Part IA - Dynamics and Relativity

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Basic concepts
Space and time, frames of reference, Galilean transformations. Newton's laws. Dimensional analysis. Examples of forces, including gravity, friction and Lorentz. [4]

Newtonian dynamics of a single particle
Equation of motion in Cartesian and plane polar coordinates. Work, conservative forces and potential energy, motion and the shape of the potential energy function; stable equilibria and small oscillations; effect of damping.

Angular velocity, angular momentum, torque.

Orbits: the $u(\theta)$ equation; escape velocity; Kepler's laws; stability of orbits; motion in a repulsive potential (Rutherford scattering). Rotating frames: centrifugal and coriolis forces. *Brief discussion of Foucault pendulum.* [8]

Newtonian dynamics of systems of particles
Momentum, angular momentum, energy. Motion relative to the centre of mass; the two body problem. Variable mass problems; the rocket equation. [2]

Rigid bodies
Moments of inertia, angular momentum and energy of a rigid body. Parallel axes theorem. Simple examples of motion involving both rotation and translation (e.g. rolling). [3]

Special relativity
The principle of relativity. Relativity and simultaneity. The invariant interval. Lorentz transformations in $(1+1)$-dimensional spacetime. Time dilation and length contraction. The Minkowski metric for $(1+1)$-dimensional spacetime. Lorentz transformations in $(3+1)$ dimensions. 4-vectors and Lorentz invariants. Proper time. 4-velocity and 4-momentum. Conservation of 4-momentum in particle decay. Collisions. The Newtonian limit. [7]

Contents

1 Newtonian dynamics of particles

We start with a few basic definitions:

Definition (Particle)**.** An *particle* is an object of insignificant size. It can be regarded as a point. It has a *mass* $m > 0$, and *electric charge* q.

Its position at time t is described by its *position vector*, $\mathbf{r}(t)$ or $\mathbf{x}(t)$ with respect to an origin O.

Definition (Frame of reference)**.** A *frame of reference* is choice of coordinate axes for $\mathbf{r}$. The axes may be fixed, moving, or accelerating relative to another frame.

With a frame of reference, we can write $\mathbf{r}$ in cartesian coordinates as (x, y, z)

Definition (Velocity)**.** The *velocity* of the particle is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

and is tangent to the path or trajectory.

Definition (Acceleration)**.** The *acceleration* of the particle is

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{dt^2}.$$

Definition (Momentum)**.** The *momentum* of a particle is

$$\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}.$$

m is the *inertial mass* of the particle, and measures its reluctance to accelerate (c.f. Newton's Second Law)

1.1 Newton's laws of motion

We begin by stating Newton's three laws of motion:

Law (Newton's First Law of Motion)**.** A body remains at rest, or moves uniformly in a straight line, unless acted on by a force. (This is in fact Galileo's Law of Inertia)

Law (Newton's Second Law of Motion)**.** The rate of change of momentum of a body is equal to the force acting on it (in both magnitude and direction).

Law (Newton's Third Law of Motion)**.** To every action there is an equal and opposite reaction: the forces of two bodies on each other are equal and in opposite directions.

The first law might seem redundant given the second if interpreted literally. Therefore, we should be interpreting it in the different way:

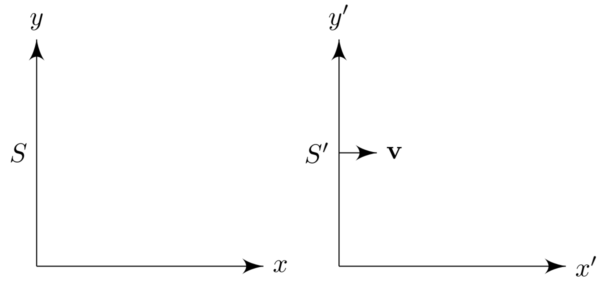
Note that the first law isn't always true. Take yourself as a frame of reference. When you move around your room, things will seem like they are moving around (relative to you). When you sit down, they stop moving. However, on second thought, this is because you, the frame of reference, is accelerating, not the objects. The first law only holds in frames that are themselves not accelerating. We call these *inertial frames.*

Definition (Inertial frames)**.** *Inertial frames* are frames of references in which the frames themselves are not accelerating. Newton's Laws only hold in inertial frames.

The we can take the first law to assert that inertial frames exists. Even though the Earth itself is rotating and orbiting the sun, for most purposes, any fixed place on the Earth counts as an inertial frame.

