Mathematical Tripos Part IA **Lent Term 2015**
Vector Calculus **Prof B C Allanach**

Example Sheet 1

1 Sketch the curve in the plane given parametrically by

$$\mathbf{r}(u) \;=\; (\,x(u),\,y(u)\,) \;=\; (\,a\cos^3 u,\,a\sin^3 u\,) \qquad \text{with} \qquad 0 \leqslant u \leqslant 2\pi\ .$$

Calculate its tangent vector $\mathrm{d}\mathbf{r}/\mathrm{d}u$ at each point and hence find its total length.

2 In three dimensions, use suffix notation and the summation convention to show that

$$\text{(i)}\quad \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}\ ; \qquad\qquad \text{(ii)}\quad \boldsymbol{\nabla} r^n = n r^{n-2}\mathbf{x}\ ,$$

where $\mathbf{a}$ is any constant vector and $r = |\mathbf{x}|$.

Given a function $f(\mathbf{r})$ in two dimensions, use the Chain Rule to express its partial derivatives with respect to Cartesian coordinates (x, y) in terms of its partial derivatives with respect to polar coordinates (ρ, ϕ). From the relationship between the basis vectors in these coordinate systems, deduce that

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi\ .$$

3 Evaluate explicitly each of the line integrals

$$\int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z)\,,$$

along (i) the straight line path from the origin to $x = y = z = 1$, and (ii) the parabolic path given parametrically by $x = t$, $y = t$, $z = t^2$ from $t = 0$ to $t = 1$.

For which of these integrals do the two paths give the same results, and why?

4 Consider forces $\mathbf{F} = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G} = (3x^2y^2z, 2x^3yz, x^3y^2)$. Compute the work done, given by the line integrals $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ and $\int \mathbf{G}\cdot\mathrm{d}\mathbf{r}$, along the following paths, each of which consist of straight line segments joining the specified points: (i) $(0,0,0) \to (1,1,1)$; (ii) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$; (iii) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$.

5 A curve C is given parametrically in Cartesian coordinates by

$$\mathbf{r}(t) \;=\; \big(\cos(\sin nt)\cos t\,,\ \cos(\sin nt)\sin t\,,\ \sin(\sin nt)\big)\,, \qquad 0 \leqslant t \leqslant 2\pi\,,$$

where n is some fixed integer. Using spherical polar coordinates, or otherwise, sketch or describe the curve. Show that

$$\int_C \mathbf{H}\cdot\mathrm{d}\mathbf{r} = 2\pi\ , \qquad \text{where} \qquad \mathbf{H}(\mathbf{r}) = \left(-\frac{y}{x^2+y^2},\quad \frac{x}{x^2+y^2},\quad 0\right)$$

and C is traversed in the direction of increasing t. Can $\mathbf{H}(\mathbf{r})$ be written as the gradient of a scalar function? Comment on your results.

6 Obtain the equation of the plane which is tangent to the surface $z = 3x^2 y \sin(\pi x/2)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

7 Use the substitution $x = r\cos\theta$, $y = \frac{1}{2} r \sin\theta$, to evaluate

$$\int_A \frac{x^2}{x^2 + 4y^2}\, \mathrm{d}A\,,$$

where A is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

8 The closed curve C in the $z = 0$ plane consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The area enclosed by C is A. Show, by calculating the integrals explicitly, that

$$\int_C (x^2 y\, \mathrm{d}x + xy^2\, \mathrm{d}y) = \int_A (y^2 - x^2)\, \mathrm{d}A = \frac{104}{105} a^4\,.$$

where C is traversed anticlockwise.

9 The region A is bounded by the segments $x = 0$, $0 \leqslant y \leqslant 1$; $y = 0$, $0 \leqslant x \leqslant 1$; $y = 1$, $0 \leqslant x \leqslant \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1 - x)$. Consider a mapping into the (x, y) plane from the (u, v) plane defined by the transformation $x = u^2 - v^2, \quad y = 2uv$. Sketch A and also the two regions in the (u, v) plane which are mapped into it. Hence evaluate

$$\int_A \frac{\mathrm{d}A}{(x^2 + y^2)^{1/2}}\,.$$

10 By using a suitable change of variables, calculate the volume within an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\,.$$

11 A tetrahedron V has vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Find the centre of volume, defined by

$$\frac{1}{V} \int_V \mathbf{x}\, \mathrm{d}V\,.$$

12 A solid cone is bounded by the surface $\theta = \alpha$ in spherical polar coordinates and the surface $z = a$. Its mass density is $\rho_0 \cos\theta$. By evaluating a volume integral find the mass of the cone.

