Steepest Descent Method for Minimizing the Peak Sidelobe Level of Hopped-Frequency Waveform

Dehua Zhao*, Yinsheng Wei, Yongtan Liu
Department of Electronic Engineering
Harbin Institute of Technology
Harbin, China
*E-mail: userzhaodehua@126.com

***Abstract*—Hopped-frequency waveform achieves high range resolution by transmitting a burst of narrowband pulses spanning the desired bandwidth. As its frequencies are irregularly or even sparsely distributed, hopped-frequency waveform could obtain a wider bandwidth with fixed pulses number and shows stronger anti-jamming ability than the linear stepped-frequency waveform; however, an undesired attribute associated is its high range sidelobe which would mask weak echo. In this paper, we optimally design the transmitted frequencies of the hopped-frequency waveform to suppress the peak sidelobe level (PSL). The considered design problem is formulated into a finite minimax optimization and a gradient-based descent method is proposed to solve this problem. This descent method uses the gradients of the largest several sub-functions of the considered minimax problem to fix a descent direction, and computes the step-length under the greedy criterion. Through iterative operation, this method reduces the function value monotonically and thus guarantees the convergence. Numeric results indicate that the proposed method can effectively suppress the PSL of the hopped- frequency waveform.**

Keywords—steepest descent method; minimax optimization; hopped frequency waveform; sidelobes suppression

I. Introduction

The linear stepped-frequency waveform are widely used in radar and communication system, because they offer considerable performance improvement by exploiting frequency diversity and are easily generated [1, 2]. The stepped-frequency waveform, by applying an inverse fast Fourier transform (IFFT) to a burst of narrowband pulses, achieves a wide bandwidth without requiring excessive system complexity and cost. However, the compressed pulse of a stepped-frequency waveform is sinc-type; and it has a high first sidelobe (-13.2dB), which needs to be suppressed by spectral weighting or mismatch method [3] at the expense of mainlobe width and signal-to-noise ratio. Moreover, due to the periodic response of IFFT at a digital frequency of 2π radians per sample, stepped-frequency waveform will suffer from large grating lobes when the frequency step is larger than the bandwidth of a single pulse. As a result, stepped-frequency waveform can only achieve a limited bandwidth given the number of pulses. Unlike the stepped-frequency waveform, hopped-frequency waveform uses irregularly or even randomly frequency hopping pattern [4]. As a result, periodic grating lobes are transformed into sidelobes and to achieve a wider bandwidth with fixed pulses number becomes possible. Moreover, the incoherent diversity and flexibility in frequency domain grant hopped-frequency waveform several advantages, such as resistance to narrowband interference, low probability of intercept and sidelobe suppression potential [4, 5].

This work was supported by the National Natural Science Foundation of China under Grant no. 60602039.

For hopped-frequency waveform, an additional component is added to its range sidelobes due to its irregularly distributed frequencies; and achieving a low sidelobe level poses a special challenge. This challenge can be partly addressed by applying a density tapered design. For example, [6, 7] proposed a nonlinear stepped-frequency waveform whose frequencies density would approximate a particular window function. [8] proposed a piecewise linear stepped-frequency waveform and achieved the tapered design by tuning the frequency step of each segment. These methods above are generally easy for implementation, however their results are far from optimality in terms of sidelobe level and they cannot control the shape of the compressed pulse accurately either.

In this paper, we considered the optimal sidelobe design of hopped-frequency waveforms. Our design goal is to find the optimal transmitted frequencies to minimize the peak sidelobe level (PSLL) with accurate control of the sidelobes pattern. We formulate the problem into a finite minimax optimization over a continuous space, like

$$\begin{aligned} &\boldsymbol{x} = \arg\min G(\boldsymbol{x}) \\ &\text{s.t.}\quad \boldsymbol{x}_{\text{lb}} \le \boldsymbol{x} \le \boldsymbol{x}_{\text{ub}} \end{aligned} \tag{1}$$

