Part IB - Electromagnetism

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Electromagnetism and Relativity
Review of Special Relativity; tensors and index notation. Lorentz force law. Electromagnetic tensor. Lorentz transformations of electric and magnetic fields. Currents and the conservation of charge. Maxwell equations in relativistic and non-relativistic forms. [5]

Electrostatics
Gauss's law. Application to spherically symmetric and cylindrically symmetric charge distributions. Point, line and surface charges. Electrostatic potentials; general charge distributions, dipoles. Electrostatic energy. Conductors. [3]

Magnetostatics
Magnetic fields due to steady currents. Ampre's law. Simple examples. Vector potentials and the Biot-Savart law for general current distributions. Magnetic dipoles. Lorentz force on current distributions and force between current-carrying wires. Ohm's law. [3]

Electrodynamics
Faraday's law of induction for fixed and moving circuits. Electromagnetic energy and Poynting vector. 4-vector potential, gauge transformations. Plane electromagnetic waves in vacuum, polarization. [5]

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1 Introduction

Electromagnetism is important.

1.1 Charge and Current

The strength of the electromagnetic force experienced by a particle is determined by its *(electric) charge*. The SI unit of charge is the *Columb*. In this course, we assume that the charge can be any real number. However, at the fundamental level, charge is quantised. All particles carry charge $q = ne$ with $n \in \mathbb{Z}$, [1] and the basic unit $e \approx 1.6 \times 10^{-19}$C. For example, the electron has $n = -1$, proton has $n = +1$, neutron $= n = 0$.

In this course, it will be more useful to talk about *charge density* $\rho(\mathbf{x}, t)$. This is the charge per unit volume. The total charge in a region V is

$$Q = \int_V \rho(\mathbf{x}, t)\ \mathrm{d}^3 x$$

The motion of charge is described by the *current* density $\mathbf{J}(\mathbf{x}, t)$. For any surface S, the integral

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

counts the charge per unit time passing through S. I is called the current, and $\mathbf{J}$ is the "current per unit area".

Intuitively, if the charge distribution $\rho(\mathbf{x}, t)$ has velocity $\mathbf{v}(x, t)$, then (neglecting relativistic effects), we have

$$\mathbf{J} = \rho \mathbf{v}$$

Example. A wire is a cylinder of cross-sectional area A. Suppose there are n electrons per unit volume. Then

$$\begin{aligned}
\rho &= nq = -ne \\
\mathbf{J} &= nq\mathbf{v} \\
I &= nqvA
\end{aligned}$$

It is well know that charge is conserved. However, we can have a stronger statement- charge is conserved locally: it is not possible that a charge in a box disappears and instantaneously appears on the moon. If it disappears in the box, it must have moved to somewhere nearby.

This is captured by the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

The charge Q in some region V is

$$Q = \int_V \rho\ \mathrm{d}^3 x$$

[1] Actually quarks have $n = \pm 1/3$ or $\pm 2/3$, but when we study quarks, electromagnetism becomes insignificant compared to the strong force. So for all practical purposes we can have $n \in \mathbb{Z}$.

So

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_V \frac{\partial \rho}{\partial t}\, d^3x = -\int_V \nabla \cdot \mathbf{J}\, \mathrm{d}^3x = -\int_S \mathbf{J} \cdot \mathrm{d}S$$

where the last equality is by the divergence theorem. i.e. the rate of change in charge is the rate of charge flowing out of the point.

We can take $V = \mathbb{R}^3$, the whole of space. If there are no currents at infinity, then

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

So the continuity equation implies the conservation of charge.

1.2 Forces and Fields

All forces are mediated by *fields*. In physics a *field* is a dynamical quantity which takes values at every point in space and time. The electromagnetic force is mediated by two fields:

- electric field $\mathbf{E}(\mathbf{x}, t)$
- magnetic field $\mathbf{B}(\mathbf{x}, t)$

Each of these fields is itself a 3-vector. There are two aspects to the force:

- Particles create fields
- Fields move particles

The second aspect is governed by the Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The first aspect is governed by the *Maxwell equations.*

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{E}}{\partial t} &= 0 \\
\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j}
\end{aligned}$$

where

- $\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{m}^{-3}\,\mathrm{kg}^{-1}\,\mathrm{s}^2\,\mathrm{C}^2$ is the electric constant
- $\mu_0 = 4\pi \times 10^{-6}\,\mathrm{m}\,\mathrm{kg}\,\mathrm{C}^{-2}$ is the magnetic constant

These equations are special in a mathematical way. In fact, using quantum mechanics and relativity, it can be shown that these are the only equations that can possibly describe electromagnetism