Part IA - Probability

Definitions

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Basic concepts
Classical probability, equally likely outcomes. Combinatorial analysis, permutations and combinations. Stirling's formula (asymptotics for $\log n!$ proved). [3]

Axiomatic approach
Axioms (countable case). Probability spaces. Inclusion-exclusion formula. Continuity and subadditivity of probability measures. Independence. Binomial, Poisson and geometric distributions. Relation between Poisson and binomial distributions. Conditional probability, Bayes's formula. Examples, including Simpson's paradox. [5]

Discrete random variables
Expectation. Functions of a random variable, indicator function, variance, standard deviation. Covariance, independence of random variables. Generating functions: sums of independent random variables, random sum formula, moments.

Conditional expectation. Random walks: gambler's ruin, recurrence relations. Difference equations and their solution. Mean time to absorption. Branching processes: generating functions and extinction probability. Combinatorial applications of generating functions. [7]

Continuous random variables
Distributions and density functions. Expectations; expectation of a function of a random variable. Uniform, normal and exponential random variables. Memoryless property of exponential distribution. Joint distributions: transformation of random variables (including Jacobians), examples. Simulation: generating continuous random variables, independent normal random variables. Geometrical probability: Bertrand's paradox, Buffon's needle. Correlation coefficient, bivariate normal random variables. [6]

Inequalities and limits
Markov's inequality, Chebyshev's inequality. Weak law of large numbers. Convexity: Jensen's inequality for general random variables, AM/GM inequality.

Moment generating functions and statement (no proof) of continuity theorem. Statement of central limit theorem and sketch of proof. Examples, including sampling. [3]

Contents

1 Introduction **3**

2 Classical probability **4**
2.1 Classical probability 4
2.2 Sample space and events 4

3 Combinatorial analysis **5**
3.1 Counting 5
3.2 Sampling with or without replacement 5
3.3 Sampling with or without regard to ordering 5
3.4 Four cases of enumerative combinatorics 5

4 Stirling's formula **6**
4.1 Multinomial coefficient 6
4.2 Stirling's formula 6

5 Axiomatic approach **7**
5.1 Boole's inequality 7
5.2 Inclusion-exclusion formula 7

1 Introduction

2 Classical probability

2.1 Classical probability

Definition (Classical probability)**.** *Classical probability* applies in a situation when there are a finite number of equally likely outcome.

2.2 Sample space and events

Definition (Sample space)**.** The set of all possible outcomes is the *sample space*, Ω. We can lists the outcomes as $\omega_1, \omega_2, \cdots \in \Omega$. Each $\omega \in \Omega$ is an *outcome*.

Definition (Event)**.** A subset of Ω is called an *event*.

Definition (Set notations)**.** Given any two events $A, B \subseteq \Omega$,

- The *complement* of A is $A^C = A' = \bar{A} = \omega \setminus A$.
- "A or B" is the set $A \cup B$.
- "A and B" is the set $A \cap B$.
- A and B are *mutually exclusive* or *disjoint* if $A \cap B = \emptyset$.
- $A \subseteq B$ means $A \Rightarrow B$.

Definition (Probability)**.** Suppose $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$. Let $A \subseteq \Omega$ be an event. Then the *probability* of A is

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } \omega} = \frac{|A|}{N}$$

3 Combinatorial analysis

3.1 Counting

3.2 Sampling with or without replacement

Definition (1. Sampling with replacement)**.** When we sample with replacement, after choosing at item, it is put back and can be chosen again. Then any sampling function f satisfies sampling with replacement.

Definition (2. Sampling without replacement)**.** After choosing an item, we burn it and cannot choose it again. Then f must be an injective function, and clearly we must have $X \geq n$.

3.3 Sampling with or without regard to ordering

3.4 Four cases of enumerative combinatorics

4 Stirling's formula

4.1 Multinomial coefficient

Definition (Multinomial coefficient)**.** A *multinomial coefficient* is

$$\binom{n}{n_1, n_2, \cdots, n_x} = \binom{n}{n_1}\binom{n - n_1}{n_2}\cdots\binom{n - n_1 \cdots - n_{x-1}}{n_x} = \frac{n!}{n_1!n_2!\cdots n_x!}.$$

It is the number of ways to distribute x items into n positions with replacement, in which the ith position has n_i items.

4.2 Stirling's formula

5 Axiomatic approach

Definition (Probability space)**.** A *probability space* is a triple $(\Omega, \mathcal{F}, P)$. Ω is the *sample space*, $\mathcal{F}$ is a collection of subsets of Ω. $P : \mathcal{F} \to [0, 1]$ is the *probability measure*. $\mathcal{F}$ has to satisfy the following axioms:

(i) $\emptyset, \Omega \in \mathbf{F}$.

(ii) $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$.

(iii) $A_1, A_2, \cdots, \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} \in \mathcal{F}$.

And P has to satisfy the following *Kolmogorov axioms*:

(i) $o \leq P(A) \leq 1$ for all $A \in \mathcal{F}$

(ii) $P(\Omega) = 1$

(iii) For any countable collection of events $A_1, A_2, \cdots$ which are disjoint, i.e. $A_i \cap A_j = \emptyset$ for all i, j, then

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i).$$

We say $P(A)$ is the probability of the event A.

Definition (Probability distribution)**.** Let $\Omega = \{\omega_1, \omega_2, \cdots\}$. Choose $\{p_1, p_2, \cdots,\}$ such that $\sum_i^\infty = 1$. Let $p(\omega_i) = p_i$. Then define

$$P(A) = \sum_{\omega_i \in A} p(\omega_i).$$

This $P(A)$ satisfies the above axioms, and $p_1, p_2, \cdots$ is the *probability distribution*

5.1 Boole's inequality

5.2 Inclusion-exclusion formula