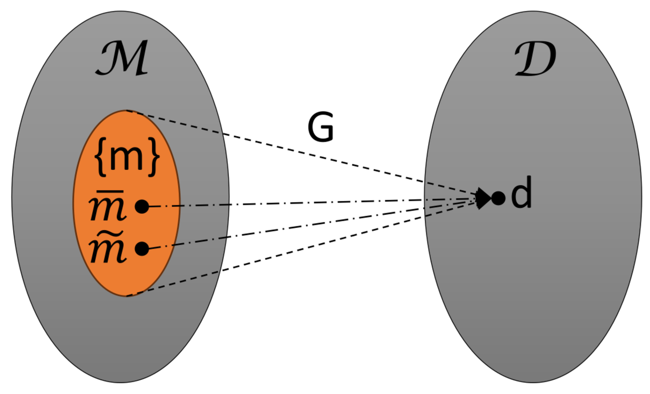
- Certain local averages can be better constrained than others.
- Different properties of an unknown model (such as gradients or Fourier coefficients) can be constrained through an appropriate choice of target kernels.
- The method can be used to obtain families of discretized models by choosing as target kernels a set of basis functions.

We assume:

- Error-free data. The introduction of data errors would not change the main points of this presentation, but it would lead to considerably more convoluted mathematics.
- A prior model norm bound[7] to move the interpretation from the resolving kernels to the target kernels.

### Model and Data space

- The model space can be continuous or discrete (here we assume continuous).
- When our data depends on more physical parameters (such as shear and compressional wave speeds, density, etc.), each physical parameter has its own model space, and the total model space is given by their direct sum.
- Data space is always $\mathbb{R}^N$ (N is number of data)




### Forward mapping

- The forward mapping G (also known as model-data mapping) is assumed to be:

$$d_i = \sum_j \int_{\Omega_j} K_i^j m^j d\Omega_j$$

where $\Omega_j$ are some domains and $m^j$ is the $j^{th}$ physical parameter.

- Such relations appear often when a problem is linearized using perturbation theory[8].
- More generally, we may assume the relation (brackets denote inner product):

$$d_i = \sum_j \langle K_j^i, m^j \rangle_{\mathcal{M}}$$

- $\mathcal{M}$ - Model Space
- $\mathcal{D}$ - Data Space
- $\mathcal{P}$ - Property Space
- $G$ - Forward Map
- $\mathcal{T}$ - Property Map
- d - Data
- $\tilde{m}$ - Regularised solution
- $\bar{m}$ - True model solution
- $U_M$ - Subset of norm bounded models

### Model Norm Bound

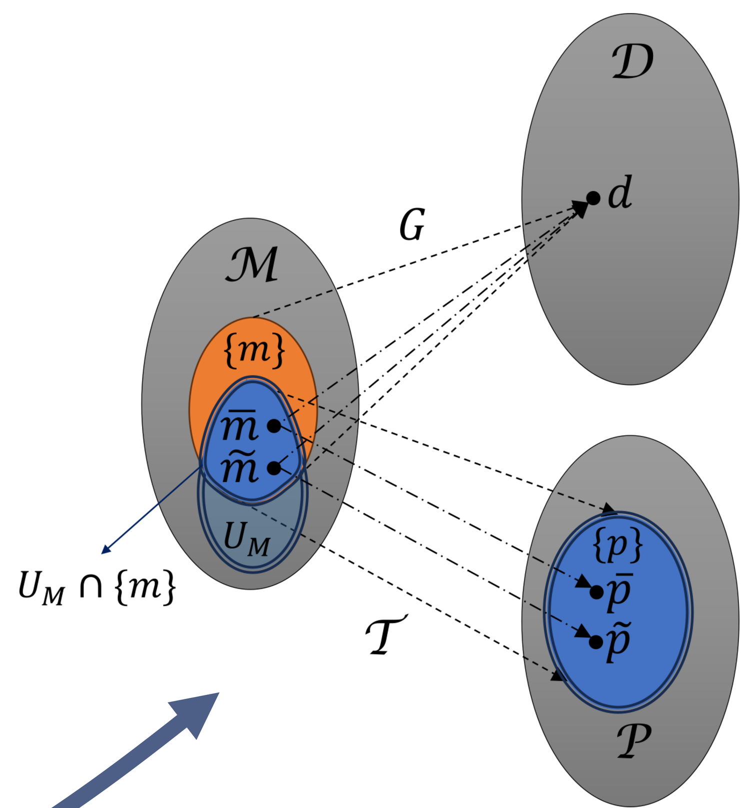
- We assume:

