Part IA - Vector Calculus

Lectured by B. Allanach

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Curves in $\mathbb{R}^3$
Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1]

Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$
Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

Vector operators
Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical *and general orthogonal curvilinear* coordinates.

Divergence, curl and ∇^2 in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical *and general orthogonal curvilinear* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

Integration theorems
Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

Laplace's equation
Laplace's equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

Cartesian tensors in $\mathbb{R}^3$
Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

Contents

0 Introduction

In the differential equations class, we learnt how to do calculus in one dimension (mostly). However, (apparently) the world has more than one dimension. We live in a 3 (or 4) dimensional worlds, and string theorists think that the world have more than 10 dimensions. In this course, we are mostly going to learn about doing calculus in many dimensions. In the last few lectures, we are going to learn about Cartesian tensors, which is a generalization of vectors.

Note that throughout the course (and lecture notes), summation convention is implied unless otherwise stated.

1 Derivatives and coordinates

1.1 Differentiable functions $\mathbb{R} \to \mathbb{R}^n$

Definition. A *vector function* is a function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^n$.

While we used to define a derivative as a limit and a function as differentiable if the derivative exists, we now want to do it in a way that can capture differentiability in a way that is easily extensible to vector functions.

We say a function $f(x)$ is differentiable if, when we perturb its argument x slightly by δx, then the change in $f(x)$ is proportional to δx, where "proportional to" means equal to "something" times δx, and the "something" can be anything from a scalar to a matrix. Then we call that "something" the derivative.

In particular, for vector functions, we define:

Definition (Derivative of vector function)**.** A vector function $\mathbf{F}(x)$ is *differentiable* if

$$\delta\mathbf{F} \stackrel{\text{def}}{=} \mathbf{F}(x + \delta x) - \mathbf{F}(x) = \mathbf{F}'(x)\delta u + o(\delta x)$$

for some $\mathbf{F}'(x)$. $\mathbf{F}'(x)$ is called the *derivative* of $\mathbf{F}(x)$.

Clearly, if $\mathbf{F}'(x)$ exists, then it is given by

$$\mathbf{F}' = \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{1}{\delta x}[\mathbf{F}(x + \delta x) - \mathbf{F}(x)].$$

Using differential notation, the differentiability condition can be written as

$$\mathrm{d}\mathbf{F} = \mathbf{F}'(x)\mathrm{d}x.$$

Given a basis $\mathbf{e}_i$ that is independent of x, vector differentiation is performed componentwise, i.e.

Proposition.

$$\mathbf{F}'(x) = F_i'(x)\mathbf{e}_i.$$

Leibnitz identities hold for the products of scalar and vector functions.

Proposition.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(f\mathbf{g}) &= \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{g} + f\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g} \cdot \mathbf{h}) &= \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \cdot \mathbf{h} + \mathbf{g} \cdot \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g} \times \mathbf{h}) &= \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \times \mathbf{h} + \mathbf{g} \times \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}
\end{aligned}$$

Note that the order of multiplication must be retained in the case of the cross product.

Example. Consider a particle with mass m. It has position $\mathbf{r}(t)$, velocity $\dot{\mathbf{r}}(t)$ and acceleration $\ddot{\mathbf{r}}$. Its momentum is $\mathbf{p} = m\dot{\mathbf{r}}(t)$.
Note: that derivatives with respect to t are usually denoted by dots instead of dashes.

If $\mathbf{F}(\mathbf{r})$ is the force on a particle, then Newton's second law states that

$$\dot{\mathbf{p}} = m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}).$$

We can define the angular momentum about the origin to be

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}.$$

If we want to know how the angular momentum changes over time, then

$$\dot{\mathbf{L}} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + m\mathbf{r} \times \ddot{\mathbf{r}} = m + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F}.$$

which is the *torque* of $\mathbf{F}$ about the origin.

1.2 Differentiable functions $\mathbb{R}^n \to \mathbb{R}$

Definition. A *scalar function* is a function $f : \mathbb{R}^n \to \mathbb{R}$.

Before we define the derivative of a scalar function, we have to first define what it means to take a limit of a vector.

Definition (Limit of vector)**.** The *limit of vectors* is defined using the norm. So $\mathbf{v} \to \mathbf{c}$ iff $|\mathbf{v} - \mathbf{c}| \to 0$.