1.2 Galilean transformations

Inertial frames aren't unique. If S is an inertial frame, then any other frame S' in uniform motion relative to S is also inertial:



Assuming the frames coincide at $t = 0$, then

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

Generally, the position vector transforms as

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t,$$

where $\mathbf{v}$ is the (constant) velocity of S' relative to S. The velocity and acceleration transform as follows:

$$\begin{aligned} \dot{\mathbf{r}}' &= \dot{\mathbf{r}} - \mathbf{v} \\ \ddot{\mathbf{r}}' &= \ddot{\mathbf{r}} \end{aligned}$$

Definition (Galilean boost)**.** A Galilean boost is a change in frame of reference by

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} - \mathbf{v}t \\ t' &= t \end{aligned}$$

for a fixed, constant $\mathbf{v}$.

In addition to Galilean boosts, we can construct new inertial frames by applying (any combination of) the following:

– Translations of space:

$$\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$$

– Translations of time:

$$t' = t - t_0$$

– Rotation (and reflection):

$$\mathbf{r}' = R\mathbf{r}$$

with $R \in O(3)$.

These transformations together generate the Galilean group.

Law (Galilean relativity)**.** The *principle of relativity* asserts that the laws of physics are the same in inertial frames.

This implies that physical processes work the same

– at every point of space

– at all times

– in whichever direction we face

– at whatever constant velocity we travel.

In other words, the equations of Newtonian physics must have *Galilean invariance*.

Since the laws of physics are the same regardless of your velocity, velocity must be a *relative concept*, and there is no such thing as an "absolute velocity" that all inertial frames agree on.

However, all inertial frames must agree on whether you are accelerating or not (even though they need not agree on the direction of acceleration since you can rotate your frame). So acceleration is an *absolute* concept.

1.3 Newton's Second Law

Law. The *equation of motion* for a particle subject to a force F is

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

where $\mathbf{p} = m\mathbf{v} = m\ddot{\mathbf{r}}$ is the (linear) momentum of the particle. We say m is the (inertial) mass of the particle, which is a measure of its reluctance to accelerate.

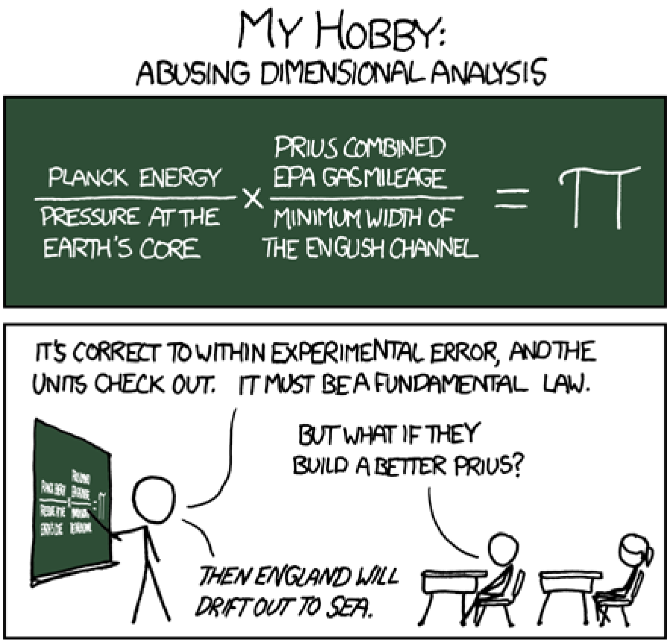
Usually, m is constant. Then

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}.$$

If $\mathbf{F}$ is specified as a function of $\mathbf{r}, \dot{\mathbf{r}}$ and t, then we have a second-order ordinary differential equation for $\mathbf{r}$.

To determine the solution, we need to specify the initial values of $\mathbf{r}$ and $\dot{\mathbf{r}}$, i.e. the initial position and velocity. The trajectory of the particle is then uniquely determined for all future (and past) times.

2 Dimensional Analysis



xkcd (Randall Munroe) CC-BY-NC 2.5

Physical quantities are not pure numbers, but have *dimensions.* In any equation, the dimensions have to be consistent.

For many problems in dynamics, three basic dimensions are sufficient:

- length, L
- mass, M
- time, T

The dimensions of a physical quantity X, denoted as $[X]$ are expressible uniquely in terms of L, M and T, e.g.

