Part IA - Vector Calculus

Theorems

Lectured by B. Allanach

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Curves in $\mathbb{R}^3$
Parameterised curves and arc length, tangents and normals to curves in $\mathbb{R}^3$, the radius of curvature. [1]

Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$
Line integrals. Surface and volume integrals: definitions, examples using Cartesian, cylindrical and spherical coordinates; change of variables. [4]

Vector operators
Directional derivatives. The gradient of a real-valued function: definition; interpretation as normal to level surfaces; examples including the use of cylindrical, spherical *and general orthogonal curvilinear* coordinates.

Divergence, curl and ∇^2 in Cartesian coordinates, examples; formulae for these operators (statement only) in cylindrical, spherical *and general orthogonal curvilinear* coordinates. Solenoidal fields, irrotational fields and conservative fields; scalar potentials. Vector derivative identities. [5]

Integration theorems
Divergence theorem, Green's theorem, Stokes's theorem, Green's second theorem: statements; informal proofs; examples; application to fluid dynamics, and to electromagnetism including statement of Maxwell's equations. [5]

Laplace's equation
Laplace's equation in $\mathbb{R}^2$ and $\mathbb{R}^3$: uniqueness theorem and maximum principle. Solution of Poisson's equation by Gauss's method (for spherical and cylindrical symmetry) and as an integral. [4]

Cartesian tensors in $\mathbb{R}^3$
Tensor transformation laws, addition, multiplication, contraction, with emphasis on tensors of second rank. Isotropic second and third rank tensors. Symmetric and antisymmetric tensors. Revision of principal axes and diagonalization. Quotient theorem. Examples including inertia and conductivity. [5]

Contents

1 Introduction

2 Derivatives and coordinates

2.1 Differentiable functions $\mathbb{R} \to \mathbb{R}^n$

Proposition.

$$\mathbf{F}'(x) = F_i'(x)\mathbf{e}_i.$$

Proposition.

$$\frac{\mathrm{d}}{\mathrm{d}t}(f\mathbf{g}) = \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{g} + f\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\cdot\mathbf{h}) = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\cdot\mathbf{h} + \mathbf{g}\cdot\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\times\mathbf{h}) = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\times\mathbf{h} + \mathbf{g}\times\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$

Note that the order of multiplication must be retained in the case of the cross product.

2.2 Differentiable functions $\mathbb{R}^n \to \mathbb{R}$

Theorem. The gradient is

$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i$$

Theorem (Chain rule)**.** Given a function $f(\mathbf{r}(u))$,

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r} = \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

2.3 Differentiable functions $\mathbb{R}^n \to \mathbb{R}^m$

Theorem. The derivative of $\mathbf{F}$ is given by

$$M_{ji} = \frac{\partial y_j}{\partial x_i}.$$

2.4 Chain rule

Theorem (Chain rule)**.** Suppose $g : \mathbb{R}^p \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that the coordinates of the vectors in $\mathbb{R}^p, \mathbb{R}^n$ and $\mathbb{R}^m$ are u_a, x_i and y_r respectively. By the chain rule,

$$\frac{\partial y_r}{\partial u_a} = \frac{\partial y_r}{\partial x_i}\frac{\partial x_i}{\partial u_a},$$

with summation implied. Writing in matrix form,

$$M(f \circ g)_{ra} = M(f)_{ri}M(g)_{ia}.$$

Alternatively, in operator form,

$$\frac{\partial}{\partial u_a} = \frac{\partial x_i}{\partial u_a}\frac{\partial}{\partial x_i}.$$

2.5 Inverse functions

2.6 Coordinate systems

3 Curves and Line

3.1 Parametrised curves, lengths and arc length

Proposition. Let s denote the arclength of a curve $\mathbf{r}(u)$. Then

$$\frac{\mathrm{d}s}{\mathrm{d}u} = \pm\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| = \pm|\mathbf{r}'(u)|.$$

With the sign determining whether it is in the direction of increasing or decreasing arclength.

3.2 Line integrals of vector fields

3.3 Sums of curves and integrals

3.4 Gradients and Differentials

3.4.1 Line integrals and Gradients

Theorem. If $\mathbf{F} = \nabla f(\mathbf{r})$, then

$$\int_C \mathbf{F} \cdot \mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

In particular, the line integral does NOT depend on the curve, but the end points only. This is the vector counterpart of the fundamental theorem of calculus. A special case is when C is a closed curve, then $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

3.4.2 Differentials

Proposition. If $F = \nabla \mathbf{f}$ for some f, then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

This is because both are equal to $\partial^2 f / \partial x_i \partial x_j$.

Proposition.

$$\begin{aligned} \mathrm{d}(\lambda f + \mu g) &= \lambda \mathrm{d}f + \mu \mathrm{d}g \\ \mathrm{d}(fg) &= (\mathrm{d}f)g + f(\mathrm{d}g) \end{aligned}$$

3.5 Work and potential energy

4 Integration in $\mathbb{R}^2$ and R^3

4.1 Integrals over subsets of $\mathbb{R}^2$

4.1.1 Definition as the limit of as sum