Part IA - Differential Equations
Theorems with Proof

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Basic calculus
Informal treatment of differentiation as a limit, the chain rule, Leibnitz's rule, Taylor series, informal treatment of O and o notation and l'Hôpitals rule; integration as an area, fundamental theorem of calculus, integration by substitution and parts. [3]

Informal treatment of partial derivatives, geometrical interpretation, statement (only) of symmetry of mixed partial derivatives, chain rule, implicit differentiation. Informal treatment of differentials, including exact differentials. Differentiation of an integral with respect to a parameter. [2]

First-order linear differential equations
Equations with constant coefficients: exponential growth, comparison with discrete equations, series solution; modelling examples including radioactive decay.

Equations with non-constant coefficients: solution by integrating factor. [2]

Nonlinear first-order equations
Separable equations. Exact equations. Sketching solution trajectories. Equilibrium solutions, stability by perturbation; examples, including logistic equation and chemical kinetics. Discrete equations: equilibrium solutions, stability; examples including the logistic map. [4]

Higher-order linear differential equations
Complementary function and particular integral, linear independence, Wronskian (for second-order equations), Abel's theorem. Equations with constant coefficients and examples including radioactive sequences, comparison in simple cases with difference equations, reduction of order, resonance, transients, damping. Homogeneous equations. Response to step and impulse function inputs; introduction to the notions of the Heaviside step-function and the Dirac delta-function. Series solutions including statement only of the need for the logarithmic solution. [8]

Multivariate functions: applications
Directional derivatives and the gradient vector. Statement of Taylor series for functions on $\mathbb{R}^n$. Local extrema of real functions, classification using the Hessian matrix. Coupled first order systems: equivalence to single higher order equations; solution by matrix methods. Non-degenerate phase portraits local to equilibrium points; stability.

Simple examples of first- and second-order partial differential equations, solution of the wave equation in the form $f(x+ct)+g(x-ct)$. [5]

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1 Differentiation

1.1 Differentiation

1.2 Small o and big O notations

Proposition.

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$$

Proof. We have

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{o(h)}{h}$$

by the definition of the derivative and the small o notation. The result follows. $\square$

1.3 Methods of differentiation

Theorem (Chain rule)**.** Given $f(x) = F(g(x))$, then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}F}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}.$$

Proof. Assuming that $\frac{\mathrm{d}g}{\mathrm{d}x}$ exists and is therefore finite, we have

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0}\frac{F(g(x+h)) - F(g(x))}{h} \\
&= \lim_{h\to 0}\frac{F[g(x) + hg'(x) + o(h)] - F(g(x))}{h} \\
&= \lim_{h\to 0}\frac{F(g(x)) + (hg'(x) + o(h))F'(g(x)) + o(hg'(x) + o(h)) - F(g(x))}{h} \\
&= \lim_{h\to 0} g'(x)F'(g(x)) + \frac{o(h)}{h} \\
&= g'(x)F'(g(x)) \\
&= \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}
\end{aligned}$$

$\square$

Theorem (Product Rule)**.** Give $f(x) = u(x)v(x)$. Then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Theorem (Leibniz's Rule)**.** Given $f = uv$, then

$$f^{(n)}(x) = \sum_{r=0}^{n}\binom{n}{r}u^{(r)}v^{(n-r)},$$

where $f^{(n)}$ is the n-th derivative of f.

1.4 Taylor's theorem

Theorem (Taylor's Theorem)**.** For n-times differentiable f, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + E_n,$$

where $E_n = o(h^n)$ as $h \to 0$. If $f^{(n+1)}$ exists, then $E_n = O(h^{n+1})$.

1.5 L'Hopital's rule

Theorem (L'Hopital's Rule)**.** Let $f(x)$ and $g(x)$ be differentiable at x_0, and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$. Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. From the Taylor's Theorem, we have $f(x) = f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)$, and similarly for $g(x)$. Thus

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x)}{g(x)} &= \lim_{x \to x_0} \frac{f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)}{g(x_0) + (x - x_0)g'(x_0) + o(x - x_0)} \\
&= \lim_{x \to x_0} \frac{f'(x_0) + \frac{o(x - x_0)}{x - x_0}}{g'(x_0) + \frac{o(x - x_0)}{x - x_0}} \\
&= \lim_{x \to x_0} \frac{f'(x)}{g'(x)}
\end{aligned}$$

□

2 Integration

2.1 Integration

Theorem (Fundamental Theorem of Calculus)**.** Let $F(x) = \int_a^x f(t)\ \mathrm{d}t$. Then $F'(x) = f(x)$.

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}F(x) &= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\ \mathrm{d}t - \int_a^x f(t)\ \mathrm{d}t\right] \\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\ \mathrm{d}t \\
&= \lim_{h\to 0}\frac{1}{h}[f(x)h + O(h^2)] \\
&= f(x)
\end{aligned}$$

□

2.2 Methods of integration

Theorem (Integration by parts)**.**

$$\int uv'\ \mathrm{d}x = uv - \int vu'\ \mathrm{d}x.$$

Proof. From the product rule, we have $(uv)' = uv' + u'v$. Integrating the whole expression and rearranging gives the formula above. □

3 Partial differentiation

3.1 Partial differentiation

Theorem. $f_{xy} = f_{yx}$.

