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Risk measures in a quantile regression credibility framework with Fama/French data applications

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A B S T R A C T

In this paper we extend the idea of embedding the classical credibility model into risk measures, as was presented by Pitselis (2016), to the idea of embedding regression credibility into risk measures. The resulting credible regression risk measures capture the risk of individual insurer's contract (in finance, the individual asset return portfolio) as well as the portfolio risk consisting of several similar but not identical contracts (in finance, several similar portfolios of asset returns), which are grouped together to share the risk. In insurance, credibility plays a special role of spreading the risk. In financial terminology, credibility plays a special role of diversification of risk. For each model, regression credibility models are established and the robustness of these models is investigated. Applications to Fama/French financial portfolio data are also presented.

1. Introduction

During the last decade, there is a tremendous development of risk measures with applications in finance and actuarial science. The value at risk (*VaR*) and conditional tail expectation (*CTE*) or expected shortfall (*ES*) have become two popular measures of market risk associated with an asset or portfolio of assets. More specifically, the *VaR* has been chosen by the Basel Committee on Banking Supervision, as the benchmark of risk measurement for capital requirements. Both *VaR* and *CTE* have been used by financial institutions for asset management and minimization of risk, and have been rapidly developed as analytic tools to assess riskiness of trading activities. The new regulatory insurance system, Solvency II requires the calculation of *VaR* and other risk measures to determine the insurance risk.

Artzner et al. (1999) proposed the family of coherent measures and Wang et al. (1997) the family of insurance risks. Valdez (2004) suggested the tail conditional variance (*TCV*) risk measure in order to measure the variability along the right tail of its distribution. Landsman and Valdez (2005) examined the tail conditional expectation for exponential dispersion models. Dhaene et al. (2002, 2006) considered the problem of how to evaluate risk measures for sums of the non-independent random variables based on the concept of comonotonicity with applications in actuarial science and finance. Furman and Landsman (2006) proposed a new premium called tail variance premium (*TVP*) and suggested a number of risk measures associated with it. Heyde et al. (2007) proposed the data-based risk measure, called natural risk statistics as a new family of risk measures and characterized a set of new axioms. In addition to tail conditional expectation they proposed a special case of natural risk measure, the tail condition median (*TCM*).

In insurance and financial markets it is essential to investigate the relation between different variables. Regression analysis is an important technique, because it can be used to explain and forecast variables of interest in relation to other financial components. We consider a situation where there exists information composed of economic variables $X_1, \ldots, X_n$, which can be considered as a set of predictors for a variable of interest Y, i.e., we are interested to estimate a risk measure of Y conditional on observed values of predictors $X_1, \ldots, X_n$. This means that we consider ways to estimate risk measures for a single asset at given market conditions (market index, interest rate, etc.). For example, Fama and French (1992, 1993) used time series regression to identify common risk factors in the returns on stocks and bonds.

Risk measures may be defined in a regression type setting and may depend on the state of the economy, since economic and market conditions vary from time to time. This requires risk managers to focus on the conditional distributions of profit and loss (P/L), which take a full survey of current information on

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the investment environment, such as macroeconomic, financial, and political environments, in forecasting future market values, volatilities, and correlations. In view of the time-varying nature of the distribution of financial returns, Engle and Manganelli (2004) proposed a conditional autoregressive specification for VaR_t, so called, conditional autoregressive value at risk (*CAViaR*), i.e., they provided a formula for calculating VaR_t (value at risk at time t) as a function of variables known at time $t-1$.

While least squares regression is adequate in many situations, it underestimates losses in cases where a decision maker is risk averse. New legislations, stricter rules indicate that some changes over time occurred across the claim distribution. Therefore, it is essential to examine these changes at different points of the distribution. The least squares estimators investigate only changes in the mean when the entire shape of the distribution may change dramatically. Quantile regression estimation may be more efficient from the ordinary least squares when the distribution is not normal [see Buchinsky, 1998]. Therefore, quantile regression may be more appropriate than least squares estimation in the context of the financial and insurance industry. Taylor (2008a) used quantile regression to derive estimators for the quantile and the expected shortfall. Taylor (2008b) also proposed the exponentially weighted quantile regression (*EWQR*) for estimating time varying quantiles. The *EWQR* cost function can be used as the basis for estimating the time-varying expected shortfall associated with the *EWQR* quantile forecast.