Comments to: B.C.Allanach@damtp.cam.ac.uk

Example Sheet 2

1 A circular helix is given by

$$\mathbf{r}(u) = (\, a\cos u,\, a\sin u,\, cu\,)\,.$$

Calculate the tangent $\mathbf{t}$, curvature κ, principal normal $\mathbf{n}$, binormal $\mathbf{b}$, and torsion τ.

2 Show that a curve in the plane, $\mathbf{r}(t) = (\, x(t),\, y(t),\, 0\,)$, has curvature

$$\kappa(t) = |\, \dot{x}\ddot{y} - \dot{y}\ddot{x}\,| \,/\, (\, \dot{x}^2 + \dot{y}^2\,)^{3/2}\,.$$

Find the minimum and maximum curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b > 0)$.

3 Let $\psi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show, using index notation, that

$$\boldsymbol{\nabla}\cdot(\psi\mathbf{v}) = (\boldsymbol{\nabla}\psi)\cdot\mathbf{v} + \psi\,\boldsymbol{\nabla}\cdot\mathbf{v}\,, \qquad \boldsymbol{\nabla}\times(\psi\mathbf{v}) = (\boldsymbol{\nabla}\psi)\times\mathbf{v} + \psi\,\boldsymbol{\nabla}\times\mathbf{v}\,.$$

Evaluate (using index notation where necessary) the divergence and the curl of the following:

$$r\,\mathbf{x}\,, \quad \mathbf{a}(\mathbf{x}\cdot\mathbf{b})\,, \quad \mathbf{a}\times\mathbf{x}\,, \quad \mathbf{x}/r^3\,,$$

where $r = |\mathbf{x}|$, and $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors.

4 Use suffix notation to show that

$$\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{v}) + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u} - \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{u}) - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}\,.$$

for vector fields $\mathbf{u}$ and $\mathbf{v}$. Show also that $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\frac{1}{2}u^2) - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})$.

5 Check, by calculating its curl, that the force field

$$\mathbf{F} = \left(\, 3x^2\tan z - y^2 \mathrm{e}^{-xy^2}\sin y\,,\; (\cos y - 2xy\sin y)\,\mathrm{e}^{-xy^2}\,,\; x^3\sec^2 z\,\right)$$

is *conservative*. Find the most general scalar potential for $\mathbf{F}$ and hence, or otherwise, find the work done by the force as it acts on a particle moving from $(0,0,0)$ to $(1,\pi/2,\pi/4)$.

6 Verify that the vector field

$$\mathbf{u} = e^x(x\cos y + \cos y - y\sin y)\,\mathbf{i} + e^x(-x\sin y - \sin y - y\cos y)\,\mathbf{j}$$

is *irrotational* and express it as the gradient of a scalar field ϕ. Check that $\mathbf{u}$ is also *solenoidal* and show that it can be written as the curl of a vector field $\psi\mathbf{k}$, for some function ψ.

7 **(a)** The vector field $\mathbf{B}(\mathbf{x})$ is everywhere parallel to the normals to a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that

$$\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = 0\,.$$

7 **(b)** The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $|\mathbf{H}|^{-3}\,|\,\mathbf{H}\times(\mathbf{H}\cdot\boldsymbol{\nabla})\mathbf{H}\,|$.

8 Consider the line integral

$$\oint_C -x^2y\,\mathrm{d}x \;+\; xy^2\,\mathrm{d}y$$

for C a closed curve traversed anti-clockwise in the xy plane.
(i) Evaluate this integral when C is a circle with radius R and centre the origin. Use Green's Theorem to relate the results for $R = b$ and $R = a$ to an area integral over the region $a^2 \leqslant x^2{+}y^2 \leqslant b^2$, and calculate the area integral directly.
(ii) Now suppose that C is the boundary of a square, with centre the origin, and sides of length ℓ. Show that the line integral is independent of the orientation of the square in the plane.