$$\|\bar{m}\|_{\mathcal{M}} \le M$$

- The norm bound can be obtained by finding some functions $b_j$ such that:

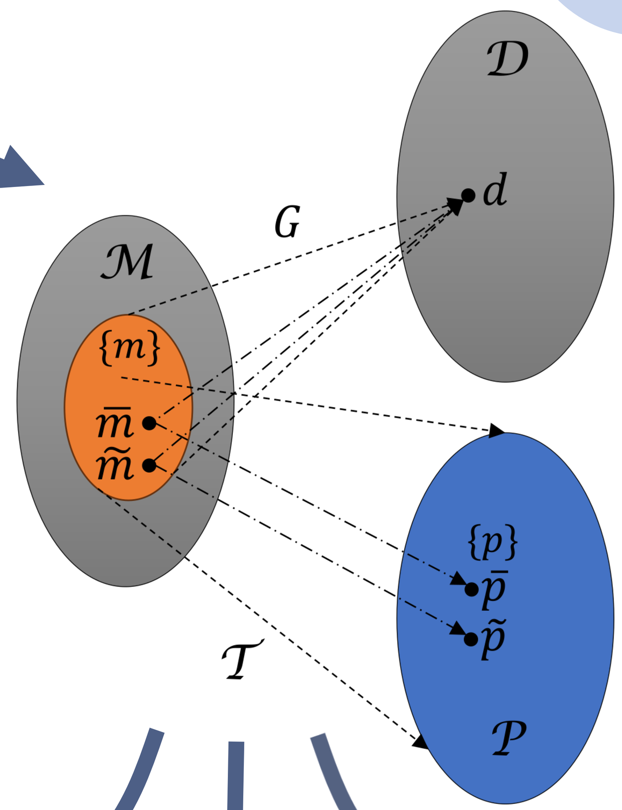
$$|m^j| \le b_j$$

- Without the norm bound we can only find an approximate property of the true model. With the norm bound we can find bounds for the true property[7].
- The functions $b_j$ are normally found from physical arguments.
- This prior information is weaker than a classic regularization.




Define Property

Apply Norm Bound



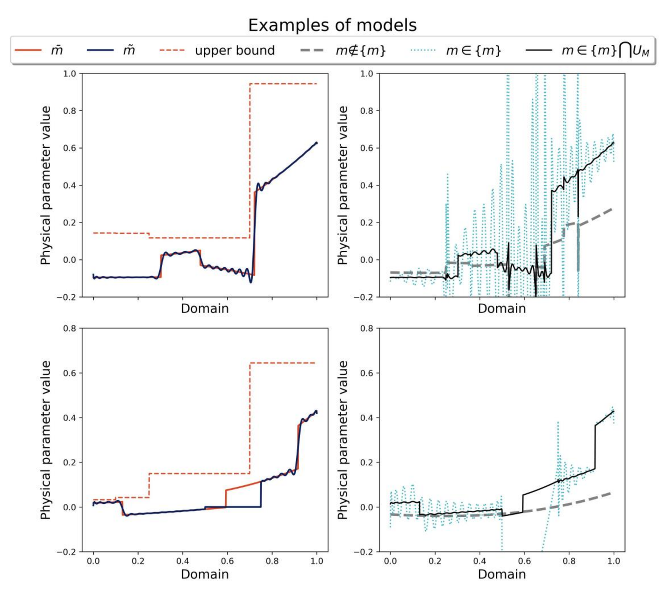




**Figure 1**
Examples of models

### Property mapping

- The property mapping $\mathcal{T}$ extracts a property from a model.
- A property might be a local average, a gradient, or any other set of values described by:

$$p^{(k)} = \sum_j \int_\Omega T^{(k),j} m^j d\Omega$$

where $T_i^j$ are called target kernels.