Definition (Gradient of scalar function)**.** A scalar function $f(\mathbf{r})$ is *differentiable* at $\mathbf{r}$ if

$$\delta f \overset{\text{def}}{=} f(\mathbf{r} + \delta\mathbf{r}) - f(\mathbf{r}) = (\nabla f) \cdot \delta\mathbf{r} + o(\delta\mathbf{r})$$

for some vector ∇f, the *gradient* of f at $\mathbf{r}$.

Taking $\delta\mathbf{r} = h\mathbf{n}$, with $\mathbf{n}$ a unit vector,

$$f(\mathbf{r} + h\mathbf{n}) - f(\mathbf{r}) = \nabla f \cdot (h\mathbf{n}) + o(h),$$

which gives

Definition (Directional derivative)**.** The *directional derivative* of f along $\mathbf{n}$ is

$$\mathbf{n} \cdot \nabla f = \lim_{h \to 0} \frac{1}{h}[f(\mathbf{r} + h\mathbf{n}) - f(\mathbf{r})],$$

It refers to how fast f changes when we move in the direction of $\mathbf{n}$.

In particular, the directional derivative is maximized when $\mathbf{n}$ is in the direction of ∇f. So ∇f points in the direction of greatest slope.

To evaluate ∇f, suppose we have an orthonormal basis $\mathbf{e}_i$. Setting $\mathbf{n} = \mathbf{e}_i$ in the above equation, we obtain

$$\mathbf{e}_i \cdot \nabla f = \lim_{h \to 0} \frac{1}{h}[f(\mathbf{r} + h\mathbf{e}_i) - f(\mathbf{r})] = \frac{\partial f}{\partial x_i}.$$

Hence

Theorem. The gradient is

$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i$$

Hence we can write the condition of differentiability as

$$\delta f = \frac{\partial f}{\partial x_i}\delta x_i + o(\delta x).$$

In differential notation, we write

$$df = \nabla f \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial x_i}\mathrm{d}x_i,$$

which is the chain rule for partial derivatives.

Example. Take $f(x, y, z) = x + e^{xy}\sin z$. Then

$$\begin{aligned}\nabla f &= \left(\frac{\partial f}{\varphi x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\\ &= (1 + ye^{xy}\sin z, xe^{xy}\sin z, e^{xy}\cos z)\end{aligned}$$

At $(x, y, z) = (0, 1, 0)$, $\nabla f = (1, 0, 1)$. So f increases/decreases most rapidly for $\mathbf{n} = \pm\frac{1}{\sqrt{2}}(1, 0, 1)$ with a rate of change of $\pm\sqrt{2}$. There is no change in f if $\mathbf{n}$ is perpendicular to $\pm\frac{1}{\sqrt{2}}(1, 0, 1)$.

Now suppose we have a scalar function $f(\mathbf{r})$ and we want to consider the rate of change along a path $\mathbf{r}(u)$. A change δu produces a change $\delta\mathbf{r} = \mathbf{r}'\delta u + o(\delta u)$, and

$$\delta f = \nabla f \cdot \delta\mathbf{r} + o(|\delta\mathbf{r}|) = \nabla f \cdot \mathbf{r}'(u)\delta u + o(\delta u).$$

This shows that f is differentiable as a function of u and

Theorem (Chain rule)**.** Given a function $f(\mathbf{r}(u))$,

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

1.3 Differentiable functions $\mathbb{R}^n \to \mathbb{R}^m$

So far, we have only considered functions $\mathbb{R} \to \mathbb{R}^n$ and $\mathbb{R}^n \to \mathbb{R}$. We can also consider vector fields:

Definition (Vector field)**.** A *vector field* is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$.

Definition (Derivative of vector field)**.** A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if

$$\delta\mathbf{F} \stackrel{def}{=} \mathbf{F}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{F}(\delta\mathbf{x}) = M\delta\mathbf{x} + o(\delta\mathbf{x})$$

for some $n \times m$ matrix M. M is the *derivative* of $\mathbf{F}$.

Given an arbitrary function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ that maps $\mathbf{x} \mapsto \mathbf{y}$ and a choice of basis, we can write $\mathbf{F}$ as a set of m functions $y_j = F_j(\mathbf{x})$ such that $\mathbf{y} = (y_1, y_2, \cdots, y_m)$. Then

$$\mathrm{d}y_j = \frac{\partial F_j}{\partial x_i}\mathrm{d}x_i.$$

and we can write the derivative as

Theorem. The derivative of $\mathbf{F}$ is given by

$$M_{ji} = \frac{\partial y_j}{\partial x_i}.$$

Definition. A function is *smooth* if it can be differentiated any number of times, i.e. if all partial derivatives exist and are totally symmetric in i, j and k (i.e. the differential operation is commutative).

