Part IA - Differential Equations

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Basic calculus
Informal treatment of differentiation as a limit, the chain rule, Leibnitz's rule, Taylor series, informal treatment of O and o notation and l'Hôpitals rule; integration as an area, fundamental theorem of calculus, integration by substitution and parts. [3]

Informal treatment of partial derivatives, geometrical interpretation, statement (only) of symmetry of mixed partial derivatives, chain rule, implicit differentiation. Informal treatment of differentials, including exact differentials. Differentiation of an integral with respect to a parameter. [2]

First-order linear differential equations
Equations with constant coefficients: exponential growth, comparison with discrete equations, series solution; modelling examples including radioactive decay.

Equations with non-constant coefficients: solution by integrating factor. [2]

Nonlinear first-order equations
Separable equations. Exact equations. Sketching solution trajectories. Equilibrium solutions, stability by perturbation; examples, including logistic equation and chemical kinetics. Discrete equations: equilibrium solutions, stability; examples including the logistic map. [4]

Higher-order linear differential equations
Complementary function and particular integral, linear independence, Wronskian (for second-order equations), Abel's theorem. Equations with constant coefficients and examples including radioactive sequences, comparison in simple cases with difference equations, reduction of order, resonance, transients, damping. Homogeneous equations. Response to step and impulse function inputs; introduction to the notions of the Heaviside step-function and the Dirac delta-function. Series solutions including statement only of the need for the logarithmic solution. [8]

Multivariate functions: applications
Directional derivatives and the gradient vector. Statement of Taylor series for functions on $\mathbb{R}^n$. Local extrema of real functions, classification using the Hessian matrix. Coupled first order systems: equivalence to single higher order equations; solution by matrix methods. Non-degenerate phase portraits local to equilibrium points; stability.

Simple examples of first- and second-order partial differential equations, solution of the wave equation in the form $f(x + ct) + g(x - ct)$. [5]

Contents

1 Differentiation

1.1 Differentiation

Definition (Derivative of function)**.** The *derivative* of a function $f(x)$ with respect to x, interpreted as the rate of change of $f(x)$ with x, is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

A function $f(x)$ is differentiable at x if the limit exists (i.e. the left-hand and right-hand limits are equal).

Example. $f(x) = |x|$ is not differentiable at $x = 0$ as $\lim_{h \to 0^+} \frac{|h| - |0|}{h} = 1$ and $\lim_{h \to 0^-} \frac{|h| - |0|}{h} = -1$.

Notation. We write $\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$. Also, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = f''(x)$.
Note: The notation f' is the derivative w.r.t. the argument, e.g. $f'(2x) = \frac{\mathrm{d}f}{\mathrm{d}(2x)}$

1.2 Small o and big O notations

Definition (O and o notations)**.**

(i) $f(x) = o(g(x))$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$. Intuitively, $f(x)$ is much smaller than $g(x)$.

(ii) $f(x) = O(g(x))$ as $x \to x_0$ if $\frac{f(x)}{g(x)}$ is bounded as $x \to x_0$. Intuitively, $f(x)$ is about as big as $g(x)$.

Note: $\lim_{x \to x_0} \dfrac{f(x)}{g(x)}$ need not exist for $f(x) = O(g(x))$.

Example.

- $x = o(\sqrt{x})$ as $x \to 0$ and $\sqrt{x} = o(x)$ as $x \to \infty$.
- $\sin 2x = O(x)$ as $x \to 0$ as $\sin\theta \approx \theta$ for small θ.
- $\sin 2x = O(1)$ as $x \to \infty$ even though the limit does not exist.

Note: $f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))$.

Proposition.

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$$

Proof. We have

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{o(h)}{h}$$

by the definition of the derivative and the small o notation. The result follows. □

1.3 Methods of differentiation

Theorem (Chain rule)**.** Given $f(x) = F(g(x))$, then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}F}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}.$$

Proof. Assuming that $\frac{\mathrm{d}g}{\mathrm{d}x}$ exists and is therefore finite, we have

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0}\frac{F(g(x+h)) - F(g(x))}{h} \\
&= \lim_{h\to 0}\frac{F[g(x) + hg'(x) + o(h)] - F(g(x))}{h} \\
&= \lim_{h\to 0}\frac{F(g(x)) + (hg'(x) + o(h))F'(g(x)) + o(hg'(x) + o(h)) - F(g(x))}{h} \\
&= \lim_{h\to 0} g'(x)F'(g(x)) + \frac{o(h)}{h} \\
&= g'(x)F'(g(x)) \\
&= \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}
\end{aligned}$$

□

Theorem (Product Rule)**.** Give $f(x) = u(x)v(x)$. Then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Theorem (Leibniz's Rule)**.** Given $f = uv$, then

$$f^{(n)}(x) = \sum_{r=0}^{n}\binom{n}{r}u^{(r)}v^{(n-r)},$$

where $f^{(n)}$ is the n-th derivative of f.

1.4 Taylor's theorem

Theorem (Taylor's Theorem)**.** For n-times differentiable f, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + E_n,$$

where $E_n = o(h^n)$ as $h \to 0$. If $f^{(n+1)}$ exists, then $E_n = O(h^{n+1})$.

Note: This only gives a local approximation around x. This does not tell anything about values of f far from x.

An alternative form of the sum above is:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) + E_n.$$

When the limit as $n \to \infty$ is taken, the Taylor series of $f(x)$ about the point $x = x_0$ is obtained.

1.5 L'Hopital's rule

Theorem (L'Hopital's Rule)**.** Let $f(x)$ and $g(x)$ be differentiable at x_0, and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$. Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. From the Taylor's Theorem, we have $f(x) = f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)$, and similarly for $g(x)$. Thus

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x)}{g(x)} &= \lim_{x \to x_0} \frac{f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)}{g(x_0) + (x - x_0)g'(x_0) + o(x - x_0)} \\
&= \lim_{x \to x_0} \frac{f'(x_0) + \frac{o(x - x_0)}{x - x_0}}{g'(x_0) + \frac{o(x - x_0)}{x - x_0}} \\
&= \lim_{x \to x_0} \frac{f'(x)}{g'(x)}
\end{aligned}$$

□

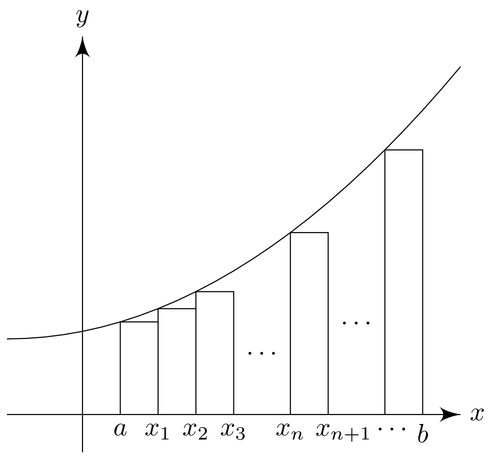
2 Integration

2.1 Integration

Definition (Integral)**.** An *integral* is the limit of a sum, e.g.

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{\Delta x \to 0} \sum_{n=0}^{N} f(x_n)\Delta x.$$

For example, we can take $\Delta x = \frac{b-a}{N}$ and $x_n = a + n\Delta x$. Note that an integral need not be defined with this particular Δx and x_n. The term "integral" simply refers to any limit of a sum. (The usual integrals we use are a special kind known as Riemann integral, c.f. Analysis I) Pictorially, we have



The area under the graph from x_n to x_{n+1} is $f(x_n)\Delta x + O(\Delta x^2)$. Provided that f is differentiable, the total area under the graph from a to b is

$$\lim_{N \to \infty} \sum_{n=0}^{N-1} (f(x_n)\Delta x) + N \cdot O(\Delta x^2) = \lim_{N \to \infty} \sum_{n=0}^{N-1} (f(x_n)\Delta x) + O(\Delta x) = \int_a^b f(x)\ \mathrm{d}x$$

Theorem (Fundamental Theorem of Calculus)**.** Let $F(x) = \int_a^x f(t)\ \mathrm{d}t$. Then $F'(x) = f(x)$.

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} F(x) &= \lim_{h \to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\ \mathrm{d}t - \int_a^x f(t)\ \mathrm{d}t\right] \\
&= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\ \mathrm{d}t \\
&= \lim_{h \to 0} \frac{1}{h}[f(x)h + O(h^2)] \\
&= f(x)
\end{aligned}$$

□

Similarly, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b f(t)\ \mathrm{d}t = -f(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\ \mathrm{d}t = f(g(x))g'(x).$$

Notation. We write $\int f(x)\ \mathrm{d}x = \int^x f(t)\ \mathrm{d}t$, where the unspecified lower limit gives rise to the constant of integration.

2.2 Methods of integration

Example (Integration by substitution)**.** Consider $\int \frac{1-2x}{\sqrt{x-x^2}}\ \mathrm{d}x$. Write $u = x - x^2$ and $\mathrm{d}u = (1-2x)\ \mathrm{d}x$. Then the integral becomes

$$\int \frac{\mathrm{d}u}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{x-x^2} + C.$$

Trigonometric substitution can be performed with reference to the following table: if the function in the 2nd column is found in the integrand, perform the substitution in the 3rd column and simplify using the identity in the first column:

Useful identity	Part of integrand	Substitution
$\cos^\theta + \sin^\theta = 1$	$\sqrt{1-x^2}$	$x = \sin\theta$
$1 + \tan^2\theta = \sec^2\theta$	$1 + x^2$	$x = \tan\theta$
$\cosh^2 u - \sinh^2 u = 1$	$\sqrt{x^2-1}$	$x = \cosh u$
$\cosh^2 u - \sinh^2 u = 1$	$\sqrt{1+x^2}$	$x = \sinh u$
$1 - \tanh^2 u = sech^2 u$	$1 - x^2$	$x = \tanh u$

Example. Consider $\int \sqrt{2x - x^2}\ \mathrm{d}x = \int \sqrt{1-(x-1)^2}\ \mathrm{d}x$. Let $x - 1 = \sin\theta$ and thus $\mathrm{d}x = \cos\theta\ \mathrm{d}\theta$. The expression becomes

$$\begin{aligned}\int \cos^2\theta\ \mathrm{d}\theta &= \int \frac{\cos 2\theta + 1}{2}\ \mathrm{d}\theta \\ &= \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C \\ &= \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{2x-x^2} + C.\end{aligned}$$

Theorem (Integration by parts)**.**

$$\int uv'\ \mathrm{d}x = uv - \int vu'\ \mathrm{d}x.$$

Proof. From the product rule, we have $(uv)' = uv' + u'v$. Integrating the whole expression and rearranging gives the formula above. □

Example. Consider $\int_0^\infty xe^{-x}\mathrm{d}x$. Let $u = x$ ad $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. We have

$$\begin{aligned}\int_0^\infty xe^{-x}\ \mathrm{d}x &= [-xe^{-x}]_0^\infty + \int_0^\infty e^{-x}\ \mathrm{d}x \\ &= 0 + [e^{-x}]_0^\infty \\ &= 1\end{aligned}$$

Example (Integration by Parts)**.** Consider $\int \log x \, \mathrm{d}x$. Let $u = \log x$ and $v' = 1$. Then $u' = \frac{1}{x}$ and $v = x$. So we have

$$\begin{aligned} \int \log x \, \mathrm{d}x &= x \log x - \int \mathrm{d}x \\ &= x \log x - x + C \end{aligned}$$

3 Partial differentiation

3.1 Partial differentiation

Definition (Partial derivative)**.** Given a function of several variables $f(x, y)$, the *partial derivative* of f with respect to x is the rate of change of f as x varies, keeping y constant. It is given by

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Example. Consider $f(x, y) = x^2 + y^3 + e^{xy^2}$. Computing the partial derivative is equivalent to computing the regular derivative with the other variables treated as constants. e.g.

$$\left.\frac{\partial f}{\partial y}\right|_y = 2x + y^2 e^{xy^2}.$$

Second and mixed partial derivatives can also be computed:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= 2 + y^4 e^{xy^2} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2ye^{xy^2} + 2xy^3 e^{xy^2}
\end{aligned}$$

Notation. If the variables to be kept constant are not given, e.g. simply $\frac{\partial f}{\partial x}$, it is assumed that ALL other variables are being held constant.

We also write $f_x = \frac{\partial f}{\partial x}$ and $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

Theorem. $f_{xy} = f_{yx}$.

3.2 Chain rule

Consider an arbitrary displacement in any direction $(x, y) \to (x + \delta x, y + \delta y)$. We have

$$\begin{aligned}
\delta f &= f(x + \delta x, y + \delta y) - f(x, y) \\
&= f(x + \delta x, y + \delta y) - f(x + \delta x, y) + f(x + \delta x, y) - f(x, y) \\
&= f_y(x + \delta x, y)\delta y + o(\delta y) + f_x(x, y)\delta x + o(\delta x) \\
&= (f_x(x, y) + o(1))\delta y + o(\delta y) + f_x(x, y)\delta x + o(\delta x) \\
\delta f &= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + o(\delta x, \delta y)
\end{aligned}$$

Take the limit as $\delta x, \delta y \to 0$, we have

Theorem (Chain rule for partial derivatives)**.**

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y.$$

Given this form, we can sum the differentials to obtain the integral form:

$$\int \mathrm{d}f = \int \frac{\partial f}{\partial x}\mathrm{d}x + \int \frac{\partial f}{\partial y}\mathrm{d}y,$$

or divide by another small quantity. e.g. to find the slope along the path $(x(t), y(t))$, we can divide by $\mathrm{d}t$ to obtain

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

If we use the arclength s as parameter instead of t, we have

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right)\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \hat{s}\cdot\nabla f,$$

which is known as the directional derivative (c.f. Section 7).

Alternatively, the path may also be given by $y = y(x)$. So $f = f(x, y(x))$. Then the slope along the path is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \left.\frac{\partial f}{\partial x}\right|_y + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The chain rule can also be used for the change of independent variables, e.g. change to polar coordinates $x = x(r, \theta)$, $y = y(r, \theta)$. Then

$$\left.\frac{\partial f}{\partial \theta}\right|_r = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial \theta}\right|_r + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial \theta}\right|_r.$$

3.3 Implicit differentiation

Consider the contour surface of a function $F(x, y, z)$ given by $F(x, y, z) = \text{const}$. This implicitly defines $z = z(x, y)$. e.g. If $F(x, y, z) = xy^2 + yz^2 + z^5x = 5$, then we can have $x = \frac{5 - yz^2}{y^2 + z^5}$. Even though $z(x, y)$ cannot be found explicitly (involves solving quintic equation), the derivatives of $z(x, y)$ can still be found by differentiating $F(x, y, z) = \text{const}$ w.r.t. x holding y constant. e.g.

$$\begin{aligned}
\frac{\partial}{\partial x}(xy^2 + yz^2 + z^5x) &= \frac{\partial}{\partial x}5 \\
y^2 + 2yz\frac{\partial z}{\partial x} + z^5 + 5z^4x\frac{\partial z}{\partial x} &= 0 \\
\frac{\partial z}{\partial x} &= -\frac{y^2 + z^5}{2yz + 5z^4x}
\end{aligned}$$

In general, we can derive the following formula:

Theorem (Multi-variable implicit differentiation)**.** Given an equation

$$F(x, y, z) = c$$

for some constant c, we have

$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{(\partial F)/(\partial x)}{(\partial F)/(\partial z)}$$

Proof.

$$\begin{aligned}
\mathrm{d}F &= \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z \\
\left.\frac{\partial F}{\partial x}\right|_y &= \frac{\partial F}{\partial x}\left.\frac{\partial x}{\partial x}\right|_y + \frac{\partial F}{\partial y}\left.\frac{\partial y}{\partial x}\right|_y + \frac{\partial F}{\partial z}\left.\frac{\partial z}{\partial x}\right|_y = 0 \\
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\left.\frac{\partial z}{\partial x}\right|_y &= 0 \\
\left.\frac{\partial z}{\partial x}\right|_y &= -\frac{(\partial F)/(\partial x)}{(\partial F)/(\partial z)}
\end{aligned}$$

□

3.4 Differentiation of an integral w.r.t. parameter in the integrand

Consider a family of functions $f(x,c)$. Define $I(b,c) = \int_a^b f(x,c)\mathrm{d}x$. Then we have $\frac{\partial I}{\partial b} = f(x,c)$. On the other hand, we have

$$\begin{aligned}
\frac{\partial I}{\partial c} &= \lim_{\delta c\to 0}\frac{1}{\delta c}\left[\int_0^b f(x,c+\delta c)\mathrm{d}x - \int_0^b f(x,c)\;\mathrm{d}x\right] \\
&= \lim_{\delta c\to 0}\int_0^b \frac{f(x,c+\delta c)-f(x,c)}{\delta c}\;\mathrm{d}x \\
&= \int_0^b \lim_{\delta c\to 0}\frac{f(x,c+\delta c)-f(x,c)}{\delta c}\;\mathrm{d}x \\
&= \int_0^b \frac{\partial f}{\partial c}\;\mathrm{d}x
\end{aligned}$$

If $I(b(x),c(x)) = \int_0^{b(x)} f(y,c(x))\mathrm{d}y$, then by the chain rule, we have

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \frac{\partial I}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}x} + \frac{\partial I}{\partial c}\frac{\mathrm{d}c}{\mathrm{d}x} = f(b,c)b'(x) + c'(x)\int_0^b \frac{\partial f}{\partial c}\;\mathrm{d}y.$$

Theorem (Differentiation under the integral sign)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{b(x)} f(x,c(x))\;\mathrm{d}x = f(b,c)b'(x) + c'(x)\int_0^b \frac{\partial f}{\partial c}\;\mathrm{d}y$$

Example. Let $I = \int_0^1 e^{-\lambda x^2}\mathrm{d}x$. Then

$$\frac{\mathrm{d}I}{\mathrm{d}\lambda} = \int_0^1 -x^2e^{-\lambda x^2}\;\mathrm{d}x.$$

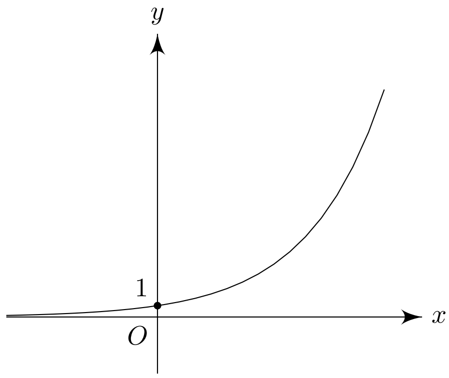
If $I = \int_0^\lambda e^{-\lambda x^2}\mathrm{d}x$. Then

$$\frac{\mathrm{d}I}{\mathrm{d}\lambda} = e^{-\lambda^3} + \int_0^1 -x^2e^{-\lambda x^2}\;\mathrm{d}x.$$

4 First-order differential equations

4.1 The exponential function

Consider a function $f(x) = a^x$, where $a > 0$ is constant.