where $G(\boldsymbol{x}) = \max\{g_m(\boldsymbol{x})|m=1,\cdots,M\}$ is the objective function, with sub-functions $g_m(\boldsymbol{x})$: $S\in\mathbb{R}^{\mathrm{N}}\rightarrow\mathbb{R}$. $\boldsymbol{x}_{\text{ub}}$ and $\boldsymbol{x}_{\text{lb}}$ are the up and low bound of the solution vector. The inequality sign shall be understood in a componentwise sense. The max-function $G(\boldsymbol{x})$ is not continuous differentiable, or in other words not smooth. This poses special difficulties in the efforts to utilize the general gradient descent methods in minimax optimization. Specifically, at point where two or more sub-functions $g_m(\boldsymbol{x})$ are equal to the objective function $G(\boldsymbol{x})$, the first partial derivatives of $G(\boldsymbol{x})$ are discontinuous, even if all the sub-functions, i.e. $g_m(\boldsymbol{x})$, $\forall m$, have continuous first partial derivatives.

In [9], we have made an effort to use the gradient of the largest sub-function of the considered minimax problem to guide the optimizing. This direction however is not very effective in reducing the objective function value when the largest several sub-functions approximate the same, as this direction may not improve these sub-functions simultaneously. As a result, up-hill moves occur frequently in its optimizing process [9]. Moreover, in general gradient descent method, the norm of the gradient of the objective function is often used in a stopping criterion. This is however impossible in [9], since the gradient of the largest sub-function may not vanish at a local/global optimal in a minimax problem. To deal with this, an intuitive control on the search step-length has to be introduced to guarantee the convergence in [9]. In this paper, we present a new gradient-based descent method for minimax optimization. This descent method uses the gradients of the largest several sub-objectives to fix a descent direction for the objective function. By carefully selecting the search step-length (under the greedy criterion), the proposed algorithm can reduce the objective function value monotonously.

This reminder of this paper is organized as follows. In Section II, we present the signal model and formulate the waveform design into a minimax optimization problem. In Section III, we present the steepest descent method for minimax problem and give notes for applying the proposed method to hopped-frequency waveform design. Section IV reports the simulation results to validate the proposed method. And the last section draws the conclusion.

II. System Model and Problem Formulation

A. Signal Defination

Similar to linear stepped-frequency pulses, the hopped-frequency waveform consists of a burst of N narrowband sub-pulses, each of duration T, separated in time by a pulse repetition interval (PRI) T_r. In one signal repetition interval T_s ($T_s = NT_r$), the hopped-frequency waveform can be expressed in complex mode as

$$s(t) = \sum_{n=1}^{N} A e^{j2\pi f_c t} e^{j2\pi f_n t} \mathrm{rect}_T\left(t-(n-1)T_r - T/2\right) \quad ,0 \le t \le T_s \tag{2}$$

where A denotes the amplitude, f_c the carrier frequency, $f_n \in [0,B]$ the frequency of the nth sub-pulse with B being the total bandwidth, and

$$\mathrm{rect}_T(t) = \begin{cases} 1 & -T/2 \le t \le T/2 \\ 0 & \text{others} \end{cases} \tag{3}$$

Fig. 1 gives an illustration on the frequencies of a transmitted hopped-frequency waveform.

In this paper, we focus on the masking effect removal within a PRI and thus only need to consider the auto-response sidelobes. Accordingly, the autocorrelation function (ACF) of (2) is given by [9]

$$\chi(\tau) = \frac{1}{N}\left(1-\frac{|\tau|}{T}\right)\sum_{n=1}^{N} e^{j2\pi f_n \tau} \quad ,|\tau| < T \tag{4}$$

where τ denotes the round-trip delay. In the rest of the paper, we will consider the magnitude squared of (4) as the expression of ACF, namely

$$\begin{aligned} R(\tau) &= \chi(\tau)\chi^*(\tau) \\ &= \frac{1}{N^2}\left(1-\frac{|\tau|}{T}\right)^2 \sum_{n=1}^{N}\sum_{m=1}^{N} \cos\left(2\pi\left(f_n - f_m\right)\tau\right) \quad ,|\tau| < T \end{aligned} \tag{5}$$

B. Optimization Problem Formulation

The ACF (5) is completely determined by the frequency combination $\boldsymbol{f}=[f_1 \cdots f_N]^T$ given pulses number N, pulses duration T and spectrum width B. And the goal of interest is to find the optimal frequency combination to meet a prescribed ACF specification.

