Part IA - Analysis I

Theorems

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Limits and convergence
Sequences and series in R and C. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

Continuity
Continuity of real- and complex-valued functions defined on subsets of $\mathbb{R}$ and $\mathbb{C}$. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

Differentiability
Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from $\mathbb{R}$ to $\mathbb{R}$; Lagranges form of the remainder. Complex differentiation. [5]

Power series
Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. *Direct proof of the differentiability of a power series within its circle of convergence*. [4]

Integration
Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

Contents

1 The real number system

Lemma. Let $\mathbb{F}$ be an ordered field and $x \in \mathbb{F}$. Then $x^2 \geq 0$.

Lemma (Archimedean property v1)**.** Let $\mathbb{F}$ be an ordered field with the least upper bound property. Then the set $\{1, 2, 3, \cdots\}$ is not bounded above. (Note that these need not refer to natural numbers. We simply define 1 to be the multiplicative identity, $2 = 1 + 1$, $3 = 1 + 2$ etc.)

2 Convergence of sequences

Lemma (Archimedean property v2)**.** $1/n \to 0$.

Lemma. Every eventually bounded sequence is bounded.

2.1 Sums, products and quotients

Lemma (Sums of sequences)**.** If $a_n \to a$ and $b_n \to b$, then

(i) $a_n + b_n \to a + b$

Lemma (Scalar multiplication of sequences)**.** Let $a_n \to a$ and $\lambda \in \mathbb{R}$. Then $\lambda a_n \to \lambda a$.

Lemma. Let a_n be bounded $b_n \to 0$. Then $a_n b_n \to 0$.

Lemma. Every convergent sequence is bounded.

Lemma (Product of sequences)**.** Let $a_n \to a$ and $b_n \to b$. Then $a_n b_n \to ab$.

Lemma (Quotient of sequences)**.** Let (a_n) be a sequence such that $\forall n \neq 0$. Suppose that $a_n \to a$ and $a \neq 0$. Then $1/a_n \to 1/a$.

Corollary. If $a_n \to a, b_n \to b$, $b_n, b \neq 0$. Then $a_n/b_n = a/b$.

Lemma (Sandwich rule)**.** Let (a_n) and (b_n) be sequences that both converge to a limit x. Suppose that $a_n \leq c_n \leq b_n$ for every n. Then $c_n \to x$.

2.2 Monotone-sequences property

Lemma. Least upper bound property $\Rightarrow$ monotone-sequences property.

Lemma. Monotone-sequences property. $\Rightarrow$ Archimedean property.

Lemma. Monotone sequences property $\Rightarrow$ least upper bound property.

Lemma. Let (a_n) be a sequence and suppose that $a_n \to a$. If $\forall n$, $a_n \leq x$, then $a \leq x$.

Lemma. A sequence can have at most 1 limit.

Lemma (Nested intervals property)**.** Let $\mathbb{F}$ be an ordered field with the monotone sequences property. Let $I_1 \supseteq I_2 \supseteq \cdots$ be closed bounded non-empty intervals. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proposition. $\mathbb{R}$ is uncountable.

Theorem (Bolzano-Weierstrass theorem)**.** Let $\mathbb{F}$ be an ordered field with the monotone sequences property (i.e. $\mathbb{F} = \mathbb{R}$).

Then every bounded sequence has a convergent subsequence.

2.3 Cauchy sequences

Lemma. Every convergent sequence is Cauchy.

Lemma. Let (a_n) be a Cauchy sequence with a subsequence (a_{n_k}) that converges to a. Then $a_n \to a$.

Theorem (The general principle of convergence)**.** Let $\mathbb{F}$ be an ordered field with the monotone-sequence property. Then every Cauchy sequence of $\mathbb{F}$ converges.

Lemma. Let $\mathbb{F}$ be an ordered field with the Archimedean property such that every Cauchy sequence converges. The $\mathbb{F}$ satisfies the monotone sequences property.

2.4 Limit supremum and infimum

Lemma. Let (a_n) be a sequnece. The following two statements are equivalent:

- $a_n \to a$
- $\limsup a_n = \liminf a_n = a$.

3 Convergence of infinite sums

Lemma. If $\sum_{n=1}^{\infty} a_n$ converges. Then $a_n \to 0$.

Lemma. Suppose that $a_n \geq 0$ for every n and the partial sums S_n are bounded above. Then $\sum_{n=1}^{\infty} a_n$ converges.

Lemma (Comparison test)**.** Let (a_n) and (b_n) be non-negative sequences, and suppose that $\exists C, N$ such that $\forall n \geq N$, $a_n \leq CB_n$.Then if $\sum b_n$ converges, then so does $\sum a_n$.