$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}\\ &= a^x\lim_{h\to 0}\frac{a^h-1}{h}\\ &= \lambda a^x\\ &= \lambda f(x)\end{aligned}$$

where $\lambda = \lim_{h\to 0}\dfrac{a^h-1}{a} = f'(0) =$const.

Definition (Exponential function)**.** $\exp(x) = e^x$ is the unique function f satisfying $f'(x) = f(x)$ and $f(0) = 1$.

Using this property, we can prove that $e = \lim_{k\to\infty}\left(1+\dfrac{1}{k}\right)^k$.

Then if $y = a^x = e^{x\ln a}$, then $y' = e^{x\ln a}\ln a = a^x\ln a$ and $\lambda = \ln a$.

Notation. $\ln z \equiv \log_e z = \log z$

4.2 Homogeneous linear ordinary differential equations

Definition (Eigenfunction)**.** An *eigenfunction* under the differential operator is a function whose functional form is unchanged by the operator. Only its magnitude is changed. i.e.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lambda f$$

Example. e^{mx} is an eigenfunction since $\frac{\mathrm{d}}{\mathrm{d}x}e^{mx} = me^{mx}$.

Definition (Linear differential equation)**.** A differential equation is *linear* if the dependent variable (y, y', y'' etc.) appears only linearly.

Definition (Homogeneous differential equation)**.** A differential equation is *homogeneous* if $y = 0$ is a solution.

Definition (Differential equation with constant coefficients)**.** A differential equation has *constant coefficients* if the independent variable x does not appear explicitly.

Definition (First-order differential equation)**.** A differential equation is *first-order* if only first derivatives are involved.

Theorem. Any linear, homogeneous, ordinary differential equation with constant coefficients has solutions of the form e^{mx}.

Example. Given $5\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$. Then

$$\begin{aligned} y &= e^{mx} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= me^{mx} \\ 5me^{mx} - 3e^{mx} &= 0 \end{aligned}$$

Since this must hold for all values of x, there exists some value x for which $e^{mx} \neq 0$ and we can divide by e^{mx} (note in this case this justification is not necessary, because e^{mx} is never equal to 0. However, we should justify as above if we are dividing, say, x^m). Thus $5m - 3 = 0$ and $m = 3/5$. So $y = e^{3x/5}$ is a solution.

(i) Because the equation is linear and homogeneous, any multiple of a solution is also a solution. Therefore $y = Ae^{3x/5}$ is a solution for any value of A.

(ii) An n^{th}-order linear differential equation has n and only n independent solutions. So $y = Ae^{3x/5}$ is indeed the most general solution.

We can determine A by applying a given boundary condition.

4.2.1 Discrete equations

$5y' - 3y = 0$ with $y = y_0$ at $x = 0$ gives a unique function y. We can approximate this by considering discrete steps of length h between x_n and x_{n+1}. (Using the simple Euler numerical scheme,) we have

$$5\frac{y_{n+1} - y_n}{h} - 3y_n \approx 0.$$

Rearranging the terms, we have $y_{n+1} \approx (1 + \frac{3}{5}h)y_n$. (c.f. compound interest formula). Applying the relation successively, we have

$$\begin{aligned} y_n &= \left(1 + \frac{3}{5}h\right) y_{n-1} \\ &= \left(1 + \frac{3}{5}h\right)\left(1 + \frac{3}{5}h\right) y_{n-2} \\ &= \left(1 + \frac{3}{5}h\right)^n y_0 \end{aligned}$$

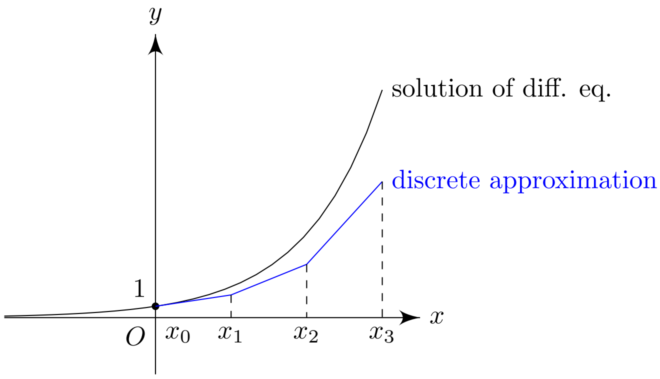
For each given value of x, choose $h = x/n$, so

$$y_n = y_0 \left(1 + \frac{3}{5}(x/n)\right)^n.$$

Taking the limit as $n \to \infty$, we have

$$y(x) = \lim_{n\to\infty} y_0 \left(1 + \frac{3x/5}{n}\right)^n = y_0 e^{3x/5}.$$

in agreement with the solution of the differential equation.



4.2.2 Series solution

We can also try to find a solution in the form of a Taylor Series $y = \sum_{n=0}^{\infty} a_n x^n$. We have $y' = \sum a_n n x^{n-1}$. Substituting these into our equation

$$\begin{aligned} 5y' - 3y &= 0 \\ 5(xy') - 3x(y) &= 0 \\ \sum a_n(5n - 3x)x^n &= 0 \end{aligned}$$

Consider the coefficient of x^n: $5na_n - 3a_{n-1} = 0$. Since this holds for all values of n, when $n = 0$, we get $0a_0 = 0$. This tells that a_0 is arbitrary. If $n > 0$, then

$$a_n = \frac{3}{5n} a_{n-1} = \frac{3^2}{5^2} \frac{1}{n(n-1)} a_{n-2} = \cdots = \left(\frac{3}{5}\right)^n \frac{1}{n!} a_0.$$

Therefore we have

$$y = a_0 \sum_{n=0}^{\infty} \left(\frac{3x}{5}\right)^n \frac{1}{n!} \left[= a_0 e^{3x/5}\right].$$

4.3 Forced (inhomogeneous) equations

4.3.1 Constant forcing

Example. Consider $5y' - 3y = 10$. We can spot that there is a equilibrium (constant) solution $y = y_p = -\frac{10}{3}$ with $y_p' = 0$.

The particular solution y_p is a solution of the ODE. Now suppose the general solution is $y = y_p + y_c$. We see that $5y_c' - 3y_c = 0$. So y_c satisfies the homogeneous equation we already solved, and

$$y = -\frac{10}{3} + Ae^{3x/5}.$$

Note: Any boundary conditions to determine A must be applied to the full solution y and not the complementary function y_c.

4.3.2 Eigenfunction forcing

When the R.H.S. of the equation is an eigenfunction of the differential operator.

Example. In a radioactive rock, isotope A decays into isotope B at a rate proportional to the number a of remaining nuclei A, and B also decays at a rate proportional to the number b of remaining nuclei B. Determine $b(t)$.

We have

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= -k_a a\\ \frac{\mathrm{d}b}{\mathrm{d}t} &= k_a a - k_b b.\end{aligned}$$

Solving the first equation, we obtain $a = a_0 e^{-k_a t}$. Then we have

$$\frac{\mathrm{d}b}{\mathrm{d}t} + k_b b = k_a a_0 e^{-k_a t}.$$

We usually put the variables involving b on the left hand side and the others on the right. We call the right-hand term $k_a a_0 e^{-k_a t}$ the *forcing term*.

Note that the forcing term is an eigenfunction of the differential operator on the LHS. So that suggests that we can try a particular integral $b_p = Ce^{-k_a t}$. Substituting it in, we obtain

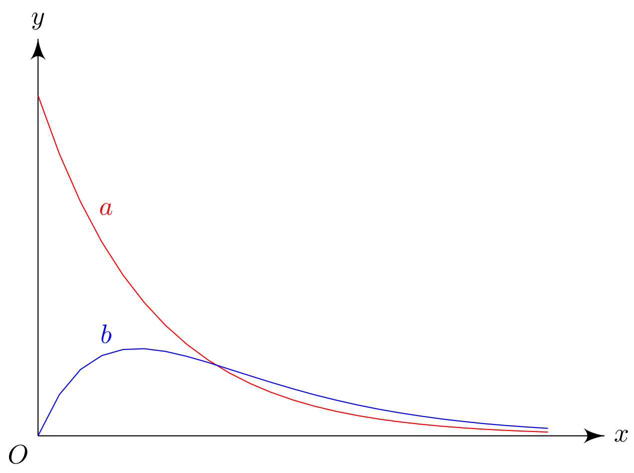
$$\begin{aligned}-k_a C + kB_C &= k_a a_0\\ C &= \frac{k_a}{k_b - k_a} a_0.\end{aligned}$$

Then write $b = b_p + b_c$. We get $b_c' + k_b b_c = 0$ and $b_c = De^{-k_b t}$. All together, we have the general solution

$$b = \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + De^{-k_b t}.$$

Assume the following boundary condition: $b = 0$ when $t = 0$, in which case we can find

$$b = \frac{k_a}{k_b - k_a} a_0 \left(e^{-k_a t} - e^{-k_b t}\right).$$



The isotope ratio is

$$\frac{b}{a} = \frac{k_a}{k_b - k_a}\left[1 - e^{(k_a - k_b)t}\right].$$

So given the current ratio b/a, with laboratory determined rates k_a and k_b, we can determine the value of t, i.e. the age of the rock.

4.4 Non-constant coefficients

Consider the general form of equation

$$a(x)y' + b(x)y = c(x).$$

Divide by $a(x)$ to get the standard form

$$y' + p(x)y = f(x).$$

We solve this by multiplying an integrating factor $\mu(x)$ to obtain $(\mu)y' + (\mu p)y = \mu f$.

We want to choose a μ such that the left hand side is equal to $(\mu y)'$. By the product rule, we want $\mu p = \mu'$, i.e.

$$\begin{aligned}
p &= \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x}\\
\int p\ \mathrm{d}x &= \int \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x}\ \mathrm{d}x\\
&= \int \frac{1}{\mu}\ \mathrm{d}u\\
&= \ln\mu (+C)\\
\mu &= \exp\left(\int p\ \mathrm{d}x\right)
\end{aligned}$$

Then by construction, we have $(\mu y)' = \mu f$ and thus

$$y = \frac{\int \mu f\ \mathrm{d}x}{\mu},\ \text{where } \mu = \exp\left(\int p\ \mathrm{d}x\right)$$

Example. Consider $xy' + (1-x)y = 1$. To obtain it in standard form, we have $y' + \frac{1-x}{x} y = \frac{1}{x}$. We have $\mu = \exp\left(\int (\frac{1}{x} - 1)\ \mathrm{d}x\right) = e^{\ln x - x} = xe^{-x}$. Then

$$\begin{aligned} y &= \frac{\int xe^{-x}\frac{1}{x}\ \mathrm{d}x}{xe^{-x}} \\ &= \frac{-e^{-x} + C}{xe^{-x}} \\ &= \frac{-1}{x} + \frac{C}{x}e^x \end{aligned}$$

Suppose that we have a boundary condition y is finite at $x = 0$. We've got $y = \frac{Ce^x - 1}{x}$. We have to ensure that $Ce^x - 1 \to 0$ as $x \to 0$. Thus $C = 1$, and by L'Hopital's rule, $y \to 1$ as $x \to 0$.

4.5 Non-linear equations

In general, a first-order equation has the form

$$Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x, y) = 0.$$

4.5.1 Separable equations

Definition (Separable equation)**.** A first-order differential equation is *separable* if it can be manipulated into the following form:

$$q(y)\ \mathrm{d}y = p(x)\ \mathrm{d}x.$$

in which case the solution can be found by integration

$$\int q(y)\ \mathrm{d}y = \int p(x)\ \mathrm{d}x.$$

Example.

$$\begin{aligned} (x^2y - 3y)\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy^2 &= 4x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4x + 2xy^2}{x^2y - 3y} \\ &= \frac{2x(2 + y^2)}{y(x^2 - 3)} \\ \frac{y}{2 + y^2}\ \mathrm{d}y &= \frac{2x}{x^2 - 3}\ \mathrm{d}x \\ \int \frac{y}{2 + y^2}\ \mathrm{d}y &= \int \frac{2x}{x^2 - 3}\ \mathrm{d}x \\ \frac{1}{2}\ln(2 + y^2) &= \ln(x^2 - 3) + C \\ \ln\sqrt{2 + y^2} &= \ln A(x^2 - 3) \\ \sqrt{y^2 + 2} &= A(x^2 - 3) \end{aligned}$$

4.5.2 Exact equations

Definition (Exact equation)**.** $Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x,y) = 0$ is an *exact equation* iff the differential form $Q(x,y)\ \mathrm{d}y + P(x,y)\ \mathrm{d}x$ is *exact*, i.e. there exists a function $f(x,y)$ for which

$$\mathrm{d}f = Q(x,y)\ \mathrm{d}y + P(x,y)\ \mathrm{d}x$$

If $P(x,y)\ \mathrm{d}x + Q(x,y)\ \mathrm{d}y$ is an exact differential of f, then $\mathrm{d}f = P(x,y)\ \mathrm{d}x + Q(x,y)\ \mathrm{d}y$. But by the chain rule, $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$ and this equality holds for any displacements $\mathrm{d}x, \mathrm{d}y$. So

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q.$$

From this we have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

We know that the two mixed 2nd derivatives are equal. So

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Definition (Simply-connected domain)**.** A domain $\mathcal{D}$ is simply-connected if it is connected and any closed curve in $\mathcal{D}$ can be shrunk to a point in $\mathcal{D}$ without leaving $\mathcal{D}$.

Example. A disc in 2D is simply-connected. A disc with a "hole" in the middle is not simply-connected because a loop around the hole cannot be shrunk into a point. Similarly, a sphere in 3D is simply-connected but a torus is not.

Theorem. (Converse of above result) If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ through a simply-connected domain $\mathcal{D}$, then $P\ \mathrm{d}x + Q\ \mathrm{d}y$ is an exact differential of a single-valued function in $\mathcal{D}$.

If the equation is exact, then the solution is simply $f =$ constant, and we can find f by integrating $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$.

