Part IA - Vector Calculus
Definitions

Lectured by B. Allanach

Lent 2015

Curves in $\mathbb{R}^3$
Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1]

Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$
Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

Vector operators
Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical *and general orthogonal curvilinear* coordinates.

Divergence, curl and ∇^2 in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical *and general orthogonal curvilinear* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

Integration theorems
Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

Laplace's equation
Laplace's equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

Cartesian tensors in $\mathbb{R}^3$
Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

Contents

1 Introduction

2 Derivatives and coordinates

2.1 Differentiable functions $\mathbb{R} \to \mathbb{R}^n$

Definition. A *vector function* is a function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^n$.

Definition (Derivative of vector function)**.** A vector function $\mathbf{F}(x)$ is *differentiable* if

$$\delta \mathbf{F} \stackrel{\text{def}}{=} \mathbf{F}(x + \delta x) - \mathbf{F}(x) = \mathbf{F}'(x)\delta u + o(\delta x)$$

for some $\mathbf{F}'(x)$. $\mathbf{F}'(x)$ is called the *derivative* of $\mathbf{F}(x)$.

Clearly, if $\mathbf{F}'(x)$ exists, then it is given by

$$\mathbf{F}' = \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{1}{\delta x}[\mathbf{F}(x + \delta x) - \mathbf{F}(x)].$$

2.2 Differentiable functions $\mathbb{R}^n \to \mathbb{R}$

Definition. A *scalar function* is a function $f : \mathbb{R}^n \to \mathbb{R}$.

Definition (Limit of vector)**.** The *limit of vectors* is defined using the norm. So $\mathbf{v} \to \mathbf{c}$ iff $|\mathbf{v} - \mathbf{c}| \to 0$.

Definition (Gradient of scalar function)**.** A scalar function $f(\mathbf{r})$ is *differentiable* at $\mathbf{r}$ if

$$\delta f \stackrel{\text{def}}{=} f(\mathbf{r} + \delta \mathbf{r}) - f(\mathbf{r}) = (\nabla f) \cdot \delta \mathbf{r} + o(\delta \mathbf{r})$$

for some vector ∇f, the *gradient* of f at $\mathbf{r}$.

Definition (Directional derivative)**.** The *directional derivative* of f along $\mathbf{n}$ is

$$\mathbf{n} \cdot \nabla f = \lim_{h \to 0} \frac{1}{h}[f(\mathbf{r} + h\mathbf{n}) - f(\mathbf{r})],$$

It refers to how fast f changes when we move in the direction of $\mathbf{n}$.

2.3 Differentiable functions $\mathbb{R}^n \to \mathbb{R}^m$

Definition (Vector field)**.** A *vector field* is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$.

Definition (Derivative of vector field)**.** A vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if

$$\delta \mathbf{F} \stackrel{def}{=} \mathbf{F}(\mathbf{x} + \delta \mathbf{x}) - \mathbf{F}(\delta \mathbf{x}) = M \delta \mathbf{x} + o(\delta \mathbf{x})$$

for some $n \times m$ matrix M. M is the *derivative* of $\mathbf{F}$.

Definition. A function is *smooth* if it can be differentiated any number of times, i.e. if all partial derivatives exist and are totally symmetric in i, j and k (i.e. the differential operation is commutative).

2.4 Chain rule

2.5 Inverse functions

2.6 Coordinate systems

3 Curves and Line

3.1 Parametrised curves, lengths and arc length

Definition (Parametrisation of curve)**.** Given a curve C in $\mathbb{R}^n$, a *parametrisation* of it is a continuous and invertible function $\mathbf{r}: D \to \mathbb{R}^n$ for some $D \subseteq \mathbb{R}$ whose image is C.

$\mathbf{r}'(u)$ is a vector tangent to the curve at each point. A parametrization is *regular* if $\mathbf{r}'(u) \neq 0$ for all u.

Definition (Scalar line element)**.** We say $\mathrm{d}s = \pm|\mathbf{r}'(u)|\mathrm{d}u$ is a *scalar line element* on C.

3.2 Line integrals of vector fields

Definition (Line integral)**.** The *line integral* of a smooth vector field $\mathbf{F}(\mathbf{r})$ along a path C parametrised by $\mathbf{r}(u)$ with along the direction (orientation) $\mathbf{r}(\alpha) \to \mathbf{r}(\beta)$ is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_\alpha^\beta \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}'(u)\mathrm{d}u.$$

We say $\mathrm{d}\mathbf{r} = \mathbf{r}'(u)\mathrm{d}u$ is the *line element* on C. Note that the upper and lower limits of the integral are the end point and start point respectively, and β is not necessarily larger than α.

Definition (Closed curve)**.** A *closed curve* is a curve with the same start and end point. The line integral along a closed curve is written as $\oint$ and is called the *circulation* of $\mathbf{F}$ around C.