- Target kernels determine what property is extracted from the model.
- More generally we have:

$$p^{(k)} = \sum_j \langle T^{(k),j}, m^j \rangle_{\mathcal{M}}$$

### Problem and Solution

- We want:

$$\bar{p}^{(k)} = \sum_j \int_{\Omega_j} T^{(k),j} \bar{m}^j d\Omega_j$$

Given that:

$$d_i = \sum_j \int_{\Omega_j} K_j^i m^j d\Omega_j$$

and

$$\|\bar{m}\|_{\mathcal{M}} \le M$$

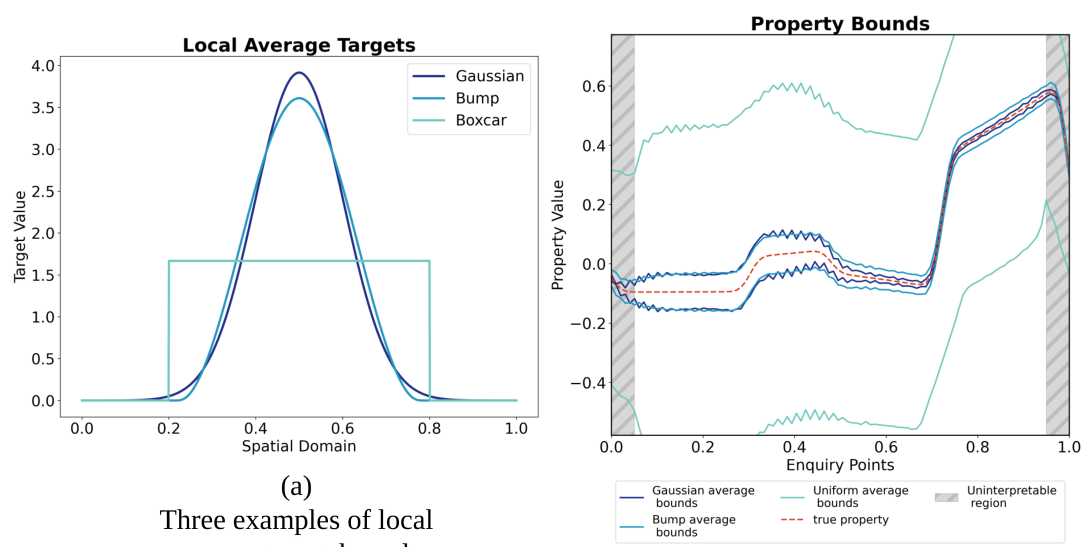
- Solution is given by:

$$\bar{p}^{(k)} \in \tilde{p}^{(k)} + [-\epsilon^{(k)}, +\epsilon^{(k)}]$$

where $\tilde{p}$ is the property of the least norm solution, and $\epsilon^k$ is the maximum error on the property.

## Local Averages

- Any averaging weight function can be used as a target kernel to extract a local average of a model[8,9].
- We use BG SOLA to constrain 3 types of local averages from the physical parameter $m^1$ (Fig. 1a) under the norm bound shown in Fig. 1a.
- For the target kernels we use 100 evenly distributed weight functions of width 0.2. The weight functions used are Gaussian, Bump function, and boxcar function (Fig. 2a).
- Uniform local averages are poorly constrained compared to Gaussian and bump averages (Fig. 2b).
- Bump averages offer smoother property bounds and better interpretability thanks to their compact support (Fig. 2b).