Example.

$$6y(y-x)\frac{\mathrm{d}y}{\mathrm{d}x} + (2x - 3y^2) = 0.$$

We have

$$P = 2x - 3y^2, \quad Q = 6y(y-x).$$

Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -6y$. So the differential form is exact. We now have

$$\frac{\partial f}{\partial x} = 2x - 3y^2, \quad \frac{\partial f}{\partial y} = 6y^2 - 6xy.$$

Integrating the first equation, we have

$$f = x^2 - 3xy^2 + h(y).$$

Note that since it was a partial derivative w.r.t. x holding y constant, the "constant" term can be any function of y. Differentiating the derived f w.r.t y, we have

$$\frac{\partial f}{\partial y} = -6xy + h'(y).$$

Thus $h'(y) = 6y^2$ and $h(y) = 2y^3 + C$, and

$$f = x^2 - 3xy^2 + 2y^3 + C.$$

Since the original equation was $\mathrm{d}f = 0$, we have $f =$ constant. Thus the final solution is

$$x^2 - 3xy^2 + 3y^3 = C.$$

4.6 Solution curves (trajectories)

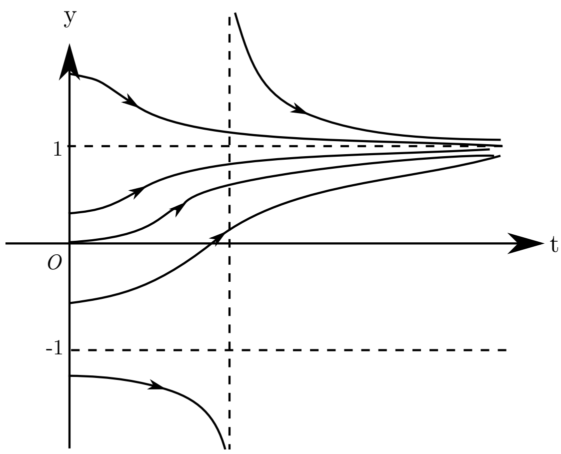
Example. Consider the first-order equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t(1 - y^2).$$

We can solve it to obtain

$$\begin{aligned}
\frac{\mathrm{d}y}{1 - y^2} &= t\ \mathrm{d}t \\
\frac{1}{2}\ln\frac{1+y}{1-y} &= \frac{1}{2}t^2 + C \\
\frac{1+y}{1-y} &= Ae^{t^2} \\
y &= \frac{A - e^{-t^2}}{A + e^{-t^2}}
\end{aligned}$$

We can plot the solution for different values of A and obtain the following graph:

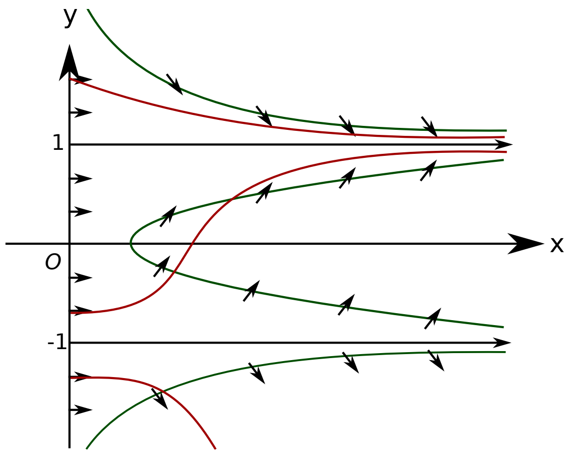


But can we understand the nature of the family of solutions without solving the equation? Can we sketch the graph without solving it?

We can spot that $y = \pm 1$ are two constant solutions and we can plot them first. We also note (and mark on our graph) that $y' = 0$ at $t = 0$ for any y.

Then notice that for $t > 0$, $y' > 0$ if $-1 < y < 1$. Otherwise, $y' < 0$.

Now we can find *isoclines*, which are curves along which $\frac{\mathrm{d}y}{\mathrm{d}t}$ (i.e. f) is constant: $t(1 - y^2) = D$ for some constant D. Then $y^2 = 1 - D/t$. After marking a few isoclines, can sketch the approximate form of our solution:



In general, we sketch the graph of a differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$$

by locating constant solutions (and determining stability: see below) and isoclines.

4.7 Fixed (equilibrium) points and stability

Definition (Equilibrium/fixed point)**.** An *equilibrium point* or a *fixed point* of a differential equation is a solution with $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ for all t. This happens when $y = c$ for some constant c.

Definition (Stability of fixed point)**.** An equilibrium is *stable* if when y is deviated slightly from the constant solution $y = c$, $y \to c$ as $t \to \infty$. An equilibrium is *unstable* if the deviation grows as $t \to \infty$.

Example. Refer to the differential equation above ($y' = t(1 - y^2)$). We see that the solutions converge towards $y = 1$, and it is a stable fixed point. They diverge from $y = -1$, and this is an unstable fixed point.

4.7.1 Perturbation analysis

Perturbation analysis is used to determine stability. Suppose $y = a$ is a fixed point of $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$, so $f(a, t) = 0$. Write $y = a + \epsilon(t)$, when $\epsilon(t)$ is a small perturbation from $y = a$. We will later assume that ϵ is arbitrarily small. Putting this into the differential equation, we have

$$\begin{aligned}
\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} &= f(a + \epsilon, t) \\
&= f(a, t) + \epsilon\frac{\partial f}{\partial y}(a, t) + O(\epsilon^2) \\
&= \epsilon\frac{\partial f}{\partial y}(a, t) + O(\epsilon^2)
\end{aligned}$$

Note that this is a Taylor expansion valid when $\epsilon \ll 1$. Thus $O(\epsilon^2)$ can be neglected and

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} \cong \epsilon \frac{\partial f}{\partial y}.$$

This approximation is called a linearization of the differential equation.

Example. Continuing the example above, we have

$$\frac{\partial f}{\partial y} = -2yt = \begin{cases} -2t & \text{at } y = 1 \\ 2y & \text{at } y = -1 \end{cases}.$$

At $y = 1$, $\dot{\epsilon} = -2t\epsilon$ and $\epsilon = \epsilon_0 e^{-t^2}$. Since $\epsilon \to 0$ as $t \to \infty$, $y \to 1$. So $y = 1$ is a stable fixed point.

On the other hand, if we consider $y = -1$, then $\dot{\epsilon} = 2t\epsilon$ and $\epsilon = \epsilon_0 e^{t^2}$. Since $\epsilon \to \infty$ as $t \to \infty$, $y = -1$ is unstable.

Note: Technically $\epsilon \to \infty$ is not a correct statement, since the approximation used is only valid for small ϵ. But we can be sure that the perturbation grows (even if not $\to \infty$) as t increases.

4.7.2 Autonomous systems

Definition (Autonomous system)**.** An *autonomous system* is a system in the form $\dot{y} = f(y)$, where the derivative is only (explicitly) dependent on y.

When, near a fixed point $y = a$, where $f(a) = 0$, write $y = a + \epsilon(t)$. Then $\dot{\epsilon} = \epsilon \frac{\mathrm{d}f}{\mathrm{d}y}(a) = k\epsilon$ for some constant k. Then $\epsilon = \epsilon_0 e^{kt}$. The stability of the system then depends on the solely on sign of k.

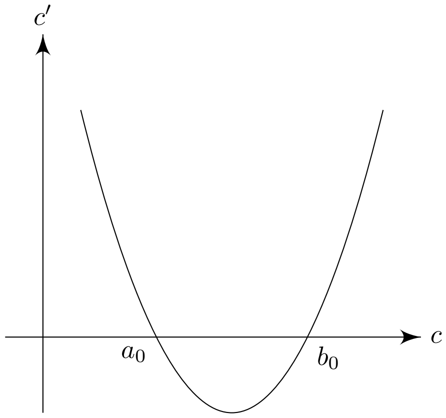
Example. Consider a chemical reaction NaOH + HCl $\to$ H_2O + NaCl. We have

	NaOH	+	HCl	$\to$	H_2O	+	NaCl
Number of molecules	a		b		c		c
Initial number of molecules	a_0		b_0		0		0

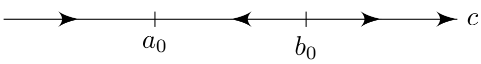
If the reaction is in dilute solution, then the reaction rate is proportional to ab. Thus

$$\begin{aligned} \frac{\mathrm{d}c}{\mathrm{d}t} &= \lambda ab \\ &= \lambda(a_0 - c)(b_0 - c) \\ &= f(c) \end{aligned}$$

We can plot $\frac{\mathrm{d}c}{\mathrm{d}t}$ as a function of c, and wlog $a_0 < b_0$.



We can also plot a *phase portrait*, which is a plot of the dependent variable only, where arrows show the evolution with time,



We can clearly see that the fixed point $c = a_0$ is stable while $c = b_0$ is unstable.
Note: We can solve the equation explicitly to obtain

$$c = \frac{a_0 b_0 [1 - e^{-(b_0 - a_0)\lambda t}]}{b_0 - a_0 e^{-\lambda(b_0 - a_0)t}}$$

Note: This example makes no sense physically whatsoever because it assumes that we can have negative values of a and b. Clearly in reality we cannot have, say, -1 mol of NaOH, and only solutions for $c \leq a_0$ are physically attainable, and in this case, any solution will tend towards $c = a_0$.

4.7.3 Logistic Equation

The logistic equation is a simple model of population dynamics. Suppose we have a population of size y. It has a birth rate αy and a death rate βy. With this model, we obtain

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= (\alpha - \beta)y \\ y &= y_0 e^{(\alpha - \beta)t}\end{aligned}$$

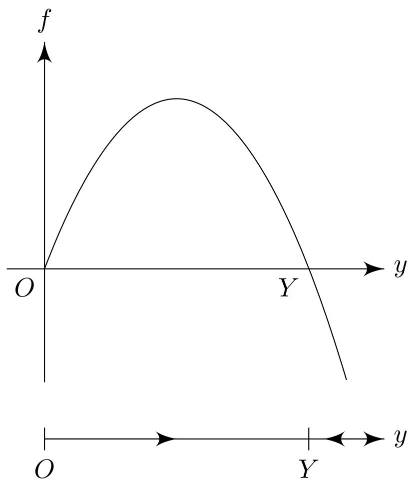
Our population increases or decreases exponentially depending on whether the birth rate exceeds death rate or vice versa.

However, in reality, there is fighting for limited resources. The probability of some piece of food (resource) being found is $\propto y$. The probability of the same piece of food being found by two individuals is $\propto y^2$. If food is scarce, they fight (to the death), so death rate due to fighting (competing) is γy^2 for some γ. So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= (\alpha - \beta)y - \gamma y^2 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= ry\left(1 - \frac{y}{Y}\right),\end{aligned}$$

where $r = \alpha - \beta$ and $Y = r/\gamma$. This is the differential logistic equation. Note that it is separable and can be solved explicitly.

However, we find the phase portrait instead. If $r = \alpha - \beta > 0$, then the graph is a quadratic parabola, and we see that Y is a stable fixed point.



Now when the population is small, we have

$$\dot{y} \simeq ry$$

So the population grows exponentially. Eventually, the stable equilibrium Y is reached.

4.8 Discrete equations (Difference equations)

Since differential equations are approximated numerically by computers with discrete equations, it is important to study the behaviour of discrete equations (and their difference with continuous counterparts).

In the logistic equation, the evolution of species may occur discretely (e.g. births in spring, deaths in winter), and we can consider the population at certain time intervals (e.g. consider the population at the end of each month). We might have a model in the form

$$x_{n+1} = \lambda x_n(1 - x_n).$$

This can be derived from the continuous equation with a discrete approximation

$$\begin{aligned}
\frac{y_{n+1} - y_n}{\Delta t} &= ry_n\left(1 - \frac{y_n}{Y}\right)\\
y_{n+1} &= y_n + r\Delta t y_n\left(1 - \frac{y_n}{Y}\right)\\
&= (1 + r\Delta t)y_n - \frac{r\Delta t}{Y}y_n^2\\
&= (1 + r\Delta t)y_n\left[1 - \left(\frac{r\Delta t}{1 + r\Delta t}\right)\frac{y_n}{Y}\right]
\end{aligned}$$

Write

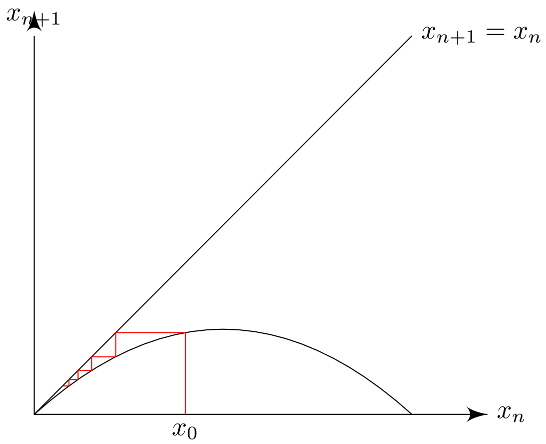
$$\lambda = 1 + r\Delta t, \qquad x_n = \left(\frac{r\Delta t}{1 + r\Delta t}\right)\frac{y_n}{Y},$$

then

$$x_{n+1} = \lambda x_n(1 - x_n).$$

This is the discrete logistic equation or logistic map. It is of the general form $x_{n+1} = f(x_n)$.

If $\lambda < 1$, then deaths exceed births and the population decays to zero.

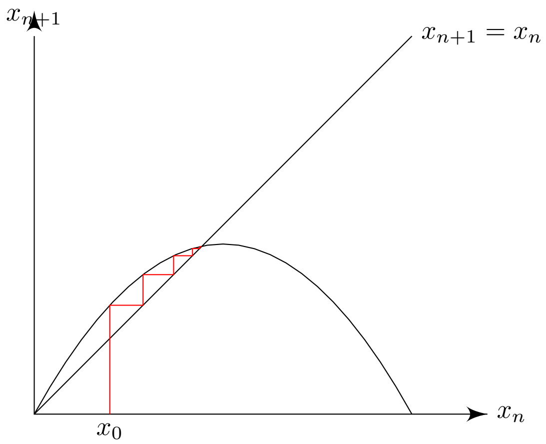


We see that $x = 0$ is a fixed point.

In general, to find fixed points, we solve for $x_{n+1} = x_n$, i.e. $f(x_n) = x_n$. For the logistics map, we have

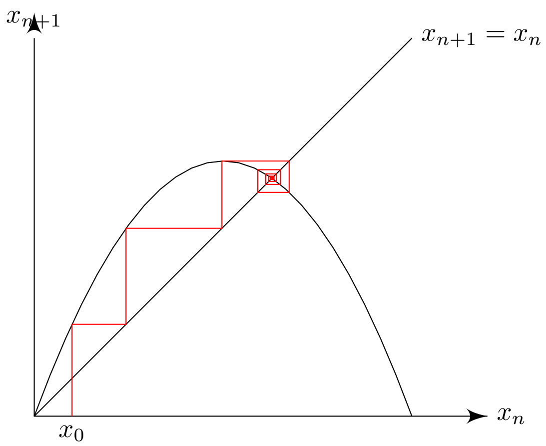
$$\begin{aligned}\lambda x_n(1 - x_n) &= x_n\\ x_n[1 - \lambda(1 - x_n)] &= 0\\ x_n &= 0 \text{ or } x_n = 1 - \frac{1}{\lambda}\end{aligned}$$

When $1 < \lambda < 2$, we have



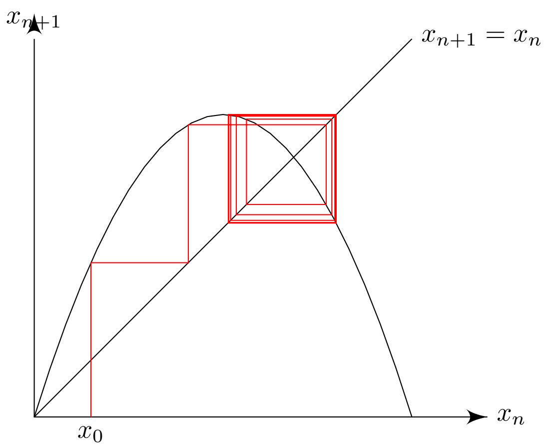
We see that $x_n = 0$ is an unstable fixed point and $x_n = 1 - \frac{1}{\lambda}$ is a stable fixed point.

When $2 < \lambda < 3$, we have

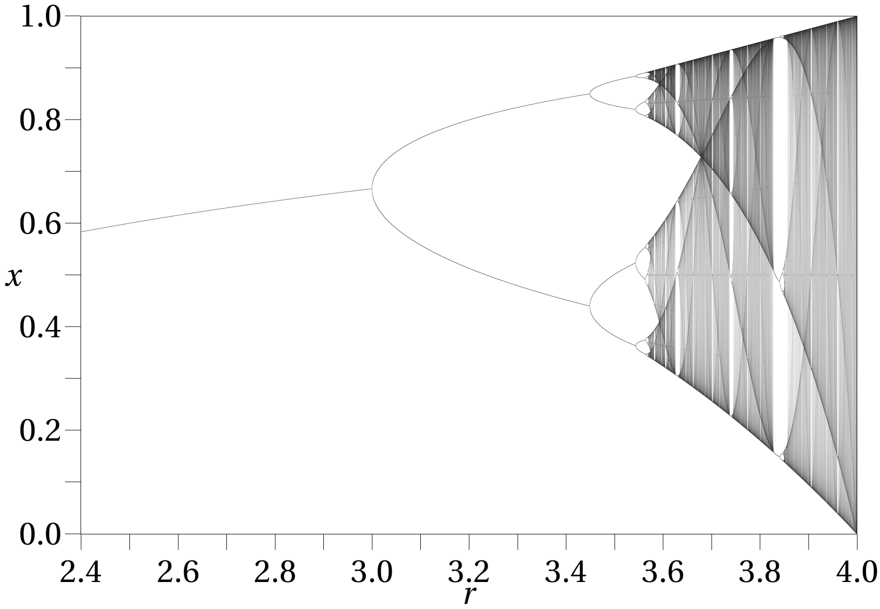


There is an oscillatory convergence to $x_n = 1 - \frac{1}{\lambda}$.

