Part IA - Analysis I

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Limits and convergence
Sequences and series in R and C. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

Continuity
Continuity of real- and complex-valued functions defined on subsets of $\mathbb{R}$ and $\mathbb{C}$. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

Differentiability
Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from $\mathbb{R}$ to $\mathbb{R}$; Lagranges form of the remainder. Complex differentiation. [5]

Power series
Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. *Direct proof of the differentiability of a power series within its circle of convergence*. [4]

Integration
Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

Contents

1 The real number system

One can define real numbers as "decimals" consisting of infinitely many digits. However, while this is legitimate, it is not a convenient definition to work with. Instead, we define the real numbers to be "an ordered field with the least upper bound property", and show that the decimals form "an ordered field with the least upper bound property" if we really want to.

Definition (Field)**.** A *field* is a set X with two binary operations $+$ and $\times$ that satisfies all the familiar properties satisfied by addition and multiplication in $\mathbb{Q}$, namely

- Associativity: $\forall a, b, c \in X$, $a+(b+c) = (a+b)+c$ and $a\times(b\times c) = (a\times b)\times c$
- Commutativity: $\forall a, b \in X$, $a + b = b + a$ and $a \times b = b \times a$
- Identity: $\exists 0, 1 \in X$ such that $\forall a$, $a + 0 = a$ and $a \times 1 = a$.
- Inverses: $\forall a \in X$, $\exists(-a) \in X$ such that $a + (-a) = 0$. If $a \neq 0$, then $\exists a^{-1}$ such that $a \times a^{-1} = 1$.
- Distributivity: $\forall a, b, c \in F$, $a \times (b + c) = (a \times b) + (a \times c)$

Example. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, integers mod p, $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$.

Definition (Totally ordered set)**.** An *(totally) ordered set* is a set X with a relation $<$ that satisfies

(i) Transitivity: if $x, y, z \in X$, $x < y$ and $y < z$, then $x < z$

(ii) Trichotomy: if $x, y \in X$, exactly one of $x < y, x = y, y < x$ holds

Definition (Ordered field)**.** An *ordered field* is a field $\mathbb{F}$ with a relation $<$ that makes $\mathbb{F}$ into an ordered set such that

(i) if $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$

(ii) if $x, y, z \in \mathbb{F}$, $x < y$ and $z > 0$, then $xz < yz$

Lemma. Let $\mathbb{F}$ be an ordered field and $x \in \mathbb{F}$. Then $x^2 \geq 0$.

Proof. By trichotomy, either $x < 0$, $x = 0$ or $x > 0$. If $x = 0$, then $x^2 = 0$. So $x^2 \geq 0$. If $x > 0$, then $x^2 > 0 \times x = 0$. If $x < 0$, then $x - x < 0 - x$. So $0 < -x$. But then $x^2 = (-x)^2 > 0$. □

Definition (Least upper bound)**.** Let X be an ordered set and let $A \subseteq X$. An *upper bound* for A is an element $x \in X$ such that $\forall a \in A (a \leq x)$. If A has an upper bound, then we say that A is *bounded above*.

An upper bound x for A is a *least upper bound* or *supremum* if nothing smaller that x is an upper bound. That is, we need

(i) $\forall a \in A (a \leq x)$

(ii) $\forall y < x (\exists a \in A (a \geq y))$

We usually write $\sup A$ for the supremum of A when it exists. If $\sup A \in A$, then we call it $\max A$, the maximum of A.

Example. Let $X = \mathbb{Q}$. Then the supremum of $(0,1)$ is 1. The set $\{x : x^2 < 2\}$ is bounded above by 2, but has no supremum (even though $\sqrt{2}$ seems like a supremum, we are in $\mathbb{Q}$ and $\sqrt{2}$ is non-existent!).

$\max[0,1] = 1$ but $(0,1)$ has no maximum because the supremum is not in $(0,1)$.

We can think of the supremum as a point we can get arbitrarily close to in the set but cannot pass through.

Definition (Least upper bound property)**.** An ordered field has the *least upper bound property* if every non-empty subset of $\mathbb{F}$ that is bounded above has a supremum.

Obvious modifications give rise to definitions of lower bound, greatest lower bound (or infimum) etc. It is simple to check that an ordered field with the least upper bound property has the greatest lower bound property.

Lemma (Archimedean property v1))**.** Let $\mathbb{F}$ be an ordered field with the least upper bound property. Then the set $\{1, 2, 3, \cdots\}$ is not bounded above. (Note that these need not refer to natural numbers. We simply define 1 to be the multiplicative identity, $2 = 1 + 1$, $3 = 1 + 2$ etc.)

Proof. If it is bounded above, then it has a supremum x. But then $x - 1$ is not an upper bound. So we can find $n \in \{1, 2, 3, \cdots\}$ such that $n > x - 1$. But then $n + 1 > x$ but x is supposed to be an upper bound. $\square$

While the Archimedean property seems to be trivially true for all ordered fields even if they are not bounded above, actually there are ordered fields in which the integers are not bounded above.

For example, consider the field of rational functions, i.e. functions in the form $\frac{P(x)}{Q(x)}$ with $P(x), Q(x)$ being polynomials. We order two functions $\frac{P(x)}{Q(x)}, \frac{R(s)}{S(x)}$ as follows: these two functions intersect only finitely many times because $P(x)S(x) = R(x)Q(x)$ has only finitely many roots. After the last intersection, the function whose value is greater counts as the greater function. It can be checked that these form an ordered field.

