Part II - Logic and Set Theory

Theorems with Proof

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Ordinals and cardinals
Well-orderings and order-types. Examples of countable ordinals. Uncountable ordinals and Hartogs' lemma. Induction and recursion for ordinals. Ordinal arithmetic. Cardinals; the hierarchy of alephs. Cardinal arithmetic. [5]

Posets and Zorn's lemma
Partially ordered sets; Hasse diagrams, chains, maximal elements. Lattices and Boolean algebras. Complete and chain-complete posets; fixed-point theorems. The axiom of choice and Zorn's lemma. Applications of Zorn's lemma in mathematics. The well-ordering principle. [5]

Propositional logic
The propositional calculus. Semantic and syntactic entailment. The deduction and completeness theorems. Applications: compactness and decidability. [3]

Predicate logic
The predicate calculus with equality. Examples of first-order languages and theories. Statement of the completeness theorem; *sketch of proof*. The compactness theorem and the Lowenheim-Skolem theorems. Limitations of first-order logic. Model theory. [5]

Set theory
Set theory as a first-order theory; the axioms of ZF set theory. Transitive closures, epsilon-induction and epsilon-recursion. Well-founded relations. Mostowski's collapsing theorem. The rank function and the von Neumann hierarchy. [5]

Consistency
Problems of consistency and independence [1]

Contents

1 Propositional calculus

1.1 Semantic implication

Proposition.

(i) If v and v' are valuations with $v(p) = v'(p)$ for all $p \in P$, then $v = v'$.

(ii) For any function $W : P \to \{0, 1\}$, there is a valuation v such that $v(p) = w(p)$ for all $p \in L$, i.e. we can extend w to a full valuation.

This means "A valuation is determined by its values on P, and any values will do".

Proof. (i) Recall that L is defined inductively. We are given that $v(p) = v'(p)$ on L_1. Then for all $p \in L_2$, p must be in the form $q \Rightarrow r$ for $q, r \in L_1$. Then $v(q \Rightarrow r) = v(p \Rightarrow q)$ since the value of v is uniquely determined by the definition. So for all $p \in L_2$, $v(p) = v'(p)$.

Continue inductively to show that $v(p) = v'(p)$ for all $p \in L_n$ for any n.

(ii) Set v to agree with w for all $p \in P$, and set $v(\perp) = 0$. Then define v on L_n inductively according to the definition.

□

1.2 Syntactic implication

Proposition (Deduction theorem)**.** Let $S \subset L$ and $p, q \in L$. Then $S \vdash (p \Rightarrow q)$ if and only if $S \cup p \vdash q$.

"$\vdash$ behaves like the connective $\Rightarrow$ in the language"

Proof. ($\Rightarrow$) Given a proof of $p \Rightarrow q$ from S, append the lines

- p Hypothesis
- q MP

to obtain a proof of q from $S \cup \{q\}$.

($\Leftarrow$) Let $t_1, t_2, \cdots, t_n = q$ be a proof of q from $S \cup \{p\}$. We'll show that $S \vdash p \Rightarrow t_i$ for all i.

We consider different possibilities of t_i:

- t_i is an axiom: Write down
  - $t_i \Rightarrow (p \Rightarrow t_i)$ (Axiom 1)
  - t_i Axiom
  - $p \Rightarrow t_i$ MP
- $t_i \in S$: Write down
  - $t_i \Rightarrow (p \Rightarrow t_i)$ (Axiom 1)
  - t_i Hypothesis
  - $p \Rightarrow t_i$ MP

To get $S \models (p \Rightarrow t_i)$

– $t_i = p$: Write down our proof of $p \Rightarrow p$ from our example above.

– t_i is obtained by MP: we have some $j, k < i$ such that $t_k = (t_k \Rightarrow t_i)$. We can assume that $S \vdash (p \Rightarrow t_j)$ and $S \vdash (p \Rightarrow t_k)$ by induction on i. Now we can write down

◦ $[p \Rightarrow (t_j \Rightarrow t_i)] \Rightarrow [(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)]$	Axiom 2
◦ $p \Rightarrow (t_j \Rightarrow t_i)$	Known already
◦ $(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)$	MP
◦ $p \Rightarrow t_j$	Known already
◦ $p \Rightarrow t_i$	MP

to get $S \models (p \Rightarrow t_i)$.

This is why Axiom 2 is as it is - it enables us to prove the deduction theorem. □

Proposition (Soundness theorem)**.** If $S \vdash t$, then $S \models t$.

Proof. Given valuation v with $v(s) = 1$ for all $s \in S$, we need to show that $v(t) = 1$. But $v(p) = 1$ for all axioms p, and $v(p) = 1$ for all $p \in S$, and if $v(p) = 1$ and $v(p \Rightarrow q) = 1$, then $v(q) = 1$. Hence each line t_i in a proof $t_1, \cdots, t_n$ of from S has $v(t_i) = 1$. □

Theorem (Model existence theorem)**.** If $S \models \bot$, then $S \vdash \bot$. i.e., if S has no model, then S is consistent. i.e. If S is consistent, then S has a model.
Note: Some books call this the "completeness theorem", because the rest of the completeness theorem follows trivially from this.