In this paper we show how regression risk measures can be embedded within the framework of credibility theory analogous to Hachemeister (1975) model. We introduce new risk measures, so called regression credible risk measures that are extensions of the classical credible risk measures model introduced by Pitselis (2013, 2016) and can be applied to insurance and financial data. Kudryavtsev (2009) used quantile regression for rate-making and Kim and Jeon (2013) proposed a credibility approach based on truncating (or trimming) the original data and the properties of the trimmed mean were examined from a risk measure perspective. The linear empirical Bayes estimation of quantiles discussed by Maritz (1989).

Traditionally, credibility is an experience rating technique to determine insurance premiums for a group of insurance contracts for which we have some claim experience for that group and a lot more experience for a larger group of contracts that are similar but not exactly the same. Bühlmann (1967) established the theoretical foundation of modern credibility theory, presented as a distribution free credibility estimation. Bühlmann and Straub (1970) generalized Bühlmann's classical credibility model, in the sense that the weight of a contract may vary in time. Hachemeister (1975), extended the classical credibility of Bühlmann and Straub (1970) by introducing a regression technique. For detailed description of credibility theory the reader may refer to the books of Goovaerts et al. (1990), Bühlmann's and Gisler (2005) and Herzog (2010). Here, we extend the application of credibility techniques to risk measures of a collective of somehow heterogeneous insurance contracts or financial portfolios of assets, which are grouped together to spread the risk.

By presenting risk measures in a quantile regression credibility framework, actuaries and risk managers can be benefited from the following: (1) By grouping together a collective of somehow heterogeneous contracts or financial portfolios (portfolios of assets) the risk can be spread in a way that credibility plays a special role of diversification of risks. (2) Quantile regression describes how the conditional distribution of Y depends on the covariates X at each quantile, enabling one to obtain a more complete description of how the conditional distribution of Y given $X=x$ depends on x. (3) With statistical packages (e.g. *R*), quantile regression coefficients as well as other important statistics for risk measures' interpretation are easily calculated. (4) Credible risk measures provide more informative tools than the usual risk measures (i.e. *VaR*, *CTE*) in capturing the individual insurer's risk and industry's risk.

The paper is organized as follows. In Section 2 we provide a brief introduction of quantile regression and parameter estimation. In Section 3 we illustrate the VaR risk measure related to quantile regression and the way that it can be embedded into credibility framework by introducing the credible regression value at risk (*CrRVaR*). The credible regression conditional tail expectation (*CrQRTE*) is presented in Section 4 and the credible regression tail conditional median (*CrRTCM*) in Section 5. Finally, the credible quantile regression tail expectation (*CrQRTE*) is presented in Section 6. Applications to Fama/French financial portfolio data are illustrated in Section 7 and some concluding remarks are presented in Section 8.

2. Quantile regression

In this section, we provide the technique of quantile regression that lately became very popular in both actuarial and finance risk management. We first describe the quantile regression and then provide the quantile regression coefficients through the solution of a minimization problem, as well as, the variance–covariance matrix of coefficients. There is a parallelism between the asymptotic behavior of ordinary sample quantiles in the location model and regression quantiles in the linear model suggesting a straightforward extension of ordinary quantiles in a location model to a more general class of linear models in which the conditional quantiles have a linear form (Koenker and Bassett, 1978; Bassett and Koenker, 1982).

While the ordinary sample quantiles are equally spaced on the interval $[0, 1]$ with each distinct order statistics containing in the interval of length exactly $1/n$, in the quantile regression the lengths of the intervals for $p \in [0, 1]$ are irregular and depend on the design matrix as well as the values of the response variable. In the regression pairs of points play the role of order statistics and serve to define the estimated linear conditional quantile functions (Koenker, 2005).

