Part IA - Vector Calculus
Theorems

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Lent 2015

Curves in $\mathbb{R}^3$
Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1]

Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$
Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

Vector operators
Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical *and general orthogonal curvilinear* coordinates.

Divergence, curl and ∇^2 in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical *and general orthogonal curvilinear* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

Integration theorems
Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

Laplace's equation
Laplace's equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

Cartesian tensors in $\mathbb{R}^3$
Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

Contents

1 Introduction

2 Derivatives and coordinates

2.1 Differentiable functions $\mathbb{R} \to \mathbb{R}^n$

Proposition.
$$\mathbf{F}'(x) = F_i'(x)\mathbf{e}_i.$$

Proposition.

$$\frac{\mathrm{d}}{\mathrm{d}t}(f\mathbf{g}) = \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{g} + f\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g} \cdot \mathbf{h}) = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \cdot \mathbf{h} + \mathbf{g} \cdot \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g} \times \mathbf{h}) = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \times \mathbf{h} + \mathbf{g} \times \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$

Note that the order of multiplication must be retained in the case of the cross product.

2.2 Differentiable functions $\mathbb{R}^n \to \mathbb{R}$

Theorem. The gradient is
$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i$$

Theorem (Chain rule)**.** Given a function $f(\mathbf{r}(u))$,

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

2.3 Differentiable functions $\mathbb{R}^n \to \mathbb{R}^m$

Theorem. The derivative of $\mathbf{F}$ is given by

$$M_{ji} = \frac{\partial y_j}{\partial x_i}.$$