When $\lambda > 3$, we have a limit cycle, in which x_n oscillates between 2 values, i.e. $x_{n+2} = x_n$. When $\lambda = 1 + \sqrt{6} \approx 3.449$, we have a 4-cycle, and so on.



We can have the following plot of the stable solutions for different values of λ (plotted as r in the horizontal axis)



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Note: The fixed point still exists after $x > 3$, but is no longer stable. Similarly, the 2-cycles still exist after $\lambda = 1 + \sqrt{6}$, but it is not stable.

5 Second-order differential equations

Note that most methods in this section applies to higher order differential equations as well.

5.1 Constant coefficients

The general form of the equation is

$$ay'' + by' + cy = f(x).$$

We solve this in two steps:

(i) Find the complementary functions which satisfy the homogeneous equation $ay'' + by' + cy = 0$.

(ii) Find a particular solution that satisfies the full equation.

5.1.1 Complementary functions

Recall that $e^{\lambda x}$ is an eigenfunction of the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$. Hence it is also an eigenfunction of the second derivative $\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$.

If the complementary function has the form $y_c = e^{\lambda x}$, then $y_c' = \lambda e^{\lambda x}$ and $y_c'' = \lambda^2 e^{\lambda x}$. Substituting into the differential equation gives

Definition (Characteristic equation)**.** The *characteristic equation* of a (second-order) differential equation $ay'' + by' + c = 0$ is

$$a\lambda^2 + b\lambda + c = 0.$$

In this case there are two solutions to the characteristic equation, giving (in principle) two complementary functions $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$.

If λ_1 and λ_2 are distinct, then y_1 and y_2 are linearly independent and complete - they form a basis of the solution space. The (most) general complementary function is

$$y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

Example. $y'' - 5y' + 6y = 0$. Try $y = e^{\lambda x}$. The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$. Then $\lambda = 2$ or 3. So the general solution is $y = Ae^{2x} + Be^{3x}$. *Note*: A and B can be complex constants.

Example (Simple harmonic motion)**.** $y'' + 4y = 0$. Try $y = e^{\lambda x}$. The characteristic equation is $\lambda^2 + 4 = 0$, with solutions $\lambda = \pm 2i$. Then our general solution is $y = Ae^{2ix} + Be^{-2ix}$. However, if this is in a case of simple harmonic motion in physics, we want the function to be real (or *look* real). We can write

$$\begin{aligned} y &= A(\cos 2x + i\sin 2x) + B(\cos 2x - i\sin 2x) \\ &= (A + B)\cos 2x + i(A - B)\sin 2x \\ &= \alpha\cos 2x + \beta\sin 2x \end{aligned}$$

where $\alpha = A + B$ and $\beta = i(A - B)$, and α and β are independent constants. *Note*: In effect, we have changed the basis from $\{e^{2ix}, e^{-2ix}\}$ to $\{\cos 2x, \sin 2x\}$.

Example. (Degeneracy) $y'' - 4y' + 4y = 0$. (Seeing this as an oscillation, we have the y'' and $4y$ from the previous example, and $-4y'$ acts as a damping term)

Try $y = e^{\lambda x}$. We have $\lambda^2 - 4\lambda + 4 = 0$ and $(\lambda - 2)^2 = 0$. So $\lambda = 2$ or 2. But e^{2x} and e^{2x} are clearly not linearly independent. We have only managed to find one basis function of the solution space, but a second order equation has a 2 dimensional solution space. We need to find a second solution.

We can perform *detuning*. We can separate the two functions found above from each other by considering $y'' - 4y' + (4 - \epsilon^2)y = 0$. This turns into the equation we want to solve as $\epsilon \to 0$. Try $y = e^{\lambda x}$. We obtain $\lambda^2 - 4\lambda + 4 - \epsilon^2$. The two roots are $\lambda = 2 \pm \epsilon$. Then

$$\begin{aligned} y &= Ae^{(2+\epsilon)x} + Be^{(2-\epsilon)X} \\ &= e^{2x}[Ae^{\epsilon x} + Be^{-\epsilon x}] \end{aligned}$$

Taking the limit as $\epsilon \to 0$, we use the Taylor expansion of $e^{\epsilon x}$ to obtain

$$y = e^{2x}[(A + B) + \epsilon x(A - B) + O(A\epsilon^2, B\epsilon^2)]$$

Choose $(A + B) = \alpha$ and $\epsilon(A - B) = \beta$. This is perfectly valid for any non-zero ϵ. Then $A = \frac{1}{2}(\alpha + \frac{\beta}{\epsilon})$ and $B = \frac{1}{2}(\alpha - \frac{\beta}{\epsilon})$. So we have

$$y = e^{2x}[\alpha + \beta x + O(A\epsilon^2, B\epsilon^2)]$$

Seeing α and β as fixed constants independent of ϵ, A and B diverge as $\epsilon \to 0$. Since $A = O(\frac{1}{\epsilon})$, $O(A\epsilon^2) = O(\epsilon)$, and similarly for B. So

$$\begin{aligned} y &= e^{2x}[\alpha + \beta x + O(\epsilon)] \\ &= e^{2x}(\alpha + \beta x) \end{aligned}$$

In this way, we have derived two separate basis functions. In general, if $y_1(x)$ is a degenerate complementary function of a linear differential equation with constant coefficients, then $y_2(x) = xy_1(x)$ is an independent complementary function.

5.1.2 Second complementary function

In general (not necessarily constant coefficients), we can find a second complementary function associated with a degenerate solution of the homogeneous equation by looking for a solution in the form $y_2(x) = v(x)y_1(x)$, where $y_1(x)$ is the degenerate solution we found.

Example. Consider $y'' - 4y + 4y = 0$. We have $y_1 = e^{2x}$. We try $y_2 = ve^{2x}$

$$\begin{aligned} y_2' &= (v' + 2v)e^{2x} \\ y_2'' &= (v'' + 4v'' + 4v)e^{2x}. \end{aligned}$$

Substituting into the original equation gives

$$\begin{aligned} (v'' + 4v' + 4v) - 4(v' + 2v) + 4v &= 0 \\ v'' &= 0 \\ v' &= \beta \\ v &= \alpha + \beta x. \end{aligned}$$

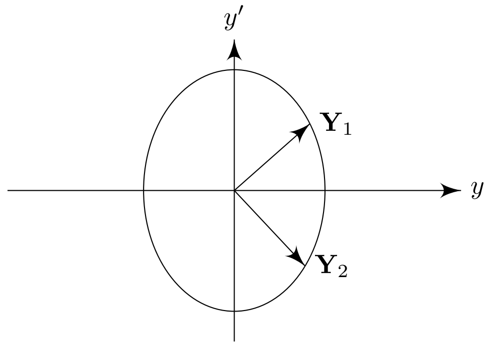
So $y_2 = (Ax + B)e^{2x}$.

5.1.3 Phase space

A differential equation of nth order, e.g. $a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$, determines the nth derivative $y^{(n)}(x_0)$ (and hence all higher derivatives) in terms of $y(x_0), y'(x_0), \cdots, y^{(n-1)}(x_0)$ at any point x_0. Therefore given the first $n-1$ derivatives of y at x_0, we know the Taylor series about x_0, which we can use to extend y to the neighbourhood of x_0.

So we can define a solution vector $\mathbf{Y}(x) = (y(x), y'(x), f \cdots, y^{n-1}(x))$ for each value of x. As x varies, $\mathbf{Y}(x)$ traces out a trajectory in the *phase space*.

Example. Consider $y'' + 4y = 0$. The solutions are $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Thus $y_1' = -2\sin 2x$ and $y_2' = 2\cos 2x$. The solution vectors of the complementary functions are $\mathbf{Y}_1 = (\cos 2x, -2\sin 2x)$ and $\mathbf{Y}_2 = (\sin 2x, 2\cos 2x)$. We can plot them as follows:



with $\mathbf{Y}_1$ and $\mathbf{Y}_2$ tracing out the same curve but starting at different points.

Note that each point $\mathbf{Y}$ in the phase space determines a second derivative of y, which then gives the derivative of $\mathbf{Y}$. This allows us to trace out a curve in the phase space, which then determines the whole solution.

Note that this phase space is a 2-dimensional space, and we can take the two complementary functions Y_1 and Y_2 as basis vectors for the phase space at each particular value of x.

Definition (Wronskian)**.** Given a differential equations with solutions y_1, y_2, the *Wronskian* is the determinant

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Definition (Independent solutions)**.** Two solutions $y_1(x)$ and $y_2(x)$ are *independent* solutions of the differential equation if and only if $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent as vectors in the phase space, i.e. iff the Wronskian is non-zero.

In our example, we have $W(x) = 2\cos^2 2x + 2\sin^2 2x = 2 \neq 0$ for all x.

Example. In our earlier example, $y_1 = e^{2x}$ and $y_2 = xe^{2x}$. We have

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x}(1 + 2x - 2x) = e^{4x} \neq 0.$$

Theorem (Abel's Theorem)**.** Given an equation $y'' + p(x)y' + q(x)y = 0$, either $W = 0$ for all x, or $W \neq 0$ for all x. i.e. iff two solutions are independent for some particular x, then they are independent for all x.

Proof. If y_1 and y_2 are both solutions, then

$$\begin{aligned} y_2(y_1'' + py_1' + qy_1) &= 0 \\ y_1(y_2'' + py_2' + qy_2) &= 0 \end{aligned}$$

Subtracting the two equations, we have

$$y_1y_2'' - y_2y_1'' + p(y_1y_2' - y_2y_1') = 0$$

Note that $W = y_1y_2' - y_2y_1'$ and $W' = y_1y_2'' + y_1'y_2' - (y_2'y_1' + y_2y_1'') = y_1y_2'' - y_2y_1''$

$$\begin{aligned} W' + P(x)W &= 0 \\ W(x) &= W_0e^{-\int P\,\mathrm{d}x}, \end{aligned}$$

Where $W_0 = \text{const}$. Since the exponential function is never zero, either $W_0 = 0$, in which case $W = 0$, or $W_0 \neq 0$ and $W \neq 0$ for any value of x. □

Note: Any linear nth-order homogeneous differential equation can be written in the form $\mathbf{Y}' + A\mathbf{Y} = 0$, a system of first-order equations. It can then be shown that $W' + \operatorname{tr}(A)W = 0$, and $W = W_0e^{-\int \operatorname{tr} A\,\mathrm{d}x}$. So Abel's theorem holds.

5.2 Particular integrals

We now consider integrals of the form $ay'' + by' + cy = f(x)$.

5.2.1 Guessing

If the forcing terms are simple, we can easily "guess" the form of the particular integral.

$f(x)$	$y_p(x)$
e^{mx}	Ae^{mx}
$\sin kx$ $\cos kx$	$A\sin kx + B\cos kx$
polynomial $p_n(x)$	$q_n(x) = a_nx^n + \cdots + a_1x + a_0$

It is important to remember that the equation is linear, so we can superpose solutions and consider each forcing term separately.

Example. Consider $y'' - 4y' + 6y = 2x + e^{4x}$. To obtain the forcing term $2x$, we need a first order polynomial $ax + b$, and to get e^{4x} we need ce^{4x}. Thus we can guess

$$\begin{aligned} y_p &= ax + b + ce^{4x} \\ y_p' &= a + 4ce^{4x} \\ y_p'' &= 16ce^{4x} \end{aligned}$$

Substituting in, we get

$$16ce^{4x} - 5(a + 4ce^{4x}) + 6(ax + b + ce^{4x}) = 2x + e^{4x}$$

Comparing coefficients of similarly functions, we have

$$\begin{aligned} 16c - 20c + 6c = 1 \Rightarrow c &= \frac{1}{2} \\ 6a = 2 \Rightarrow a &= \frac{1}{3} \\ -5a + 6b = 0 \Rightarrow b &= \frac{5}{18} \end{aligned}$$

Since the complementary function is $y_c = Ae^{3x} + Be^{2x}$, the general solution is $y = Ae^{3x} + Be^{2x} + \frac{1}{2}e^{4x} + \frac{1}{3}x + \frac{5}{18}$.
Note: Any boundary condition to determine A and B must be applied to the full solution, not the complementary function

5.2.2 Resonance

Consider $\ddot{y} + \omega_0^2 y = \sin \omega_0 t$. The complementary solution is $y_c = A \sin \omega_0 t + B \cos w_0 t$. We notice that the forcing is linearly dependent on the complementary functions. So if we guess a particular integral $y_p = C \sin \omega_0 t + D \cos \omega_0 t$, we'll simply find $\ddot{y}_p + \omega_0^2 y_p = 0$, so we can't balance the forcing.
Note: This is an example of a simple harmonic oscillator being forced at its natural frequency.

We can *detune* our forcing away from the natural frequency, and consider $\ddot{y} + \omega_0 y = \sin \omega t$ with $\omega \neq \omega_0$. Try

$$y_p = C(\sin \omega t - \sin \omega_0 t).$$

We have

$$\ddot{y}_p = C(-\omega^2 \sin \omega t + \omega_0^2 \sin \omega_0 t).$$

Substituting into the differential equation, we have $C(\omega_0^2 - \omega^2) = 1$. Then

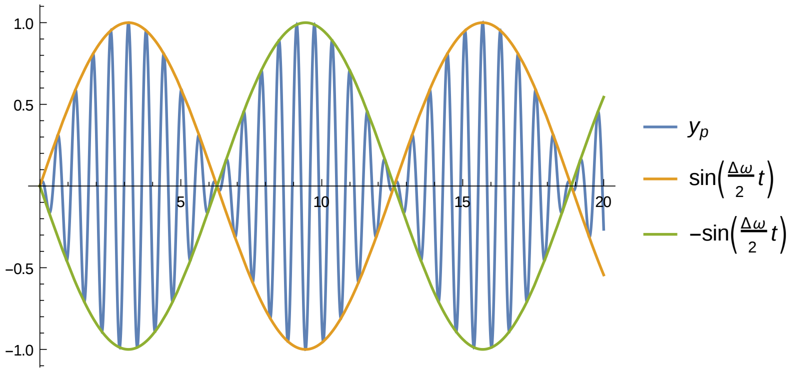
$$y_p = \frac{\sin \omega t - \sin \omega_0 t}{\omega_0^2 - \omega^2}.$$

We can simplify this to

$$y_p = \frac{2}{\omega_0^2 - \omega^2} \cos\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega - \omega_0}{2} t\right)$$

We let $\omega_0 - \omega = \Delta\omega$. Then

$$y_p = \frac{-2}{(2\omega + \Delta\omega)\Delta\omega} \cos\left[\left(\omega + \frac{\Delta\omega}{2}\right) t\right] \sin\left(\frac{\Delta\omega}{2} t\right).$$



The wavelength of the sin function has order $O(\frac{1}{\Delta\omega})$ and cos has wavelength $O(\frac{1}{\omega_0})$. If the forcing frequency is close to the natural frequency, then we get beating. As $\Delta\omega \to 0$, the wavelength of the beating envelope $\to \infty$ and we just have the initial linear growth.

Mathematically, since $\sin\theta \approx \theta$ as $\theta \to 0$, as $\Delta\omega \to 0$, we have

$$y_p \to \frac{-t}{2\omega_0}\cos\omega_0 t.$$

In general, if the forcing is a linear combination of complementary functions, then the particular integral is proportional to t (the independent variable) times the non-resonant guess.

5.2.3 Variation of parameters

Let $y_1(x)$ and $y_2(x)$ be linearly independent complementary functions of the ODE $y'' + p(x)y' + q(x)y = f(x)$. The solution vectors $\mathbf{Y}_1 = (y_1, y_1')$ and $\mathbf{Y}_2 = (y_2, y_2')$ form a basis of the solution space. So write

$$\mathbf{Y}_p(x) = u(x)\mathbf{Y}_1(x) + v(x)\mathbf{Y}_2(x)$$

Component-wise, we have

$$y_p = uy_1 + vy_2 \tag{a}$$
$$y_p' = uy_1' + vy_2' \tag{b}$$

From the second equation, we have

$$y_p'' = (uy_1'' + u'y_1') + (vy_2'' + v'y_2') \tag{c}$$

If we use consider (c) + p(b) + q(a), we have $y_1'u' + y_2'v' = f$.

