

Part II - Logic and Set Theory
Definitions

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1 Propositional calculus

Definition (Propositions). Let P be a set of *primitive propositions*. These are a bunch of (meaningless) symbols, that are usually interpreted to take a truth value. Usually, any symbol (composed of alphabets and subscripts) is in the set of primitive propositions.

The set of *propositions*, written as L or $L(P)$, is defined inductively by

- (i) If $p \in P$, then $p \in L$.
- (ii) $\perp \in L$, where \perp is “false” (also a meaningless symbol).
- (iii) If $p, q \in L$, then $p \Rightarrow q \in L$.

Definition (Logical symbols).

$\neg p$	(“not p ”)	is an abbreviation for	$(p \Rightarrow \perp)$
$p \wedge q$	(“ p and q ”)	is an abbreviation for	$\neg(p \Rightarrow (\neg q))$
$p \vee q$	(“ p or q ”)	is an abbreviation for	$(\neg p) \Rightarrow q$

1.1 Semantic implication

Definition (Valuation). A *valuation* on L is a function $v : L \rightarrow \{0, 1\}$ such that:

- $v(\perp) = 0$,
- $v(p \Rightarrow q) = \begin{cases} 0 & \text{if } v(p) = 1, v(q) = 0, \\ 1 & \text{otherwise} \end{cases}$

We interpret $v(p)$ to be the truth value of p , with 0 denoting “false” and 1 denoting “true”.

Note that we do not impose any restriction of $v(p)$ when p is a primitive proposition (that is not \perp).

Definition (Tautology). t is a *tautology*, written as $\models t$, if $v(t) = 1$.

Definition (Semantic entailment). For $S \subseteq L$, $t \in L$, we say S *entails* t , S *semantically implies* t or $S \models t$ if, for any v such that $v(s) = 1$ for all $s \in S$, $v(t) = 1$.

“Whenever all of S is true, t is true as well.”

Definition (Truth and model). If $v(t) = 1$, then we say that t is *true* in v , or v is a *model* of t . For $S \subseteq L$, a valuation v is a *model* of S if $v(s) = 1$. Then $\models t$ means $\emptyset \models t$.

1.2 Syntactic implication

Definition (Proof and syntactic entailment). For any $S \subseteq L$, a *proof* of t from S is a finite sequence t_1, t_2, \dots, t_n of propositions, with $t_n = t$, such that each t_i is one of the following:

- (i) An axiom
- (ii) A member of S

(iii) A proposition t_i such that there exist $j, k < i$ with $t_j = (t_k \Rightarrow t_i)$.

If there is a proof of t from S , we say that S *proves* or *syntactically entails* t , written $S \vdash t$.

If $\emptyset \vdash t$, say t is a *theorem* and write $\vdash t$.

In a proof of t from S , t is the *conclusion* and S is the set of *hypothesis* or *premises*.