

Part IA - Probability

Definitions

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Lent 2015

Basic concepts

Classical probability, equally likely outcomes. Combinatorial analysis, permutations and combinations. Stirling's formula (asymptotics for $\log n!$ proved). [3]

Axiomatic approach

Axioms (countable case). Probability spaces. Inclusion-exclusion formula. Continuity and subadditivity of probability measures. Independence. Binomial, Poisson and geometric distributions. Relation between Poisson and binomial distributions. Conditional probability, Bayes's formula. Examples, including Simpson's paradox. [5]

Discrete random variables

Expectation. Functions of a random variable, indicator function, variance, standard deviation. Covariance, independence of random variables. Generating functions: sums of independent random variables, random sum formula, moments.

Conditional expectation. Random walks: gambler's ruin, recurrence relations. Difference equations and their solution. Mean time to absorption. Branching processes: generating functions and extinction probability. Combinatorial applications of generating functions. [7]

Continuous random variables

Distributions and density functions. Expectations; expectation of a function of a random variable. Uniform, normal and exponential random variables. Memoryless property of exponential distribution. Joint distributions: transformation of random variables (including Jacobians), examples. Simulation: generating continuous random variables, independent normal random variables. Geometrical probability: Bertrand's paradox, Buffon's needle. Correlation coefficient, bivariate normal random variables. [6]

Inequalities and limits

Markov's inequality, Chebyshev's inequality. Weak law of large numbers. Convexity: Jensen's inequality for general random variables, AM/GM inequality.

Moment generating functions and statement (no proof) of continuity theorem. Statement of central limit theorem and sketch of proof. Examples, including sampling. [3]

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1 Introduction

2 Classical probability

2.1 Classical probability

Definition (Classical probability). *Classical probability* applies in a situation when there are a finite number of equally likely outcome.

2.2 Sample space and events

Definition (Sample space). The set of all possible outcomes is the *sample space*, Ω . We can lists the outcomes as $\omega_1, \omega_2, \dots \in \Omega$. Each $\omega \in \Omega$ is an *outcome*.

Definition (Event). A subset of Ω is called an *event*.

Definition (Set notations). Given any two events $A, B \subseteq \Omega$,

- The *complement* of A is $A^C = A' = \bar{A} = \Omega \setminus A$.
- “ A or B ” is the set $A \cup B$.
- “ A and B ” is the set $A \cap B$.
- A and B are *mutually exclusive* or *disjoint* if $A \cap B = \emptyset$.
- $A \subseteq B$ means $A \Rightarrow B$.

Definition (Probability). Suppose $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$. Let $A \subseteq \Omega$ be an event. Then the *probability* of A is

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } \Omega} = \frac{|A|}{N}$$

3 Combinatorial analysis

3.1 Counting

3.2 Sampling with or without replacement

Definition (1. Sampling with replacement). When we sample with replacement, after choosing an item, it is put back and can be chosen again. Then any sampling function f satisfies sampling with replacement.

Definition (2. Sampling without replacement). After choosing an item, we burn it and cannot choose it again. Then f must be an injective function, and clearly we must have $X \geq n$.

3.3 Sampling with or without regard to ordering

3.4 Four cases of enumerative combinatorics

4 Stirling's formula

4.1 Multinomial coefficient

Definition (Multinomial coefficient). A *multinomial coefficient* is

$$\binom{n}{n_1, n_2, \dots, n_x} = \binom{n}{n_1} \binom{n - n_1}{n_2} \dots \binom{n - n_1 - \dots - n_{x-1}}{n_x} = \frac{n!}{n_1! n_2! \dots n_x!}.$$

It is the number of ways to distribute x items into n positions with replacement, in which the i th position has n_i items.

4.2 Stirling's formula