The functions we will consider will be smooth except where things obviously go wrong (e.g. $f(x) = 1/x$ at $x = 0$).

1.4 Chain rule

Theorem (Chain rule)**.** Suppose $g : \mathbb{R}^p \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that the coordinates of the vectors in $\mathbb{R}^p, \mathbb{R}^n$ and $\mathbb{R}^m$ are u_a, x_i and y_r respectively. By the chain rule,

$$\frac{\partial y_r}{\partial u_a} = \frac{\partial y_r}{\partial x_i}\frac{\partial x_i}{\partial u_a},$$

with summation implied. Writing in matrix form,

$$M(f \circ g)_{ra} = M(f)_{ri} M(g)_{ia}.$$

Alternatively, in operator form,

$$\frac{\partial}{\partial u_a} = \frac{\partial x_i}{\partial u_a}\frac{\partial}{\partial x_i}.$$

1.5 Inverse functions

Suppose $g, f : \mathbb{R}^n \to \mathbb{R}^n$ are inverse functions, i.e. $g \circ f = f \circ g = \mathrm{id}$. Suppose that $f(\mathbf{x}) = \mathbf{u}$ and $g(\mathbf{u}) = \mathbf{x}$.

Since the derivative of the identity function is the identity matrix (if you differentiate $\mathbf{x}$ wrt to $\mathbf{x}$, you get 1), we must have

$$M(f \circ g) = I.$$

Therefore we know that

$$M(g) = M(f)^{-1}.$$

We derive this result more formally by noting

$$\frac{\partial u_b}{\partial u_a} = \delta_{ab}.$$

So by the chain rule,

$$\frac{\partial u_b}{\partial x_i}\frac{\partial x_i}{\partial u_a} = \delta_{ab},$$

i.e. $M(f \circ g) = I$.

In the $n = 1$ case, it is the familiar result that $\mathrm{d}u/\mathrm{d}x = 1/(\mathrm{d}x/\mathrm{d}u)$.

Example. For $n = 2$, write $u_1 = \rho$, $u_2 = \varphi$ and let $x_1 = \rho\cos\varphi$ and $x_2 = \rho\sin\varphi$. Then the function used to convert between the coordinate systems is $g(u_1, u_2) = (u_2 \cos u_1, u_2 \sin u_1)$

Then

$$M(g) = \begin{pmatrix} \partial x_1/\partial\rho & \partial x_1/\partial\varphi \\ \partial x_2/\partial\rho & \partial x_2/\partial\varphi \end{pmatrix} \qquad = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix}$$

We can invert the relations between (x_1, x_2) and (ρ, φ) to obtain

$$\varphi = \tan^{-1}\frac{x_2}{x_1}$$
$$\rho = \sqrt{x_1^2 + x_2^2}$$

We can calculate

$$M(f) = \begin{pmatrix} \partial\rho/\partial x_1 & \partial\rho/\partial x_2 \\ \partial\varphi/\partial x_1 & \partial\varphi/\partial x_2 \end{pmatrix} = M(g)^{-1}.$$

These matrices are known as Jacobians matrices, and their determinants are known as the Jacobians.

Note that

$$\det M(f) \det M(g) = 1.$$

1.6 Coordinate systems

Now we can apply the results above the changes of coordinates on Euclidean space. Suppose x_i are the coordinates are Cartesian coordinates. Then we can define an arbitrary new coordinate system u_a in which each coordinate u_a is a function of $\mathbf{x}$. For example, we can define the plane polar coordinates ρ, φ by

$$x_1 = \rho\cos\varphi, \quad x_2 = \rho\sin\varphi.$$

However, note that ρ and φ are not components of a position vector, i.e. they are not the "coefficients" of basis vectors like $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ are. But we can associate related basis vectors that point to directions of increasing ρ and φ:

$$\mathbf{e}_\rho = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2, \quad \mathbf{e}_\varphi = -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2.$$

However, these basis vectors vary with position and are undefined at the origin.

In three dimensions, we have cylindrical polars and spherical polars.