3.2 Chain rule

Theorem (Chain rule for partial derivatives)**.**

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y.$$

Given this form, we can sum the differentials to obtain the integral form:

$$\int \mathrm{d}f = \int \frac{\partial f}{\partial x}\mathrm{d}x + \int \frac{\partial f}{\partial y}\mathrm{d}y,$$

or divide by another small quantity. e.g. to find the slope along the path $(x(t), y(t))$, we can divide by $\mathrm{d}t$ to obtain

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

3.3 Implicit differentiation

Theorem (Multi-variable implicit differentiation)**.** Given an equation

$$F(x, y, z) = c$$

for some constant c, we have

$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{(\partial F)/(\partial x)}{(\partial F)/(\partial z)}$$

Proof.

$$\begin{aligned}
\mathrm{d}F &= \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z \\
\left.\frac{\partial F}{\partial x}\right|_y &= \frac{\partial F}{\partial x}\left.\frac{\partial x}{\partial x}\right|_y + \frac{\partial F}{\partial y}\left.\frac{\partial y}{\partial x}\right|_y + \frac{\partial F}{\partial z}\left.\frac{\partial z}{\partial x}\right|_y = 0 \\
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\left.\frac{\partial z}{\partial x}\right|_y &= 0 \\
\left.\frac{\partial z}{\partial x}\right|_y &= -\frac{(\partial F)/(\partial x)}{(\partial F)/(\partial z)}
\end{aligned}$$

□

3.4 Differentiation of an integral w.r.t. parameter in the integrand

Theorem (Differentiation under the integral sign)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{b(x)} f(x, c(x))\,\mathrm{d}x = f(b, c)b'(x) + c'(x)\int_0^b \frac{\partial f}{\partial c}\,\mathrm{d}y$$

4 First-order differential equations

4.1 The exponential function

4.2 Homogeneous linear ordinary differential equations

Theorem. Any linear, homogeneous, ordinary differential equation with constant coefficients has solutions of the form e^{mx}.

4.2.1 Discrete equations

4.2.2 Series solution

4.3 Forced (inhomogeneous) equations

4.3.1 Constant forcing

4.3.2 Eigenfunction forcing

4.4 Non-constant coefficients

4.5 Non-linear equations

4.5.1 Separable equations

4.5.2 Exact equations

Theorem. (Converse of above result) If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ through a simply-connected domain $\mathcal{D}$, then $P\ \mathrm{d}x + Q\ \mathrm{d}y$ is an exact differential of a single-valued function in $\mathcal{D}$.

4.6 Solution curves (trajectories)

4.7 Fixed (equilibrium) points and stability

4.7.1 Perturbation analysis

4.7.2 Autonomous systems

4.7.3 Logistic Equation

4.8 Discrete equations (Difference equations)

5 Second-order differential equations

5.1 Constant coefficients

5.1.1 Complementary functions

5.1.2 Second complementary function

5.1.3 Phase space

Theorem (Abel's Theorem)**.** Given an equation $y'' + p(x)y' + q(x)y = 0$, either $W = 0$ for all x, or $W \neq 0$ for all x. i.e. iff two solutions are independent for some particular x, then they are independent for all x.

Proof. If y_1 and y_2 are both solutions, then

$$\begin{aligned} y_2(y_1'' + py_1' + qy_1) = 0 \\ y_1(y_2'' + py_2' + qy_2) = 0 \end{aligned}$$

Subtracting the two equations, we have

$$y_1 y_2'' - y_2 y_1'' + p(y_1 y_2' - y_2 y_1') = 0$$

Note that $W = y_1 y_2' - y_2 y_1'$ and $W' = y_1 y_2'' + y_1' y_2' - (y_2' y_1' + y_2 y_1'') = y_1 y_2'' - y_2 y_1''$

$$\begin{aligned} W' + P(x)W = 0 \\ W(x) = W_0 e^{-\int P \, \mathrm{d}x}, \end{aligned}$$

Where $W_0 = \text{const}$. Since the exponential function is never zero, either $W_0 = 0$, in which case $W = 0$, or $W_0 \neq 0$ and $W \neq 0$ for any value of x. $\square$

5.2 Particular integrals

5.2.1 Guessing

5.2.2 Resonance

5.2.3 Variation of parameters

5.3 Linear equidimensional equations

5.3.1 Solving by eigenfunctions

5.3.2 Solving by substitution

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6 Series solutions

6.1 Ordinary points

6.2 Regular singular points

6.3 Behaviour near $x = x_0$

7 Directional derivative

7.1 Stationary points

7.2 Taylor series for multi-variable functions

7.3 Classification of stationary points

7.3.1 Determination of definiteness

Proposition. H is positive definite if and only if the signature is $+, +, \cdots, +$. H is negative definite if and only if the signature is $-, +, \cdots, (-1)^n$. Otherwise, H is indefinite.

7.3.2 Contours of $f(x, y)$

8 Systems of linear differential equations

8.1 Nonlinear dynamical systems

8.1.1 Equilibrium (fixed) points

8.1.2 Stability

9 Partial differential equations (PDEs)

9.1 First-order wave equation

9.2 Second-order wave equation

9.3 The diffusion equation