9 Verify Stokes's Theorem for the hemispherical surface $r = 1$, $z \geqslant 0$, and the vector field

$$\mathbf{F}(\mathbf{r}) \;=\; (\,y,\,-x,\,z\,)\,.$$

10 Let $\mathbf{F}(\mathbf{r}) = (\,x^3{+}3y{+}z^2,\; y^3,\; x^2{+}y^2{+}3z^2\,)$, and let S be the *open* surface

$$1 - z \;=\; x^2 + y^2, \qquad 0 \leqslant z \leqslant 1\,.$$

Use the divergence theorem (and cylindrical polar coordinates) to evaluate $\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$.

Verify your result by calculating the integral directly. [You should find that the vector area element is $\mathrm{d}\mathbf{S} = (\,2\rho\cos\phi\,,\,2\rho\sin\phi\,,\,1\,)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,$.]

11 By applying the divergence theorem to the vector field $\mathbf{a}\times\mathbf{A}$, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{A}(\mathbf{x})$ is a vector field, show that

$$\int_V \boldsymbol{\nabla}\times\mathbf{A}\,\mathrm{d}V = -\int_S \mathbf{A}\times\mathrm{d}\mathbf{S}\,,$$

where the surface S encloses the volume V.

Verify this result when S is the sphere $|\mathbf{x}| = R$ and $\mathbf{A} = (z, 0, 0)$ in Cartesian coordinates.

12 By applying Stokes's theorem to the vector field $\mathbf{a}\times\mathbf{F}$, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{F}(\mathbf{x})$ is a vector field, show that

$$\oint_C \mathrm{d}\mathbf{x}\times\mathbf{F} \;=\; \int_S (\mathrm{d}\mathbf{S}\times\boldsymbol{\nabla})\times\mathbf{F}\,,$$

where the curve C bounds the open surface S.

Verify this result when C is the unit square in the xy plane with opposite vertices at $(0,0,0)$ and $(1,1,0)$ and $\mathbf{F}(\mathbf{x}) = \mathbf{x}$.

Comments to: B.C.Allanach@damtp.cam.ac.uk

Mathematical Tripos Part IA **Lent Term 2015**
Vector Calculus **Prof B C Allanach**

Example Sheet 3

1 (i) Write down the operator $\boldsymbol{\nabla}$ in Cartesian coordinates and in spherical polars. Calculate the gradient of

$$\psi = Ez = Er\cos\theta$$

in both coordinate systems (E is a constant) and check that your answers agree.
(ii) Apply the standard formulas in Cartesian, cylindrical, and spherical polar coordinates to calculate, in three ways, the curl of the following vector field (with B a constant):

$$\mathbf{A} = \tfrac{1}{2}B(-y\,\mathbf{e}_x + x\,\mathbf{e}_y) = \tfrac{1}{2}B\rho\,\mathbf{e}_\phi = \tfrac{1}{2}Br\sin\theta\,\mathbf{e}_\phi\ .$$

2 In cylindrical polar coordinates,

$$\boldsymbol{\nabla} = \mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z} \qquad \text{and} \qquad \frac{\partial\mathbf{e}_\rho}{\partial\phi} = \mathbf{e}_\phi\,, \quad \frac{\partial\mathbf{e}_\phi}{\partial\phi} = -\mathbf{e}_\rho\,,$$

while all other derivatives of the basis vectors are zero. Derive expressions for $\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times\mathbf{A}$, where $\mathbf{A} = A_\rho\mathbf{e}_\rho + A_\phi\mathbf{e}_\phi + A_z\mathbf{e}_z$, and also for $\nabla^2\psi$, where ψ is a scalar field.

