

Part IA - Probability

Theorems

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Basic concepts

Classical probability, equally likely outcomes. Combinatorial analysis, permutations and combinations. Stirling's formula (asymptotics for $\log n!$ proved). [3]

Axiomatic approach

Axioms (countable case). Probability spaces. Inclusion-exclusion formula. Continuity and subadditivity of probability measures. Independence. Binomial, Poisson and geometric distributions. Relation between Poisson and binomial distributions. Conditional probability, Bayes's formula. Examples, including Simpson's paradox. [5]

Discrete random variables

Expectation. Functions of a random variable, indicator function, variance, standard deviation. Covariance, independence of random variables. Generating functions: sums of independent random variables, random sum formula, moments.

Conditional expectation. Random walks: gambler's ruin, recurrence relations. Difference equations and their solution. Mean time to absorption. Branching processes: generating functions and extinction probability. Combinatorial applications of generating functions. [7]

Continuous random variables

Distributions and density functions. Expectations; expectation of a function of a random variable. Uniform, normal and exponential random variables. Memoryless property of exponential distribution. Joint distributions: transformation of random variables (including Jacobians), examples. Simulation: generating continuous random variables, independent normal random variables. Geometrical probability: Bertrand's paradox, Buffon's needle. Correlation coefficient, bivariate normal random variables. [6]

Inequalities and limits

Markov's inequality, Chebyshev's inequality. Weak law of large numbers. Convexity: Jensen's inequality for general random variables, AM/GM inequality.

Moment generating functions and statement (no proof) of continuity theorem. Statement of central limit theorem and sketch of proof. Examples, including sampling. [3]

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1 Introduction

2 Classical probability

2.1 Classical probability

2.2 Sample space and events

3 Combinatorial analysis

3.1 Counting

Theorem (Fundamental rule of counting). Suppose we have to make r multiple choices in sequence. There are m_1 possibilities for the first choice, m_2 possibilities for the second etc. Then the total number of choices is $m_1 \times m_2 \times \cdots m_r$.

3.2 Sampling with or without replacement

3.3 Sampling with or without regard to ordering

3.4 Four cases of enumerative combinatorics

4 Stirling's formula

4.1 Multinomial coefficient

4.2 Stirling's formula

Proposition. $\log n! \sim n \log n$

Theorem (Stirling's formula). As $n \rightarrow \infty$,

$$\log \left(\frac{n! e^n}{n^{n+\frac{1}{2}}} \right) = \log \sqrt{2\pi} + O\left(\frac{1}{n}\right)$$

Corollary.

$$n! \sim \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n}$$

Proposition ((non-examinable)). We use the $1/12n$ term from the proof above to get a better approximation:

$$\sqrt{2\pi n} n^{n+1/2} e^{-n} + \frac{1}{12n+1} \leq n! \leq \sqrt{2\pi n} n^{n+1/2} e^{-n} + \frac{1}{12n}.$$