- [area] $= L^2$
- [density] $= L^{-3}M$
- [velocity] $= LT^{-1}$
- [acceleration] $= LT^{-2}$
- [force] $= LMT^{-2}$ since the dimensions must satisfy the equation $F = ma$.
- [energy] $= L^2MT^{-2}$, e.g. consider $E = mv^2/2$.

Physical constants also have dimensions, e.g. $[G] = L^3M^{-1}T^{-2}$ by $F = GMm/r^2$.

We can only take sums and products of terms that have dimensions, and if we sum two terms, they must have the same dimension. More complicated functions of dimensional quantities are not allowed, e.g. e^x makes no sense if x has a dimension, since

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots$$

and if x had a dimension, we would be summing up terms of different dimensions.

2.1 Units

A *unit* is a convenient standard physical quantity. In the SI system, there are base units corresponding to the basics dimensions. The three we need are

- meter (m) for length
- kilogram (kg) for mass
- second (s) for time

A physical quantity can be expressed as a pure number times a unit with the correct dimensions, e.g.

$$G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}.$$

It is important to realize that SI units are chosen arbitrarily for historical reasons only. The equation of physics must work in any consistent system of units. This is captured by the fact that physical equations must be dimensionally consistent.

2.2 Scaling

Suppose we believe that a physical quantity Y depends on 3 other physical quantities X_1, X_2, X_3, i.e. $Y = Y(X_1, X_2, X_3)$. Let their dimensions be as follows:

- $[Y] = L^a M^b T^c$
- $[X_i] = L^{a_i} M^{b_i} T^{c_i}$

Suppose further that we know that the relationship is a power law, i.e.

$$Y = C X_1^{p_1} X_2^{p_2} X_3^{p_3},$$

where C is a dimensionless constant (i.e. pure number). Since the dimensions must work out, we know that

$$\begin{aligned} a &= p_1 a_1 + p_2 a_2 + p_3 a_3 \\ b &= p_1 b_1 + p_2 b_2 + p_3 b_3 \\ c &= p_1 c_1 + p_2 c_2 + p_3 c_3 \end{aligned}$$

for which there is a unique solution provided that the dimensions of X_1, X_2 and X_3 are independent.

Note that if X_i are not independent, e.g. $X_1^2 X_2$ is dimensionless, then our law can involve more complicated terms such as $\exp(X_1^2 X_2)$ since the argument to exp is dimensionless.

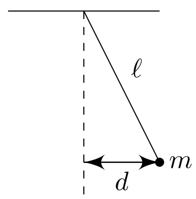
In general, if the dimensions of X_i are not independent, order them so that the independent terms $[X_1], [X_2], [X_3]$ are at the front. For each of the remaining variables, form a dimensionless quantity $\lambda_i = X_i X_1^{q_1} X_2^{q_2} X_3^{q_3}$. Then the relationship must be of the form

$$Y = f(\lambda_4, \lambda_5, \cdots) X_1^{p_1} X_2^{p_2} X_3^{p_3}.$$

where f is a dimensionless function of the dimensionless variables.

Formally, we have the *Buckingham's Pi theorem*.

Example (Simple pendulum).



We want to find the period P. We know that P could depend on

- mass m: $[m] = M$
- length ℓ: $[\ell] = L$
- gravity g: $[g] = LT^{-2}$
- initial displacement d: $[d] = L$

and of course $[P] = T$.

We observe that m, ℓ, g have independent dimensions, and with the fourth, we can form the dimensionless group d/ℓ. So the relationship must be of the form

$$P = f\left(\frac{d}{l}\right) m^{p_1} \ell^{p_2} g^{p_3},$$

where f is a dimensionless function. For the dimensions to balance,

$$T = M^{p_1} L^{p_2} L^{p_3} T^{-2p_3}.$$

So $p_1 = 0, p_2 = -p_3 = 1/2$. Then

$$P = f\left(\frac{d}{\ell}\right) \sqrt{\frac{\ell}{g}}.$$

While we cannot find the exact formula, if ℓ is quadrupled and d is also quadrupled, the P will double.

3 Forces

3.1 Force and potential energy in one dimension

Consider a particle of mass m moving in a straight line with position $x(t)$. Suppose $F = F(x)$, i.e. depends on position only. We define the potential energy as follows:

Definition (Potential energy)**.** Given a force field $F = F(x)$, we define the *potential energy* to be a function $V(x)$ such that

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

or

$$V = -\int F\,\mathrm{d}x.$$

V includes an arbitrary additive constant.