Least squares estimation of mean regression models describes how the conditional mean of Y depends on the covariates X. If the model predicts that the $\boldsymbol{\beta}$ coefficients change with quantile p, then we have evidence of heterogeneity in the population. This heterogeneity is often the result of unequal variances (heteroscedasticity). The quantile regression is inherently robust to contamination of the response observations, while it can be quite sensitive to contamination of the design observation x_i [see Koenker, 2005].

2.1. Quantile regression estimation

We consider a general form of the linear quantile regression model. Let $Y_1, Y_2, \ldots$ be independent random variables with distribution functions $F_1, F_2, \ldots$ and suppose that the pth conditional quantile function,

$$Q_{Y_i}(p|\boldsymbol{x}_i) = \boldsymbol{x}_i'\boldsymbol{\beta}_p, \tag{2.1}$$

is linear in the covariate vector $\boldsymbol{x}_i$ and $\boldsymbol{\beta}_p$ is a vector to be estimated. The conditional distribution functions of the Y_is can be written as

$$P(Y_i \le y|\boldsymbol{x}_i) = F_{Y_i}(y|\boldsymbol{x}_i) \quad i = 1, \ldots, n, \tag{2.2}$$

and

$$Q_{Y_i}(p|\boldsymbol{x}_i) = F_{Y_i}^{-1}(p|\boldsymbol{x}_i) = \xi_i^p. \tag{2.3}$$

Then we can consider the pth sample quantile $\widehat{\xi}_i^p$ that provides a convenient realization for the least square estimation of the model

$$\widehat{\xi}_i^p = \boldsymbol{x}_i'\boldsymbol{\beta}_p + u_i, \tag{2.4}$$

where u_i is the error term. For independent observations satisfying Conditions A1 and A2, (see Koenker, 2005, p. 120) the u_is have asymptotic variances

$$\sigma^2_{\xi_i^p} = \frac{p(1-p)}{n[f_i(F_i^{-1}(p))]^2} = \frac{p(1-p)}{n[f_i(\xi_i^p)]^2}, \tag{2.5}$$

where the density function $f_i(.)$ is continuous with continuous derivative in some neighborhood of ξ_i^p and it is different than zero, (for details see Buchinsky, 1998).

With quantile regression we can show how various financial characteristics are different at different quantiles. Thus, the quantile regression method involves allowing the marginal effects to change for claims at different points in the conditional distribution by estimating $\boldsymbol{\beta}_p$ using several different values of p, $p \in (0, 1)$. It is in this way that quantile regression allows for parameter heterogeneity across different types of claims.

In general, the pth quantile regression coefficient $\widehat{\boldsymbol{\beta}}_p, 0 < p < 1$, can be defined as any solution to the minimization problem [see Koenker and Bassett, 1978].

$$\min_{\boldsymbol{\beta}} \frac{1}{n}\sum_{i=1}^{n} \rho_p(u_{p_i}) = \min_{(\boldsymbol{\beta})} \frac{1}{n}\Bigg(\sum_{i:y_i\geq \boldsymbol{x}_i'\boldsymbol{\beta}} p|Y_i - \boldsymbol{x}_i'\boldsymbol{\beta}| + \sum_{i:Y_i<\boldsymbol{x}_i'\boldsymbol{\beta}} (1-p)|Y_i - \boldsymbol{x}_i'\boldsymbol{\beta}|\Bigg), \tag{2.6}$$

where $\rho_p(t) = (p - I(t < 0))t$ is a check function, and $I(.)$ is the indicator function. For the derivation of the covariance of $\widehat{\boldsymbol{\beta}}_p$ and other details see Koenker (2005) or Buchinsky (1998).

3. Credible regression value at risk (*CrRVaR*)

In this section we provide a brief introduction to *VaR* risk measures related to quantile regression that have recently been developed in the actuarial and financial literature and the way that they can be embedded into credibility framework.

