Part IA - Analysis I

Theorems with Proof

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Limits and convergence
Sequences and series in R and C. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

Continuity
Continuity of real- and complex-valued functions defined on subsets of $\mathbb{R}$ and $\mathbb{C}$. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

Differentiability
Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from $\mathbb{R}$ to $\mathbb{R}$; Lagranges form of the remainder. Complex differentiation. [5]

Power series
Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. *Direct proof of the differentiability of a power series within its circle of convergence*. [4]

Integration
Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

Contents

1 The real number system

Lemma. Let $\mathbb{F}$ be an ordered field and $x \in \mathbb{F}$. Then $x^2 \geq 0$.

Proof. By trichotomy, either $x < 0$, $x = 0$ or $x > 0$. If $x = 0$, then $x^2 = 0$. So $x^2 \geq 0$. If $x > 0$, then $x^2 > 0 \times x = 0$. If $x < 0$, then $x - x < 0 - x$. So $0 < -x$. But then $x^2 = (-x)^2 > 0$. □

Lemma (Archimedean property v1))**.** Let $\mathbb{F}$ be an ordered field with the least upper bound property. Then the set $\{1, 2, 3, \cdots\}$ is not bounded above. (Note that these need not refer to natural numbers. We simply define 1 to be the multiplicative identity, $2 = 1 + 1$, $3 = 1 + 2$ etc.)

Proof. If it is bounded above, then it has a supremum x. But then $x - 1$ is not an upper bound. So we can find $n \in \{1, 2, 3, \cdots\}$ such that $n > x - 1$. But then $n + 1 > x$ but x is supposed to be an upper bound. □

2 Convergence of sequences

Lemma (Archimedean property v2). $1/n \to 0$.

Proof. Let $\varepsilon > 0$. We want to find an N such that $|1/N - 0| = 1/N < \varepsilon$. So pick N such that $N > 1/\varepsilon$. This exists such an N by the Archimedean property v1. Then for all $n > N$, we have $0 < 1/n \leq 1/N < \varepsilon$. So $|1/n - 0| \to \varepsilon$. $\square$

Lemma. Every eventually bounded sequence is bounded.

Proof. Let C and N be such that $\forall n \geq N$ $|a_n| \leq C$. Then $\forall n \in \mathbb{N}$, $|a_n| \leq \max\{|a_1|, \cdots, |a_{N-1}|, C\}$. $\square$

2.1 Sums, products and quotients

Lemma (Sums of sequences). If $a_n \to a$ and $b_n \to b$, then

(i) $a_n + b_n \to a + b$

Proof. Let $\varepsilon > 0$. We want to show that $\exists N$ such that $\forall n \geq N$, $|a_n + b_n - (a+b)| < \varepsilon$. We know that a_n is very close to a and b_n is very close to b. So their sum must be very close to $a + b$.

Formally, since $a_n \to a$ and $b_n \to b$, we can find N_1, N_2 such that $\forall n \geq N_1$, $|a_n - a| < \varepsilon/2$ and $\forall n \geq N_2$, $|b_n - b| < \varepsilon/2$.

Now let $N = \max\{N_1, N_2\}$. Then by the triangle inequality,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon.$$

$\square$

Lemma (Scalar multiplication of sequences). Let $a_n \to a$ and $\lambda \in \mathbb{R}$. Then $\lambda a_n \to \lambda a$.

Proof. If $\lambda = 0$, then the result is trivial.

Otherwise, let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon/|\lambda|$. So $|\lambda a_n - \lambda a| < \epsilon$. $\square$

Lemma. Let a_n be bounded $b_n \to 0$. Then $a_n b_n \to 0$.

Proof. Let $C \neq 0$ be such that $\forall n:\ |a_n| \leq C$. Let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|b_n| < \varepsilon/C$. Then $|a_n b_n| < \varepsilon$. $\square$

Lemma. Every convergent sequence is bounded.

Proof. Let $a_n \to l$. Then $\exists N : \forall n \geq N$, $|a_n - l| \leq 1$. So $|a_n| \leq |l| + 1$. So a_n is eventually bounded, and therefore bounded. $\square$

Lemma (Product of sequences). Let $a_n \to a$ and $b_n \to b$. Then $a_n b_n \to ab$.

Proof. Let $c_n = a_n - a$ and $d_n = b_n - b$. Then $a_n b_n = (a + c_n)(b + d_n) = ab + ad_n + bc_n + c_n d_n$.

