Part IA - Groups
Definitions

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Examples of groups
Axioms for groups. Examples from geometry: symmetry groups of regular polygons, cube, tetrahedron. Permutations on a set; the symmetric group. Subgroups and homomorphisms. Symmetry groups as subgroups of general permutation groups. The Möbius group; cross-ratios, preservation of circles, the point at infinity. Conjugation. Fixed points of Möbius maps and iteration. [4]

Lagranges theorem
Cosets. Lagranges theorem. Groups of small order (up to order 8). Quaternions. Fermat-Euler theorem from the group-theoretic point of view. [5]

Group actions
Group actions; orbits and stabilizers. Orbit-stabilizer theorem. Cayley's theorem (every group is isomorphic to a subgroup of a permutation group). Conjugacy classes. Cauchy's theorem. [4]

Quotient groups
Normal subgroups, quotient groups and the isomorphism theorem. [4]

Matrix groups
The general and special linear groups; relation with the Möbius group. The orthogonal and special orthogonal groups. Proof (in $\mathbb{R}^3$) that every element of the orthogonal group is the product of reflections and every rotation in $\mathbb{R}^3$ has an axis. Basis change as an example of conjugation. [3]

Permutations
Permutations, cycles and transpositions. The sign of a permutation. Conjugacy in S_n and in A_n. Simple groups; simplicity of A_5. [4]

Contents

1 Groups and homomorphisms

1.1 Groups

Definition (Binary operation)**.** A *(binary) operation* is a way of combining two elements to get a new element. Formally, it is a map $* : A \times A \to A$.

Definition (Group)**.** A *group* is a set G with a binary operation $*$ satisfying the following axioms:

0. (Closure) $\forall a, b \in G, a * b \in G$
1. (Identity) $\exists e \in G(\forall a \in G(a * e = e * a = a))$
2. (Inverse) $\forall a \in G(\exists a^{-1} \in G(a * a^{-1} = a^{-1} * a = e))$
3. (Associativity) $\forall a, b, c \in G((a * b) * c = (a * (b * c)))$

Definition (Abelian group)**.** A group is *abelian* if it satisfies

4. (Commutativity) $\forall a, b \in G(a * b = b * a)$

Definition (Order of group)**.** The *order* of the group, denoted as $|G|$, is the number of elements in G. A group is a finite group is the order is finite.

Definition (Subgroup)**.** A *subgroup* $H \leq G$ is a subset $H \subseteq G$ such that H with the restricted operation $*$ from G is also a group.

1.2 Homomorphisms

Definition (Function)**.** Given 2 sets X, Y, a *function* $f : X \to Y$ sends each $x \in X$ to a particular $f(x) \in Y$. X is called the domain and Y is the co-domain.

Definition (Composition of functions)**.** The *composition* of two functions is a function you get by applying one after another. In particular, if $f : X \to Y$ and $G : Y \to Z$, then $g \circ f : X \to Z$ with $g \circ f(x) = g(f(x))$.

Definition (Injective functions)**.** A function f is *injective* if it hits everything at most once, i.e.

$$\forall x, y \in X(f(x) = f(y) \Rightarrow x = y)$$

Definition (Surjective functions)**.** A function is *surjective* if it hits everything at least once, i.e.

$$\forall y \in Y(\exists x \in X(f(x) = y))$$

Definition (Bijective functions)**.** A function is *bijective* if it is both injective and surjective. i.e. it hits everything exactly once. Note that a function has an inverse iff it is bijective.

Definition (Group homomorphism)**.** Let $(G, *)$ and $(H, \times)$ be groups. A function $f : G \to H$ is a *group homomorphism* iff

$$\forall g_1, g_2 \in G : f(g_1) \times f(g_2) = f(g_1 * g_2),$$

i.e. they "preserve group properties"

Definition (Group isomorphism)**.** *Isomorphisms* are bijective homomorphisms. 2 groups are *isomorphic* if there exists an isomorphism between them. We write $G \cong H$.

