

Part IA - Probability

Theorems with Proof

Lectured by R. Weber

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Basic concepts

Classical probability, equally likely outcomes. Combinatorial analysis, permutations and combinations. Stirling's formula (asymptotics for $\log n!$ proved). [3]

Axiomatic approach

Axioms (countable case). Probability spaces. Inclusion-exclusion formula. Continuity and subadditivity of probability measures. Independence. Binomial, Poisson and geometric distributions. Relation between Poisson and binomial distributions. Conditional probability, Bayes's formula. Examples, including Simpson's paradox. [5]

Discrete random variables

Expectation. Functions of a random variable, indicator function, variance, standard deviation. Covariance, independence of random variables. Generating functions: sums of independent random variables, random sum formula, moments.

Conditional expectation. Random walks: gambler's ruin, recurrence relations. Difference equations and their solution. Mean time to absorption. Branching processes: generating functions and extinction probability. Combinatorial applications of generating functions. [7]

Continuous random variables

Distributions and density functions. Expectations; expectation of a function of a random variable. Uniform, normal and exponential random variables. Memoryless property of exponential distribution. Joint distributions: transformation of random variables (including Jacobians), examples. Simulation: generating continuous random variables, independent normal random variables. Geometrical probability: Bertrand's paradox, Buffon's needle. Correlation coefficient, bivariate normal random variables. [6]

Inequalities and limits

Markov's inequality, Chebyshev's inequality. Weak law of large numbers. Convexity: Jensen's inequality for general random variables, AM/GM inequality.

Moment generating functions and statement (no proof) of continuity theorem. Statement of central limit theorem and sketch of proof. Examples, including sampling. [3]

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1 Introduction

2 Classical probability

2.1 Classical probability

2.2 Sample space and events

3 Combinatorial analysis

3.1 Counting

Theorem (Fundamental rule of counting). Suppose we have to make r multiple choices in sequence. There are m_1 possibilities for the first choice, m_2 possibilities for the second etc. Then the total number of choices is $m_1 \times m_2 \times \cdots m_r$.

3.2 Sampling with or without replacement

3.3 Sampling with or without regard to ordering

3.4 Four cases of enumerative combinatorics

4 Stirling's formula

4.1 Multinomial coefficient

4.2 Stirling's formula

Proposition. $\log n! \sim n \log n$

Proof. Note that

$$e^n = 1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!} + \cdots.$$

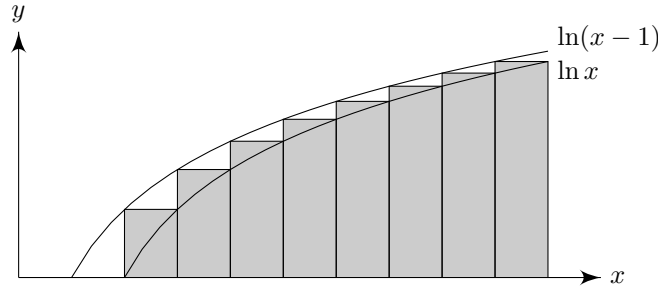
So $1 \leq n^n/n! \leq e^n$. Note that

$$\log n! = \sum_{k=1}^n \log k.$$

Now we claim that

$$\int_1^n \log x \, dx \leq \sum_{k=1}^n \log k \leq \int_1^{n+1} \log x \, dx.$$

This is true by considering the diagram:



Perform the integral to obtain

$$n \log n - n + 1 \leq \log n! \leq (n+1) \log(n+1) - n;$$

Divide both sides by $n \log n$ and let $n \rightarrow \infty$. Both sides tend to 1. So

$$\frac{\log n!}{n \log n} \rightarrow 1.$$

□

Theorem (Stirling's formula). As $n \rightarrow \infty$,

$$\log \left(\frac{n! e^n}{n^{n+\frac{1}{2}}} \right) = \log \sqrt{2\pi} + O \left(\frac{1}{n} \right)$$