Now note that we derived the equation of y_p' from the vector equation. This must be equal to what we get if we differentiate (a). By (a)$'$ - (b), we obtain $y_1u' + y_2v' = 0$. Now we have two simultaneous equations for u' and v'.

We can, for example, write them in matrix form as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Inverting the left matrix, we have

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ f \end{pmatrix}$$

So $u' = -\frac{y_2}{W}f$ and $v' = \frac{y_1}{W}f$.

Example. $y'' + 4y = \sin 2x$. We know that $y_1 = \sin 2x$ and $y_2 = \cos 2x$. $W = -2$. We write

$$\begin{aligned} y_p &= u\sin 2x + v\cos 2x \\ y_p' &= 2u\cos 2x - 2v\sin 2x \\ y_p'' &= 2u'\cos 2x - 4u\sin 2x - 2v'\sin 2x - 4v\cos 2x \end{aligned}$$

We obtain

$$u' = \frac{\cos 2x\sin 2x}{2} = \frac{\sin 4x}{4},\quad v' = \frac{-\sin^2 2x}{2} = \frac{\cos 4x - 1}{4}$$

So

$$u = -\frac{\cos 4x}{16},\quad v = \frac{\sin 4x}{16} - \frac{x}{4}$$

Therefore

$$y_p = \frac{1}{16}(-\cos 4x\sin 2x + \sin 4x\cos 2x - \frac{x}{4}\cos 2x) = \frac{1}{16}\sin 2x - \frac{x}{4}\cos 2x$$

Note that $-\frac{1}{4}x\cos 2x$ is what we found previously by detuning, and $\frac{1}{16}\sin 2x$ is a complementary function, so the results agree.

5.3 Linear equidimensional equations

Note: These are often called homogeneous equations, but this is confusing as it has the same name as those with no forcing term.

Definition (Equidimensional equation)**.** An equation is *equidimensional* if it has the form

$$ax^2y'' + bxy' + cy = f(x),$$

where a, b, c are constants. It is "equidimensional" since if a, b, c are dimensionless numbers and x, y have dimensions of, say pressure and length, then the three terms on the left have the same dimension.

5.3.1 Solving by eigenfunctions

Note that $y = x^k$ is an eigenfunction of $x\frac{\mathrm{d}}{\mathrm{d}x}$. We can try an eigenfunction $y = x^k$. We have $y' = kx^{k-1}$ and thus $xy' = kx^k = ky$; and $y'' = k(k-1)x^{k-2}$ and $x^2y'' = k(k-1)x^k$.

Substituting in, we have

$$ak(k-1) + bk + c = 0,$$

which we can solve, in general, to give two roots k_1 and k_2, and $y_c = Ax^{k_1} + Bx^{k_2}$.

5.3.2 Solving by substitution

Alternatively, we can make a substitution $z = \ln x$. Then we can show that

$$a^2\frac{\mathrm{d}^2y}{dz^2} + (b-a)\frac{\mathrm{d}y}{\mathrm{d}z} + cy = f(e^z).$$

This turns an equidimensional equation into an equation with constant coefficients. The characteristic equation for solutions in the form $y = e^{\lambda z}$ is of form $a^2\lambda^2 + (b-a)\lambda + c = 0$, which we can rearrange to become $a\lambda(\lambda-1) + b\lambda + c = 0$. So $\lambda = k_1, k_2$.

Then the complementary function is $y_c = Ae^{k_1 z} + be^{k_2 z} = x^{k_1} + x^{k_2}$.

5.3.3 Degenerate solutions

If the roots of the characteristic equation are equal, then $y_c = \{e^{kz}, ze^{kz}\} = \{x^k, x^k \ln x\}$. Similarly, if there is a resonant forcing proportional to x^{k_1} (or x^{k_2}), then there is a particular integral of the form $x^{k_1}\ln x$.

5.4 Difference equations

Consider an equation of the form

$$ay_{n+2} + by_{n+1} + cy_n = f_n.$$

We can solve in a similar way to differential equations, by exploiting linearity and eigenfunctions.

We can think of the difference operator $D[y_n] = y_{n+1}$. This has an eigenfunction $y_n = k^n$. We have $D[y_n] = D[k^n] = k^{n+1} = k \cdot k^n = ky_n$.

5.4.1 Complementary functions

To solve the difference equation, first look for complementary functions satisfying

$$ay_{n+2} + by_{n+1} + cy_n = 0$$

We try $y_n = k^n$ to obtain

$$\begin{aligned} ak^{n+2} + bk^{n+1} + ck^n &= 0 \\ ak^2 + bk + c &= 0 \end{aligned}$$

from which we can determine k. So the general complementary function is $y_n^c = Ak_1^n + Bk_2^n$ if $k_1 \neq k_2$. If they are equal, then $y_n^c = (A + Bn)k^n$.

5.4.2 Particular integrals

Guesswork:

f_n	y_n^p
k^n	Ak^n if $k \neq k_1, k_2$
k_1^n	nk_1^n
n^p	$An^p + Bn^{p-1} + \cdots + Cn + D$

Example. (Fibonacci sequence) The Fibonacci sequence defined by

$$y_n = y_{n-1} + y_{n-2}$$

with $y_0 = y_1 = 1$.

We can write this as

$$y_{n+2} - y_{n+1} - y_n = 0$$

We try $y_n = k^n$. Then $k^2 - k - 1 = 0$. Then

$$\begin{aligned} k^2 - k - 1 &= 0 \\ k &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

We write $k = \varphi_1, \varphi_2$. Then $y_n = A\varphi_1^n + B\varphi_2^n$. Our initial conditions give

$$\begin{aligned} A + B &= 1 \\ A\varphi_1 + B\varphi_2 &= 1 \end{aligned}$$

We get $A = \dfrac{\varphi_1}{\sqrt{5}}$ and $B = \dfrac{-\varphi_2}{\sqrt{5}}$. So

$$y_n = \frac{\varphi_1^{n+1} - \varphi_2^{n+1}}{\sqrt{5}} = \frac{\varphi_1^{n+1} - \left(\frac{-1}{\varphi_1}\right)^{n+1}}{\sqrt{5}}$$

5.5 Transients and damping

In many physical systems, there is some sort of restoring force and some damping, e.g. car suspension system.

Consider a car of mass M with a vertical force $F(t)$ acting on it (e.g. mouse jumping on the car). We can consider the wheels to be springs ($F = kx$) with a "shock absorber" ($F = l\dot{x}$).

$$M\ddot{x} = F(t) - kx - l\dot{x}.$$

So we have

$$\ddot{x} + \frac{l}{M}\dot{x} + \frac{k}{M}x = F(t).$$

Note that if we don't have the damping and the forcing, we end up with a simple harmonic motion with angular frequency $\sqrt{k/M}$. Write $t = \tau\sqrt{M/k}$, where τ is dimensionless. The timescale $\sqrt{M/k}$ is proportional to the period of the undamped, unforced system (or 1 over its natural frequency). Then we obtain

$$\ddot{x} + 2\kappa\dot{x} + x = f(\tau)$$

where, $\dot{x}$ means $\frac{\mathrm{d}x}{d\tau}$, $\kappa = \frac{l}{2\sqrt{kM}}$ and $f = \frac{F}{k}$.

By this substitution, we are now left with only one parameter κ instead of the original three (M, l, k).

5.5.1 Free (natural) response $f = 0$

$$\ddot{x} + 2\kappa\dot{x} + x = 0$$

We try $x = e^{\lambda\tau}$

$$\lambda + 2\kappa\lambda + 1 = 0$$

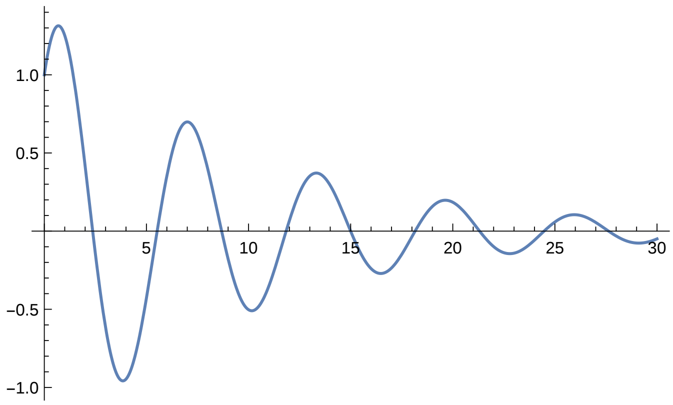
$$\begin{aligned}\lambda &= -\kappa \pm \sqrt{\kappa^2 - 1}\\ &= -\lambda_1, -\lambda_2\end{aligned}$$

where λ_1 and λ_2 have positive real parts.

5.5.2 Underdamping

If $\kappa < 1$, we have $x = e^{-\kappa\tau}(A \sin\sqrt{1-\kappa^2}\tau + B\cos\sqrt{1-\kappa^2}\tau)$.

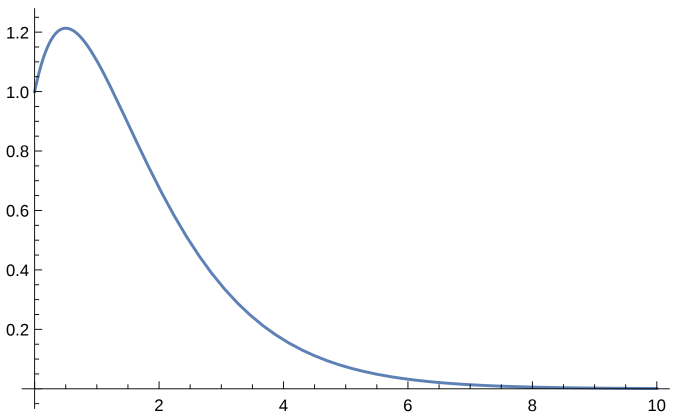
The period is $\frac{2\pi}{\sqrt{1-\kappa^2}}$ and its amplitude decays in a characteristic of $O(\frac{1}{\kappa})$. Note that the damping increases the period. As $\kappa \to 1$, then the oscillation period $\to \infty$.



5.5.3 Critically damping

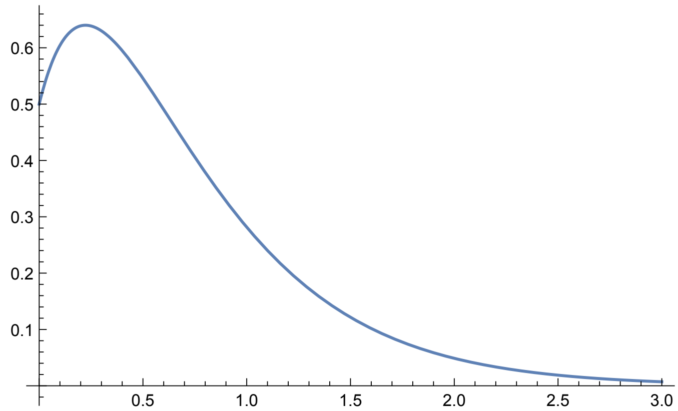
If $\kappa = 1$, then $x = (A + B\tau)e^{-\kappa\tau}$.

The rise time and decay time are both $O(\frac{1}{\kappa}) = O(1)$. So the dimensional rise and decay times are $O(\sqrt{M/k})$.



5.5.4 Overdamping

If $\kappa > 1$, then $x = Ae^{-\lambda_1 \tau} + Be^{-\lambda_2 \tau}$ with $\lambda_1 < \lambda_2$. Then the decay time is $O(1/\lambda_1)$ and the rise time is $O(1/\lambda_2)$.



Note that it is possible to get a large initial increase in amplitude.

5.5.5 Forcing

In a forced system, the complementary functions typically determine the short-time transient response, while the particular integral determines the long-time (asymptotic) response. For example, if $f(\tau) = \sin \tau$, $x_p = C \sin \tau + D \cos \tau$. In this case, $x_p = -\frac{1}{2\kappa} \cos \tau$.

The general solution is $x = Ae^{-\lambda_1 \tau} + Be^{-\lambda \tau} - \frac{1}{2\kappa} \cos \tau \sim -\frac{1}{2\kappa} \cos \tau$ as $\lambda \to \infty$ since $\mathrm{Re}(\lambda_{1,2}) > 0$.

Note: The forcing response is out of phase with the forcing.

5.6 Impulses and point forces

5.6.1 Dirac delta function

Consider a ball bouncing on the ground. When the ball hits the ground at some time T, it experiences a force from the ground for some short period of time. The force on the ball exerted by the ground in $F(t)$ is 0 for most of the time, except during the short period $(T - \epsilon, T + \epsilon)$.

Often we don't know (or we don't wish to know) the details of $F(t)$ but we can note that it only acts for a short time of $O(\epsilon)$ that is much shorter than the overall time $O(t_2 - t_1)$ of the system. It is convenient mathematically to imagine the force acting instantaneously at time $t = T$, i.e. consider the limit $\epsilon \to 0$.

Newton's second law gives $m\ddot{x} = F(t) - mg$. While we cannot solve it, we can integrate the equation from $T - \epsilon$ to $T + \epsilon$. So

$$\int_{T-\epsilon}^{T+\epsilon} m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \, \mathrm{d}t = \int_{T-\epsilon}^{T+\epsilon} F(t) \, \mathrm{d}t - \int_{T-\epsilon}^{T+\epsilon} \mathrm{d}t$$

$$\left[m \frac{\mathrm{d}x}{\mathrm{d}t} \right]_{T-\epsilon}^{T+\epsilon} = I - 2\epsilon m g$$

$$\Delta p = I - O(\epsilon)$$

Where Δp is the change in momentum and the impulse $I = \int_{T-\epsilon}^{T+\epsilon} F(t)\ \mathrm{d}t$ is the area under the force curve. Note that the impulse I is the only property of F that influences the macroscopic behaviour of the system. If the contact time 2ϵ is small, we'll neglect it and write

$$\Delta p = I$$

The only feature of $F(t; \epsilon)$ (we introduce the variable ϵ so that we can we can take the limit as $\epsilon \to 0$) is its integral, i.e. area under the force curve.

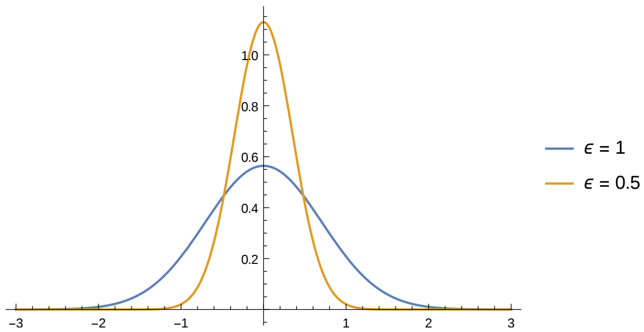
WLOG, assume $T = 0$ for easier mathematical treatment. We can consider a family of functions $D(t; \epsilon)$ such that

$$\lim_{\epsilon \to 0} D(t; \epsilon) = 0 \text{ for all } t \neq 0$$
$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} D(t; \epsilon)\ \mathrm{d}t = 1$$

So we can replace the force in our example in being replaced by $ID(t; \epsilon)$.

For example, we can choose

$$D(t; \epsilon) = \frac{1}{\epsilon\sqrt{\pi}} e^{-t^2/\epsilon^2}$$



This has height $O(1/\epsilon)$ and width $O(\epsilon)$.

It can be checked that this satisfied the properties listed above. Note that as $\epsilon \to 0$, $D(0; \epsilon) \to \infty$. Therefore $\lim_{\epsilon \to 0} D(t; \epsilon)$ does not exist.

Definition (Dirac delta function)**.** The *Dirac delta function* is defined by

$$\delta(x) = \lim_{\epsilon \to 0} D(x; \epsilon)$$

on the understanding that we can only use its integral properties. For example, when we write

$$\int_{-\infty}^{\infty} g(x)\delta(x)\ \mathrm{d}x,$$

we actually mean

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} g(x)D(x; \epsilon)\ \mathrm{d}x.$$

In fact, this is equal to $g(0)$.

More generally, $\int_a^b g(x)\delta(x - c)\ \mathrm{d}x = g(c)$ if $c \in (a, b)$ and 0 otherwise, provided g is continuous at $x = c$.