Definition (Image of homomorphism)**.** If $f : G \to H$ is a homomorphism, then the *image* of f is

$$\operatorname{im} f = f(G) = \{f(g) : g \in G\}.$$

Definition (Kernel of homomorphism)**.** The *kernel* of f, written as

$$\ker f = f^{-1}(\{e_H\}) = \{g \in G : f(g) = e_H\}.$$

1.3 Cyclic groups

Definition (Cyclic group C_n)**.** A group G if *cyclic* if $\exists a \in G(\forall b \in G(\exists n \in \mathbb{Z}(b = a^n)))$, i.e. every element is some power of a. Such an a is called a generator of G.

Definition (Order of element)**.** The *order* of an element a is the smallest integer n such that $a^n = e$. If k doesn't exist, a has infinite order. Write $\operatorname{ord}(a)$ for the order of a.

Definition (Exponent of group)**.** The *exponent* of a group G is the smallest integer n such that $\forall a(a^n = e)$.

1.4 Dihedral groups

Definition (Dihedral groups D_{2n})**.** Dihedral groups are the symmetries of a regular n-gon. It contains n rotations (including the identity symmetry, i.e. rotation by $0°$) and n reflections. All rotations are generated by $r = \frac{360°}{n}$. r has order n. Any reflection has order 2.

Now consider any reflection s. Then r and s generate the whole group. We have

$$\begin{aligned} D_{2n} &= \langle r, s | r^n = e = s^2, srs^{-1} = r^{-1} \rangle \\ &= \{e, r, r^2, \cdots r^{n-1}, s, rs, r^2 s, \cdots r^{n-1} s\} \end{aligned}$$

Note that we have $sr = r^{-1}s$ and $sr^k = r^{-k}s = r^{n-k}s$.

1.5 Direct products of groups

Definition (Direct product of groups)**.** Given two groups $(G_1, *_1)$ and $(G_2, *_2)$, we can define a set $G_1 \times G_2 = \{(g_1, g_2) : g_i \in G_i\}$ and an operation $(a_1, a_2) * (b_1, b_2) = (a_1 *_1 b_1, a_2 *_2 b_2)$. This forms a group.

2 Symmetric group I

Definition (Permutation)**.** A *permutation* of X is a bijection from a set X to X itself. The set of all permutations on X is $\operatorname{Sym} X$.

Definition (Symmetric group S_n)**.** If X is finite, say $|X| = n$ (usually use $X = \{1, 2, \cdots n\}$), we write $\operatorname{Sym} X = S_n$. The is THE *symmetric group* of degree n.

Definition (k-cycles and transpositions)**.** We call $(a_1\ a_2\ a_3 \cdots a_k)$ *k-cycles*. 2-cycles are called *transpositions*. Two cycles are *disjoint* if no number appears in both cycles.

Definition (Cycle type)**.** Write a permutation $\sigma \in S_n$ in disjoint cycle notation. The *cycle type* is the list of cycle lengths. This is unique up to re-ordering. We often (but not always) leave out singleton cycles.

2.1 Sign of permutations

Definition (Sign of permutation)**.** Viewing $\sigma \in S_n$ as a product of transpositions, $\sigma = \tau_1 \cdots \tau_l$, we call $\operatorname{sgn}(\sigma) = (-1)^l$. If $\operatorname{sgn}(\sigma) = 1$, we call σ an even permutation. If $\operatorname{sgn}(\sigma) = -1$, we call σ an odd permutation.

Definition (Alternating group A_n)**.** The *alternating group* A_n is the kernel of sgn, i.e. the even permutations. Since A_n is a kernel of a group homomorphism, $A_n \leq S_n$.

3 Lagrange's Theorem

Definition (Cosets)**.** Let $H \leq G$ and $a \in G$. Then the set $aH = \{ah : h \in H\}$ is a *left coset* of H and $Ha = \{ha : h \in H\}$ is a *right coset* of H.

Definition (Partition)**.** Let X be a set, and $X_1, \cdots X_n$ be subsets of X. The X_i are called a *partition* of X if $\bigcup X_i = X$ and $X_i \cap X_j = \emptyset$ for $i \neq j$. i.e. every element is in exactly one of X_i.

Definition (Index of a subgroup)**.** The *index* of H in G ($|G : H|$) is the number of left cosets in G.

Definition (Equivalence relation)**.** An *equivalence relation* $\sim$ is a relation that is reflexive, symmetric and transitive. i.e.

