Part IA - Vector Calculus
Definitions

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Curves in $\mathbb{R}^3$
Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1]

Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$
Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

Vector operators
Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical *and general orthogonal curvilinear* coordinates.

Divergence, curl and ∇^2 in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical *and general orthogonal curvilinear* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

Integration theorems
Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

Laplace's equation
Laplace's equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

Cartesian tensors in $\mathbb{R}^3$
Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

Contents

1 Introduction

2 Derivatives and coordinates

2.1 Differentiable functions $\mathbb{R} \to \mathbb{R}^n$

Definition. A *vector function* is a function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^n$.

Definition (Derivative of vector function)**.** A vector function $\mathbf{F}(x)$ is *differentiable* if

$$\delta\mathbf{F} \stackrel{\text{def}}{=} \mathbf{F}(x + \delta x) - \mathbf{F}(x) = \mathbf{F}'(x)\delta u + o(\delta x)$$

for some $\mathbf{F}'(x)$. $\mathbf{F}'(x)$ is called the *derivative* of $\mathbf{F}(x)$.

Clearly, if $\mathbf{F}'(x)$ exists, then it is given by

$$\mathbf{F}' = \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{1}{\delta x}[\mathbf{F}(x + \delta x) - \mathbf{F}(x)].$$

2.2 Differentiable functions $\mathbb{R}^n \to \mathbb{R}$

Definition. A *scalar function* is a function $f : \mathbb{R}^n \to \mathbb{R}$.

Definition (Limit of vector)**.** The *limit of vectors* is defined using the norm. So $\mathbf{v} \to \mathbf{c}$ iff $|\mathbf{v} - \mathbf{c}| \to 0$.

Definition (Gradient of scalar function)**.** A scalar function $f(\mathbf{r})$ is *differentiable* at $\mathbf{r}$ if

$$\delta f \stackrel{\text{def}}{=} f(\mathbf{r} + \delta\mathbf{r}) - f(\mathbf{r}) = (\nabla f) \cdot \delta\mathbf{r} + o(\delta\mathbf{r})$$

for some vector ∇f, the *gradient* of f at $\mathbf{r}$.

Definition (Directional derivative)**.** The *directional derivative* of f along $\mathbf{n}$ is

$$\mathbf{n} \cdot \nabla f = \lim_{h \to 0} \frac{1}{h}[f(\mathbf{r} + h\mathbf{n}) - f(\mathbf{r})],$$

It refers to how fast f changes when we move in the direction of $\mathbf{n}$.

2.3 Differentiable functions $\mathbb{R}^n \to \mathbb{R}^m$

Definition (Vector field)**.** A *vector field* is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$.

Definition (Derivative of vector field)**.** A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if

$$\delta\mathbf{F} \stackrel{def}{=} \mathbf{F}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{F}(\delta\mathbf{x}) = M\delta\mathbf{x} + o(\delta\mathbf{x})$$

for some $n \times m$ matrix M. M is the *derivative* of $\mathbf{F}$.

Definition. A function is *smooth* if it can be differentiated any number of times, i.e. if all partial derivatives exist and are totally symmetric in i, j and k (i.e. the differential operation is commutative).

2.4 Chain rule

2.5 Inverse functions

2.6 Coordinate systems

3 Curves and Line

3.1 Parametrised curves, lengths and arc length

Definition (Parametrisation of curve)**.** Given a curve C in $\mathbb{R}^n$, a *parametrisation* of it is a continuous and invertible function $\mathbf{r} : D \to \mathbb{R}^n$ for some $D \subseteq \mathbb{R}$ whose image is C.

$\mathbf{r}'(u)$ is a vector tangent to the curve at each point. A parametrization is *regular* if $\mathbf{r}'(u) \neq 0$ for all u.

Definition (Scalar line element)**.** We say $\mathrm{d}s = \pm|\mathbf{r}'(u)|\mathrm{d}u$ is a *scalar line element* on C.

3.2 Line integrals of vector fields

Definition (Line integral)**.** The *line integral* of a smooth vector field $\mathbf{F}(\mathbf{r})$ along a path C parametrised by $\mathbf{r}(u)$ with along the direction (orientation) $\mathbf{r}(\alpha) \to \mathbf{r}(\beta)$ is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_\alpha^\beta \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}'(u)\mathrm{d}u.$$

We say $\mathrm{d}\mathbf{r} = \mathbf{r}'(u)\mathrm{d}u$ is the *line element* on C. Note that the upper and lower limits of the integral are the end point and start point respectively, and β is not necessarily larger than α.

Definition (Closed curve)**.** A *closed curve* is a curve with the same start and end point. The line integral along a closed curve is written as $\oint$ and is called the *circulation* of $\mathbf{F}$ around C.

3.3 Sums of curves and integrals

Definition (Piecewise smooth curve)**.** A *piecewise smooth curve* is a curve $C = C_1 + C_2 + \cdots + C_n$ with all C_i smooth with regular parametrisations. The line integral over a piecewise smooth C is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

3.4 Gradients and Differentials

3.4.1 Line integrals and Gradients

Definition (Conservative vector field)**.** If $\mathbf{F} = \nabla f$ for some f, the $\mathbf{F}$ is called a *conservative vector field.*

3.4.2 Differentials

Definition (Exact differential)**.** A differential $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is *exact* if there is an f such that $\mathbf{F} = \nabla f$. Then

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

3.5 Work and potential energy

Definition (Work and potential energy)**.** If $\mathbf{F}(\mathbf{r})$ is a force, then $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is the *work done* by the force along the curve C. It is the limit of a sum of terms $\mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{r}$, i.e. the force along the direction of $\delta\mathbf{r}$.

Definition (Potential energy)**.** Given a conservative force $= -\nabla V$, $V()$ is the *potential energy*. Then

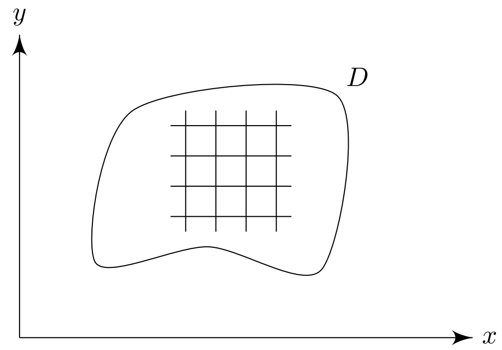
$$\int_C \mathbf{F} \cdot \mathrm{d} = V(\mathbf{a}) - V(\mathbf{b}).$$

4 Integration in $\mathbb{R}^2$ and R^3

4.1 Integrals over subsets of $\mathbb{R}^2$

4.1.1 Definition as the limit of as sum

Definition (Surface integral)**.** Let $D \subseteq \mathbb{R}^2$. Let $\mathbf{r}(x, y)$ be in Cartesian coordinates. We can approximate D by N disjoint subsets of simple shapes, e.g. triangles, parallelograms. These shapes are labelled by I and have areas δA_i. Each of these are small enough to be contained in a disc of diameter ℓ.



Assume that as $\ell \to 0$ and $N \to \infty$, the union of the small sets $\to D$. For a function $f(\mathbf{r})$, we define the *surface integral* as

$$\int_D f(\mathbf{r})\ \mathrm{d}A = \lim_{\ell \to 0} \sum_I f(\mathbf{r}_i) \delta A_i.$$

where $\mathbf{r}_i$ is some point within each subset A_i. The integral *exists* if the limit is well-defined (i.e. the same regardless of what A_i and $\mathbf{r}_i$ we choose before we take the limit) and exists.