# Part II - Logic and Set Theory
## Definitions

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**Ordinals and cardinals**
Well-orderings and order-types. Examples of countable ordinals. Uncountable ordinals and Hartogs' lemma. Induction and recursion for ordinals. Ordinal arithmetic. Cardinals; the hierarchy of alephs. Cardinal arithmetic. [5]

**Posets and Zorn's lemma**
Partially ordered sets; Hasse diagrams, chains, maximal elements. Lattices and Boolean algebras. Complete and chain-complete posets; fixed-point theorems. The axiom of choice and Zorn's lemma. Applications of Zorn's lemma in mathematics. The well-ordering principle. [5]

**Propositional logic**
The propositional calculus. Semantic and syntactic entailment. The deduction and completeness theorems. Applications: compactness and decidability. [3]

**Predicate logic**
The predicate calculus with equality. Examples of first-order languages and theories. Statement of the completeness theorem; *sketch of proof*. The compactness theorem and the Lowenheim-Skolem theorems. Limitations of first-order logic. Model theory. [5]

**Set theory**
Set theory as a first-order theory; the axioms of ZF set theory. Transitive closures, epsilon-induction and epsilon-recursion. Well-founded relations. Mostowski's collapsing theorem. The rank function and the von Neumann hierarchy. [5]

**Consistency**
*Problems of consistency and independence* [1]

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# 1 Propositional calculus

**Definition** (Propositions)**.** Let $P$ be a set of *primitive propositions.* These are a bunch of (meaningless) symbols, that are usually interpreted to take a truth value. Usually, any symbol (composed of alphabets and subscripts) is in the set of primitive propositions.

The set of *propositions*, written as $L$ or $L(P)$, is defined inductively by

(i) If $p \in P$, then $p \in L$.

(ii) $\perp \in L$, where $\perp$ is "false" (also a meaningless symbol).

(iii) If $p, q \in L$, then $p \Rightarrow q) \in L$.

**Definition** (Logical symbols)**.**

| | | | |
|---|---|---|---|
| $\neg p$ | ("not $p$") | is an abbreviation for | $(p \Rightarrow \perp)$ |
| $p \wedge q$ | ("$p$ and $q$") | is an abbreviation for | $\neg(p \Rightarrow (\neg q))$ |
| $p \vee q$ | ("$p$ or $q$") | is an abbreviation for | $(\neg p) \Rightarrow q$ |

## 1.1 Semantic implication

**Definition** (Valuation)**.** A *valuation* on $L$ is a function $v : L \to \{0, 1\}$ such that:

– $v(\perp) = 0$,

– $v(p \Rightarrow q) = \begin{cases} 0 & \text{if } v(p) = 1, v(q) = 0, \\ 1 & \text{otherwise} \end{cases}$

We interpret $v(p)$ to be the truth value of $p$, with 0 denoting "false" and 1 denoting "true".

Note that we do not impose any restriction of $v(p)$ when $p$ is a primitive proposition (that is not $\perp$).

**Definition** (Tautology)**.** $t$ is a *tautology*, written as $\models t$, if $v(t) = 1$.

**Definition** (Semantic entailment)**.** For $S \subseteq L$, $t \in L$, we say $S$ *entails* $t$, $S$ *semantically implies* $t$ or $S \models t$ if, for any $v$ such that $v(s) = 1$ for all $s \in S$, $v(t) = 1$.

"Whenever all of $S$ is true, $t$ is true as well."

**Definition** (Truth and model)**.** If $v(t) = 1$, then we say that $t$ is *true* in $v$, or $v$ is a *model* of $t$. For $S \subseteq L$, a valuation $v$ is a *model* of $S$ if $v(s) = 1$. Then $\models t$ means $\emptyset \models t$.

## 1.2 Syntactic implication

**Definition** (Proof and syntactic entailment)**.** For any $S \subseteq L$, a *proof* of $t$ from $S$ is a finite sequence $t_1, t_2, \cdots t_n$ of propositions, with $t_n = t$, such that each $t_i$ is one of the following:

(i) An axiom

(ii) A member of $S$

(iii) A proposition $t_i$ such that there exist $j, k < i$ with $t_j = (t_k \Rightarrow t_i)$.

If there is a proof of $t$ from $S$, we say that $S$ *proves* or *syntactically entails* $t$, written $S \vdash t$.

If $\emptyset \vdash t$, say $t$ is a *theorem* and write $\vdash t$.

In a proof of $t$ from $S$, $t$ is the *conclusion* and $S$ is the set of *hypothesis* or *premises*.

**Definition** (Consistent)**.** $S$ is *inconsistent* if $S \vdash \perp$. $S$ is *consistent* if it is not inconsistent.