Part IA - Analysis I
Theorems

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Limits and convergence
Sequences and series in R and C. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

Continuity
Continuity of real- and complex-valued functions defined on subsets of $\mathbb{R}$ and $\mathbb{C}$. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

Differentiability
Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from $\mathbb{R}$ to $\mathbb{R}$; Lagranges form of the remainder. Complex differentiation. [5]

Power series
Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. *Direct proof of the differentiability of a power series within its circle of convergence*. [4]

Integration
Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

Contents

1   The real number system

Lemma. Let $\mathbb{F}$ be an ordered field and $x \in \mathbb{F}$. Then $x^2 \geq 0$.

Lemma (Archimedean property v1)**.** Let $\mathbb{F}$ be an ordered field with the least upper bound property. Then the set $\{1, 2, 3, \cdots\}$ is not bounded above. (Note that these need not refer to natural numbers. We simply define 1 to be the multiplicative identity, $2 = 1 + 1$, $3 = 1 + 2$ etc.)

2 Convergence of sequences

Lemma (Archimedean property v2). $1/n \to 0$.

Lemma. Every eventually bounded sequence is bounded.

2.1 Sums, products and quotients

Lemma (Sums of sequences). If $a_n \to a$ and $b_n \to b$, then

(i) $a_n + b_n \to a + b$

Lemma (Scalar multiplication of sequences). Let $a_n \to a$ and $\lambda \in \mathbb{R}$. Then $\lambda a_n \to \lambda a$.

Lemma. Let a_n be bounded $b_n \to 0$. Then $a_n b_n \to 0$.

Lemma. Every convergent sequence is bounded.

Lemma. Let $a_n \to a$ and $b_n \to b$. Then $a_n b_n \to ab$.