3 The vector field $\mathbf{B}(\mathbf{x})$ is defined in cylindrical polar coordinates ρ, ϕ, z by

$$\mathbf{B}(\mathbf{x}) = \rho^{-1}\mathbf{e}_\phi\ , \qquad \rho\neq 0\ .$$

Calculate $\boldsymbol{\nabla}\times\mathbf{B}$ using the formula for curl in cylindrical polars. Evaluate $\oint_C \mathbf{B}\cdot d\mathbf{x}$, where C is the circle $z=0$, $\rho=1$ and $0\leqslant\phi\leqslant 2\pi$. Is your answer consistent with Stokes's Theorem?

4 The scalar field $\varphi(r)$ depends only on $r=|\mathbf{x}|$, where $\mathbf{x}$ is the position vector in three dimensions. Use Cartesian coordinates, index notation, and the chain rule to show that

$$\boldsymbol{\nabla}\varphi = \varphi'(r)\,\frac{\mathbf{x}}{r}\,, \qquad \nabla^2\varphi = \varphi''(r) + \frac{2}{r}\,\varphi'(r)\,.$$

Find the solution of $\nabla^2\varphi = 1$ which is defined on the region $r\leqslant a$ and which satisfies $\varphi(a)=1$.

5 **(a)** Using Cartesian coordinates x, y, find all solutions of Laplace's equation in two dimensions of the form $\varphi(x,y) = f(x)e^{\alpha y}$ with α a constant. Hence find a solution on the region $0\leqslant x\leqslant a$ and $y\geqslant 0$ with boundary conditions:

$$\varphi(0,y) = \varphi(a,y) = 0\ , \quad \varphi(x,0) = \lambda\sin(\pi x/a)\ , \quad \varphi\to 0 \text{ as } y\to\infty \quad (\lambda \text{ a const})\,.$$

(b) Using the formula for ∇^2 in polar coordinates r,θ, verify that Laplace's equation in the plane has solutions $\varphi(r,\theta) = A\,r^\alpha\cos\beta\theta$, if α and β are related appropriately. Hence find solutions on the following regions, with the given boundary conditions (λ a const):

$$\text{(i)}\quad r\leqslant a\,,\quad \varphi(a,\theta)=\lambda\cos\theta\,; \qquad \text{(ii)}\quad r\geqslant a\,,\quad \varphi(a,\theta)=\lambda\cos\theta\,,\quad \varphi\to 0 \text{ as } r\to\infty\,;$$

$$\text{(iii)}\quad a\leqslant r\leqslant b\,,\quad \frac{\partial\varphi}{\partial n}(a,\theta)=0\,,\quad \varphi(b,\theta)=\lambda\cos 2\theta\ .$$

6 Consider a complex-valued function $f = \varphi(x,y) + i\psi(x,y)$ satisfying $\partial f/\partial\bar z = 0$, where $\partial/\partial\bar z = \partial/\partial x + i\partial/\partial y$. Show that $\nabla^2\varphi = \nabla^2\psi = 0$. Show also that a curve on which φ is constant is orthogonal to a curve on which ψ is constant at a point where they intersect.
Find φ and ψ when $f = ze^z$, where $z = x+iy$, and compare with question **6** on Sheet 2.

7 Use Gauss's flux method to find the gravitational field $\mathbf{g}(\mathbf{r})$ due to a spherical shell of matter with density

$$\rho(r) = \begin{cases} 0 & \text{for } 0 \leqslant r \leqslant a, \\ \rho_0 r/a & \text{for } a < r < b, \\ 0 & \text{for } r \geqslant b. \end{cases}$$

Now find the gravitational potential $\varphi(r)$ *directly* from Poisson's equation by writing down the general, spherically symmetric solution to Laplace's equation in each of the intervals $0<r<a$, $a<r<b$ and $r>b$, and adding a particular integral where necessary. Assume that φ is not singular at the origin, and that φ and φ' are continuous at $r=a$ and $r=b$. Check that this solution gives the same result for the gravitational field.