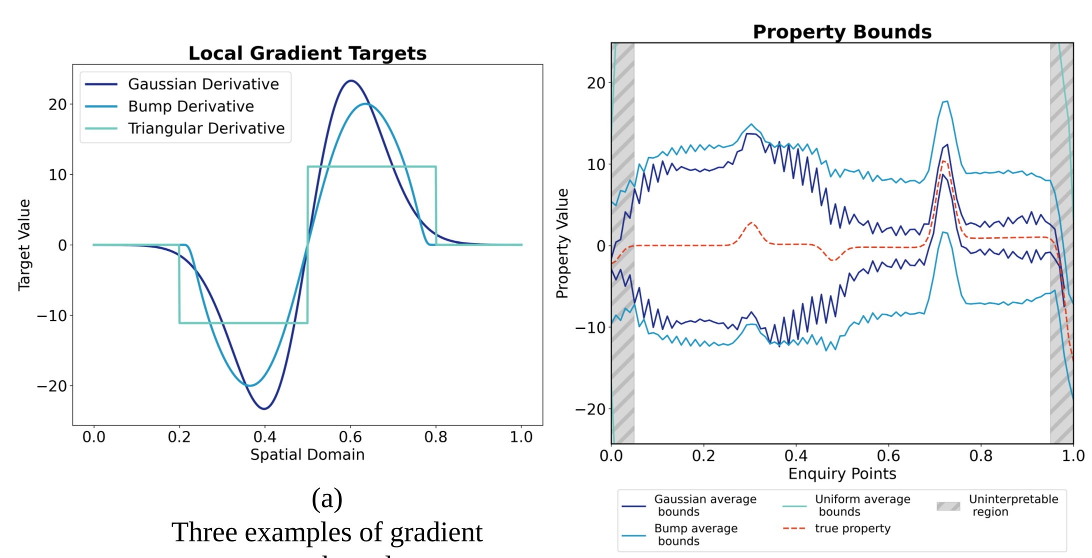

(a)
Three examples of local average target kernels

(b)
Solution $\tilde{p}$

**Figure 2**
Constraining local averages

## Gradients

- By taking the derivative of the Gaussian/Bump/Triangular average weight functions we obtain new targets that extract a locally averaged gradient of a model (Fig. 3a).
- Fig. 3b shows the three gradient types constrained. The bounds corresponding to the triangular derivative are too large to be visible. Just like in the Local averages example, the Gaussian based gradient is better constrained than the bump-based gradient. However, the bump-based gradient has smoother bounds.
- Overall, in this example, gradients are not constrained as well as local averages. In other instances, the gradients may be better constrained.




(a)
Three examples of gradient target kernels

(b)
Solution $\tilde{p}$

**Figure 3**
Constraining gradients

## Discretized Models

- Choosing a set of basis functions $\{B_k\}$ we can discretize the model space, thus removing the non-uniqueness. We can then use least squares to obtain the corresponding coefficients.
- Alternatively, we can set our target kernels equal to $\{B_k\}$ and find bounds for the coefficients using BG SOLA. We can then use these coefficients to build families of discretized models.

