Part IA - Analysis I

Theorems with Proof

Lectured by W. T. Gowers

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Limits and convergence
Sequences and series in R and C. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

Continuity
Continuity of real- and complex-valued functions defined on subsets of $\mathbb{R}$ and $\mathbb{C}$. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

Differentiability
Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from $\mathbb{R}$ to $\mathbb{R}$; Lagranges form of the remainder. Complex differentiation. [5]

Power series
Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. *Direct proof of the differentiability of a power series within its circle of convergence*. [4]

Integration
Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

Contents

1 The real number system

Lemma. Let $\mathbb{F}$ be an ordered field and $x \in \mathbb{F}$. Then $x^2 \geq 0$.

Proof. By trichotomy, either $x < 0$, $x = 0$ or $x > 0$. If $x = 0$, then $x^2 = 0$. So $x^2 \geq 0$. If $x > 0$, then $x^2 > 0 \times x = 0$. If $x < 0$, then $x - x < 0 - x$. So $0 < -x$. But then $x^2 = (-x)^2 > 0$. $\square$

Lemma (Archimedean property v1)**.** Let $\mathbb{F}$ be an ordered field with the least upper bound property. Then the set $\{1, 2, 3, \cdots\}$ is not bounded above. (Note that these need not refer to natural numbers. We simply define 1 to be the multiplicative identity, $2 = 1 + 1$, $3 = 1 + 2$ etc.)

Proof. If it is bounded above, then it has a supremum x. But then $x - 1$ is not an upper bound. So we can find $n \in \{1, 2, 3, \cdots\}$ such that $n > x - 1$. But then $n + 1 > x$ but x is supposed to be an upper bound. $\square$

2 Convergence of sequences

Lemma (Archimedean property v2)**.** $1/n \to 0$.

Proof. Let $\varepsilon > 0$. We want to find an N such that $|1/N - 0| = 1/N < \varepsilon$. So pick N such that $N > 1/\varepsilon$. This exists such an N by the Archimedean property v1. Then for all $n > N$, we have $0 < 1/n \leq 1/N < \varepsilon$. So $|1/n - 0| \to \varepsilon$. □

Lemma. Every eventually bounded sequence is bounded.

Proof. Let C and N be such that $\forall n \geq N$ $|a_n| \leq C$. Then $\forall n \in \mathbb{N}$, $|a_n| \leq \max\{|a_1|, \cdots, |a_{n-1}|, C\}$. □

2.1 Sums, products and quotients

Lemma (Sums of sequences)**.** If $a_n \to a$ and $b_n \to b$, then

(i) $a_n + b_n \to a + b$

Proof. Let $\varepsilon > 0$. We want to show that $\exists N$ such that $\forall n \geq N$, $|a_n + b_n - (a+b)| < \varepsilon$. We know that a_n is very close to a and b_n is very close to b. So their sum must be very close to $a + b$.

Formally, since $a_n \to a$ and $b_n \to b$, we can find N_1, N_2 such that $\forall n \geq N_1$, $|a_n - a| < \varepsilon/2$ and $\forall n \geq N_2$, $|b_n - b| < \varepsilon/2$.

Now let $N = \max\{N_1, N_2\}$. Then by the triangle inequality,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon.$$

□

Lemma (Scalar multiplication of sequences)**.** Let $a_n \to a$ and $\lambda \in \mathbb{R}$. Then $\lambda a_n \to \lambda a$.

Proof. If $\lambda = 0$, then the result is trivial.

Otherwise, let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon/|\lambda|$. So $|\lambda a_n - \lambda a| < \epsilon$. □

Lemma. Let a_n be bounded $b_n \to 0$. Then $a_n b_n \to 0$.

Proof. Let $C \neq 0$ be such that $\forall n: \ |a_n| \leq C$. Let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|b_n| < \varepsilon/C$. Then $|a_n b_n| < \varepsilon$. □

Lemma. Every convergent sequence is bounded.

Proof. Let $a_n \to l$. Then $\exists N : \forall n \geq N$, $|a_n - l| \leq 1$. So $|a_n| \leq |l| + 1$. So a_n is eventually bounded, and therefore bounded. □

Lemma. Let $a_n \to a$ and $b_n \to b$. Then $a_n b_n \to ab$.

Product of sequences. Let $c_n = a_n - a$ and $d_n = b_n - b$. Then $a_n b_n = (a + c_n)(b + d_n) = ab + a d_n + b c_n + c_n d_n$.

But by "sum of sequences", $c_n \to 0$ and $d_n \to 0$. So $a d_n \to 0$ and $b c_n \to 0$. Since c_n is bounded, $c_n d_n \to 0$. Hence by sum of sequences, $a_n b_n \to ab$ □

Proof. (alternative) Observe that $a_n b_n - ab = (a_n - a)b_n + (b_n - b)a$. We know that $a_n - a \to 0$ and $b_n - b \to 0$. Since (b_n) is bounded, so $(a_n - a)b_n + (b_n - b)a \to 0$. So $a_n b_n \to ab$. □