Part II - Logic and Set Theory

Theorems with Proof

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Contents

1 Propositional calculus

1.1 Semantic implication

Proposition.

(i) If v and v' are valuations with $v(p) = v'(p)$ for all $p \in P$, then $v = v'$.

(ii) For any function $W : P \to \{0, 1\}$, there is a valuation v such that $v(p) = w(p)$ for all $p \in L$, i.e. we can extend w to a full valuation.

This means "A valuation is determined by its values on P, and any values will do".

Proof. (i) Recall that L is defined inductively. We are given that $v(p) = v'(p)$ on L_1. Then for all $p \in L_2$, p must be in the form $q \Rightarrow r$ for $q, r \in L_1$. Then $v(q \Rightarrow r) = v(p \Rightarrow q)$ since the value of v is uniquely determined by the definition. So for all $p \in L_2$, $v(p) = v'(p)$.

Continue inductively to show that $v(p) = v'(p)$ for all $p \in L_n$ for any n.

(ii) Set v to agree with w for all $p \in P$, and set $v(\perp) = 0$. Then define v on L_n inductively according to the definition.

$\square$

1.2 Syntactic implication

Proposition (Deduction theorem)**.** Let $S \subset L$ and $p, q \in L$. Then $S \vdash (p \Rightarrow q)$ if and only if $S \cup p \vdash q$.

"$\vdash$ behaves like the connective $\Rightarrow$ in the language"

Proof. ($\Rightarrow$) Given a proof of $p \Rightarrow q$ from S, append the lines

- p Hypothesis
- q MP

to obtain a proof of q from $S \cup \{q\}$.

($\Leftarrow$) Let $t_1, t_2, \cdots, t_n = q$ be a proof of q from $S \cup \{p\}$. We'll show that $S \vdash p \Rightarrow t_i$ for all i.

We consider different possibilities of t_i:

- t_i is an axiom: Write down
  - $t_i \Rightarrow (p \Rightarrow t_i)$ (Axiom 1)
  - t_i Axiom
  - $p \Rightarrow t_i$ MP
- $t_i \in S$: Write down
  - $t_i \Rightarrow (p \Rightarrow t_i)$ (Axiom 1)
  - t_i Hypothesis
  - $p \Rightarrow t_i$ MP

To get $S \models (p \Rightarrow t_i)$

- $t_i = p$: Write down our proof of $p \Rightarrow p$ from our example above.
- t_i is obtained by MP: we have some $j, k < i$ such that $t_k = (t_k \Rightarrow t_i)$. We can assume that $S \vdash (p \Rightarrow t_j)$ and $S \vdash (p \Rightarrow t_k)$ by induction on i. Now we can write down
  - $[p \Rightarrow (t_j \Rightarrow t_i)] \Rightarrow [(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)]$ Axiom 2
  - $p \Rightarrow (t_j \Rightarrow t_i)$ Known already
  - $(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)$ MP
  - $p \Rightarrow t_j$ Known already
  - $p \Rightarrow t_i$ MP

  to get $S \models (p \Rightarrow t_i)$.

This is why Axiom 2 is as it is - it enables us to prove the deduction theorem. $\square$

Proposition (Soundness)**.** If $S \subset L$, $t \in L$, then if $S \vdash t$, then $S \models t$.

Proof. Given valuation v with $v(s) = 1$ for all $s \in S$, we need to show that $v(t) = 1$. But $v(p) = 1$ for all axioms p, and $v(p) = 1$ for all $p \in S$, and if $v(p) = 1$ and $v(p \Rightarrow q) = 1$, then $v(q) = 1$. Hence each line t_i in a proof $t_1, \cdots, t_n$ of from S has $v(t_i) = 1$. $\square$