This gives a convenient way of representing and making calculations involving impulsive or point forces. For example, in the previous example, we can write

$$m\ddot{x} = -mg + I\delta(t - T).$$

Example. $y'' - y = 3\delta(x - \frac{\pi}{2})$ with $y = 0$ at $x = 0, \pi$. Note that or function y is split into two parts by $x = \frac{\pi}{2}$.

First consider the region $0 \leq x < \frac{\pi}{2}$. Here the delta function is 0, and we have $y'' - y = 0$ and $y = 0$ and $x = 0$. Then $y = Ce^x + De^{-x} = A\sinh x + B\cosh x$ and obtain $B = 0$ from the boundary condition. In the region $\frac{\pi}{2} < x \leq \pi$, we again obtain $y = C\sinh(\pi - x) + D\cosh(\pi - x)$ and (from the boundary condition), $D = 0$.

When $x = \frac{\pi}{2}$, first insist that y is continuous at $x = \frac{\pi}{2}$. So $A = C$. Then note that we have to solve

$$y'' - y = 3\delta\left(x - \frac{\pi}{2}\right)$$

But remember that the delta function makes sense only in an integral. So we integrate both sides from $\frac{\pi}{2}^-$ to$\frac{\pi}{2}^+$. Then we obtain

$$[y']_{\frac{\pi}{2}^-}^{\frac{\pi}{2}^+} - \int_{\frac{\pi}{2}^-}^{\frac{\pi}{2}^+} y \, \mathrm{d}x = 3$$

Since we assume that y is well behaved, the second integral is 0. So we are left with

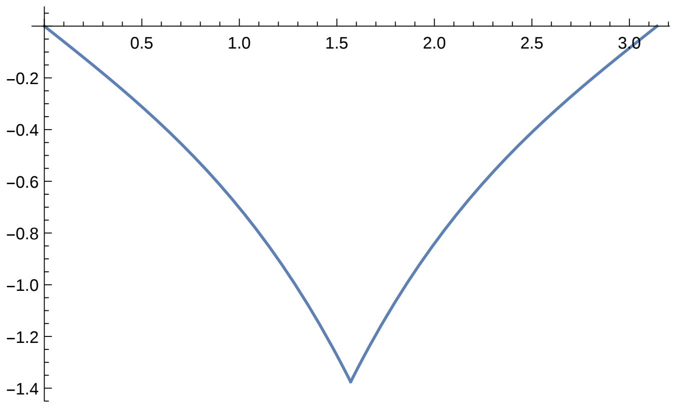
$$[y']_{\frac{\pi}{2}^-}^{\frac{\pi}{2}^+} = 3$$

So we have

$$\begin{aligned} -C\cosh\frac{\pi}{2} - A\cosh\frac{\pi}{2} &= 3 \\ A = C &= \frac{-3}{2\cosh\frac{\pi}{2}} \end{aligned}$$

Then we have

$$y = \begin{cases} \frac{-3\sinh x}{2\cosh\frac{\pi}{2}} & 0 \leq x < \frac{\pi}{2} \\ \frac{-3\sinh(\pi - x)}{2\cosh\frac{\pi}{2}} & \frac{\pi}{2} < x \leq \pi \end{cases}$$



Note that our final function has continuous y, discontinuous y' at $\frac{\pi}{2}$ and y'' infinite at $\frac{\pi}{2}$. If y were not continuous, then y' would look like a delta function and y'' would be a derivative of a delta function.

Note that the discontinuity is addressed by the highest order derivative since differentiation increases the discontinuity while integration smoothens it.

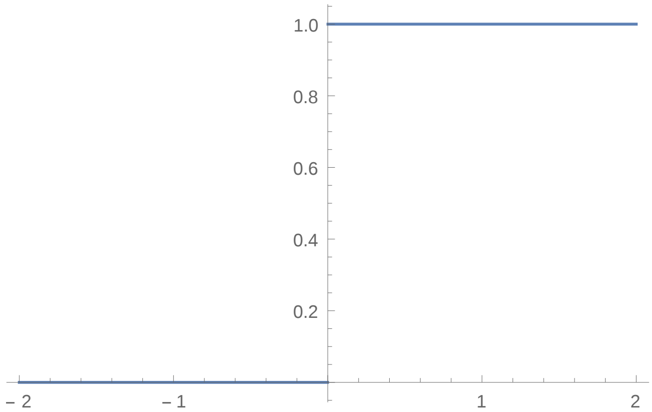
5.7 Heaviside step function

Definition (Heaviside step function)**.** Define the Heaviside step function as:

$$H(x) = \int_{-\infty}^{x} \delta(t)\ \mathrm{d}t$$

We have

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \\ \text{undefined} & x = 0 \end{cases}$$



By the fundamental theorem of calculus,

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \delta(x)$$

But remember that these functions and relationships can only be used inside integrals.

6 Series solutions

We consider equations of the form

$$p(x)y'' + q(x)y' + r(x)y = 0$$

Definition (Ordinary and singular points)**.** The point $x = x_0$ is an *ordinary point* of the differential equation if $\frac{q}{p}$ and $\frac{r}{p}$ have Taylor series about x_0 (i.e. "analytic", c.f. Complex Analysis). Otherwise, x_0 is a *singular point.*

If x_0 is a singular point but the equation can be written as

$$P(x)(x - x_0)^2 y'' + Q(x)(x - x_0)y' + R(x)y = 0,$$

and $\frac{Q}{P}$ and $\frac{R}{P}$ have Taylor series about x_0, then x_0 is a *regular singular point.*

Example. Consider the following:

(i) $(1 - x^2)y'' - 2cy' + 2y = 0$. $x = 0$ is an ordinary point. However, $x = \pm 1$ are (regular) singular points since $p(\pm 1) = 0$.

(ii) $\sin xy'' + \cos xy' + 2y = 0$. $x = n\pi$ are regular singular points while all others are ordinary.

(iii) $(1 + \sqrt{x})y'' - 2xy' + 2y = 0$. $x = 0$ is an irregular singular point because $\sqrt{x}$ is not differentiable at $x = 0$.

If x_0 is an ordinary point, then the equation is guaranteed to have two linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

i.e. Taylor series about x_0. The solution must be convergent in some neighbourhood of x_0.

If x_0 is a regular singular point, then there is at least one solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma}$$

with $a_0 \neq 0$ (to ensure σ is unique). The *index* σ can be any complex number. This is called a Frobenius series.

Alternatively, it can be nice to think of the Frobenius series as

$$\begin{aligned} y &= (x - x_0)^\sigma \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= (x - x_0)^\sigma f(x) \end{aligned}$$

where $f(x)$ is analytic and has a Taylor series.

6.1 Ordinary points

Example. Consider $(1-x^2)y'' - 2xy' + 2y = 0$. Find a series solution about $x = 0$ (which is an ordinary point).

We try $y = \sum_{n=0}^{\infty} a_n x^n$. First write the equation in the form of an equidimensional equation with polynomial coefficients: multiplying both sides by x^2, we obtain

$$\begin{aligned}(1-x)^2(x^2y'') - 2x^2(xy') + 2x^2y &= 0\\ \sum a_n[(1-x^2)n(n-1) - 2x^2n + 2x^2]x^n &= 0\\ \sum a_n[n(n-1) + (-n^2 - n + 2)x^2]x^n &= 0\end{aligned}$$

We look at the coefficient of x^n and obtain the following general recurrence relation:

$$\begin{aligned}n(n-1)a_n + [-(n-2)^2 - (n-2) + 2]a_{n-2} &= 0\\ n(n-1)a_n &= (n^2 - 3n)a_{n-2}\end{aligned}$$

Note: NEVER divide until you know it's safe!

Consider the case $n = 0$. The left hand side gives $0 \cdot a_0 = 0$ (the right hand side is 0 since $a_{n-2} = 0$). So any value of a_0 satisfies the recurrence relationship (it's a constant of integration). Similarly, a_1 is arbitrary.

For $n > 1$, we have

$$a_n = \frac{n-3}{n-1}a_{n-2}$$

In this case (but generally not), we can further simplify it to obtain:

$$\begin{aligned}a_n &= \frac{n-3}{n-1}\frac{n-5}{n-3}a_{n-4}\\ &= \frac{n-5}{n-1}a_{n-4}\\ &\ \ \vdots\end{aligned}$$

So

$$\begin{aligned}a_{2k} &= \frac{-1}{2k-1}a_0\\ a_{2k+1} &= 0\end{aligned}$$

So

$$\begin{aligned}y &= a_0[1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} - \cdots] + a_1x\\ &= a_0\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + a_1x\end{aligned}$$

Notice the logarithmic behaviour of $x = \pm 1$ which are regular singular points.

6.2 Regular singular points

Example. Consider $4xy'' + 2(1-x^2)y' - xy = 0$. Note that $x = 0$ is a singular point. However, if we multiply throughout by x to obtain an equidimensional equation, we obtain

$$4(x^2y'') + 2(1-x^2)xy' - x^2y = 0.$$

So $\frac{Q}{P} = \frac{1-x^2}{2}$ and $\frac{R}{P} = \frac{x^2}{4}$ both have Taylor series and $x = 0$ is a regular singular point. Try

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} \text{ with } a_0 \neq 0.$$

Substituting in, we have

$$\sum a_n x^{n+\sigma}[4(n+\sigma)(n+\sigma-1) + 2(1-x^2)(n+\sigma) - x^2]$$

By considering the coefficient of $x^{n+\sigma}$, we obtain the general recurrence relation

$$[4(n+\sigma)(n+\sigma-1) + 2(n+\sigma)]a_n - [2(n-2+\sigma)+1]a_{n-2} = 0.$$

Simplifying the equation gives

$$2(n+\sigma)(2n+2\sigma-1)a_n = (2n+2\sigma-3)a_{n-2}.$$

The $n = 0$ case gives the *indicial equation* for the *index* σ:

$$2\sigma(2\sigma-1)a_0 = 0.$$

Since $a_0 \neq 0$, $\sigma = 0$ or $\frac{1}{2}$. The $\sigma = 0$ solution corresponds to an analytic ("Taylor series") solution, while $\sigma = \frac{1}{2}$ corresponds to a non-analytic one.

When $\sigma = 0$, the recurrence relation becomes

$$2n(2n-1)a_n = (2n-3)a_{n-2}.$$

When $n = 0$, this gives $0 \cdot a_0 = 0$. So a_0 is arbitrary. For $n > 0$, we can divide and obtain

$$a_n = \frac{2n-3}{2n(2n-1)}a_{n-2}.$$

We can see that $a_1 = 0$ and so are subsequent odd terms.

If $n = 2k$, i.e. is even, then

$$a_{2k} = \frac{4k-3}{4k(4k-1)}a_{2k-2}$$

$$y = a_0\left[1 + \frac{1}{4\cdot 3}x^2 + \frac{5}{8\cdot 7\cdot 4\cdot 3}x^4 + \cdots\right]$$

Note that we have only found one solution in this case.

Now when $\sigma = \frac{1}{2}$, we obtain

$$(2n+1)(2n)a_n = (2n-2)a_{n-2}$$

When $n = 0$, we obtain $0 \cdot a_0 = 0$, so a_0 is arbitrary. To avoid confusion with the a_0 above, call it b_0 instead.

When $n = 1$, we obtain $6a_1 = 0$ and $a_1 = 0$ and so are subsequent odd terms. For even n,

$$a_n = \frac{n-1}{n(2n+1)} a_{n-2}$$

So

$$y = b_0 x^{1/2} \left[1 + \frac{1}{2\cdot 5}x^2 + \frac{3}{2\cdot 5\cdot 4\cdot 9}x^4 + \cdots\right]$$

6.2.1 Resonance of solutions

Note that the indicial equation has two roots σ_1, σ_2. Consider the two different cases:

(i) If $\sigma_2 - \sigma_1$ is not an integer, then there are two linearly independent Frobenius solutions

$$y = \left[(x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n\right] + \left[(x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n\right].$$

As $x \to x_0$, $y \sim (x - x_0)^{\sigma_1}$, where $\mathrm{Re}(\sigma_1) \leq \mathrm{Re}(\sigma_2)$

(ii) If $\sigma_2 - \sigma_1$ is an integer (including when they are equal), there is one solution of the form

$$y_1 = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with $\sigma_2 \geq \sigma_1$.

The other solution is (usually) in the form

$$y_2 = \ln(x - x_0) y_1 + \sum_{n=0}^{\infty} b_n (x - x_0)^n.$$

(this form arises from resonance between the two solutions, but if the resonance somehow avoids itself, we can end up with regular Frobenius series solutions).

We can substitute this form of solution into the differential equation to determine b_n.

Example. Consider $x^2 y'' - xy = 0$. $x = 0$ is a regular singular point. It is already in equidimensional form $(x^2 y'') - x(y) = 0$. Try

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with $a_0 \neq 0$. We obtain

$$\sum a_n x^{n+\sigma}[(n+\sigma)(n+\sigma-1) - x] = 0.$$

The general recurrence relation is

$$(n+\sigma)(n+\sigma-1)a_n = a_{n-1}.$$

$n = 0$ gives the indicial equation

$$\sigma(\sigma - 1) = 0$$

Then $\sigma = 0, 1$. We are guaranteed to have a solution in the form $\sigma = 1$. When $\sigma = 1$, the recurrence relation becomes

$$(n+1)na_n = a_{n-1}.$$

When $n = 0$, $0 \cdot a_0 = 0$ so a_0 is arbitrary. When $n > 0$, we obtain

$$a_n = \frac{1}{n(n+1)} a_{n-1} = \frac{1}{(n+1)(n!)^2} a_0.$$

So

$$y_1 = a_0 x \left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots\right)$$

When $\sigma = 0$, we obtain

$$n(n-1)a_n = a_{n-1}.$$

When $n = 0$, $0 \cdot a_0 = 0$ and a_0 is arbitrary. When $n = 1$, $0 \cdot a_1 = a_0$. However, $a_0 \neq 0$ by our initial constraint. Contradiction. So there is no solution in this form (If we ignore the constraint that $a_0 \neq 0$, we know that a_0 is arbitrary. But this gives exactly the same solution we found previously with $\sigma = 1$)

The other solution is thus in the form

$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n.$$

7 Directional derivative

Consider a function $f(x, y)$ and a displacement $\mathrm{d}\mathbf{s} = (\mathrm{d}x, \mathrm{d}y)$. The change in $f(x, y)$ during that displacement is

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

We can also write this as

$$\begin{aligned}\mathrm{d}f &= (\mathrm{d}x, \mathrm{d}y) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\\ &= \mathrm{d}\mathbf{s} \cdot \nabla f\end{aligned}$$

Where we define $\mathrm{d}\mathbf{s} = (\mathrm{d}x, \mathrm{d}y)$, and $\nabla f = \mathrm{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ are the Cartesian components of the *gradient* of f.

We write $\mathrm{d}\mathbf{s} = \hat{\mathbf{s}}\,\mathrm{d}s$, where $|\hat{\mathbf{s}}| = 1$. Then

Definition (Directional derivative)**.** The *directional derivative* of f in the direction of $\hat{\mathbf{s}}$ is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \nabla f.$$

Definition (Gradient vector)**.** The *gradient vector* ∇f is defined by

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \nabla f$$

It is a vector with the following properties:

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \nabla f = |\nabla f| \cos\theta$$

where θ is the angle between the displacement and ∇f. Then when $\cos\theta$ is maximized, $\frac{\mathrm{d}f}{\mathrm{d}s} = |\nabla f|$

(i) ∇f has magnitude equal to the maximum rate of change of $f(x, y)$ in the $x - y$ plane.

(ii) It has direction in which f increases most rapidly.

(iii) If $\mathrm{d}\mathbf{s}$ is a displacement along a contour of f, then

$$\frac{\mathrm{d}f}{\mathrm{d}s} = 0.$$

So $\hat{\mathbf{s}} \cdot \nabla f = 0$, i.e. ∇f is orthogonal to the contour.

7.1 Stationary points

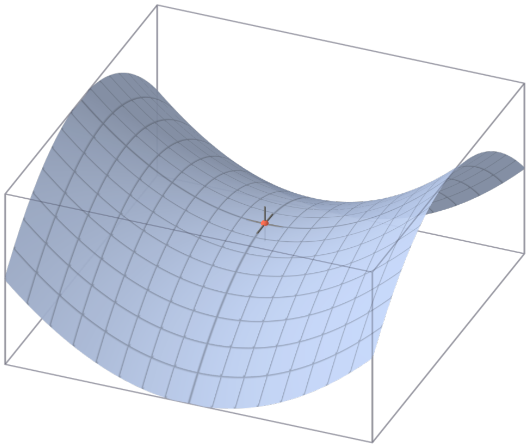
There is always (at least) one direction in which $\frac{\mathrm{d}f}{\mathrm{d}s} = 0$, namely the direction parallel to the contour of f. However, local maxima and minima have

$$\frac{\mathrm{d}f}{\mathrm{d}s} = 0$$

for *all* directions, i.e. $\hat{\mathbf{s}} \cdot \nabla f = 0$ for all $\hat{\mathbf{s}}$, i.e. $\nabla f = \mathbf{0}$. Then we know that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

However, apart from maxima and minima, in 3 dimensions, we also have saddle points:



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Note: contours are locally elliptical at maximum and minima, and are locally hyperbolic at saddle points.
Note: Contours can only cross at saddle points.