The equation of motion is then

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{*}$$

Proposition. Suppose the equation of a particle satisfies

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{*}$$

Then the total energy

$$E = T + V = \frac{1}{2}m\dot{x}^2 + V(x)$$

is conserved, i.e. $\dot{E} = 0$.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= m\dot{x}\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x}\\ &= \dot{x}\left(m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right)\\ &= 0\end{aligned}$$

□

Example. Consider the harmonic oscillator

$$V = \frac{1}{2}kx^2.$$

Then the equation of motion satisfy

$$m\ddot{x} = -kx.$$

This is the case of, say, Hooke's Law for a spring. But in general, small perturbations near potential wells are also harmonic oscillators.

The general solution of this is

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

with $\omega = \sqrt{k/m}$.

A and B are arbitrary constants, and are related to the initial position and velocity by $x(0) = A, \dot{x}(0) = \omega B$.

For a general potential energy $V(x)$, conservation of energy allows us to solve the problem formally:

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

Since E is a constant, from this equation, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\sqrt{\frac{2}{m}(E - V(x))}$$

$$\pm\int \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}(E - V(x))}} = t - t_0$$

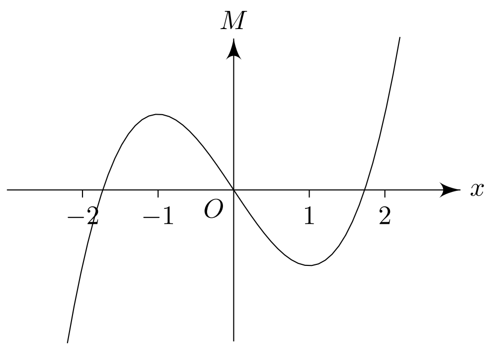
To find $x(t)$, we need to do the integral, we need to do the integral and then solve for x - not usually possible by analytical methods, but possible by numerical methods.

3.2 Motion in a potential

The graph of the potential energy $V(x)$ gives us a qualitative understanding of the dynamics, e.g. is the particle trapped or can it escape to infinity?

Example. Consider $V(x) = m(x^3 - 3x)$. Note that this can be dimensionally consistent even though we add up x^3 and $-3x$ if "3" has dimensions L^2.

We plot this as follows:



Suppose we release the particle from rest at $x = x_0$. Then $E = V(x_0)$. We can consider the different cases:

- $x_0 = \pm 1$: Particle stays there for all t.
- $-1 < x_0 < 2$: Particle oscillates back and forth in potential well
- $x_0 < -1$: Particle falls to $x = -\infty$.

- $x_0 > 2$: Particle overshoots well and continues to $x = -\infty$.
- $x_0 = 2$: Special case: The particle goes towards $x = -1$. How long does it take, and what happens next? Here $E = 2m$, we we noted previously

  $$t - t_0 = -\int \frac{\mathrm{d}x'}{\sqrt{\frac{2}{m}(E - V(x'))}}.$$

  Let $x = -1 + \varepsilon(t)$. Then

  $$\begin{aligned}\frac{2}{m}(E - V(x)) &= 4 - 2(-1+\varepsilon)^3 + 6(-1+\varepsilon)\\ &= 6\varepsilon^2 - 2\varepsilon^3.\end{aligned}$$

  So

  $$t - t_0 = -\int_3^{\varepsilon} \frac{\mathrm{d}\varepsilon'}{\sqrt{6\varepsilon'^2 - 2\varepsilon'^3}}$$

  Then the integrand is $\propto 1/\varepsilon'$ as $\varepsilon' \to 0$. So it takes infinite time to reach $\varepsilon = 0$, i.e. $x = -1$.

3.3 Equilibrium points

Definition (Equilibrium point)**.** A particle is in *equilibrium* if it has no tendency to move away. It will stay there for all time. Since $m\ddot{x} = -V'(x)$, the equilibrium points are the stationary points of the potential energy, i.e.

$$V'(x_0) = 0;$$

Consider motion near an equilibrium point. Expand V in a Taylor series:

$$V(x) \approx V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2.$$

Neglect the higher-order terms. Then the equation of motion is

$$m\ddot{x} = -V''(x_0)(x - x_0).$$

If $V''(x_0) > 0$, then V has a local minimum at x_0, and we have the potential of the harmonic oscillator. The equilibrium point is *stable* and the particle oscillates with angular frequency

$$\omega = \sqrt{\frac{V''(x_0)}{m}}.$$

This is only valid for small oscillations. This shows that small oscillations near stable equilibriums are stable.