Peak sidelobe level is one of the most commonly used measures for sidelobes. In general, our encountered design goal is to minimize the PSLL, which favors equal-ripple or uniform sidelobes. However, it may be also useful to generate waveforms with non-uniform prescribed sidelobes. For example, as the echo intensity (round-trip case) is reversely proportional to the biquadrate of radial range, we expect the compressed pulse to have a decreasing response, such that the weak echo from distant target would not be drown in the sidelobes of a close one. Another example, in tracking a plurality of densely distributed targets, it is advisable to have greatly reduced response over the close region of the compressed pulse.

To achieve a prescribed ACF specification, we can minimize the largest discrepancy between the ACF and the desired ones. And thus the optimization problem can be formulated as:

$$\begin{aligned} &\boldsymbol{f} = \arg\min \max_{|\tau|>D} \left[R(\tau;\boldsymbol{f}) - d(\tau)\right]^2 \\ &\text{s.t.} \quad -B/2 \le \boldsymbol{f} \le B/2 \end{aligned} \tag{6}$$

where $d(\tau)$ ($|\tau| \ge D$)is the target function with the desired sidelobes pattern, D is the prescribed mainlobe width. The problem (6) is an infinite minimax optimization, and it can be transformed into a finite minimax optimization by finely discretizing the objective function over τ.

III. Steepest Descent Method for Minimax Optimization

Gradient-based descent methods are very popular in optimization field due to its good convergence rate. When applying the gradient-based descent methods, it commonly requires the objective function handled having continuous first partial derivatives. This premise however is not satisfied in a minimax optimization. Specifically, at point where two or more sub-functions $g_m(\boldsymbol{x})$ are equal to the objective function $G(\boldsymbol{x})$, the first partial derivatives of $G(\boldsymbol{x})$ are discontinuous, even if all the sub-functions, i.e. $g_m(\boldsymbol{x})$, $\forall m$, have continuous first partial derivatives.

To address the above difficulty, we proposed a new gradient descent method for minimax optimization. The proposed method is iterative and it fixes an alternative descent direction for the considered objective function, i.e. the max-function $G(\boldsymbol{x})$ (1), in each iteration. By carefully selecting the searching step-length, the proposed method enjoys good convergence rate.

A. Basic Idea

The basic idea of the proposed method is to search along a direction that can reduce the largest several sub-functions in each iteration. This descent direction can be obtained by exploiting the gradients of the largest several sub-functions.

For example, in certain iteration we set a threshold δ, and we pick out those sub-functions $g_m(\boldsymbol{x})$, which satisfy

$$G(\boldsymbol{x}) - g_m(\boldsymbol{x}) < \delta \tag{7}$$

Suppose that L sub-functions are selected and they are stacked into a vector function $\boldsymbol{F}(\boldsymbol{x})$: $S \in \mathbb{R}^N \rightarrow \mathbb{R}^L$, then the desired descent direction $\boldsymbol{v} \in \mathbb{R}^N$ that can reduce $\boldsymbol{F}(\boldsymbol{x})$ should satisfy

$$\boldsymbol{JF}(\boldsymbol{x})\boldsymbol{v} \in (-\mathbb{R}_{++})^L \tag{8}$$

where $\boldsymbol{JF}(\boldsymbol{x})$ is the Jacobian of $\boldsymbol{F}(\boldsymbol{x})$, $\mathbb{R}$++ is the set of strictly positive real numbers.

This descent direction can be obtained by solving the following linear programming [10],

$$\begin{aligned} &\boldsymbol{v} = \arg\min\max\left\{\left(\boldsymbol{JF}(\boldsymbol{x})\boldsymbol{v}\right)_l \mid l = 1,\cdots,L\right\} \\ &\text{s.t. } |\boldsymbol{v}_n| < 1 \ \ , n = 1,\cdots,N \end{aligned} \tag{9}$$

With this direction at hand, an Armijo-like rule could be used to get a proper step-length η, which satisfy

$$\boldsymbol{F}(\boldsymbol{x} + \eta\boldsymbol{v}/\|\boldsymbol{v}\|_2) \le \boldsymbol{F}(\boldsymbol{x}) \tag{10}$$

The point $\boldsymbol{x}+\eta\boldsymbol{v}/\|\boldsymbol{v}\|_2$ however still cannot guarantee an improvement on $G(\boldsymbol{x})$, especially when the threshold δ is relative small. To address it, a scaling factor $\alpha \in (0, 1]$ shall be further introduced to ensure greedy, namely

$$G(\boldsymbol{x} + \alpha\eta\boldsymbol{v}/\|\boldsymbol{v}\|_2) < G(\boldsymbol{x}) \tag{11}$$

This will result in a new point, and the scheme above shall be repeated until convergence.