(i) Reflexive: $\forall x(x \sim x)$

(ii) Symmetric: $\forall x, y(x \sim y \Rightarrow y \sim x)$

(iii) Transitive $\forall x, y, z[(x \sim y) \wedge (y \sim z) \Rightarrow x \sim z]$

Definition (Equivalence class)**.** Given an equivalence relation $\sim$ on A, the *equivalence class* of a is

$$[a]_\sim = [a] = \{b \in A | a \sim b\}$$

Definition (Euler totient function)**.** (Euler totient function) $\phi(n) = |U_n|$.

3.1 Small groups

3.2 Left and right cosets

4 Quotient groups

4.1 Normal subgroups

Definition (Normal subgroup)**.** A subgroup K of G is a *normal subgroup* if $\forall a \in G(\forall k \in K(aka^{-1} \in K))$. We write $K \lhd G$. This is equivalent to:

(i) $\forall a \in G(aK = Ka)$, i.e. left coset = right coset

(ii) $\forall a \in G(aKa^{-1} = K$ (c.f. conjugacy classes)

4.2 Quotient groups

Definition (Quotient group)**.** Given a group G and a normal subgroup K, the *quotient group* or *factor group* of G by K, written as G/K, is the set of (left) cosets of K in G under the operation $aK * bK = (ab)K$.

4.3 The Isomorphism Theorem

Definition (Simple group)**.** A group is *simple* if it has no non-trivial proper normal subgroup, i.e. only $\{e\}$ and G are normal subgroups.

5 Group actions

5.1 Group acting on sets

Definition (Group action)**.** Let X be a set and G be a group. An *action* of G on X is a function $\theta : G \times X \to X$ satisfying

0. $\forall g \in G, x \in X[\theta(g, x) \in X]$.
1. $\forall x \in X[\theta(e, x) = x]$.
2. $\forall g, h \in G, x \in X[\theta(g, \theta(h, x)) = \theta(gh, x)]$

i.e. given an element $g \in G$ and an $x \in X$, g "acts on" x to give an element $\theta(g, x) \in X$ (the two conditions ensure that the group properties of G are not destroyed)

Definition (Kernel of action)**.** The *kernel* of an action G on X is the kernel of φ, i.e. all g such that $\theta_g = 1_X$.

Definition (Faithful action)**.** An action is *faithful* if the kernel is just $\{e\}$.

5.2 Orbits and Stabilizers

Definition (Orbit of action)**.** Given an action G on X, the *orbit* of an element $x \in X$ is

$$\mathrm{orb}(x) = G(x) = \{y \in X : \exists g \in G(g(x) = y)\}.$$

Intuitively, it is the elements that x can possibly get mapped to.

Definition (Stabilizer of action)**.** The *stabilizer* of x is

$$\mathrm{stab}(x) = G_x = \{g \in G : g(x) = x\} \subseteq G.$$

Intuitively, it is the elements in G that do not change x.

Definition (Transitive action)**.** An action G on X is *transitive* if $\forall x(\mathrm{orb}(x) = X)$, i.e. you can reach any element from any element.

5.3 Important actions

Definition (Conjugation of element)**.** The *conjugation* of $a \in G$ by $b \in G$ is given by $bab^{-1} \in G$.

Definition (Center of group)**.** The *center* of G is the elements that commute with all other elements.

$$Z(g) = \{g \in G : \forall a(gag^{-1} = a)\} = \{g \in G : \forall a(ga = ag)\}.$$

It is sometimes written as $C(G)$ instead of $Z(G)$.

Definition (Conjugacy classes and centralizers)**.** The *conjugacy classes* are the orbits of the conjugacy action.

$$\mathrm{ccl}(a) = \{b \in G : \exists g \in g(gag^{-1} = b\}.$$

The *centralizers* are the stabilizers of this action, i.e. elements that commute with a.

$$C_G(a) = \{g \in G : gag^{-1} = a\} = \{g \in G : ga = ag\}.$$

Definition (Normalizer of subgroup)**.** The *normalizer* of a subgroup is the stabilizer of the (group) conjugation action.