3.1. Value at risk (VaR)

VaR is an example of risk measures used in practice, is a very popular risk measure and was actually in use by actuaries long before it was reinvented for investment banking. In actuarial contexts it is known as the quantile risk measure or quantile premium principle, (see Hardy, 2006). *VaR* measure is usually defined in terms of net wins or P/L and therefore ignores the difference between money at one date and money at a different date, which, for small time periods and a single currency, may be acceptable. It uses quantiles, which require us to pay attention to discontinuities and intervals of quantile numbers.

Definition 3.1. The VaR^p risk measure is defined as the p-quantile of a loss distribution for some prescribed confidence level $p \in (0, 1)$. In words, the VaR^p specifies a level of excessive losses such that the probability of a loss larger than ξ_p is less than p.

Definition 3.2. The pth conditional quantile function of Y given $X = x$, for $0 < p < 1$, is defined as $Q_p(Y|X = x) = \inf\{y : F(y|x) \geq p\}$.

Definition 3.3. In a linear regression setting the value at risk measure is defined as the p-quantile of a loss distribution for some prescribed confidence level $p \in (0, 1)$, i.e.

$$\xi^p_{Y|\boldsymbol{X}} = VaR^p(Y|\boldsymbol{X} = \boldsymbol{x}) = Q_p(Y|\boldsymbol{X} = \boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{\beta}_p \tag{3.1}$$

specifies a level of excessive losses such that the probability of a loss larger than $VaR^p(Y|\boldsymbol{X} = \boldsymbol{x})$ is less than p.

3.2. Credible quantile regression

In the following, we will show how quantile unweighted regression can be applied to the credibility model with one contract (or one portfolio of assets) and $i = 1, \dots, n$ years of claims experience, or other characteristics that corresponding to stochastic parameter Θ. Suppose that for given Θ, let $\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}} = (\hat{\xi}^p_{Y_1|\boldsymbol{x}_1}, \dots, \hat{\xi}^p_{Y_n|\boldsymbol{x}_n})'$ be the conventional estimator of the conditional p-quantile (VaR^p) of $\boldsymbol{Y}$ given $\boldsymbol{X}$ in a regression setting. We shall assume that given Θ, the bias of $\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}$ is negligible, i.e., $E(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}|\Theta) = \boldsymbol{\Xi}^p_{\boldsymbol{Y}|\boldsymbol{X}}(\Theta)$. Furthermore, we assume the following:

(i) $\boldsymbol{\Xi}^p_{\boldsymbol{Y}|\boldsymbol{X}}(\Theta) = E(\widehat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}|\Theta) = \boldsymbol{X}\boldsymbol{\beta}_p(\Theta)$, where $\boldsymbol{\beta}_p(\Theta)$ is an unknown regression vector,

(ii) $\mathrm{Cov}(\widehat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}|\Theta) = \sigma^2_{\xi^p_{\boldsymbol{Y}|\boldsymbol{X}}}(\Theta)\boldsymbol{I}_n$.

Then we can define the first two moments of the vector of quantile regression coefficients and the mean variance (structural parameters) as

$$\boldsymbol{\beta}_p = E[\boldsymbol{\beta}_p(\Theta)], \qquad \boldsymbol{A}_p = \mathrm{Cov}[\boldsymbol{\beta}_p(\Theta)],$$
$$s^2_{\xi^p_{\boldsymbol{Y}|\boldsymbol{X}}} = E[\sigma^2_{\xi^p_{\boldsymbol{Y}|\boldsymbol{X}}}(\Theta)]. \tag{3.2}$$

Theorem 3.1. *Under the above assumptions, the best linear estimate of the posterior parameter vector* $E[\boldsymbol{\Xi}^p_{\boldsymbol{Y}|\boldsymbol{X}}(\Theta)|\widehat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}]$ *(*$\boldsymbol{X}$ *is fixed and known) is*

$$\boldsymbol{\Xi}_p^{Cred} = \boldsymbol{\beta}_p + \boldsymbol{Z}_p[\widehat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}} - E(\widehat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}})], \tag{3.3}$$

with a risk measure credibility factor

$$\boldsymbol{Z}_p = \boldsymbol{A}_p\boldsymbol{X}'(\boldsymbol{X}\boldsymbol{A}_p\boldsymbol{X}' + s^2_{\xi^p_{\boldsymbol{Y}|\boldsymbol{X}}}\boldsymbol{I}_n)^{-1}. \tag{3.4}$$