8 From an integral theorem, derive (one of *Green's Identities*):

$$\int_V (\psi\nabla^2\varphi - \varphi\nabla^2\psi)\,\mathrm{d}V = \int_{\partial V} \left(\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right)\mathrm{d}S\,.$$

9 Let $\rho(\mathbf{x})$ be a function on a volume V and $f(\mathbf{x})$ a function on its boundary $S = \partial V$. Show that a solution $\varphi(\mathbf{x})$ to the following problem is unique:

$$\nabla^2\varphi - \varphi = \rho \;\text{ on }\; V\,, \qquad \frac{\partial\varphi}{\partial n} = f \;\text{ on }\; S\,.$$

10 The functions $u(\mathbf{x})$ and $v(\mathbf{x})$ on V satisfy $\nabla^2 u = 0$ on V and $v = 0$ on ∂V. Show that

$$\int_V \boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v\;\mathrm{d}V = 0\,.$$

Now if $w(\mathbf{x})$ is a function on V with $u = w$ on ∂V, show, by considering $v = w - u$, that

$$\int_V |\boldsymbol{\nabla}w|^2\;\mathrm{d}V \geqslant \int_V |\boldsymbol{\nabla}u|^2\;\mathrm{d}V\,.$$

11 Show that there is at most one solution $\varphi(\mathbf{x})$ to Laplace's equation in a volume V with the boundary condition given in terms of functions $f(\mathbf{x})$ and $g(\mathbf{x})$ by

$$g\frac{\partial\varphi}{\partial n} + \varphi = f \;\text{ on }\; \partial V\,,$$

assuming $g(\mathbf{x}) \geqslant 0$ on ∂V. Find a non-zero solution of Laplace's equation on $|\mathbf{x}| \leqslant 1$ which satisfies the boundary condition above with $f = 0$ and $g = -1$ on $|\mathbf{x}| = 1$.

12 Maxwell's equations for electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0\,, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t\,,$$
$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0\,, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\,\mathbf{j} + \epsilon_0\mu_0\partial\mathbf{E}/\partial t\,,$$

where $\rho(\mathbf{x}, t)$ and $\mathbf{j}(\mathbf{x}, t)$ are the charge density and current, and ϵ_0 and μ_0 are constants. Show that these imply the conservation equation $\boldsymbol{\nabla}\cdot\mathbf{j} = -\partial\rho/\partial t$. Show also that if $\mathbf{j}$ is zero then

$$U = \tfrac{1}{2}\left(\epsilon_0\mathbf{E}^2 + \mu_0^{-1}\mathbf{B}^2\right) \quad\text{and}\quad \mathbf{P} = \mu_0^{-1}\,\mathbf{E}\times\mathbf{B} \qquad\text{satisfy}\qquad \boldsymbol{\nabla}\cdot\mathbf{P} = -\partial U/\partial t\,.$$

Comments to: B.C.Allanach@damtp.cam.ac.uk

Mathematical Tripos Part IA **Lent Term 2015**
Vector Calculus **Prof B C Allanach**

Example Sheet 4

1 The current J_i due to an electric field E_i is given by $J_i = \sigma_{ij} E_j$, where σ_{ij} is the conductivity tensor. In a certain coordinate system,

$$(\sigma_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Show that there is a direction along which no current flows, and find the direction(s) along which the current flow is largest, for an electric field of fixed magnitude.

2 Given vectors $\mathbf{u} = (1,0,1)$, $\mathbf{v} = (0,1,-1)$ and $\mathbf{w} = (1,1,0)$, find all components of the second-rank and third-rank tensors defined by

$$T_{ij} = u_i v_j + v_i w_j \; ; \qquad S_{ijk} = u_i v_j w_k - v_i u_j w_k + v_i w_j u_k - w_i v_j u_k + w_i u_j v_k - u_i w_j v_k \; .$$

3 Using the transformation law for a second-rank tensor T_{ij}, show that the quantities

$$\alpha = T_{ii}\,, \quad \beta = T_{ij}T_{ji}\,, \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are the same in all Cartesian coordinate systems. If T_{ij} is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are roots of the cubic equation

$$\lambda^3 - \alpha\lambda^2 + \tfrac{1}{2}(\alpha^2 - \beta)\lambda - \tfrac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0 \; .$$

4 If $u_i(\mathbf{x})$ is a vector field, show that $\partial u_i / \partial x_j$ transforms as a second-rank tensor.
If $\sigma_{ij}(\mathbf{x})$ is a second-rank tensor field, show that $\partial \sigma_{ij}/\partial x_j$ transforms as a vector.