But by "sum of sequences", $c_n \to 0$ and $d_n \to 0$. So $ad_n \to 0$ and $bc_n \to 0$. Since c_n is bounded, $c_n d_n \to 0$. Hence by sum of sequences, $a_n b_n \to ab$ $\square$

Proof. (alternative) Observe that $a_n b_n - ab = (a_n - a)b_n + (b_n - b)a$. We know that $a_n - a \to 0$ and $b_n - b \to 0$. Since (b_n) is bounded, so $(a_n - a)b_n + (b_n - b)a \to 0$. So $a_n b_n \to ab$. $\square$

Lemma (Quotient of sequences)**.** Let (a_n) be a sequence such that $\forall n \neq 0$. Suppose that $a_n \to a$ and $a \neq 0$. Then $1/a_n \to 1/a$.

Proof. We have
$$\frac{1}{a_n} - \frac{1}{a} = \frac{a - a_n}{a a_n}.$$
We want to show that $1/(aa_n)$ is bounded: Since $a_n \to a$, $\exists N \, \forall n \geq N$, $|a_n - a| \leq a_2$. Then $\forall n \geq N$, $|a_n| \geq |a|/2$. Then $|1/(a_n a)| \leq 2/|a|^2$. So $1/(a_n a)$ is bounded. So $(a - a_n)/(aa_n) \to 0$ and the result follows. $\square$

Corollary. If $a_n \to a, b_n \to b$, $b_n, b \neq 0$. Then $a_n/b_n = a/b$.

Proof. We know that $1/b_n \to 1/b$. So the result follows by the product rule. $\square$

Lemma (Sandwich rule)**.** Let (a_n) and (b_n) be sequences that both converge to a limit x. Suppose that $a_n \leq c_n \leq b_n$ for every n. Then $c_n \to x$.

Proof. Let $\varepsilon > 0$. We can find N such that $\forall n \geq N$, $|a_n - x| < \varepsilon$ and $|b_n - x| < \varepsilon$.

The $\forall n \geq N$, we have $x - \varepsilon < a_n \leq c_n \leq b_n < x + \varepsilon$. So $|c_n - x| < \varepsilon$. $\square$

2.2 Monotone-sequences property

Lemma. Least upper bound property $\Rightarrow$ monotone-sequences property.

Proof. Let (a_n) be an increasing sequence and let C an upper bound for (a_n). The C is an upper bound for the set $\{a_n : n \in \mathbb{N}\}$. By the least upper bound property, it has a supremum s.

Let $\varepsilon > 0$. Since $s = \sup\{a_n : n \in \mathbb{N}\}$, there exists an N such that $a_N > s - \varepsilon$. The $\forall n \geq N$, we have $s - \varepsilon < a_n \leq a_n \leq s$. So $|a_n - s| < \epsilon$. $\square$

Lemma. Monotone-sequences property. $\Rightarrow$ Archimedean property.

Proof. We prove version 2, i.e. that $1/n \to 0$.

Since $1/n > 0$ and is decreasing, by MSP, in converges. Let δ be the limit. We must have $\delta \geq 0$, since if $\delta < 0$, then there would exist n with $3\delta/2 < 1/n < \delta/2 < 0$. Contradiction.

If $\delta > 0$, then we can find N such that $1/N < 2\delta$. But then for all $n \geq 4N$, we have $1/n \leq 1/(4N) < \delta/2$. Contradiction. Therefore $\delta = 0$. $\square$

Lemma. Monotone sequences property $\Rightarrow$ least upper bound property.

Proof. Let A be a non-empty set that's bounded above. Pick u_0, v_0 such that u_0 is not an upper bound for A and v_0 is an upper bound. Now do a repeated bisection: having chosen u_n and v_n such that u_n is not an upper bound and v_n is, if $(u_n + v_n)/2$ is an upper bound, then let $u_{n+1} = u_n$, $v_{n+1} = (u_n + v_n)/2$. Otherwise, let $u_{n+1} + \frac{u_n + v_n}{2}$, $v_{n+1} = v_n$.

Then $u_0 \leq u_1 \leq u_2 \leq \cdots$ and $v_0 \geq v_1 \geq v_2 \geq \cdots$. Then
$$v_n - u_n = \frac{v_0 - u_0}{2^n} \to 0.$$

Note that here we used the Archimidean property since to prove $1/2^n \to 0$, we sandwich it with $1/n$. But to show $1/n \to 0$, we need the Archimedean property.