Proof. A convenient choice of an approximate forecast vector is a linear function of the observables for $l = 1, \dots, L,\ i = 1, 2, \dots, n$,

$$g_l(\widehat{\xi}^p_{Y_1|\boldsymbol{x}_1}, \dots, \widehat{\xi}^p_{Y_n|\boldsymbol{x}_n}) = z^p_{0l} + \sum_{i=1}^{n} z^p_{il}\hat{\xi}^p_{Y_i|\boldsymbol{x}_i}, \tag{3.5}$$

where the coefficients z^p_{il}, henceforth called credibility coefficients, are adjusted so as to minimize

$$E\left[\beta^l_p(\Theta) - z^p_{0l} - \sum_{i=1}^{n} z^p_{il}\hat{\xi}^p_{Y_i|\boldsymbol{x}_i}\right]^2 \tag{3.6}$$

and $\beta^l_p(\Theta)$ is the lth element of $\boldsymbol{\beta}_p(\Theta) = [\beta^1_p(\Theta), \dots, \beta^L_p(\Theta)]'$. Taking the derivative of (3.6) with respect to z^p_{0l} and z^p_{rl}, $r = 1, \dots, n,\ l = 1, 2, \dots, L$, we obtain the optimal values of these coefficients given by n normal equations of the form,

$$\mathrm{Cov}[\beta^l_p(\Theta), \hat{\xi}^p_{Y_r|\boldsymbol{x}_r}] = \sum_{i=1}^{n} z^p_{il}\mathrm{Cov}[\hat{\xi}^p_{Y_i|\boldsymbol{x}_i}, \hat{\xi}^p_{Y_r|\boldsymbol{x}_r}], \tag{3.7}$$

with the z_{i0} determined so as to make the forecast (3.5) unbiased, i.e.,

$$z^p_{0l} = E[\beta^l_p(\Theta)] - \sum_{i=1}^{n} z^p_{il}E(\hat{\xi}^p_{Y_i|\boldsymbol{x}_i}). \tag{3.8}$$

Let $\boldsymbol{z}^p_0$ be the n-vector $[z^p_{i0}]'$, and $\boldsymbol{Z}_p$ the $n \times n$ matrix $[z^p_{il}|i \neq 0]$. Then with $\boldsymbol{\beta}_p(\Theta) = [\beta^1_p(\Theta), \dots, \beta^L_p(\Theta)]'$ and $\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}} = (\hat{\xi}^p_{Y_1|\boldsymbol{x}_1}, \dots, \hat{\xi}^p_{Y_n|\boldsymbol{x}_n})'$ the optimal conditions (3.7) and (3.8) can be written as

$$\boldsymbol{Z}_p\mathrm{Cov}[\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}] = \mathrm{Cov}[\boldsymbol{\beta}_p(\Theta), \hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}] \tag{3.9}$$

and

$$\boldsymbol{z}_0^p = E[\boldsymbol{\beta}_p(\Theta)] - \boldsymbol{Z}_p E(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}) \tag{3.10}$$

that implies (3.3). Then from (3.9) we have

$$\boldsymbol{Z}_p = \mathrm{Cov}[\boldsymbol{\beta}_p(\Theta), \hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}][\mathrm{Cov}(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}})]^{-1}, \tag{3.11}$$

with

$$\begin{aligned}\mathrm{Cov}[\boldsymbol{\beta}_p(\Theta), \hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}] &= \mathrm{Cov}[E(\boldsymbol{\beta}_p(\Theta)|\Theta), E(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}|\Theta)] \\ &\quad + E[\mathrm{Cov}(\boldsymbol{\beta}_p(\Theta), \hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}|\Theta)] \\ &= \mathrm{Cov}[\boldsymbol{\beta}_p(\Theta), \boldsymbol{X}\boldsymbol{\beta}_p(\Theta)] \\ &= \boldsymbol{A}_p\boldsymbol{X}' \end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\mathrm{Cov}(\widehat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}) &= \mathrm{Cov}[E(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}|\Theta)] + E[\mathrm{Cov}(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}|\boldsymbol{X}}|\Theta)] \\ &= \mathrm{Cov}[\boldsymbol{X}\boldsymbol{\beta}_p(\Theta)] + E(\sigma^2_{\xi^p_{\boldsymbol{Y}|\boldsymbol{X}}})\boldsymbol{I}_n \end{aligned} \tag{3.13}$$

that yields the risk measure credibility factor in (3.4).

