Part IA - Analysis I

Definitions

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Limits and convergence

Sequences and series in R and C. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

Continuity

Continuity of real- and complex-valued functions defined on subsets of $\mathbb{R}$ and $\mathbb{C}$. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

Differentiability

Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from $\mathbb{R}$ to $\mathbb{R}$; Lagranges form of the remainder. Complex differentiation. [5]

Power series

Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. *Direct proof of the differentiability of a power series within its circle of convergence*. [4]

Integration

Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

Contents

1 The real number system

Definition (Field)**.** A *field* is a set X with two binary operations $+$ and $\times$ that satisfies all the familiar properties satisfied by addition and multiplication in $\mathbb{Q}$, namely

- Associativity: $\forall a, b, c \in X$, $a+(b+c) = (a+b)+c$ and $a\times(b\times c) = (a\times b)\times c$
- Commutativity: $\forall a, b \in X$, $a + b = b + a$ and $a \times b = b \times a$
- Identity: $\exists 0, 1 \in X$ such that $\forall a$, $a + 0 = a$ and $a \times 1 = a$.
- Inverses: $\forall a \in X$, $\exists(-a) \in X$ such that $a + (-a) = 0$. If $a \neq 0$, then $\exists a^{-1}$ such that $a \times a^{-1} = 1$.
- Distributivity: $\forall a, b, c \in F$, $a \times (b + c) = (a \times b) + (a \times c)$

Definition (Totally ordered set)**.** An *(totally) ordered set* is a set X with a relation $<$ that satisfies

(i) Transitivity: if $x, y, z \in X$, $x < y$ and $y < z$, then $x < z$

(ii) Trichotomy: if $x, y \in X$, exactly one of $x < y, x = y, y < x$ holds

Definition (Ordered field)**.** An *ordered field* is a field $\mathbb{F}$ with a relation $<$ that makes $\mathbb{F}$ into an ordered set such that

(i) if $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$

(ii) if $x, y, z \in \mathbb{F}$, $x < y$ and $z > 0$, then $xz < yz$

Definition (Least upper bound)**.** Let X be an ordered set and let $A \subseteq X$. An *upper bound* for A is an element $x \in X$ such that $\forall a \in A(a \leq x)$. If A has an upper bound, then we say that A is *bounded above*.

An upper bound x for A is a *least upper bound* or *supremum* if nothing smaller that x is an upper bound. That is, we need

(i) $\forall a \in A(a \leq x)$

(ii) $\forall y < x(\exists a \in A(a \geq y))$

We usually write $\sup A$ for the supremum of A when it exists. If $\sup A \in A$, then we call it $\max A$, the maximum of A.

Definition (Least upper bound property)**.** An ordered field has the *least upper bound property* if every non-empty subset of $\mathbb{F}$ that is bounded above has a supremum.

Definition (Real numbers)**.** The *real numbers* is an ordered field with the least upper bound property.

2 Convergence of sequences

Definition (Sequence)**.** A *sequence* is, formally, a function $a : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$). Usually (i.e. always), we write a_n instead of $a(n)$. Instead of a, (a_n), $(a_n)_1^\infty$ or $(a_n)_{n=1}^\infty$ to indicate it is a sequence.

Definition (Convergence of sequence)**.** Let (a_n) be a sequence and $\ell \in \mathbb{R}$. Then a_n *converges to* ℓ, *tends to* ℓ, or $a_n \to \ell$, if

$$\forall \varepsilon > 0\ \exists N\ \forall n \geq N :\ |a_n - \ell| < \varepsilon.$$

Definition (Bounded sequence)**.** A sequence (a_n) is *bounded*

$$\exists C\ \forall n :\ |a_n| \leq C.$$

A sequnece is *eventually bounded* if

$$\exists C\ \exists N\ \forall n \geq N :\ |a_n| \leq C.$$

2.1 Sums, products and quotients

Definition (Monotone sequence)**.** A sequence (a_n) is *increasing* if $\forall n$, $a_n \leq a_{n+1}$. It is *strictly increasing* $a_n < a_{n+1}$ for all n. *(Strictly) decreasing* sequences are defined analogously.

A sequence is *(strictly) monotone* if it is (strictly) increasing or (strictly) decreasing.

2.2 Monotone-sequences property

Definition (Monotone sequences property)**.** An ordered field has the *monotone sequences property* if every increasing sequence that is bounded above converges.

Definition (Subsequence)**.** Let (a_n) be a sequence. A *subsequence* of (a_n) is a sequence of the form $a_{n_1}, a_{n_2}, \cdots$, where $n_1 < n_2 < \cdots$.

2.3 Cauchy sequences

Definition (Cauchy sequence)**.** A sequence (a_n) is *Cauchy* if

$$\forall \varepsilon > 0\ \exists N\ \forall p; q \geq N :\ |a_p - a_q| < \varepsilon.$$

Definition (Complete ordered field)**.** An ordered field (with an appropriate metric) in which every Cauchy sequence converges is called *complete*

2.4 Limit supremum and infimum

Definition (Limit supremum/infimum)**.** Let (a_n) be a bounded sequence. We define the *limit supremum* as

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup_{m\geq n} a_m \right).$$

To see that this exists, set $b_n = \sum_{m \geq n} a_m$. Then (b_n) is decreasing since we are taking the supremum of fewer and fewer things, and is bounded below by any lower bound for (a_n) since $b_n \geq a_n$. So it converges.

Similarly, we define the *limit infimum* as

$$\liminf_{n \to \infty} a_n = \lim_{n \to \infty} \left(\inf_{m \geq n} a_m \right).$$

3 Convergence of infinite sums

Definition (Convergence of infinite sums and partial sums)**.** Let (a_n) be a real sequence. For each N, define

$$S_N = \sum_{n=1}^{N} a_n.$$

If the sequence (S_N) converges to some limit s, then we say that

$$\sum_{n=1}^{\infty} a_n = s,$$

and we say that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges*.

We call S_N the Nth *partial sum*.