By the monotone sequences property, $u_n \to s$ (since (u_n) is bounded above by v_0). Since $v_n - u_n \to 0$, $v_n \to s$. We now show that $s = \sup A$.

If s is not an upper bound, then there exists $a \in A$ such that $a > s$. Since $v_n \to s$, then there exists m such that $v_m < a$, contradicting the fact that v_m is an upper bound.

Let $t < s$. Then since $u_n \to s$, we can find m such that $u_m > t$. So t is not an upper bound. Therefore s is the least upper bound. □

Lemma. Let (a_n) be a sequence and suppose that $a_n \to a$. If $\forall n$, $a_n \leq x$, then $a \leq x$.

Proof. If $a > x$, then set $\epsilon = a - x$. Then we can find N such that $a_N > x$. Contradiction. □

Lemma. A sequence can have at most 1 limit.

Proof. Let (a_n) be a sequence, and suppose $a_n \to x$ and $a_n \to y$. Let $\epsilon > 0$ and pick N such that $\forall n \geq N$, $|a_n - x| < \varepsilon/2$ and $|a_n - y| < \varepsilon/2$. Then $|x - y| \leq |x - a_N| + |a_N - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since ϵ was arbitrary, x must equal y. □

Lemma (Nested intervals property)**.** Let $\mathbb{F}$ be an ordered field with the monotone sequences property. Let $I_1 \supseteq I_2 \supseteq \cdots$ be closed bounded non-empty intervals. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $T_n = [a_n, b_n]$ for each n. Then $a_1 \leq a_2 \leq \cdots$ and $b_1 \geq b_2 \geq \cdots$. For each n, $a_n \leq b_n \leq b_1$. So the sequence a_n is bounded above. So by the monotone sequences property, it has a limit a. For each n, $a_n \leq a$, since if ever we had $a_n > a$, then $\forall m \geq n$, $a_m \geq a_n \Rightarrow a > a$, which is a contradiction.

Also, for each fixed n, we have that $\forall m \geq n$, $a_m \leq b_m \leq b_n$. So $a \leq b_n$. Thus, for all n, $a_n \leq a \leq b_n \Rightarrow a \in O_m$. So $a \in \bigcap_{n=1}^{\infty} I_n$. □

Proposition. $\mathbb{R}$ is uncountable.

Proof. Suppose the contrary. Let $x_1, x_2, \cdots$ be a list of all real numbers. Find an interval that does not contain x_1. Within that interval, find an interval that does not contain x_2. Continue *ad infinitum*. Then the intersection of all these intervals is non-empty, but the elements in the intersection are not in the list. Contradiction. □

Theorem (Bolzano-Weierstrass theorem)**.** Let $\mathbb{F}$ be an ordered field with the monotone sequences property (i.e. $\mathbb{F} = \mathbb{R}$).

Then every bounded sequence has a convergent subsequence.

Proof. Let u_0 and v_0 be a lower and upper bound, respectively, for a sequence $(a_n)_1^{\infty}$. By repeated bisection, we can find a sequence of intervals $[u_0, v_0] \supseteq [u_1, v_1] \supseteq [u_2, v_2] \supseteq \cdots$ such that $v_n - u_n = (v_0 - u_0)/2^n$, and such that each $[u_n, v_n]$ contains infinitely many terms of (a_n).

By the nested intervals property, $\bigcap_{n=1}^{\infty} [u_n, v_n] = \emptyset$. Let x belong to the intersection. Now pick a subsequence $a_{n_1}, a_{n_2}, \cdots$ such that $a_{n_k} \in [u_k, v_k]$. We can do this because $[u_k, v_k]$ contains infinitely many a_n, and we have only picked finitely many of them.

Let $\varepsilon > 0$. By the Archimedean property, we can find K such that $(v_0 - u_0)/2^K \leq \varepsilon$. This implies that $[u_K, v_K] \subseteq (x - \varepsilon, x + \varepsilon)$, since $x \in [u_K, v_K]$.

Then $\forall k \geq K$, $a_{n_k} \in [u_k, v_k] \subseteq [u_K, v_K] \subseteq (x-\varepsilon, x+\varepsilon)$. So $|a_{n_k} - x| < \varepsilon$. $\square$

2.3 Cauchy sequences

Lemma. Every convergent sequence is Cauchy.

Proof. Let $a_n \to a$. Let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \epsilon/2$. Then $\forall p, q \geq N$, $|a_p - a_q| \leq |a_p - a| + |a - a_q| < \epsilon/2 + \epsilon/2 = \epsilon$. $\square$

Lemma. Let (a_n) be a Cauchy sequence with a subsequence (a_{n_k}) that converges to a. Then $a_n \to a$.