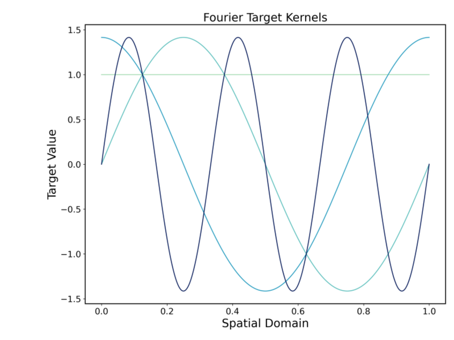
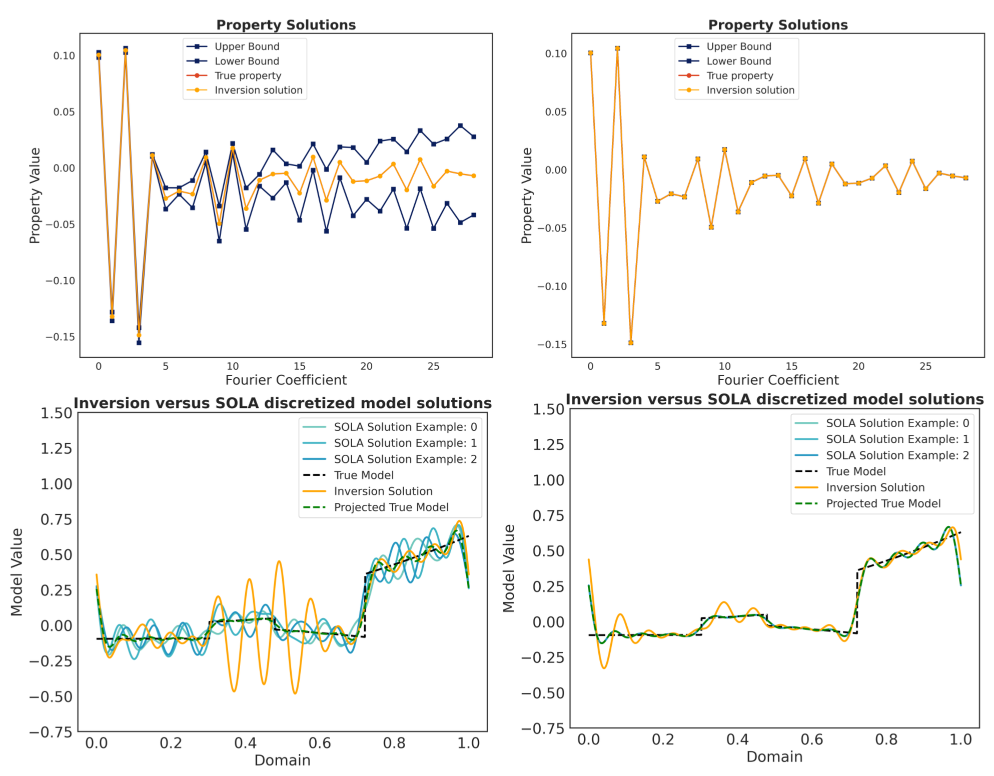



**Figure 4 (left)**
Examples of several Fourier basis functions

**Figure 5 (down)**
Fourier coefficients obtained with discretized inversion and with BG SOLA. Row 1: Fourier coefficients. Row 2: model solutions




## References

[1] Backus, G.E. and Gilbert, J.F., 1967. Numerical applications of a formalism for geophysical inverse problems. *Geophysical Journal International*, *13*(1-3), pp.247-276.
[2] Oldenburg, D.W., 1981. A comprehensive solution to the linear deconvolution problem. *Geophysical Journal International*, *65*(2), pp.331-357.
[3] Pijpers, F.P. and Thompson, M.J., 1994. The SOLA method for helioseismic inversion. *Astronomy and Astrophysics (ISSN 0004-6361), vol. 281, no. 1, p. 231-240*, *281*, pp.231-240.
[4] Zaroli, C., 2016. Global seismic tomography using Backus–Gilbert inversion. *Geophysical Supplements to the Monthly Notices of the Royal Astronomical Society*, *207*(2), pp.876-888.
[5] Zaroli, C., 2019. Seismic tomography using parameter-free Backus–Gilbert inversion. *Geophysical Journal International*, *218*(1), pp.619-630.
[6] Zaroli, C., Koelemeijer, P. and Lambotte, S., 2017. Toward seeing the Earth's interior through unbiased tomographic lenses. *Geophysical Research Letters*, *44*(22), pp.11-399.
[7] Al-Attar, D., 2021. Linear inference problems with deterministic constraints. *arXiv preprint arXiv:2104.12256*.
[8] Restelli, F., Zaroli, C. and Koelemeijer, P., 2023. Robust estimates of the ratio between S-and P-wave velocity anomalies in the Earth's mantle using normal modes. *Physics of the Earth and Planetary Interiors*, p.107135.
[9] Latallerie, F., Zaroli, C., Lambotte, S. and Maggi, A., 2022. Analysis of tomographic models using resolution and uncertainties: a surface wave example from the Pacific. *Geophysical Journal International*, *230*(2), pp.893-907.

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