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

5.4 Applications

6 Symmetric groups II

6.1 Conjugacy classes in S_n

6.2 Conjugacy classes in A_n

Definition (Splitting of conjugacy classes)**.** When $|\operatorname{ccl}_{A_n}(\sigma)| = \frac{1}{2}|\operatorname{ccl}_{S_n}(\sigma)|$, we say that the conjugacy class of σ *splits* in A_n.

7 Quaternions

Definition (Quaternions)**.** The *quaternions* is the set of matrices

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$$
$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$$

which is a subgroup of $\mathrm{GL}_2(\mathbb{C})$.

8 Matrix groups

8.1 General and special linear groups

Definition (General linear group $\mathrm{GL}_n(F)$)**.**

$$\mathrm{GL}_n(F) = \{A \in M_{n\times n}(F) : A \text{ is invertible}\}$$

is the *general linear group.*

Definition (Special linear group $\mathrm{SL}_n(F)$)**.** The *special linear group* $\mathrm{SL}_n(F)$ is the kernel of the determinant, i.e.

$$SL_n(F) = \{A \in \mathrm{GL}_n(F) : \det A = 1\}.$$

8.2 Actions of $\mathrm{GL}_n(\mathbb{C})$

8.3 Orthogonal groups

Definition (Orthogonal group $O(n)$)**.** The *orthogonal group* is

$$\mathrm{O}(n) = \mathrm{O}_n = \mathrm{O}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = I\},$$

i.e. the group of orthogonal matrices.

Definition (Special orthogonal group $SO(n)$)**.** The *special orthogonal group* is the kernel of $\det : \mathrm{O}(n) \to \{\pm 1\}$.

$$\mathrm{SO}(n) = \mathrm{SO}_n = \mathrm{SO}_n(\mathbb{R}) = \{A \in \mathrm{O}(n) : \det A = 1\}.$$

8.4 Rotations and reflections in $\mathbb{R}^2$

8.5 Unitary groups

Definition (Unitary group $U(n)$)**.** The *unitary group* is $\mathrm{U}(n) = \mathrm{U}_n = \{A \in \mathrm{GL}_n(\mathbb{C}) : A^\dagger A = I\}$.

Definition (Special unitary group $SU(n)$)**.** The *special unitary group* $\mathrm{SU}(n) = \mathrm{SU}_n$ is the kernel of $\det U(n) \to S^1$.

9 More on regular polyhedra

9.1 Symmetries of the cube

9.1.1 Rotations

9.1.2 All symmetries

9.2 Symmetries of the tetrahedron

9.2.1 Rotations

9.2.2 All symmetries

10 Möbius group

Definition (Möbius map)**.** A *Möbius map* is a map from $\mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, with $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$ when $c \neq 0$. (if $c = 0$, then $f(\infty) = \infty$)

Definition (Projective general linear group $\mathrm{PGL}_2(\mathbb{C})$)**.** (Non-examinable)The projective general linear group is

$$\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/Z.$$

10.1 Fixed points of Möbius maps

Definition (Fixed point)**.** A *fixed point* of f is a z such that $f(z) = z$.

10.2 Permutation properties of Möbius maps

Definition (Three-transitive action)**.** An action of G on X is called *three-transitive* if the induced action on $\{(x_1, x_2, x_3) \in X^3 : x_i \text{ pairwise disjoint}\}$, given by $g(x_1, x_2, x_3) = (g(x_1), g(x_2), g(x_3))$, is transitive.

This means that for any two triples x_1, x_2, x_3 and y_1, y_2, y_3 of distinct elements of X, there exists $g \in G$ such that $g(x_i) = y_i$.

If this g is always unique, then the action is called *sharply three transitive*

10.3 Cross-ratios

Definition (Cross-ratios)**.** Given four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$, their *cross-ratio* is $[z_1, z_2, z_3, z_4] = g(z_4)$, with g being the unique Möbius map that maps $z_1 \mapsto \infty, z_2 \mapsto 0, z_3 \mapsto 1$. So $[\infty, 0, 1, \lambda] = \lambda$ for any $\lambda \neq \infty, 0, 1$. We have

$$[z_1, z_2, z_3, z_4] = \frac{z_4 - z_2}{z_4 - z_1} \cdot \frac{z_3 - z_1}{z_3 - z_2}$$

(with special cases as above).

11 Projective line (non-examinable)