Cylindrical polars	Spherical polars
Conversion formulae	
$x_1 = \rho\cos\varphi$	$x_1 = r\sin\theta\cos\varphi$
$x_2 = \rho\sin\varphi$	$x_2 = r\sin\theta\sin\varphi$
$x_3 = z$	$x_3 = r\cos\theta$
Basis vectors	
$\mathbf{e}_\rho = \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2$	$\mathbf{e}_r = \cos\theta\mathbf{e}_z + \sin\theta\mathbf{e}_p$
$\mathbf{e}_\varphi = -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2$	$e_\varphi = -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2$
$\mathbf{e}_z = \mathbf{e}_3$	$\mathbf{e}_\theta = -\sin\theta\mathbf{e}_z + \cos\theta\mathbf{e}_\rho$

2 Curves and Line

2.1 Parametrised curves, lengths and arc length

Definition (Parametrisation of curve)**.** Given a curve C in $\mathbb{R}^n$, a *parametrisation* of it is a continuous and invertible function $\mathbf{r} : D \to \mathbb{R}^n$ for some $D \subseteq \mathbb{R}$ whose image is C.

$\mathbf{r}'(u)$ is a vector tangent to the curve at each point. A parametrization is *regular* if $\mathbf{r}'(u) \neq 0$ for all u.

Example. The curve

$$\frac{1}{4}x^2 + y^2 = 1, \quad y \geq 0, \quad z = 3.$$

is parametrised by $2\cos u\hat{\mathbf{i}} + \sin u\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

If we change u (and hence $\mathbf{r}$) by a small amount, then the distance $|\delta\mathbf{r}|$ is roughly equal to the change in arclength δs. So $\delta s = |\delta\mathbf{r}| + o(\delta\mathbf{r})$. Then we have

Proposition. Let s denote the arclength of a curve $\mathbf{r}(u)$. Then

$$\frac{\mathrm{d}s}{\mathrm{d}u} = \pm\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| = \pm|\mathbf{r}'(u)|.$$

With the sign determining whether it is in the direction of increasing or decreasing arclength.

Example. Consider a helix described by $\mathbf{r}(u) = (3\cos u, 3\sin u, 4u)$. Then

$$\begin{aligned}\mathbf{r}'(u) &= (-3, \sin u, 3\cos u, 4)\\ \frac{\mathrm{d}s}{\mathrm{d}u} &= |\mathbf{r}'(u)| = \sqrt{3^2 + 4^2} = 5\end{aligned}$$

So $s = 5u$. i.e. the arclength from $\mathbf{r}(0)$ and $\mathbf{r}(u)$ is $s = 5u$.

We can change parametrisation of $\mathbf{r}$ by taking an invertible smooth function $u \mapsto \tilde{u}$, and have a new parametrization $\mathbf{r}(\tilde{u}) = \mathbf{r}(\tilde{u}(u))$. Then by the chain rule,

$$\begin{aligned}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tilde{u}} \times \frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\\ \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tilde{u}} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\Big/\frac{\mathrm{d}\tilde{u}}{\mathrm{d}u}\end{aligned}$$

It is often convenient to use the arclength s as the parameter. Then the tangent vector will always have unit length since the proposition above yields

$$|\mathbf{r}'(s)| = \frac{\mathrm{d}s}{\mathrm{d}s} = 1.$$

Definition (Scalar line element)**.** We say $\mathrm{d}s = \pm|\mathbf{r}'(u)|\mathrm{d}u$ is a *scalar line element* on C.

2.2 Line integrals of vector fields

Definition (Line integral)**.** The *line integral* of a smooth vector field $\mathbf{F}(\mathbf{r})$ along a path C parametrised by $\mathbf{r}(u)$ with along the direction (orientation) $\mathbf{r}(\alpha) \to \mathbf{r}(\beta)$ is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_\alpha^\beta \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}'(u)\mathrm{d}u.$$

We say $\mathrm{d}\mathbf{r} = \mathbf{r}'(u)\mathrm{d}u$ is the *line element* on C. Note that the upper and lower limits of the integral are the end point and start point respectively, and β is not necessarily larger than α.