Proof. The idea is that we'd like to define $v : L \to \{0, 1\}$ by

$$p \mapsto \begin{cases} 1 & \text{if } p \in S \\ 0 & \text{if } p \in S \end{cases}$$

However, this is obviously going to fail, because we might have implications of S that are not in S, i.e. S is not *deductively closed* (deductively closed means $S \vdash p$ implies $p \in S$). Yet this is not a serious problem - we take the deductive closure first.

But there is a more serious problem. There might be a p with $S \nvdash p$ and $S \nvdash \neg p$. This is the case if p never appears in S. We'll try to extend S to "swallow up" half of L. i.e. we give p an arbitrary truth value, without making S inconsistent. The meat of this proof is then to see that we can "swallow up" propositions consistently.

We first try to swallow up one proposition only.
Claim. For consistent $S \subset L$ and $p \in L$, we have $S \cup \{p\}$ or $S \cup \{\neg p\}$ consistent.

Suppose instead that $S \cup \{p\} \vdash \bot$ and $S \cup \{\neg p\} \vdash \bot$.Then by the deduction theorem, $S \vdash p$ and $S \vdash \neg p$. So $S \vdash \bot$.

As a general rule of mathematics, after doing it for one thing, we do it for infinitely many things.

Now we suppose L is countable. So we can list L as $\{t_1, t_2, \cdots\}$.

Let $S_0 = S$. Then let $S_1 = S \cup \{t_1\}$ or $S \cup \{\neg t_1\}$ such that S_1 is consistent, and let $S_2 = S_1 \cup \{t_2\}$ or $S_1 \cup \{\neg t_2\}$ such that S_2 is consistent. Continue inductively.

Set $\bar{S} = S_0 \cup S_1 \cup S_2 \cdot$. Then $p \in \bar{S}$ or $\neg p \in \bar{S}$ for each $p \in l$ by construction. Also $\bar{S}$ is consistent (If $\bar{S} \vdash \perp$, then some $S_n \vdash \perp$ since proofs are finite, but all S_n are consistent). Finally, we check that $\bar{S}$ is deductively closed, since if $\bar{S} \models p$, and $p \notin \bar{S}$, then $\neg p \notin \bar{S}$. Then $\bar{S}$ would be inconsistent.

Define $v : L \to \{0, 1\}$ by

$$p \mapsto \begin{cases} 1 & \text{if } p \in \bar{S} \\ 0 & \text{if not} \end{cases}.$$

We claim this is a valuation:

$v(\perp) = 0$ as $\perp \notin \bar{S}$ (since $\bar{S}$ is consistent).

For $p \Rightarrow q$,

(i) If $v(p) = 1, v(q) = 0$, we have $p \in \bar{S}$, $q \notin \bar{S}$. We want to show $p \Rightarrow q \notin \bar{S}$. But if $p \Rightarrow q \in \bar{S}$, then $\bar{S} \vdash q$ by MP. Hence $q \in \bar{S}$ since $\bar{S}$ is deductively closed.

(ii) If $v(q) = 1$, then $q \in \bar{S}$. We want $p \Rightarrow q \in \bar{S}$. But $\vdash p \Rightarrow (p \Rightarrow q)$ (Axiom 1). So $p \Rightarrow q \in \bar{S}$ by deductive closure.

(iii) If $v(p) = 0$, then $p \notin \bar{S}$. So $\neg p \in \bar{S}$, and we want $p \Rightarrow q \in \bar{S}$. So we want to show $\neg p \vdash p \Rightarrow q$. By the deduction theorem, this is equivalent to $\{p, \neg p\} \vdash q$. But $\{p, \neg p\} \vdash \perp$. So it is enough to show $\perp \vdash q$. But $\perp \models \neg\neg q$ by Axiom 1 (since $\perp \Rightarrow (\neg q \Rightarrow \perp)$). It is a theorem that $\neg\neg q \Rightarrow q$. So $\perp \vdash q$.

□

Corollary (Adequacy theorem)**.** Let $S \subset L$, $t \in L$. Then $S \models t$ implies $S \vdash t$.

Theorem (Completeness theorem)**.** Le $S \subset L$ and $t \in L$. Then $S \models t$ if and only if $S \vdash t$.

Proof. ($\Leftarrow$): Soundness. ($\Rightarrow$) Adequacy. □

Corollary (Compactness theorem)**.** Let $S \subset L$ and $t \in L$ with $S \models t$. The some finite $S' \subset S$ has $S' \models t$.

Proof. Trivial with $\models$ replaced by $\vdash$, because proofs are finite. □

Corollary (Decidability theorem)**.** Let finite $S \subset L$, $t \in L$. Then there exists an algorithm that determines, in finite time, whether or not $S \vdash t$.

Proof. Trivial with $\vdash$ replaced by $\models$, by making a truth table. □