Corollary.

$$n! \sim \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n}$$

Proof. (non-examinable) Define

$$d_n = \log \left(\frac{n!e^n}{n^{n+1/2}} \right) = \log n! - (n + 1/2) \log n + n$$

Then

$$d_n - d_{n+1} = (n + 1/2) \log \left(\frac{n+1}{n} \right)$$

Write $t = 1/(2n + 1)$. Then

$$d_n - d_{n+1} = \frac{1}{2t} \log \left(\frac{1+t}{1-t} \right).$$

We can simplify by noting that

$$\begin{aligned} \log(1+t) - t &= -\frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \dots \\ \log(1-t) + t &= -\frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{1}{4}t^4 - \dots \end{aligned}$$

Then if we subtract the equations and divide by $2t$, we obtain

$$\begin{aligned} d_n - d_{n+1} &= \frac{1}{3}t^2 + \frac{1}{5}t^4 + \frac{1}{7}t^6 \\ &\leq \frac{1}{3}t^2 + \frac{1}{3}t^4 + \frac{1}{3}t^6 = \dots \\ &= \frac{1}{3} \frac{t^2}{1-t^2} \\ &= \frac{1}{3} \frac{1}{(2n+1)^2 - 1} \\ &= \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right) \end{aligned}$$

By summing these bounds, we know that

$$d_1 - d_n \leq \frac{1}{12} \left(1 - \frac{1}{n} \right)$$

Then we know that d_n is bounded below by $d_1 +$ something, and is decreasing since $d_n - d_{n+1}$ is positive. So it converges to a limit A .

Suppose $m > n$. Then $d_n - d_m < \left(\frac{1}{n} - \frac{1}{m} \right) \frac{1}{12} - \frac{2}{15} \frac{1}{(2n+1)^4}$ by adding back the term we removed when changing $t^4/5$ to $t^4/3$. So taking the limit as $m \rightarrow \infty$, $A < d_n < A + 1/(12n)$, with strict inequalities due to the second term.

To find A , we have a small detour to prove a formula:

Take $I_n = \int_0^{\pi/2} \sin^n \theta \, d\theta$. This is decreasing as n increases as $\sin^n \theta$ gets smaller. We also know that

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^n \theta \, d\theta \\ &= -\cos \theta \sin^{n-1} \theta \Big|_0^{\pi/2} + \int_0^{\pi/2} (n-1) \cos^2 \theta \sin^{n-2} \theta \, d\theta \\ &= (n-1)(I_n - 2 - I_n) \end{aligned}$$

So

$$I_n = \frac{n-1}{n} I_{n-2}.$$

We can directly evaluate the integral to obtain $I_0 = \pi/2$, $I_1 = 1$. Then

$$I_{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \pi/2 = \frac{(2n)!}{(2^n n!)^2} \frac{\pi}{2}$$

$$I_{2n+1} = \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}$$

So using the fact that I_n is decreasing, we know that

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}} = 1 + \frac{1}{2n} \rightarrow 1.$$

Using the approximation $n! \sim n^{n+1/2} e^{-n+A}$, where A is the limit we want to find, we can approximate

$$\frac{I_{2n}}{I_{2n+1}} = \pi(2n+1) \left[\frac{((2n)!)^2}{2^{4n+1}(n!)^4} \right] \sim \pi(2n+1) \frac{1}{n e^{2A}} \rightarrow \frac{2\pi}{e^{2A}}.$$

Since the last expression is equal to 1, we know that $A = \log \sqrt{2\pi}$ □

Proposition ((non-examinable)). We use the $1/12n$ term from the proof above to get a better approximation:

$$\sqrt{2\pi} n^{n+1/2} e^{-n} + \frac{1}{12n+1} \leq n! \leq \sqrt{2\pi} n^{n+1/2} e^{-n} + \frac{1}{12n}.$$