3.3. Empirical quantile regression credibility model

Here, we incorporate quantiles into Hachemeister's model (unweighted case). We consider the quantile regression credibility model with $j = 1, \ldots, K$ contracts and $i = 1, \ldots, n$ years of claims experience, or other characteristics corresponding to stochastic replications $\Theta_1, \ldots, \Theta_K$ of Θ.

Suppose that for given Θ_j, let $\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j} = (\widehat{\xi}^p_{Y_{1j}|\boldsymbol{x}_{1j}}, \ldots, \widehat{\xi}^p_{Y_{nj}|\boldsymbol{x}_{nj}})'$ be the conventional estimator of the conditional p-quantile of $\boldsymbol{Y}_j$ given $\boldsymbol{X}_j$ in a regression setting. We shall assume that given Θ_j, the bias of $\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}$ is negligible, i.e., $E(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \boldsymbol{\Xi}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}(\Theta_j)$, under the following assumptions:

(i) The contracts are independent [i.e. the vectors $(\Theta_j, \boldsymbol{Y}_j)$ for $j = 1, \ldots, K$] and the variables Θ_j are identically distributed,

(ii) $E(\widehat{\xi}^p_{Y_{ij}|\boldsymbol{x}_{ij}}|\Theta_j) = \widehat{VaR^p}(Y_{ij}|\boldsymbol{x}_{ij}, \Theta_j) = \boldsymbol{x}'_{ij}\boldsymbol{\beta}_p(\Theta_j)$ for $i = 1, \ldots, n$, $j = 1, \ldots, K$, where $\boldsymbol{x}_{ij}$ is the ith row of a $(n \times k)$ fixed design matrix $\boldsymbol{X}_j$ of full rank $k(< n)$ and $\boldsymbol{\beta}_p(\Theta_j)$ of unknown $k \times 1$ vector of regression parameters associated with the pth quantile. In a matrix form we have

$$\boldsymbol{\Xi}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}(\Theta_j) = E(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \boldsymbol{X}_j\boldsymbol{\beta}_p(\Theta_j)$$

(iii) and

$$\mathrm{Cov}(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \sigma^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)\boldsymbol{I}_{n\times n}.$$

The structural parameters are:

$$\boldsymbol{\beta}_p = E[\boldsymbol{\beta}_p(\Theta_j)], \qquad \boldsymbol{A}_p = \mathrm{Cov}[\boldsymbol{\beta}_p(\Theta_j)],$$
$$s^2_{\xi^p_{\boldsymbol{Y}|\boldsymbol{X}}} = E[\sigma^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)].$$

3.3.1. Quantile regression parameter estimation

For contract j, an individual estimator $\widehat{\boldsymbol{\beta}}_{pj}$, of $\widehat{\boldsymbol{\beta}}_p(\Theta_j)$, can be defined as a solution of the minimization problem

$$\min_{\boldsymbol{\beta}_j} \frac{1}{n}\sum_{i=1}^n \rho_p(u_{p_{ij}}) = \min_{(\boldsymbol{\beta}_j)} \frac{1}{n}\left(\sum^p_{i:Y_{ij}\geq \boldsymbol{x}'_{ij}\boldsymbol{\beta}_j} |Y_{ij} - \boldsymbol{x}'_{ij}\boldsymbol{\beta}_j| + \sum_{i:Y_{ij}<\boldsymbol{x}'_{ij}\boldsymbol{\beta}_j}(1-p)|Y_{ij} - \boldsymbol{x}'_{ij}\boldsymbol{\beta}_j|\right), \tag{3.14}$$

where $\rho_p(t) = (p - I(t < 0))t$ is a check function and $I(.)$ is the indicator function. Then $\widehat{\boldsymbol{\beta}}_{pj}$ can take the form [see Koenker, 2005],

$$\widehat{\boldsymbol{\beta}}_{pj} = (\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}\boldsymbol{X}'_j\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j} \tag{3.15}$$

and its conditional covariance is

$$\mathrm{Cov}(\widehat{\boldsymbol{\beta}}_{pj}|\Theta_j) = \sigma^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}. \tag{3.16}$$