7.2 Taylor series for multi-variable functions

Suppose we have a function $f(x, y)$ and a point $\mathbf{x}_0$. Now consider a finite displacement δs along a straight line in the x, y plane. Then

$$\delta s \frac{\mathrm{d}}{\mathrm{d}s} = \delta \mathbf{s} \cdot \nabla$$

The Taylor series along the line is

$$\begin{aligned} f(s) &= f(s_0 + \delta s) \\ &= f(s_0) + \delta s \frac{\mathrm{d}f}{\mathrm{d}s} + \frac{1}{2}(\delta s)^2 \frac{\mathrm{d}^2 f}{\mathrm{d}s^2} \\ &= f(s_0) + \delta \mathbf{s} \cdot \nabla f + \frac{1}{2} \delta s^2 (\hat{\mathbf{s}} \cdot \nabla)(\hat{\mathbf{s}} \cdot \nabla) f \end{aligned}$$

where

$$\begin{aligned} \delta \mathbf{s} \cdot \nabla f &= (\delta x) \frac{\partial f}{\partial x} + (\delta y) \frac{\partial f}{\partial y} \\ &= (x - x_0) \frac{\partial f}{\partial x} + (y - y_0) \frac{\partial f}{\partial y} \end{aligned}$$

and

$$
\begin{aligned}
\delta s^2(\hat{\mathbf{s}}\cdot\nabla)(\hat{\mathbf{s}}\cdot\nabla)f &= (\delta\mathbf{s}\cdot\nabla)(\delta\mathbf{s}\cdot\nabla)f\\
&= \left[\delta x\frac{\partial}{\partial x}+\delta y\frac{\partial}{\partial y}\right]\left[\delta x\frac{\partial f}{\partial x}+\delta y\frac{\partial f}{\partial y}\right]\\
&= \delta x^2\frac{\partial^2 f}{\partial x^2}+2\delta x\delta y\frac{\partial^2 f}{\partial x\partial y}+\delta y^2\frac{\partial^2 f}{\partial y^2}\\
&= \begin{pmatrix}\delta x & \delta y\end{pmatrix}\begin{pmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}
\end{aligned}
$$

Definition (Hessian matrix)**.** The *Hessian matrix* is the matrix

$$\nabla\nabla f = \begin{pmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix}$$

In conclusion, we have

$$
\begin{aligned}
f(x,y) = f(x_0,y_0) &+ (x-x_0)f_x+(y-y_0)f_y\\
&+\frac{1}{2}[(x-x_0)^2f_{xx}+2(x-x_0)(y-y_0)f_{xy}+(y-y_0)^2f_yy]
\end{aligned}
$$

In general, the coordinate-free form is

$$f(\mathbf{x}) = f(\mathbf{x_0})+\delta\mathbf{x}\cdot\nabla f(\mathbf{x}_0)+\frac{1}{2}\delta\mathbf{x}\cdot\nabla\nabla f\cdot\delta\mathbf{x}$$

where the dot in the second term represents a matrix product. Alternatively, in terms of the gradient operator (and real dot products), we have

$$f(\mathbf{x}) = f(\mathbf{x_0})+\delta\mathbf{x}\cdot\nabla f(\mathbf{x}_0)+\frac{1}{2}[\nabla(\nabla f\cdot\delta\mathbf{x})]\cdot\delta\mathbf{x}$$

7.3 Classification of stationary points

At a stationary point $\mathbf{x}_0$, we know that $\nabla f(\mathbf{x}_0)=0$. So at a point near the stationary point,

$$f(\mathbf{x})\approx f(\mathbf{x}_0)+\frac{1}{2}\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x},$$

where $H=\nabla\nabla f(\mathbf{x}_0)$ is the Hessian matrix.

At a minimum, Every point near $\mathbf{x}_0$ has $f(\mathbf{x})>f(\mathbf{x}_0)$, i.e. $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x}>0$ for all $\delta\mathbf{x}$. We say $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x}$ is *positive definite*.

Similarly, at a maximum, $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x}<0$ for all $\delta\mathbf{x}$. We say $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x}$ is *negative definite*.

At a saddle, $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x}$ is indefinite, i.e. it can be positive, negative or zero depending on the direction.

Note: If $\det H=0$, then we cannot determine the nature of the stationary point at this order (need to look at higher derivatives).

Now note that $H=\nabla\nabla f$ is symmetric (because $f_{xy}=f_{yx}$). So H can be diagonalized (c.f. Vectors and Matrices). With respect to these axes in which H

is diagonal (principal axes), we have

$$\begin{aligned}\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x} &= (\delta x, \delta y, \cdots, \delta z)\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\\ \vdots\\ \delta z\end{pmatrix}\\ &= \lambda_1(\delta x)^2 + \lambda_2(\delta y)^2 + \cdots + \lambda_n(\delta z)^2\end{aligned}$$

where $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigenvalues of H.

So for $\delta\mathbf{x}\cdot H\cdot\delta\mathbf{x}$ to be positive-definite, we need $\lambda_i > 0$ for all i. Similarly, it is negative-definite iff $\lambda_i < 0$ for all i.

If eigenvalues have mixed sign, then it is a saddle point.

7.3.1 Determination of definiteness

Definition (Signature of Hessian matrix)**.** The *signature* of H is the pattern of the signs of the subdeterminants:

$$\underbrace{f_{xx}}_{|H_1|}, \underbrace{\begin{vmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{vmatrix}}_{|H_2|}, \cdots, \underbrace{\begin{vmatrix}f_{xx} & f_{xy} & \cdots & f_{xz}\\ f_{yx} & f_{yy} & \cdots & f_{yz}\\ \vdots & \vdots & \ddots & \vdots\\ f_{zx} & f_{zy} & \cdots & f_{zz}\end{vmatrix}}_{|H_n|=|H|}$$

Proposition. H is positive definite if and only if the signature is $+, +, \cdots, +$. H is negative definite if and only if the signature is $-, +, \cdots, (-1)^n$. Otherwise, H is indefinite.

7.3.2 Contours of $f(x, y)$

Consider H in 2 dimensions, and axes in which H is diagonal. So $H = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$. Write $\mathbf{x} - \mathbf{x}_0 = (X, Y)$.

Then near $\mathbf{x}_0$, $f = \text{constant} \Rightarrow \mathbf{x}H\mathbf{x} = \text{constant}$, i.e. $\lambda_1 X^2 + \lambda_2 Y^2 = \text{constant}$. At a maximum or minimum, λ_1 and λ_2 have the same sign. So these contours are locally ellipses. At a saddle point, they have different signs and the contours are locally hyperbolae.

Example. Find and classify the stationary points of $f(x, y) = 4x^3 - 12xy + y^2 + 10y + 6$. We have

$$\begin{aligned}f_x &= 12x^2 - 12y\\ f_y &= -12x + 2y + 10\\ f_{xx} &= 24x\\ f_{xy} &= -12\\ f_{yy} &= 2\end{aligned}$$

At stationary points, $f_x = f_y = 0$. So we have

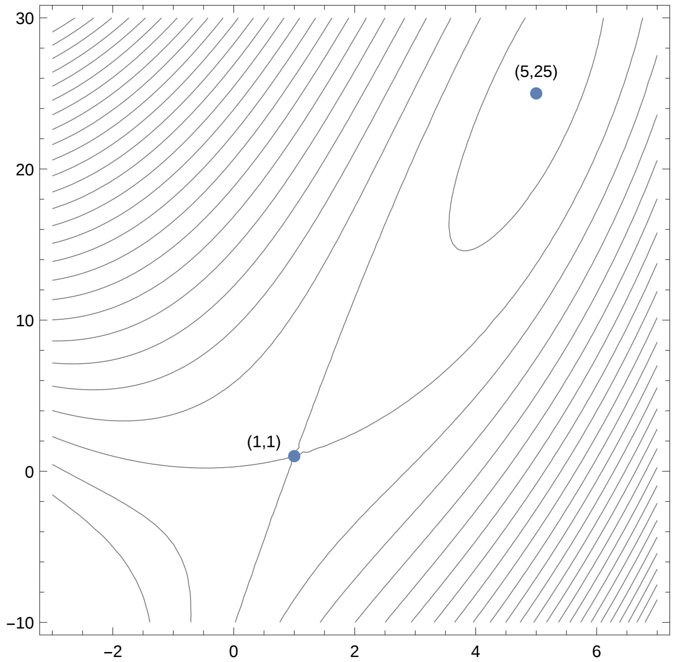
$$12x^2 - 12y = 0, \quad -12x + 2y + 10 = 0.$$

The first equation gives $y = x^2$. Substituting into the second equation, we obtain $x = 1, 5$ and $y = 1, 25$ respectively. So the stationary points are $(1, 1)$ and $(5, 25)$

To classify them, first consider $(1, 1)$. Our Hessian matrix $H = \begin{pmatrix} 24 & -12 \\ -12 & 2 \end{pmatrix}$. Our signature is $|H_1| = 24$ and $|H_2| = -96$. Since we have a $+, -$ signature, this an indefinite case and it is a saddle point.

At $(5, 25)$, $H = \begin{pmatrix} 120 & -12 \\ -12 & 2 \end{pmatrix}$ So $|H_1| = 120$ and $|H_2| = 240 - 144 = 96$. Since the signature is $+, +$, it is a minimum.

To draw the contours, we draw what the contours look like near the stationary points, and then try to join them together, noting that contours cross only at saddles.



8 Systems of linear differential equations

Consider two dependent variables $y_1(t), y_2(t)$ with

$$\begin{aligned}\dot{y}_1 &= ay_1 + by_2 + f_1(t)\\ \dot{y}_2 &= cy_1 + dy_2 + f_2(t)\end{aligned}$$

We write this in vector notation by

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \begin{pmatrix}f_1\\ f_2\end{pmatrix}$$

or $\dot{\mathbf{Y}} = M\mathbf{Y} + \mathbf{F}$. This is equivalent to the higher-order equation:

$$\begin{aligned}\ddot{y}_1 &= a\dot{y}_1 + b\dot{y}_2 + \dot{f}_1\\ &= a\dot{y}_1 + b(cy_1 + dy_2 + f_2) + \dot{f}_1\\ &= a\dot{y}_1 + bcy_1 + d(\dot{y}_1 - ay_1 - f_1) + bf_2 + \dot{f}_1\end{aligned}$$

so

$$\ddot{y}_1 - (a+d)\dot{y}_1 + (ad - bc)y_1 = bf_2 - df_1 + \dot{f}_1$$

and we know how to solve this. However, this actually complicates the equation.

So what we usually do is the other way round: if we have a high-order equation, we can do this to change it to a system of first-order equations:

If $\ddot{y} + a\dot{y} + by = f$, write $y_1 = y$ and $y_2 = \dot{y}$. Then let $\mathbf{Y} = \begin{pmatrix}y\\ \dot{y}\end{pmatrix}$

Our system of equations becomes

$$\begin{aligned}\dot{y}_1 &= y_2\\ \dot{y}_2 &= f - ay_2 - by_1\end{aligned}$$

or

$$\dot{\mathbf{Y}} = \begin{pmatrix}0 & 1\\ -b & -a\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \begin{pmatrix}0\\ f\end{pmatrix}$$

Now consider the general equation

$$\begin{aligned}\dot{\mathbf{Y}} &= M\mathbf{Y} + \mathbf{F}\\ \dot{\mathbf{Y}} - M\mathbf{Y} &= \mathbf{F}\end{aligned}$$

We first look for a complementary solution $\mathbf{Y}_c = \mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ is a constant vector. So we get

$$\lambda\mathbf{v} - M\mathbf{v} = \mathbf{0}.$$

So we have

$$M\mathbf{v} = \lambda\mathbf{v}.$$

So λ is the eigenvalue of M and $\mathbf{v}$ is the corresponding eigenvector.

We can solve this by solving the characteristic equation $\det(M - \lambda I) = 0$. Then for each λ, we find the corresponding $\mathbf{v}$.

Example.

$$\dot{\mathbf{Y}} = \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix} \mathbf{Y} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t$$

The characteristic equation of M is $\begin{vmatrix} -4-\lambda & 24 \\ 1 & -2-\lambda \end{vmatrix} = 0$, which gives $(\lambda + 8)(\lambda - 2) = 0$ and $\lambda = 2, -8$.

When $\lambda = 2$, $\mathbf{v}$ satisfies

$$\begin{pmatrix} -6 & 24 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and we obtain $\mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

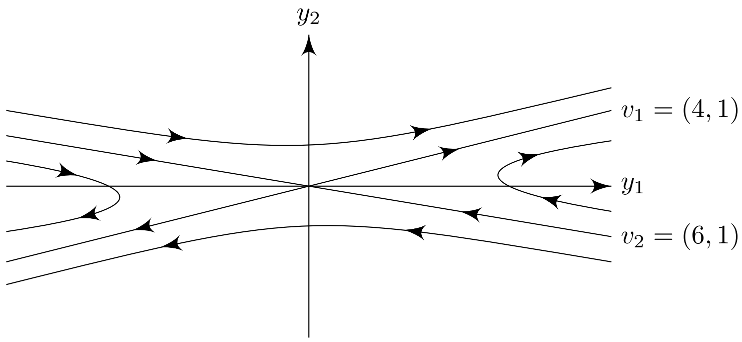
When $\lambda = -8$, we have

$$\begin{pmatrix} 4 & 24 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and $\mathbf{v}_2 = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$. So the complementary solution is

$$\mathbf{Y} = A \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B \begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t}$$

We can plot the phase-space trajectories:



To find the particular integral, we try $\mathbf{Y}_p = \mathbf{u}e^t$. Then

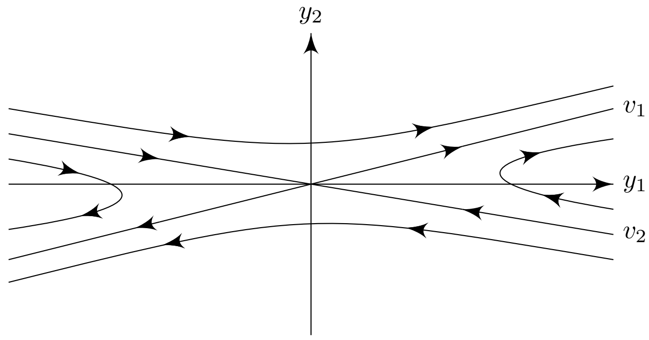
$$\begin{aligned}
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -4 & 24 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 5 & -24 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= -\frac{1}{9} \begin{pmatrix} 3 & 24 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -4 \\ -1 \end{pmatrix}
\end{aligned}$$

So the general solution is

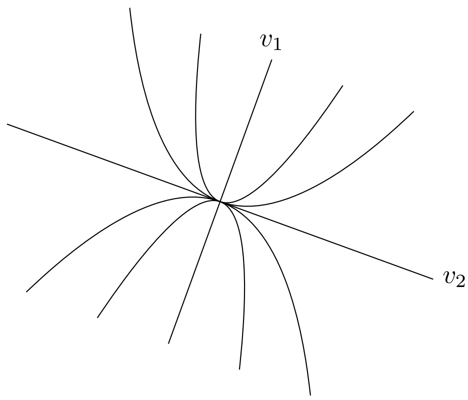
$$\mathbf{Y} = A \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + B \begin{pmatrix} -6 \\ 1 \end{pmatrix} e^{-8t} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t$$

There are three possible cases of $\dot{\mathbf{Y}} = M\mathbf{Y}$ corresponding to three different possible eigenvalues of M:

(i) If both λ_1, λ_2 are real with opposite sign ($\lambda_1\lambda_2 < 0$). wlog assume $\lambda_1 > 0$. Then there is a saddle as above:

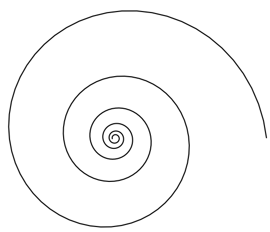


(ii) If λ_1, λ_2 are real with $\lambda_1\lambda_2 > 0$. wlog assume $|\lambda_1| \geq |\lambda_2|$. Then the phase portrait is



If both $\lambda_1, \lambda_2 < 0$, then the arrows point towards the intersection and we say there is a stable node. If both are positive, they point outwards and there is an unstable node.