If $V''(x_0) < 0$, then V has a local maximum at x_0. The equilibrium point is unstable. In this case,

$$x - x_0 \approx Ae^{\gamma t} + Be^{-\gamma t}$$

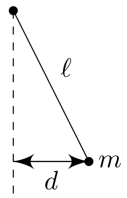
for

$$\gamma = \sqrt{\frac{-V''(x_0)}{m}}.$$

For almost all initial conditions, $A \neq 0$ and the particle will diverge from the equilibrium point, leading to a breakdown of the approximation.

If $V''(x_0) = 0$, then further work is required to determine the outcome.

Example. Consider the simple pendulum.



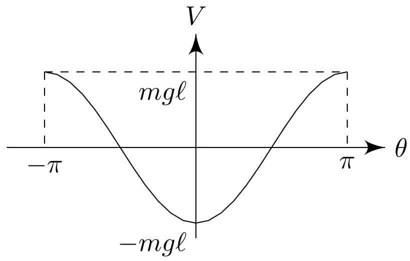
Suppose that the pendulum makes an angle θ with the vertical. The equation of motion is governed by

$$l\ddot{\theta} = -g\sin\theta.$$

The energy is

$$E = T + V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta.$$

Therefore $V \propto -\cos\theta$. We have a stable equilibrium at $\theta = 0$, and unstable equilibrium at $\theta = \pi$.



If $E > mg\ell$, then $\dot{\theta}$ never vanishes and the pendulum makes full circles.

If $E < -mg\ell < E < mg\ell$, then $\dot{\theta}$ vanishes at $\theta = \pm\theta_0$, where $0 < \theta_0 < \pi$¡ i.e. $E = -mg\ell\cos\theta_0$. The pendulum oscillates back and forth. It takes a quarter of a period to reach from $\theta = 0$ to $\theta = \theta_0$. Using the previous general solution, oscillation period P is given by

$$\frac{P}{4} = \int_0^{\theta_0} = \frac{\mathrm{d}\theta}{\sqrt{\frac{2E}{m\ell^2} + \frac{2g}{\ell}\cos\theta}}.$$

Since we know that $E = -mg\ell\cos\theta_0$, we know that

$$\frac{P}{4} = \sqrt{\frac{\ell}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\delta}{\sqrt{2\cos\theta - 2\cos\theta_0}}.$$

The integral is difficult to evaluate in general, but for small θ_0, we can use $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. So

$$P \approx \sqrt{\frac{\ell}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\theta_0^2 - \theta^2}} = 2\pi\sqrt{\frac{\ell}{g}}$$

and is independent of the amplitude θ_0. This is of course the result for the harmonic oscillator.

3.4 Force and potential energy in three dimensions

Consider a particle of mass m moving in 3D. The equation of motion is a vector equation

$$m\ddot{\mathbf{r}} = \mathbf{F}.$$

We define the kinetic energy of the particle is

$$T = \frac{1}{2}v|\mathbf{v}|^2 = \frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}.$$

If we want to know how it varies with time, we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \mathbf{F} \cdot \dot{\mathbf{r}} = \mathbf{F} \cdot \mathbf{v}.$$

This is the power.

Definition (Power)**.** The *power* is the rate at which work is done on a particle by a force. It is given by

$$P = \mathbf{F} \cdot \mathbf{v}.$$

Definition (Work done)**.** The *work done* on a particle by a force is the change in kinetic energy caused by the force. The work done on a particle moving from $\mathbf{r}_1 = \mathbf{r}(t_1)$ to $\mathbf{r}_2 = \mathbf{r}(t_2)$ along a trajectory C is the line integral

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}}\ \mathrm{d}t = \int_{t_1}^{t_2} P\ \mathrm{d}t.$$

Now we define a particular type of force known as *conservative force* that has many important properties.

Definition (Conservative force and potential energy)**.** A *conservative force* is a force field $\mathbf{F}(\mathbf{r})$ that can be written in the form

$$\mathbf{F} = -\nabla V.$$

V is the *potential energy function*.