An estimator of $E[\sigma^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}]$ is given by

$$\widehat{E}[\sigma^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}] = \frac{1}{K}\sum_{j=1}^K \widehat{\sigma}^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}, \tag{3.17}$$

or

$$\widehat{E}[\sigma^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}] = \widehat{s}^2_{\xi^p_{\boldsymbol{Y}|\boldsymbol{X}}}(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}, \tag{3.18}$$

if we assume that $\boldsymbol{X}_j$ is fixed and the same for all j, with

$$\widehat{s}^2_{\xi^p_{\boldsymbol{Y}|\boldsymbol{X}}} = \frac{1}{K}\sum_{j=1}^K \widehat{\sigma}^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j). \tag{3.19}$$

An estimator of $\boldsymbol{\beta}_p$ is given by

$$\widehat{\boldsymbol{\beta}}_p = \frac{1}{K}\sum_{j=1}^K \widehat{\boldsymbol{\beta}}_{pj} \tag{3.20}$$

and an estimator of $\boldsymbol{A}_p$ by

$$\begin{aligned}\widehat{\boldsymbol{A}}_p &= \frac{1}{K-1}\sum_{j=1}^K(\widehat{\boldsymbol{\beta}}_{pj} - \widehat{\boldsymbol{\beta}}_p)(\widehat{\boldsymbol{\beta}}_{pj} - \widehat{\boldsymbol{\beta}}_p)' \\ &\quad - \widehat{E}[\sigma^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}]. \end{aligned} \tag{3.21}$$

The term $\sigma^2_{\xi^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)$, can be estimated by an interval estimation from the $[np]$th quantile of cumulative distribution function [see Mood et al., 1974, p. 512]. Efron (1979) proposed two alternative bootstrap methods for variance estimation by: The design matrix bootstrap estimator, which provides a consistent estimator of the asymptotic matrix under more general conditions, and the error bootstrap estimator, which yields a consistent estimator only under the independence assumption. Powell (1986) considered a kernel estimator by choosing the appropriate bandwidth (see also Koenker, 2005).

Remark 1. Based on the quantile regression credibility assumptions more general estimators can be constructed, similarly as in the classical regression credibility [see Goovaerts et al., 1990]. For example, an individual estimator of $\boldsymbol{\beta}_p(\Theta_j)$ can be obtained as

$$\widehat{\boldsymbol{\beta}}_{pj} = (\boldsymbol{X}'_j\boldsymbol{C}_j^{-1}\boldsymbol{X}_j)^{-1}\boldsymbol{X}'_j\boldsymbol{C}_j^{-1}\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j} \tag{3.22}$$

where

$$\begin{aligned}\boldsymbol{C}_j = \mathrm{Cov}(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}) &= E[\mathrm{Cov}(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)] + \mathrm{Cov}[E(\hat{\boldsymbol{\xi}}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)] \\ &= s^2\boldsymbol{I} + \boldsymbol{X}'_j\boldsymbol{A}_p\boldsymbol{X}_j. \end{aligned} \tag{3.23}$$

Of course, the properties and optimality of the estimator (3.22) and the properties and optimality of other related estimators require further investigation.