B. Algorithm Implementation

Below, we list the complete steps of the steepest descent method for minimax optimization. It should be noted that, the threshold δ_k applied here is decreasing with iterations. As the search continues, more sub-functions will approximate $G(\boldsymbol{x}_k)$. If a constant threshold applied, too many sub-functions will be picked out to form the vector function $\boldsymbol{F}(\boldsymbol{x})$; this may deteriorate the search direction $\boldsymbol{v}$. Specifically this direction may not satisfy (8) strictly. Moreover, the step-length η_k may approximate zero, to make (10) hold. All these will cause the search stagnating prematurely. To avoid this, we reduce the threshold by a small amount $\Delta\delta$ in Step 4, whenever the step-length η_k is too small, such that the number of sub-functions involved in the next iteration will be reduced. Besides in (12), the 'absorbing wall' strategy is involved to confine the solution vector within the feasible space.

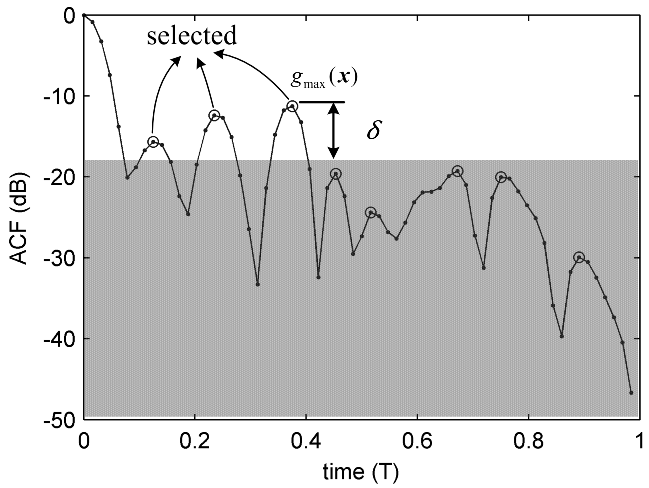


Fig. 2. Sub-functions selection. The sampling points are marked in dots while those peaks of ripples are marked in circles.

Steepest descent method for minimax optimization:

Step 1: Initialize position $\boldsymbol{x}_0$, set $k = 0$.

Step 2: Pick out those $g_m(\boldsymbol{x}_k)$, which satisfy $G(\boldsymbol{x}_k) - g_m(\boldsymbol{x}_k) < \delta_k$, and form the vector function $\boldsymbol{F}(\boldsymbol{x})$.

Step 3: Solve (9) for the direction $\boldsymbol{v}_k$ and compute a step-length η_k satisfying (10) through linear search.

Step 4: If $\eta_k < \varepsilon$, set $\delta_{k+1} = \delta_k - \Delta\delta$, otherwise $\delta_{k+1} = \delta_k$.

Step 5: Compute a scaling factor $\alpha_k \in (0, 1]$ as the maximum of $\{\alpha = 1/2^q \mid q \in \mathbb{N}, G(\boldsymbol{x}_k + \alpha\eta_k\boldsymbol{v}_k/\|\boldsymbol{v}\|_2) < G(\boldsymbol{x}_k)\}$.

Step 6: Set

$$\boldsymbol{x}_{k+1} = \max(\min(\boldsymbol{x}_k + \alpha_k\eta_k\boldsymbol{v}_k / \left\|\boldsymbol{v}_k\right\|_2,\ \boldsymbol{x}_{\text{ub}}),\ \boldsymbol{x}_{\text{lb}}) \tag{12}$$

$k = k+1$, and go to Step 2 until $\delta_k < \delta_{\min}$.

