Part II - Logic and Set Theory
Theorems

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Ordinals and cardinals
Well-orderings and order-types. Examples of countable ordinals. Uncountable ordinals and Hartogs' lemma. Induction and recursion for ordinals. Ordinal arithmetic. Cardinals; the hierarchy of alephs. Cardinal arithmetic. [5]

Posets and Zorn's lemma
Partially ordered sets; Hasse diagrams, chains, maximal elements. Lattices and Boolean algebras. Complete and chain-complete posets; fixed-point theorems. The axiom of choice and Zorn's lemma. Applications of Zorn's lemma in mathematics. The well-ordering principle. [5]

Propositional logic
The propositional calculus. Semantic and syntactic entailment. The deduction and completeness theorems. Applications: compactness and decidability. [3]

Predicate logic
The predicate calculus with equality. Examples of first-order languages and theories. Statement of the completeness theorem; *sketch of proof*. The compactness theorem and the Lowenheim-Skolem theorems. Limitations of first-order logic. Model theory. [5]

Set theory
Set theory as a first-order theory; the axioms of ZF set theory. Transitive closures, epsilon-induction and epsilon-recursion. Well-founded relations. Mostowski's collapsing theorem. The rank function and the von Neumann hierarchy. [5]

Consistency
Problems of consistency and independence [1]

Contents

1 Propositional calculus

1.1 Semantic implication

Proposition.

(i) If v and v' are valuations with $v(p) = v'(p)$ for all $p \in P$, then $v = v'$.

(ii) For any function $W : P \to \{0, 1\}$, there is a valuation v such that $v(p) = w(p)$ for all $p \in L$, i.e. we can extend w to a full valuation.

This means "A valuation is determined by its values on P, and any values will do".

1.2 Syntactic implication

Proposition (Deduction theorem)**.** Let $S \subset L$ and $p, q \in L$. Then $S \vdash (p \Rightarrow q)$ if and only if $S \cup p \vdash q$.

"$\vdash$ behaves like the connective $\Rightarrow$ in the language"

Proposition (Soundness theorem)**.** If $S \vdash t$, then $S \models t$.

Theorem (Model existence theorem)**.** If $S \models \perp$, then $S \vdash \perp$. i.e., if S has no model, then S is consistent. i.e. If S is consistent, then S has a model.
Note: Some books call this the "completeness theorem", because the rest of the completeness theorem follows trivially from this.

Corollary (Adequacy theorem)**.** Let $S \subset L$, $t \in L$. Then $S \models t$ implies $S \vdash t$.

Theorem (Completeness theorem)**.** Le $S \subset L$ and $t \in L$. Then $S \models t$ if and only if $S \vdash t$.

Corollary (Compactness theorem)**.** Let $S \subset L$ and $t \in L$ with $S \models t$. The some finite $S' \subset S$ has $S' \models t$.

Corollary (Decidability theorem)**.** Let finite $S \subset L$, $t \in L$. Then there exists an algorithm that determines, in finite time, whether or not $S \vdash t$.