(iii) If λ_1, λ_2 are complex conjugates, then we obtain a spiral



If $\text{Re}(\lambda_{1,2}) < 0$, then it spirals inwards. If $\text{Re}(\lambda_{1,2}) > 0$, then it spirals outwards. If $\text{Re}(\lambda_{1,2}) = 0$, then we have ellipses with common centers instead of spirals. We can determine whether the spiral is positive (as shown above), or negative (mirror image of the spiral above) by considering the eigenvectors.

8.1 Nonlinear dynamical systems

Consider the second-order autonomous system (i.e. t does not explicitly appear in the forcing terms on the right)

$$\begin{aligned}\dot{x} &= f(x, y)\\ \dot{y} &= g(x, y)\end{aligned}$$

It can be difficult to solve the equations, but we can learn a lot about phase-space trajectories of these solutions by studying the equilibria and their stability.

8.1.1 Equilibrium (fixed) points

Definition (Equilibrium point)**.** An *equilibrium* point is a point in which $\dot{x} = \dot{y} = 0$ at $\mathbf{x}_0 = (x_0, y_0)$.

Clearly this occurs when $f(x_0, y_0) = g(x_0, y_0) = 0$. We solve these simultaneously for x_0, y_0.

8.1.2 Stability

Write $x = x_0 + \xi$, $y = y_0 + \eta$. Then

$$\begin{aligned}\dot{\xi} &= f(x_0 + \xi, y_0 + \eta)\\ &= f(x_0, y_0) + \xi\frac{\partial f}{\partial x}(\mathbf{x}_0) + \eta\frac{\partial f}{\partial y}(\mathbf{x}_0) + O(\xi^2, \eta^2)\end{aligned}$$

So if $\xi, \eta \ll 1$,

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}f_x & f_y\\ g_x & g_y\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

This is a linear system, and we can determine its character from the eigensolutions.

Example. (Population dynamics - predator-prey system) Suppose that there are x prey and y predators. Then we have the following for the prey:

$$\dot{x} = \underbrace{\alpha x}_{\text{births - deaths}} - \underbrace{\beta x^2}_{\text{natural competition}} - \underbrace{\gamma xy}_{\text{killed by predators}}.$$

and the following for the predators:

$$\dot{y} = \underbrace{\epsilon xy}_{\text{birth/survival rate}} - \underbrace{\delta y}_{\text{natural death rate}}$$

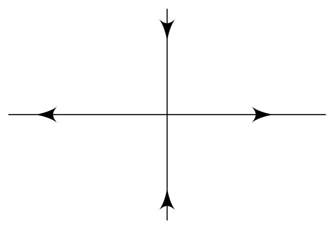
For example, let

$$\begin{aligned}\dot{x} &= 8x - 2x^2 - 2xy\\ \dot{y} &= xy - y\end{aligned}$$

We find the fixed points: $x(8-2x-2y)=0$ gives $x=0$ or $y=4-x$.
We also want $y(x-1)=0$ which gives $y=0$ or $x=1$.
So the fixed points are $(0,0),(4,0),(1,3)$.
Near $(0,0)$, we have

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix}=\begin{pmatrix}8 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

We clearly have eigenvalues $8,-1$ with the standard basis as the eigenvectors.



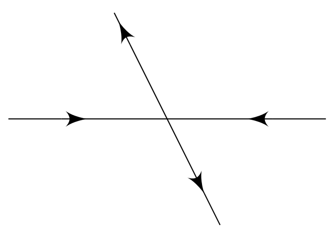
Near $(4,0)$, we have $x=4+\xi$, $y=\eta$. Instead of expanding partial derivatives, we can obtain from the equations directly:

$$\begin{aligned}\dot{\xi}&=(4+\xi)(8-8-2\xi-2\eta)\\ &=-8\xi-8\eta-2\xi^2-2\xi\eta\\ \dot{\eta}&=\eta(4+\xi-1)\\ &=3\eta+\xi\eta\end{aligned}$$

Ignoring the second-order terms, we have

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix}=\begin{pmatrix}-8 & -8\\ 0 & 3\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

The eigenvalues are -8 and 3, with associated eigenvectors $(1,0),(8,-11)$.



Near $(1,3)$, we have $x=1+\xi, y=3+\eta$. So

$$\begin{aligned}\dot{\xi}&=(1+\xi)(8-2-2\xi-6-2\eta)\\ &\approx-2\xi-2\eta\\ \dot{\eta}&=(3+\eta)(1+\xi-1)\\ &\approx 3\eta\end{aligned}$$

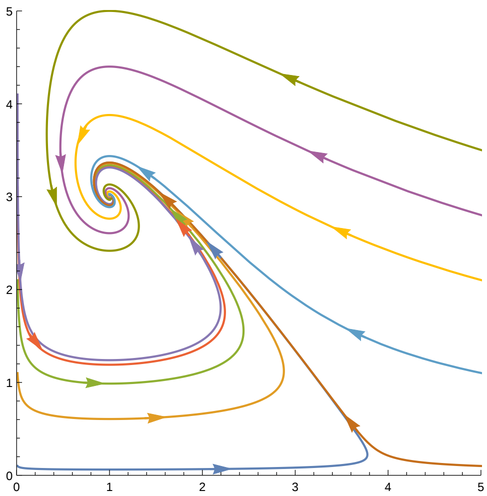
So

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix}=\begin{pmatrix}-2 & -2\\ 3 & 0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

The eigenvalues are $-1 \pm i\sqrt{5}$. Since it is complex with a negative real part, it is a stable spiral.

We can determine the chirality of the spirals by considering what happens to a small perturbation to the right $\begin{pmatrix} \xi \\ 0 \end{pmatrix}$ with $\xi > 0$. We have $\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -2\xi \\ 3\xi \end{pmatrix}$. So $\mathbf{x}$ will head top-left, and the spiral is counter-clockwise ("positive").

Overall, we have the following phase portrait:



We see that $(1, 3)$ is a stable solution in which almost all solutions spiral towards.

9 Partial differential equations (PDEs)

9.1 First-order wave equation

Consider the equation of the form

$$\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x}$$

with c constant. We write this as

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0.$$

Recall that along a path $x = x(t)$ so that $y = y(x(t), t)$,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial y}{\partial t}\\ &= \frac{\partial y}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial y}{\partial x}\end{aligned}$$

by the chain rule. Now we choose a path along which

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -c. \tag{1}$$

Along such paths,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \tag{2}$$

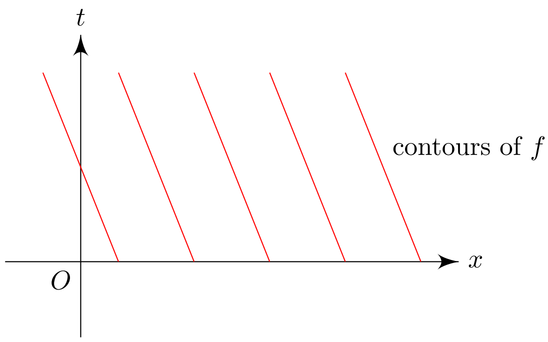
So we have replaced the original partial differential equations with a pair of ordinary differential equations.

From (2), we know that $y =$ constant along the path. From (1), we know that $x = x_0 - ct$, where x_0 is a constant. Write $x + ct = x_0$.

So we have a family of paths, each determined by x_0, and along each path, y is constant. For each value of x_0, we obtain a unique constant function along the path $y = f(x_0)$. So

$$y = f(x + ct),$$

which is the general solution of y.



Note that as we move up the time axis, we are simply taking the $t = 0$ solution and translating it to the left.

The paths we've identified are called the "characteristics" of the wave equation. In this particular example, $y = \text{const}$ along the characteristics (because the equation is unforced).

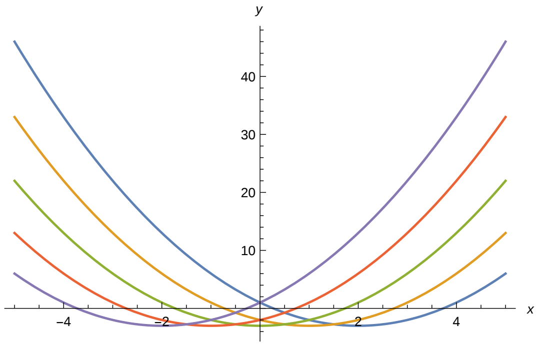
We usually have initial conditions e.g.

$$y(x, 0) = x^2 - 3$$

Since we know that $y = f(x + ct)$ and $f(x) = x^2 - 3$. So

$$y = (x + ct)^2 - 3.$$

We can plot the $x - y$ curve for different values of t to obtain this:



and see that that each solution is just a translation of the $t = 0$ version.

We can also solve forced equations, such as

$$\frac{\partial y}{\partial t} + 5\frac{\partial y}{\partial x} = e^{-t}, \quad y(x, 0) = e^{-x^2}.$$

So we obtain $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-t}$ with $\frac{\mathrm{d}x}{\mathrm{d}t}$. So $y = A - e^{-t}$ on paths $x = x_0 + 5t$.

At $t = 0$, $y = A - 1$ and $x = x_0$. So $A - 1 = e^{-x_0^2}$ from the given initial conditions., i.e. $A = 1 + e^{-x_0^2}$. So

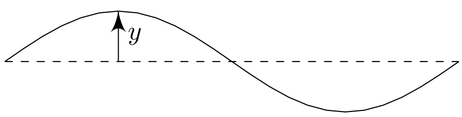
$$y = 1 + e^{-(x-5t)^2} - e^{-t}.$$

9.2 Second-order wave equation

We consider equations in the following form:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

It is obtained as followed:



Suppose that $\rho(x)$ is the mass per unit length of a string. Then the force $ma = \rho \dfrac{\partial^2 y}{\partial t^2}$ is proportional to the second derivative $\dfrac{\partial^2 y}{\partial x^2}$.

(Why is it proportional to the second derivative? It certainly cannot be proportional to y, because we get no force if we just move the whole string upwards. It also cannot be proportional to $\partial y/\partial x$: if we have a straight slope, then the force pulling upwards is the same as the force pulling downwards, and we should have no force. We have a force only if the string is curved, and curvature is measured by the second derivative)

Back to the equation: this is often known as the "hyperbolic equation" because the form resembles that of a hyperbola (which has the form $x^2 - b^2y^2 = 0$). However, the differential equation has no connections to hyperbolae whatsoever.

If c is constant, we write

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0$$
$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) y = 0$$

If $y = f(x + ct)$, then the first operator differentiates it to give a constant (as in the first-order wave equation). Then applying the second operator differentiates it to 0. So $y = f(x + ct)$ is a solution.

Since the operators are commutative, $y = f(x - ct)$ is also a solution. Since the equation is linear, the general solution is

$$y = f(x + ct) + g(x - ct).$$

This shows that the solution composes of superpositions of waves travelling to the left and waves travelling to the right.
Note: We can show that this is indeed the most general solution by substituting $\xi = x + ct$ and $\eta = x - ct$. We can show, using the chain rule, that $y_{tt} - c^2 y_{xx} \equiv -4c^2 y_{\eta\xi} = 0$. Integrating twice gives $y = f(\xi) + g(\eta)$.

How many boundary conditions do we need to have a unique solution? In ODEs, we simply count the order of the equation. In PDEs, we have to count over all variables. In this case, we need 2 boundary conditions and 2 initial conditions. For example, we can have:

- Initial conditions: at $t = 0$,

$$y = \frac{1}{1 + x^2}$$
$$\frac{\partial y}{\partial t} = 0$$

- Boundary conditions: $y \to 0$ as $x \to \pm\infty$.

We know that the solution has the form

$$y = f(x + ct) + g(x - ct).$$

The first initial condition give

$$f(x) + g(x) = \frac{1}{1 + x^2} \tag{1}$$

The second initial condition gives

$$\frac{\partial y}{\partial t} = cf'(x) - cg'(x) = 0 \tag{2}$$

From (2), we know that $f' = g'$, and $f = g+$ constant, and the constant is 0 given the boundary conditions. So from (1),

$$f(x) = g(x) = \frac{1}{2(1+x^2)}$$

So, overall,

$$y = \frac{1}{2}\left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right]$$

Where we substituted x for $x + ct$ and $x - ct$ respectively.

9.3 The diffusion equation

Heat conduction in a solid in one direction is modelled by the diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

This is known as a parabolic PDE. (parabolic because it resembles $y = ax^2$)

Here $T(x, t)$ is the temperature and the constant κ is the *diffusivity*.

Example. Consider an infinitely long bar heated at one end ($x = 0$). Note in general that "velocity" $\partial T/\partial t$ is proportional to curvature $\partial^2 T/\partial x^2$ (it is the "acceleration" that is proportional to curvature in the wave equation). In this case, instead of oscillatory, the diffusion equation is dissipative and all unevenness simply decays away.

Suppose $T(x, 0) = 0$, $T(0, t) = H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, and $T(x, t) \to 0$ as $x \to \infty$.

There is a *similarity solution* of the diffusion equation valid on an infinite domain (or our semi-infinite domain) in which $T(x, t) = \theta(\eta)$, where $\eta = \dfrac{x}{2\sqrt{\kappa t}}$.

Applying the chain rule, we have

$$\begin{aligned}
\frac{\partial T}{\partial x} &= \frac{\mathrm{d}\theta}{\mathrm{d}\eta}\frac{\partial \eta}{\partial x} \\
&= \frac{1}{2\sqrt{\kappa t}}\theta'(\eta) \\
\frac{\partial^2 T}{\partial x^2} &= \frac{1}{2\sqrt{\kappa t}}\frac{\mathrm{d}\theta'}{\mathrm{d}\eta}\frac{\partial \eta}{\partial x} \\
&= \frac{1}{4\kappa t}\theta''(\eta) \\
\frac{\partial T}{\partial t} &= \frac{\mathrm{d}\theta}{\mathrm{d}\eta}\frac{\partial \eta}{\partial t} \\
&= -\frac{1}{2}\frac{x}{2\sqrt{\kappa}}\frac{1}{t^{3/2}}\theta'(\eta) \\
&= -\frac{\eta}{2t}\theta'(\eta)
\end{aligned}$$

Putting this into the diffusion equation yields

$$\begin{aligned}
-\frac{\eta}{2t}\theta' &= \kappa\frac{1}{4\kappa t}\theta'' \\
\theta'' + 2\eta\theta' &= 0
\end{aligned}$$

This is an ordinary differential equation for $\theta(\eta)$. This can be seen as a first-order equation for θ' with non-constant coefficients. Use the integrating factor $\mu = \exp(\int 2\eta \, \mathrm{d}\eta) = e^{\eta^2}$. So

$$\begin{aligned}
(e^{\eta^2}\theta')' &= 0\\
\theta' &= Ae^{-\eta^2}\\
\theta &= A\int_0^\eta e^{-u^2}\,\mathrm{d}u + B\\
&= \alpha\,\mathrm{erf}(\eta) + B
\end{aligned}$$

where $\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2}\,\mathrm{d}u$ from statistics, and $\mathrm{erf}(\eta) \to 1$ as $\eta \to \infty$.

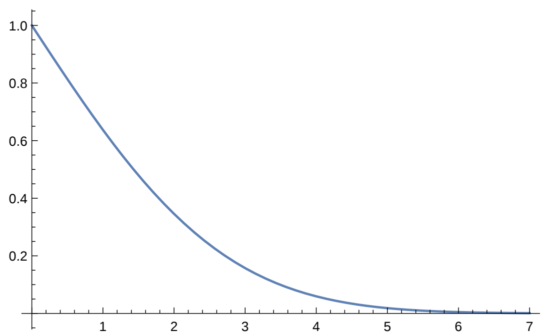
Now look at the boundary and initial conditions, (recall $\eta = x/(2\sqrt{\kappa t})$) and express them in terms ofη. As $x \to 0$, we have $\eta \to 0$. So $\theta = 1$ at $\eta = 0$.

Also, if $x \to \infty, t \to 0^+$, then $\eta \to \infty$. So $\theta \to 0$ as $\eta \to \infty$.

So $\theta(0) = 1 \Rightarrow B = 1$. Colloquially, $\theta(\infty) = 0$ gives $\alpha = -1$. So $\theta = 1 - \mathrm{erf}(\eta)$. This is also sometimes written as $\mathrm{erfc}(\eta)$, the error function complement of η. So

$$T = \mathrm{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)$$

In general, at any particular fixed time t_0, $T(x)$ looks like



with decay length $O(\sqrt{\kappa t})$. So if we actually have a finite bar of length L, we can treat it as infinite if $\sqrt{\kappa t} \ll L$, or $t \ll L^2/\kappa$