5 The fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ obey Maxwell's equations with zero charge and current. Show that the Poynting vector $\mathbf{P} = \mu_0^{-1}\, \mathbf{E}\times\mathbf{B}$ satisfies

$$\frac{1}{c^2}\frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \tfrac{1}{2}\epsilon_0\, \delta_{ij}\,(E_k E_k + c^2 B_k B_k) - \epsilon_0\,(E_i E_j + c^2 B_i B_j) \; .$$

6 The velocity field $\mathbf{u}(\mathbf{x}, t)$ of an inviscid compressible gas obeys

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \qquad \text{and} \qquad \rho\Big(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\Big) = -\boldsymbol{\nabla} p$$

where $\rho(\mathbf{x}, t)$ is the density and $p(\mathbf{x}, t)$ is the pressure. Show that

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho u^2) + \frac{\partial}{\partial x_i}(\tfrac{1}{2}\rho u^2 u_i + p u_i) = p\boldsymbol{\nabla}\cdot\mathbf{u} \qquad \text{and} \qquad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(t_{ij}) = 0$$

for a suitable symmetric tensor t_{ij}, to be determined.

7 The components of a second-rank tensor are given by a matrix A. Show that

$$A\mathbf{x} = \alpha\mathbf{x} + \boldsymbol{\omega}\times\mathbf{x} + B\mathbf{x} \quad \text{for all } \mathbf{x}\,,$$

for some scalar α, vector $\boldsymbol{\omega}$, and symmetric traceless matrix B. Find α, $\boldsymbol{\omega}$ and B when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$

8 **(a)** A tensor of rank 3 satisfies $T_{ijk} = T_{jik}$ and $T_{ijk} = -T_{ikj}$. Show that $T_{ijk} = 0$.

(b) A tensor of rank 4 satisfies $T_{jik\ell} = -T_{ijk\ell} = T_{ij\ell k}$ and $T_{ijij} = 0$. Show that

$$T_{ijk\ell} = \varepsilon_{ijp}\,\varepsilon_{k\ell q}\,S_{pq}\,, \qquad \text{where} \qquad S_{pq} = -T_{rqrp}\,,$$

9 A cuboid of uniform density and mass M has sides of lengths $2a$, $2b$ and $2c$. Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

A cube with sides of length $2a$ has uniform density, mass M, and is rotating with angular velocity $\boldsymbol{\omega}$ about an axis which passes through its centre and through a pair of opposite vertices. What is its angular momentum?

10 Evaluate the following integrals over all space, where $\gamma > 0$ and $r^2 = x_p x_p$:

$$\text{(i)} \quad \int r^{-3} e^{-\gamma r^2} x_i x_j \,\mathrm{d}V\,; \qquad \text{(ii)} \quad \int r^{-5} e^{-\gamma r^2} x_i x_j x_k \,\mathrm{d}V\,.$$

11 A tensor has components T_{ij} with respect to Cartesian coordinates x_i. If the tensor is invariant under arbitrary rotations around the x_3-axis, show that it must have the form

$$(T_{ij}) = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

12 In linear elasticity, the symmetric second-rank stress tensor σ_{ij} depends on the symmetric second-rank strain tensor e_{kl} according to $\sigma_{ij} = c_{ijkl}e_{kl}$. Explain why c_{ijkl} must be a fourth-rank tensor, assuming $c_{ijkl} = c_{ijlk}$. For an isotropic medium, use the most general possible form for c_{ijkl} (which you may quote) to show that

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}\,,$$

where λ and μ are scalars.

Invert this equation to express e_{ij} in terms of σ_{ij}, assuming $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of σ_{ij} and e_{ij} coincide.

The elastic energy density resulting from a deformation of the medium is $E = \frac{1}{2}e_{ij}\sigma_{ij}$. Show that E is strictly positive for any non-zero strain e_{ij} provided $\mu > 0$ and $\lambda > -2\mu/3$.

Comments to: B.C.Allanach@damtp.cam.ac.uk