Part IA - Dynamics and Relativity

Theorems with Proof

Lectured by G. I. Ogilvie

Lent 2015

Basic concepts

Space and time, frames of reference, Galilean transformations. Newton's laws. Dimensional analysis. Examples of forces, including gravity, friction and Lorentz. [4]

Newtonian dynamics of a single particle

Equation of motion in Cartesian and plane polar coordinates. Work, conservative forces and potential energy, motion and the shape of the potential energy function; stable equilibria and small oscillations; effect of damping.

Angular velocity, angular momentum, torque.

Orbits: the $u(\theta)$ equation; escape velocity; Kepler's laws; stability of orbits; motion in a repulsive potential (Rutherford scattering). Rotating frames: centrifugal and coriolis forces. *Brief discussion of Foucault pendulum.* [8]

Newtonian dynamics of systems of particles

Momentum, angular momentum, energy. Motion relative to the centre of mass; the two body problem. Variable mass problems; the rocket equation. [2]

Rigid bodies

Moments of inertia, angular momentum and energy of a rigid body. Parallel axes theorem. Simple examples of motion involving both rotation and translation (e.g. rolling). [3]

Special relativity

The principle of relativity. Relativity and simultaneity. The invariant interval. Lorentz transformations in $(1+1)$-dimensional spacetime. Time dilation and length contraction. The Minkowski metric for $(1+1)$-dimensional spacetime. Lorentz transformations in $(3+1)$ dimensions. 4-vectors and Lorentz invariants. Proper time. 4-velocity and 4-momentum. Conservation of 4-momentum in particle decay. Collisions. The Newtonian limit. [7]

Contents

1 Newtonian dynamics of particles

1.1 Newton's laws of motion

Law (Newton's First Law of Motion)**.** A body remains at rest, or moves uniformly in a straight line, unless acted on by a force. (This is in fact Galileo's Law of Inertia)

Law (Newton's Second Law of Motion)**.** The rate of change of momentum of a body is equal to the force acting on it (in both magnitude and direction).

Law (Newton's Third Law of Motion)**.** To every action there is an equal and opposite reaction: the forces of two bodies on each other are equal and in opposite directions.

1.2 Galilean transformations

Law (Galilean relativity)**.** The *principle of relativity* asserts that the laws of physics are the same in inertial frames.

1.3 Newton's Second Law

Law. The *equation of motion* for a particle subject to a force F is

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

where $\mathbf{p} = m\mathbf{v} = m\ddot{\mathbf{r}}$ is the (linear) momentum of the particle. We say m is the (inertial) mass of the particle, which is a measure of its reluctance to accelerate.

2 Dimensional Analysis

2.1 Units

2.2 Scaling

3 Forces

3.1 Force and potential energy in one dimension

Proposition. Suppose the equation of a particle satisfies

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{*}$$

Then the total energy

$$E = T + V = \frac{1}{2}m\dot{x}^2 + V(x)$$

is conserved, i.e. $\dot{E} = 0$.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= m\dot{x}\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x} \\ &= \dot{x}\left(m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right) \\ &= 0\end{aligned}$$

□

3.2 Motion in a potential

3.3 Equilibrium points

3.4 Force and potential energy in three dimensions

Proposition. If $\mathbf{F}$ is conservative, then the energy

$$\begin{aligned}E &= T + V \\ &= \frac{1}{2}m|\mathbf{v}|^2 + V(\mathbf{r})\end{aligned}$$

is conserved. Then the work done is equal to the change in potential energy, and is independent of the path taken between the end points.

In particular, if we travelled around a closed loop, no work is done.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{\partial V}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} \\ &= (m\mathbf{r} + \ddot{\nabla} V \cdot \dot{\mathbf{r}} \\ &= (m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \dot{\mathbf{r}} \\ &= 0\end{aligned}$$

In this case, the work done is

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C (\nabla V) \cdot \mathrm{d}\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2).$$

□

3.5 Central forces

Proposition. $\nabla r = \hat{\mathbf{r}}$.

Proof. We know that

$$r^2 = x_1^2 + x_2^2 + x_3^2.$$

Then

$$2r\frac{\partial r}{\partial x_i} = 2x_i.$$

So

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = (\hat{\mathbf{r}})_i.$$

□

Proposition. Let $\mathbf{F} = -\nabla V(r)$ be a central force. Then

$$\mathbf{F} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector in the radial direction pointing away from the origin.

Proof. Continuing the proof above,

$$(\nabla V)_u = \frac{\partial V}{\partial x_i} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\partial r}{\partial x_i} = \frac{\mathrm{d}V}{\mathrm{d}r}(\hat{\mathbf{r}})_i$$

□

Proposition. Angular momentum is conserved by a central force.

Proof.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r} \times \mathbf{F} = \mathbf{0}.$$

where the last equality comes from the fact that $\mathbf{F}$ is parallel to $\mathbf{r}$ for a central force. □

3.6 Gravity

Law. If a particle of mass M is fixed at a origin, then a second particle of mass m experiences a potential energy

$$V(r) = -\frac{GMm}{r},$$

where $G \approx 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is the *gravitational constant.*

The gravitational force experienced is then

$$F = -\nabla V = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Proposition. The gravitational potential due to many fixed masses M_i at points $\mathbf{r}_i$ is is

$$\Phi_g(\mathbf{r}) = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}.$$

Again, $V = m\Phi_g$ for a particle of mass m.

Proposition. The external gravitational potential of a spherically symmetric object of mass M is the same as that of a point particle with the same mass at the center of the object, i.e.

$$\Phi_g(r) = -\frac{GM}{r}.$$

Proof. c.f. Vector Calculus $\square$