# Part IA - Probability

## Theorems

Lectured by R. Weber

Lent 2015

**Basic concepts**
Classical probability, equally likely outcomes. Combinatorial analysis, permutations and combinations. Stirling's formula (asymptotics for $\log n!$ proved). [3]

**Axiomatic approach**
Axioms (countable case). Probability spaces. Inclusion-exclusion formula. Continuity and subadditivity of probability measures. Independence. Binomial, Poisson and geometric distributions. Relation between Poisson and binomial distributions. Conditional probability, Bayes's formula. Examples, including Simpson's paradox. [5]

**Discrete random variables**
Expectation. Functions of a random variable, indicator function, variance, standard deviation. Covariance, independence of random variables. Generating functions: sums of independent random variables, random sum formula, moments.

Conditional expectation. Random walks: gambler's ruin, recurrence relations. Difference equations and their solution. Mean time to absorption. Branching processes: generating functions and extinction probability. Combinatorial applications of generating functions. [7]

**Continuous random variables**
Distributions and density functions. Expectations; expectation of a function of a random variable. Uniform, normal and exponential random variables. Memoryless property of exponential distribution. Joint distributions: transformation of random variables (including Jacobians), examples. Simulation: generating continuous random variables, independent normal random variables. Geometrical probability: Bertrand's paradox, Buffon's needle. Correlation coefficient, bivariate normal random variables. [6]

**Inequalities and limits**
Markov's inequality, Chebyshev's inequality. Weak law of large numbers. Convexity: Jensen's inequality for general random variables, AM/GM inequality.

Moment generating functions and statement (no proof) of continuity theorem. Statement of central limit theorem and sketch of proof. Examples, including sampling. [3]

# Contents


**1 Introduction** **3**

**2 Classical probability** **4**
2.1 Classical probability . . . . . . . . . . . . . . . . . . . . . . . . . . 4
2.2 Sample space and events . . . . . . . . . . . . . . . . . . . . . . . . 4

**3 Combinatorial analysis** **5**
3.1 Counting . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
3.2 Sampling with or without replacement . . . . . . . . . . . . . . . . 5
3.3 Sampling with or without regard to ordering . . . . . . . . . . . . 5
3.4 Four cases of enumerative combinatorics . . . . . . . . . . . . . . . 5

**4 Stirling's formula** **6**
4.1 Multinomial coefficient . . . . . . . . . . . . . . . . . . . . . . . . . 6
4.2 Stirling's formula . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6

**5 Axiomatic approach** **7**
5.1 Boole's inequality . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
5.2 Inclusion-exclusion formula . . . . . . . . . . . . . . . . . . . . . . 7

# 1 Introduction

# 2 Classical probability

## 2.1 Classical probability

## 2.2 Sample space and events

# 3 Combinatorial analysis

## 3.1 Counting

**Theorem** (Fundamental rule of counting)**.** Suppose we have to make $r$ multiple choices in sequence. There are $m_1$ possibilities for the first choice, $m_2$ possibilities for the second etc. Then the total number of choices is $m_1 \times m_2 \times \cdots m_r$.

## 3.2 Sampling with or without replacement

## 3.3 Sampling with or without regard to ordering

## 3.4 Four cases of enumerative combinatorics

# 4 Stirling's formula

## 4.1 Multinomial coefficient

## 4.2 Stirling's formula

**Proposition.** $\log n! \sim n \log n$

**Theorem** (Stirling's formula)**.** As $n \to \infty$,

$$\log\left(\frac{n!e^n}{n^{n+\frac{1}{2}}}\right) = \log\sqrt{2\pi} + O\left(\frac{1}{n}\right)$$

**Corollary.**

$$n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$$

**Proposition** ((non-examinable))**.** We use the $1/12n$ term from the proof above to get a better approximation:

$$\sqrt{2\pi}n^{n+1/2}e^{-n} + \frac{1}{12n+1} \leq n! \leq \sqrt{2\pi}n^{n+1/2}e^{-n} + \frac{1}{12n}.$$

# 5 Axiomatic approach

**Theorem.**

(i) $P(\emptyset) = 0$

(ii) $P(A^C) + 1 - P(A)$

(iii) $A \subseteq B \Rightarrow P(A) \leq P(B)$

(iv) $P(A \subseteq B) = P(A) + P(B) - P(A \cap B)$.

(v) Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Then

$$P\left(\bigcup_1^\infty A_i\right) = \lim_{n\to\infty} P(A_n).$$

This states that $P$ is a continuous set function.

## 5.1 Boole's inequality

**Theorem** (Boole's inequality)**.** For any $A_1, A_2, \cdots$,

$$P\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty P(A_i).$$

## 5.2 Inclusion-exclusion formula

**Theorem** (Inclusion-exclusion formula)**.**

$$\begin{aligned}P\left(\bigcup_i^n A_i\right) &= \sum_1^n P(A_i) - \sum_{i_1<i_2} P(A_{i_1} \cap A_{j_2}) + \sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots \\ &\quad + (-1)^{n-1} P(A_1 \cap \cdots \cap A_n).\end{aligned}$$