C. Notes

When applying the proposed method to hopped-frequency waveform design, an action can/shall be taken to reduce the computation burden. As we have already stated, the problem (6) is an infinite minimax optimization. By finely discretizing the objective function over τ, the resulting discrete version would yield good fidelity, and the original problem is transformed into a finite minimax optimization. Due to the continuity of the ACF over delay τ, we only need to take those peaks of ripples into account rather than all the sampling points that satisfy (7). Consequently, the sub-function selection in hopped-frequency waveform design is slightly different, as shown in Fig. 2. Applying this modification, the number of gradients to be calculated and the scale of problem (9) are both greatly reduced.

IV. SIMULATION RESULTS

A. Equal-ripple Sidelobes Design

In the first illustrative experiment, we set the pulses number N=64, the pulse duration T = 10us; the mainlobe parameter D is 0.1us (namely 0.01T). The transmitted frequencies are

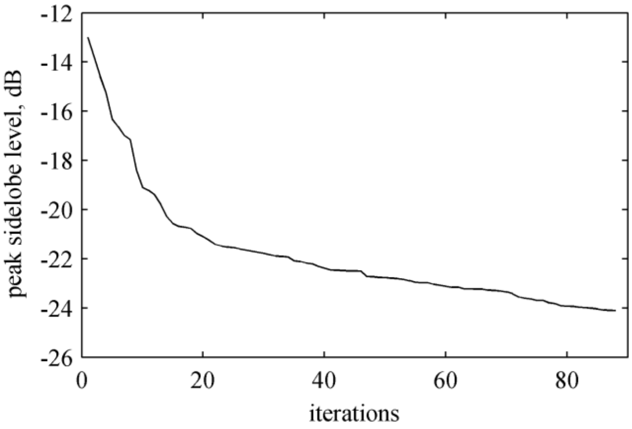
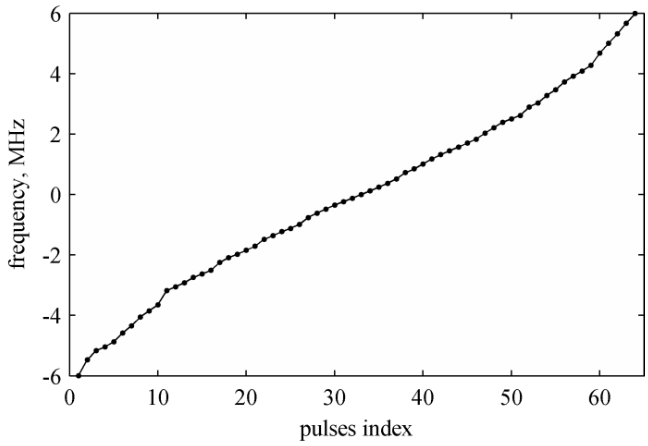
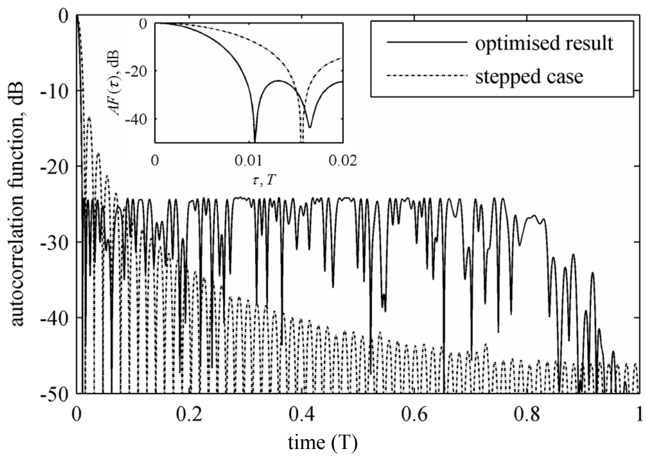


Fig. 3. Convergence curve of the proposed method for optimizing the PSL of the hopped-frequency waveform.