Proof. Let $\varepsilon > 0$. Pick N such that $\forall p, q \geq N$, $|a_p - a_q| < \varepsilon/2$. Then pick K such that $n_K \geq N$ and $|a_{n_K} - a| < \varepsilon/2$.

Then $\forall n \geq N$, we have

$$|a_n - a| \leq |a_n - a_{n_K}| + |a_{n_K} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Theorem (The general principle of convergence)**.** Let $\mathbb{F}$ be an ordered field with the monotone-sequence property. Then every Cauchy sequence of $\mathbb{F}$ converges.

Proof. Let (a_n) be a Cauchy sequence. Then it is eventually bounded, since $\exists N$, $\forall n \geq N$, $|a_n - a_N| \leq 1$ by the Cauchy condition. So it is bounded. Hence by Bolzano-Weierstrass, it has a convergent subsequence. Then (a_n) converges to the same limit. $\square$

Lemma. Let $\mathbb{F}$ be an ordered field with the Archimedean property such that every Cauchy sequence converges. The $\mathbb{F}$ satisfies the monotone sequences property.

Proof. We will show the equivalent statement that every increasnig non-Cauchy sequence is not bounded above.

Let (a_n) be an increasing sequence. If (a_n) is not Cauchy, then

$$\exists \varepsilon > 0\ \forall N\ \exists p, q > N:\ |a_p - a_q| \geq \varepsilon.$$

Since a_n is increasing, if we set $q = n$, we may deduce

$$\exists \varepsilon > 0\ \forall N\ \exists p > N:\ a_p \geq a_N + \varepsilon.$$

We can construct a subsequence $a_{n_1}, a_{n_2}, \cdots$ such that

$$a_{n_{k+1}} - a_{n_k} \geq \varepsilon.$$

Therefore

$$a_{n_k} \geq a_{n_1} + (k-1)\varepsilon.$$

So by the Archimedean property, (a_{n_k}) and hence (a_n) is unbounded. $\square$

2.4 Limit supremum and infimum

Lemma. Let (a_n) be a sequnece. The following two statements are equivalent:

- $a_n \to a$
- $\limsup a_n = \liminf a_n = a$.

Proof. If $a_n \to a$, then let $\varepsilon > 0$. Then

$$\exists n \, \forall m \geq n : \; a - \varepsilon \leq a_m \leq a + \varepsilon.$$

It follows that

$$a - \varepsilon \leq \inf_{m \geq n} a_m \leq \sup_{m \geq n} \leq a + \varepsilon.$$

Since ε was arbitrary, it follows that

$$\liminf a_n = \limsup a_n = a.$$

Conversely, if $\liminf a_n = \limsup a_n = a$, then let $\varepsilon > 0$. Then we can find n such that

$$\inf_{m \geq n} a_m > a - \varepsilon \text{ and } \sup_{m \geq n} a_m < a + \varepsilon.$$

It follows that $\forall m \geq n$, we have $|a_m - a| < \varepsilon$. $\square$

3 Convergence of infinite sums

Lemma. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Then $a_n \to 0$.

Proof. Let $\displaystyle\sum_{n=1}^{\infty} a_n = s$. Then $S_N \to s$ and $S_{N-1} \to s$. Then $a_N = S_N - S_{N-1} \to 0$. □

Lemma. Suppose that $a_n \geq 0$ for every n and the partial sums S_n are bounded above. Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. The sequence (S_n) is increasing and bounded above. So the result follows form the monotone sequences property. □

Lemma (Comparison test)**.** Let (a_n) and (b_n) be non-negative sequences, and suppose that $\exists C, N$ such that $\forall n \geq N$, $a_n \leq CB_n$.Then if $\sum b_n$ converges, then so does $\sum a_n$.

Proof. Let $M > N$. Also for each R, let $S_R = \sum_{n=1}^{R} a_n$ and $T_R = \sum_{n=1}^{R} b_n$. We want S_R to be bounded above.

$$S_M - S_N = \sum_{n=N+1}^{M} a_n \leq C \sum_{n=N+1}^{M} b_n \leq C \sum_{n=N+1}^{\infty} b_n.$$

So $\forall M \geq N$, $S_M \leq S_n + C \sum_{n=N+1}^{\infty} b_n$. Since $the S_M$ are increasing and bounded, it must converge. *Note*: N is fixed from the very beginning in the statement of the lemma. □