For example, we may be moving a particle from $\mathbf{a}$ to $\mathbf{b}$ along a curve C under a force field $\mathbf{F}$. Then we may divide the curve into many small segments $\delta\mathbf{r}$. Then for each infinitesimal segment, the force experienced is $\mathbf{F}(\mathbf{r})$ and the work done is $\mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$. Then the total work done across the curve is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}.$$

Example. Take $\mathbf{F}(\mathbf{r}) = (xe^y, z^2, xy)$ and we want find the line integral from $\mathbf{a} = (0,0,0)$ to $\mathbf{b} = (1,1,1)$. We first integrate along the curve $C_1 : \mathbf{r}(u) = (u, u^2, u^3)$. Then $\mathbf{r}'(u) = (1, 2u, 3u^2)$, and $\mathbf{F}(\mathbf{r}(u)) = (ue^{u^2}, u^6, u^3)$. Then

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{F} \cdot \mathbf{r}'(u)\mathrm{d}u \\ &= \int_0^1 ue^{u^2} + 2u^7 + 3u^5 \ \mathrm{d}u \\ &= \frac{e}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2} \\ &= \frac{e}{2} + \frac{1}{4}\end{aligned}$$

Now we try to integrate along another curve $C_2 : \mathbf{r}(t) = (t,t,t)$. So $\mathbf{r}'(t) = (1,1,1)$.

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int \mathbf{F} \cdot \mathbf{r}'(t)\mathrm{d}t) \\ &= \int_0^1 te^t + 2t^2 \ \mathrm{d}t \\ &= \frac{5}{3}\end{aligned}$$

Note that the line integral depends on the curve C in general, not just $\mathbf{a}, \mathbf{b}$.

We can also use the arclength s as parameter. Since $\mathrm{d}\mathbf{r} = \mathbf{t}\ \mathrm{d}s$, with $\mathbf{t}$ being the unit tangent vector, we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{t}\ \mathrm{d}s.$$

In addition to integrating a vector field, we can also integrate a scalar function as a function of s, $\int_C f(s)\ \mathrm{d}s$. By convention, this is calculated in the direction of increasing s. In particular, we have

$$\int_C 1\ \mathrm{d}s = \text{length of C}.$$

Definition (Closed curve)**.** A *closed curve* is a curve with the same start and end point. The line integral along a closed curve is written as $\oint$ and is called the *circulation* of $\mathbf{F}$ around C.

2.3 Sums of curves and integrals

Definition (Piecewise smooth curve)**.** A *piecewise smooth curve* is a curve $C = C_1 + C_2 + \cdots + C_n$ with all C_i smooth with regular parametrisations. The line integral over a piecewise smooth C is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Example. Take the example above, and let $C_3 = -C_2$. Then let $C = C_1 + C_3$. Then

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \left(\frac{e}{2} + \frac{1}{4}\right) - \frac{5}{3} \\ &= -\frac{17}{12} + \frac{e}{2}.\end{aligned}$$

2.4 Gradients and Differentials

2.4.1 Line integrals and Gradients

Theorem. If $\mathbf{F} = \nabla f(\mathbf{r})$, then

$$\int_C \mathbf{F} \cdot \mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

In particular, the line integral does NOT depend on the curve, but the end points only. This is the vector counterpart of the fundamental theorem of calculus. A special case is when C is a closed curve, then $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

Proof.

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = \int \nabla f \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\,\mathrm{d}u$$

for any parametrisation of C with $\mathbf{a} = \mathbf{r}(\alpha)$ and $\mathbf{b} = \mathbf{r}(\beta)$. So by the chain rule, this is equal to

$$\int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}u}(f(\mathbf{r}(u)))\,\mathrm{d}u = [f(\mathbf{r}(u))]_\alpha^\beta = f(\mathbf{b}) - f(\mathbf{a}).$$

(expand out ∇f and $\mathrm{d}\mathbf{r}/\mathrm{d}u$ if you trust that it works) □

Definition (Conservative vector field)**.** If $\mathbf{F} = \nabla f$ for some f, the $\mathbf{F}$ is called a *conservative vector field.*

2.4.2 Differentials

It is convenient to work with differentials $\mathbf{F} \cdot \mathrm{d}\mathbf{r} = F_i \mathrm{d}x_i$ as objects that can be integral along curves.

Definition (Exact differential)**.** A differential $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is *exact* if there is an f such that $\mathbf{F} = \nabla f$. Then

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

To test if this holds, we can use the necessary condition

Proposition. If $F = \nabla \mathbf{f}$ for some f, then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

This is because both are equal to $\partial^2 f / \partial x_i \partial x_j$.

For an exact differential, the result above reads

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathrm{d}f = f(\mathbf{b}) - f(\mathbf{a}).$$

Differentials can be manipulated using (for constant λ, μ):

Proposition.

$$\begin{aligned} \mathrm{d}(\lambda f + \mu g) &= \lambda \mathrm{d}f + \mu \mathrm{d}g \\ \mathrm{d}(fg) &= (\mathrm{d}f)g + f(\mathrm{d}g) \end{aligned}$$

Using these, it may be possible to find f by inspection.