Proposition. If $\mathbf{F}$ is conservative, then the energy

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2}m|\mathbf{v}|^2 + V(\mathbf{r}) \end{aligned}$$

is conserved. Then the work done is equal to the change in potential energy, and is independent of the path taken between the end points.

In particular, if we travelled around a closed loop, no work is done.

Proof.

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{\partial V}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} \\ &= (m\mathbf{r} + \ddot{\nabla} V \cdot \dot{\mathbf{r}} \\ &= (m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \dot{\mathbf{r}} \\ &= 0 \end{aligned}$$

In this case, the work done is

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C (\nabla V) \cdot \mathrm{d}\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2).$$

□

3.5 Central forces

This is a special class of conservative force, where the potential depends only on the distance from the origin.

Definition (Central force)**.** A central force is a force with a potential $V(r)$ that depends only on the distance from the origin, $r = |\mathbf{r}|$. Note that a central force can be both attractive or repulsive.

Note the following useful formula

Proposition. $\nabla r = \hat{\mathbf{r}}$.

This is because the direction in which r increases most rapidly is $\mathbf{r}$, and the rate of increase is clearly 1. This can also be proved algebraically:

Proof. We know that

$$r^2 = x_1^2 + x_2^2 + x_3^2.$$

Then

$$2r\frac{\partial r}{\partial x_i} = 2x_i.$$

So

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = (\hat{\mathbf{r}})_i.$$

□

Proposition. Let $\mathbf{F} = -\nabla V(r)$ be a central force. Then

$$\mathbf{F} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector in the radial direction pointing away from the origin.

Proof. Continuing the proof above,

$$(\nabla V)_u = \frac{\partial V}{\partial x_i} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\partial r}{\partial x_i} = \frac{\mathrm{d}V}{\mathrm{d}r}(\hat{\mathbf{r}})_i$$

□

Central forces give rise to an additional conserved quantity called *angular momentum.*

Definition (Angular momentum)**.** The *angular momentum* of a particle is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}.$$

Proposition. Angular momentum is conserved by a central force.

Proof.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r} \times \mathbf{F} = \mathbf{0}.$$

where the last equality comes from the fact that $\mathbf{F}$ is parallel to $\mathbf{r}$ for a central force. □

In general, for a non-central force, the rate of change of momentum is the *torque.*

Definition (Torque)**.** The *torque* G of a particle is the rate of change of momentum.

$$\mathbf{G} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F}.$$

Note that $\mathbf{L}$ and $\mathbf{G}$ depends on the choice of origin. For a central force, only the angular momentum about the center of the force is conserved.

3.6 Gravity

Gravity is a conservative and central force.

Law. If a particle of mass M is fixed at a origin, then a second particle of mass m experiences a potential energy

$$V(r) = -\frac{GMm}{r},$$

where $G \approx 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is the *gravitational constant.*

The gravitational force experienced is then

$$F = -\nabla V = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Since the force is negative, particles are attracted to the origin.

Definition (Gravitaional potential and field)**.** The *gravitational potential* is the gravitational potential energy per unit mass. It is

$$\Phi_g(r) = -\frac{GM}{r}.$$

Note that *potential* is different from *potential energy.*

The *gravitational field* is the force per unit mass,

$$\mathbf{g} = -\nabla\Phi_g = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

These are properties of the mass M alone.

Then the potential energy of a second particle is $V = m\Phi_g$.

Proposition. The gravitational potential due to many fixed masses M_i at points $\mathbf{r}_i$ is is

$$\Phi_g(\mathbf{r}) = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}.$$

Again, $V = m\Phi_g$ for a particle of mass m.

Proposition. The external gravitational potential of a spherically symmetric object of mass M is the same as that of a point particle with the same mass at the center of the object, i.e.

$$\Phi_g(r) = -\frac{GM}{r}.$$

Proof. c.f. Vector Calculus $\square$

Example. If you live on a spherical planet of mass M and radius R, and can move only a small distance $z \ll R$ above the surface, then

$$\begin{aligned} V(r) &= V(R+z) \\ &= -\frac{GMm}{r+z} \\ &= -\frac{GMm}{r}\left(1 - \frac{z}{R} + \cdots\right) \\ &\approx \text{const.} + \frac{GMm}{R^2} z \\ &= \text{const.} + mgz. \end{aligned}$$

where $g = GM/R^2 \approx 9.8\,\mathrm{m\,s^{-2}}$ for Earth.