(a)



(b)

Fig. 4. Result obtained by the proposed method. (a) Transmitted frequencies of the resulting waveform; (b) Range autocorrelation functions of the optimized hopped frequency waveform and the linear stepped case. The inset gives close-up picture of both waveforms' mainlobe.

restricted with scope [-6.4, 6.4]MHz. The goal is to obtain the lowest PSL, consequently the target function $d(\tau)$, $|\tau|\geq D$ is constantly zero. When applying the proposed method to problem (6), the initial threshold δ_0 is set as 3 dB, with a changing step $\Delta\delta$ = 0.1dB. The algorithm process ends when the threshold is smaller than $\delta_{\min}$ = 0.1dB. ε in Step 4 above is chosen as $10^{-4}\eta_0$ such that it would not be influenced by the absolute range of the variables.

Fig. 3 shows the convergence curve of a realization. The proposed algorithm starts with a random initial guess of $\{f_n\}$ whose PSL is -12.8 dB and convergences to -24.1 dB after 88 iterations. As the search direction and search step are carefully selected under greedy criterion, the objective function value decreases monotonically with iterations, as shown in Fig. 3. This efficiently guarantees the convergence of the proposed method (to at least a local optimum), since the objective function is always bounded from below.

Fig. 4(a) gives the transmitted frequencies of the optimized waveform, which covers a total bandwidth of 11.98 MHz. Fig. 4(b) shows the corresponding range autocorrelation functions as well as that of a linear stepped case (with a frequency step equal to $1/T$). The average frequency step of the optimized waveform is about 1.9T, however by concentrating more transmitting energy near the center of the working band, as shown in Fig. 4(a), the hopped frequency waveform avoids the range ambiguity and suppresses the first lobe (-13.2 dB), which would otherwise occur in the linear stepped case. Moreover, thanks to the wider band coverage, the hopped frequency waveform achieves a higher range resolution than the linear stepped case, as illustrated in the inset of Fig. 4(b).

B. Extension to Non-uniform Sidelobe Design

The equal-ripple design is not always the most favorable choice in practice. For example, as the echo intensity (round-trip case) is reversely proportional to the biquadrate of radial range, we expect the compressed pulse to have a decreasing response, such that the weak echo from distant target would not be drown in the sidelobes of a close one. Another example, in tracking a plurality of densely distributed targets, it is advisable to have greatly reduced response over the close region of the compressed pulse.

The non-uniform sidelobe design can be achieved with a simple modification on the target function $d(\tau)$; instead of being constant zero, the target function varies with the time delay. Fig. 5 reports two illustrative examples, precisely the waveforms with decreasing sidelobes and that with a greatly reduced response over the close region. In this two examples, the waveform parameters are as follows, the pulses number N=64, the pulse duration T = 10us, and the transmitted frequencies are restricted with scope [-3.2, 3.2]MHz. In Fig. 5(a) the target function $d(\tau)$ linearly decreases by 10dB; in Fig. 5(b) the target function $d(\tau)$ is 10dB lower in the region close to the mainlobe. In Fig. 5, the dashed lines illustrate the desired sidelobe specifications. One can see that, by optimizing the transmitted frequencies using the proposed method, the sidelobe responses get a good agreement with the specifications.

V. Conclusion

In this paper, we have addressed the optimal design of hopped-frequency waveform. The considered design goal has been to minimize the PSL under accurate control of the sidelobes