Example. Consider

$$\int_C 3x^2 y \sin z \; \mathrm{d}x + x^3 \sin z \; \mathrm{d}y + x^3 y \cos z \; \mathrm{d}z.$$

We see that if we integrate the first term with respect to x, we obtain $x^3 y \sin z$. We obtain the same thing if we integrate the second and third term. So this is equal to

$$\int_C \mathrm{d}(x^3 y \sin z) = [x^3 y \sin z]_{\mathbf{a}}^{\mathbf{b}}.$$

2.5 Work and potential energy

Definition (Work and potential energy)**.** If $\mathbf{F}(\mathbf{r})$ is a force, then $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is the *work done* by the force along the curve C. It is the limit of a sum of terms $\mathbf{F}(\mathbf{r}) \cdot \delta \mathbf{r}$, i.e. the force along the direction of $\delta \mathbf{r}$.

Consider a point particle moving under $\mathbf{F}(\mathbf{r})$ according to Newton's second law: $\mathbf{F}(\mathbf{r}) = m\ddot{\mathbf{r}}$.

Since the kinetic energy is defined as

$$K(t) = \frac{1}{2} m \dot{\mathbf{r}}^2,$$

the rate of change of energy is

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = m\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \mathbf{F} \cdot \mathbf{r}.$$

Suppose the path of particle is a curve C from $\mathbf{a} = \mathbf{r}(\alpha)$ to $\mathbf{b} = \mathbf{r}(\beta)$, Then

$$K(\beta) - K(\alpha) = \int_\alpha^\beta \frac{\mathrm{d}K}{\mathrm{d}t}\,\mathrm{d}t = \int_\alpha^\beta \mathbf{F} \cdot \dot{\mathbf{r}}\,\mathrm{d}t = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

So the work done on the particle is the change in kinetic energy.

Definition (Potential energy)**.** Given a conservative force $= -\nabla V$, $V()$ is the *potential energy.* Then

$$\int_C \mathbf{F} \cdot \mathrm{d} = V(\mathbf{a}) - V(\mathbf{b}).$$

Therefore, for a conservative force, we have $\mathbf{F} = \nabla V$, where $V(\mathbf{r})$ is the potential energy.

So the work done (gain in kinetic energy) is the loss in potential energy. So the total energy $K + V$ is conserved, i.e. constant during motion.

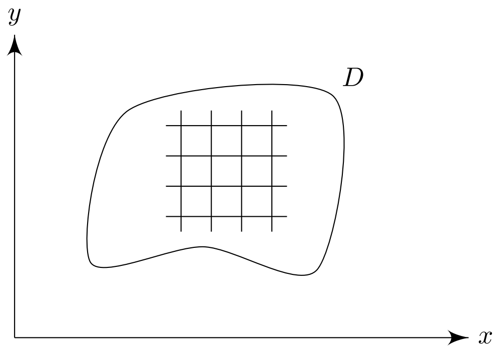
We see that energy is conserved for conservative forces. In fact, the converse is true - the energy is conserved only for conservative forces.

3 Integration in $\mathbb{R}^2$ and R^3

3.1 Integrals over subsets of $\mathbb{R}^2$

3.1.1 Definition as the limit of as sum

Definition (Surface integral)**.** Let $D \subseteq \mathbb{R}^2$. Let $\mathbf{r}(x, y)$ be in Cartesian coordinates. We can approximate D by N disjoint subsets of simple shapes, e.g. triangles, parallelograms. These shapes are labelled by I and have areas δA_i. Each of these are small enough to be contained in a disc of diameter ℓ.



Assume that as $\ell \to 0$ and $N \to \infty$, the union of the small sets $\to D$. For a function $f(\mathbf{r})$, we define the *surface integral* as

$$\int_D f(\mathbf{r})\ \mathrm{d}A = \lim_{\ell \to 0} \sum_I f(\mathbf{r}_i)\delta A_i.$$

where $\mathbf{r}_i$ is some point within each subset A_i. The integral *exists* if the limit is well-defined (i.e. the same regardless of what A_i and $\mathbf{r}_i$ we choose before we take the limit) and exists.

If we take $f = 1$, then the surface integral is the area of D.