In this field, the integers are the constant functions $1, 2, 3, \cdots$, but it is not bounded above since the function x is greater than all of them.

2 Convergence of sequences

Definition (Sequence)**.** A *sequence* is, formally, a function $a : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$). Usually (i.e. always), we write a_n instead of $a(n)$. Instead of a, (a_n), $(a_n)_1^\infty$ or $(a_n)_{n=1}^\infty$ to indicate it is a sequence.

Definition (Convergence of sequence)**.** Let (a_n) be a sequence and $\ell \in \mathbb{R}$. Then a_n *converges to* ℓ, *tends to* ℓ, or $a_n \to \ell$, if

$$\forall \varepsilon > 0\, \exists N\, \forall n \geq N :\ |a_n - \ell| < \varepsilon.$$

One can think of $\exists N : \forall n \geq N$ as saying "eventually always", or as "from some point on". So the definition means, if $a_n \to \ell$, then given any ε, there eventually, everything in the sequence is within ε of ℓ.

Lemma (Archimedean property v2)**.** $1/n \to 0$.

Proof. Let $\varepsilon > 0$. We want to find an N such that $|1/N - 0| = 1/N < \varepsilon$. So pick N such that $N > 1/\varepsilon$. This exists such an N by the Archimedean property v1. Then for all $n > N$, we have $0 < 1/n \leq 1/N < \varepsilon$. So $|1/n - 0| \to \varepsilon$. □

Note that the red parts correspond to the *definition* of a sequence.

Definition (Bounded sequence)**.** A sequence (a_n) is *bounded*

$$\exists C\, \forall n :\ |a_n| \leq C.$$

A sequnece is *eventually bounded* if

$$\exists C\, \exists N\, \forall n \geq N :\ |a_n| \leq C.$$

Lemma. Every eventually bounded sequence is bounded.

Proof. Let C and N be such that $\forall n \geq N$ $|a_n| \leq C$. Then $\forall n \in \mathbb{N}$, $|a_n| \leq \max\{|a_1|, \cdots, |a_{n-1}|, C\}$. □

2.1 Sums, products and quotients

Lemma (Sums of sequences)**.** If $a_n \to a$ and $b_n \to b$, then

(i) $a_n + b_n \to a + b$

Proof. Let $\varepsilon > 0$. We want to show that $\exists N$ such that $\forall n \geq N$, $|a_n + b_n - (a+b)| < \varepsilon$. We know that a_n is very close to a and b_n is very close to b. So their sum must be very close to $a + b$.

Formally, since $a_n \to a$ and $b_n \to b$, we can find N_1, N_2 such that $\forall n \geq N_1$, $|a_n - a| < \varepsilon/2$ and $\forall n \geq N_2$, $|b_n - b| < \varepsilon/2$.

Now let $N = \max\{N_1, N_2\}$. Then by the triangle inequality,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon.$$

□

Lemma (Scalar multiplication of sequences)**.** Let $a_n \to a$ and $\lambda \in \mathbb{R}$. Then $\lambda a_n \to \lambda a$.

Proof. If $\lambda = 0$, then the result is trivial.

Otherwise, let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon/|\lambda|$. So $|\lambda a_n - \lambda a| < \epsilon$. $\square$

Lemma. Let a_n be bounded $b_n \to 0$. Then $a_n b_n \to 0$.

Proof. Let $C \neq 0$ be such that $\forall n: \ |a_n| \leq C$. Let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|b_n| < \varepsilon/C$. Then $|a_n b_n| < \varepsilon$. $\square$

Lemma. Every convergent sequence is bounded.

Proof. Let $a_n \to l$. Then $\exists N : \forall n \geq N$, $|a_n - l| \leq 1$. So $|a_n| \leq |l| + 1$. So a_n is eventually bounded, and therefore bounded. $\square$

Lemma. Let $a_n \to a$ and $b_n \to b$. Then $a_n b_n \to ab$.

Product of sequences. Let $c_n = a_n - a$ and $d_n = b_n - b$. Then $a_n b_n = (a + c_n)(b + d_n) = ab + ad_n + bc_n + c_n d_n$.

But by "sum of sequences", $c_n \to 0$ and $d_n \to 0$. So $ad_n \to 0$ and $bc_n \to 0$. Since c_n is bounded, $c_n d_n \to 0$. Hence by sum of sequences, $a_n b_n \to ab$ $\square$

The proof can be discovered as follows: We know that a_n and b_n get close to a and b. So we can write a_n as a plus some error term, and similarly for b. We realize that the error term of the product is $ad_n + bc_n + c_n d_n$. Then we prove lemmas to show that each of the error terms tend to 0.

Proof. (alternative) Observe that $a_n b_n - ab = (a_n - a)b_n + (b_n - b)a$. We know that $a_n - a \to 0$ and $b_n - b \to 0$. Since (b_n) is bounded, so $(a_n - a)b_n + (b_n - b)a \to 0$. So $a_n b_n \to ab$. $\square$

Note that we no longer write "Let $\varepsilon > 0$". In the beginning, we have no lemmas proven. So we must prove everything from first principles and use the definition. However, after we have proven the lemmas, we can simply use them instead of using first principles. This is similar to in calculus, where we use first principles to prove the product rule and chain rule, then no longer use first principles afterwards.