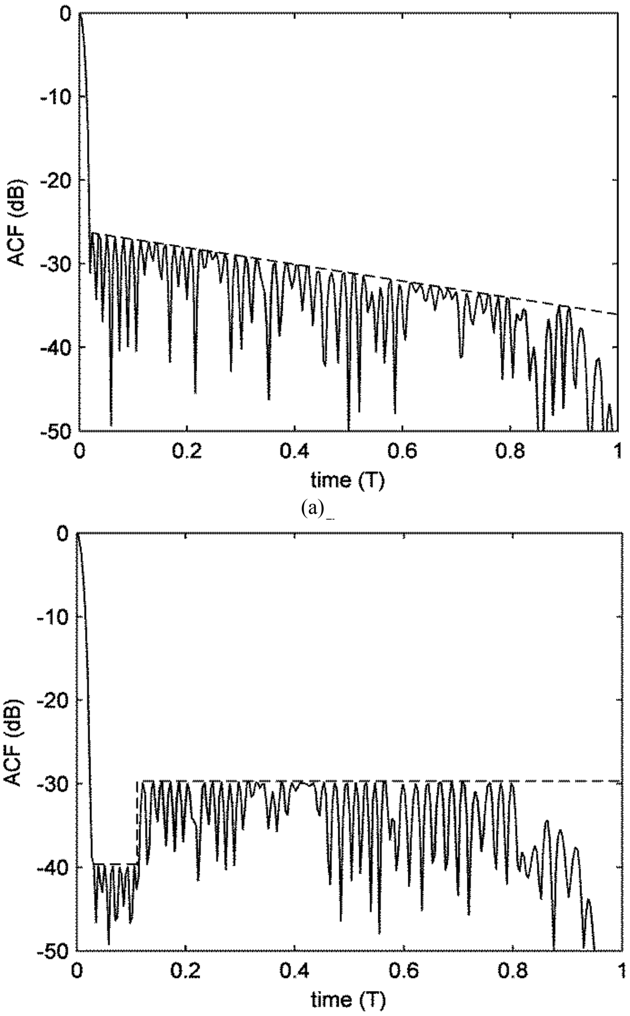


Fig. 5. Design examples with non-uniform sidelobes. The desired sidelobe specifications are illustrated with dashed lines. (a), Result with linear decreasing sidelobes. (b), Result with a much lower response on close region.

pattern. The problem has been formulated into a finite minimax optimization. In order to solve it, we have proposed a novel gradient-based descent method. This method fixes a descent direction by exploiting the gradients of the largest several sub-functions of the considered minimax problem and computes the search step under greedy criterion. As the proposed method can reduce the objective function value monotonously, it convergence is effectively gurarnteed. In the illustrative experiments, the proposed method is applied to the hopped-frequency waveform design to achieve different sidelobes pattern. Numeric results show that by optimally design the transmitted frequencies of the hopped-frequency waveform, a reduced PSL and better resolution can be achieved with a fixed number of transmitted frequencies; besides the shape of the compressed pulse can be also controlled accurately.

References

[1] D. R. Wehner, *High resolution radar*, 2nd ed., Norwood, MA: Artech House, 1995, pp. 197-236.

[2] F. Gini, A. D. Maio, and L. Patton, *Waveform Design and Diversity for Advanced Radar Systems*. London, UK: Institution of Engineering and Technology, 2012.

[3] H. Rohling, and W. Plagge, “Mismatched-filter design for periodic binary phased signals,” *IEEE Trans. Aerosp. Electron. Syst.,* vol. 25, no. 6, pp. 890-897, Nov. 1989.

[4] S. R. J. Axelsson, “Analysis of Random Step Frequency Radar and Comparison With Experiments,” *IEEE Trans. Geosci. Remote Sens.,* vol. 45, no. 4, pp. 890-904, Apr. 2007.

[5] N. M. Filiol, C. Plett, T. A. Riley, and M. A. Copeland, “An interpolated frequency-hopping spread-spectrum transceiver,” *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.,* vol. 45, no. 1, pp. 3-12, Jan. 1998.

[6] D. J. Rabideau, "Nonlinear synthetic wideband waveforms," in *Proc.* 2002 *IEEE Radar Conf., Long Beach, CA, 2002,*pp. 212-219.

[7] B. M. Keel, J. A. Saffold, M. R. Walbridge, and J. Chadwick, "Non-linear stepped chirp waveforms with sub-pulse processing for range sidelobe suppression," in *Proc. SPIE - Radar Sensor Technol. III.*, Orlando,FL, 1998, pp. 87-98.

[8] I. Gladkova, “Analysis of Stepped-Frequency Pulse Train Design,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. 45, no. 4, pp. 1251-1261, Oct., 2009.

[9] D. Zhao, and Y. Wei, “Adaptive gradient search for optimal sidelobe design of hopped-frequency waveform,” *IET Radar Sonar Navig.,* vol. 8, no. 4, pp. 275-281, Apr. 2014.

[10] J. Fliege, and B. F. Svaiter, “Steepest descent methods for multicriteria optimization,” *Math. Method Oper. Res.,* vol. 51, no. 3, pp. 479-494, Aug. 2000.