Theorem 3.2 (*Linearized Non-Homogeneous Quantile Regression Credibility Estimator*). *Under the above assumptions, the best quantile linearized credibility estimator of* $E[\boldsymbol{\beta}_p(\Theta_j)|\hat{\boldsymbol{\xi}}^p_{\mathbf{Y}_j|\mathbf{X}_j}]$ *is given by*

$$\boldsymbol{B}^{Cred}_{pj} = \boldsymbol{Z}_{pj}\widehat{\boldsymbol{\beta}}_{pj} + (\boldsymbol{I} - \boldsymbol{Z}_{pj})\boldsymbol{\beta}_p, \tag{3.24}$$

where $\widehat{\boldsymbol{\beta}}_{pj}$ *is the p-quantile regression coefficient for contract* $j(j = 1, \ldots, K)$ *and* $\boldsymbol{Z}_{pj}$ *is a risk measure credibility factor defined as*

$$\boldsymbol{Z}_{pj} = \boldsymbol{A}_p\left(\boldsymbol{A}_p + E[\sigma^2_{\xi^p_{\mathbf{Y}_j|\mathbf{X}_j}}(\Theta_j)(\boldsymbol{X}_j'\boldsymbol{X}_j)^{-1}]\right)^{-1}. \tag{3.25}$$

Proof. For the proof of the theorem we proceed similarly as in De Vylder (1976). We consider an arbitrary $L \times 1$ vector $\boldsymbol{C}$ having as components linear combinations of 1 and the observables $\xi^p_{\mathbf{Y}_j|\mathbf{X}_j}$ in the following form

$$\boldsymbol{C} = \boldsymbol{c}_0 + \boldsymbol{c}_1^{L\times n}\hat{\boldsymbol{\xi}}^p_{\mathbf{Y}_j|\mathbf{X}_j}, \tag{3.26}$$

with the lth element as

$$C_l = c_{0l} + \sum_{j=1}^{K}\sum_{i=1}^{n} c_{ijl}\widehat{\xi}^p_{Y_{ij}|\boldsymbol{x}_{ij}}. \tag{3.27}$$

Then consider $\boldsymbol{B}^{Cred}_{pj} + a\boldsymbol{C}$ and let

$$\begin{aligned} Q(a) = {} & E\{[\boldsymbol{\beta}_p(\Theta_j) - \boldsymbol{B}^{Cred}_{pj} - a\boldsymbol{C}]' \\ & \times \boldsymbol{W}[\boldsymbol{\beta}_p(\Theta_j) - \boldsymbol{B}^{Cred}_{pj} - a\boldsymbol{C}]\}, \end{aligned} \tag{3.28}$$

for a symmetric positive matrix $\boldsymbol{W}$. The theorem holds in case $\frac{dQ(a)}{da}|^{a=0} = 0$ for every $\boldsymbol{C}$. Taking the derivative of (3.28) with respect to a and then letting $a = 0$ we obtain

$$\frac{dQ(a)}{da}|^{a=0} = -2E\{[\mathbf{C}'\boldsymbol{W}\boldsymbol{\beta}_p(\Theta_j) - \mathbf{C}'\boldsymbol{W}\boldsymbol{B}^{Cred}_{pj}]\} = 0. \tag{3.29}$$

Inserting $\boldsymbol{B}^{Cred}_{pj}$ from (3.24) in (3.29) we have to prove

$$\begin{aligned} 0 &= E\{[\mathbf{C}'\boldsymbol{W}\boldsymbol{\beta}_p(\Theta_j) - \mathbf{C}'\boldsymbol{W}\boldsymbol{B}^{Cred}_{pj}]\} \\ &= E\{[\mathbf{C}'\boldsymbol{W}[\boldsymbol{\beta}_p(\Theta_j) - \boldsymbol{\beta}_p - \boldsymbol{Z}_{pj}(\widehat{\boldsymbol{\beta}}_{pj} - \boldsymbol{\beta}_p)]\} \\ &= E\{[(\boldsymbol{c}_0 + \boldsymbol{c}_1\hat{\boldsymbol{\xi}}^p_{\mathbf{Y}_j|\mathbf{X}_j})'\boldsymbol{W}[\boldsymbol{\beta}_p(\Theta_j) - \boldsymbol{\beta}_p - \boldsymbol{Z}_{pj}(\widehat{\boldsymbol{\beta}}_{pj} - \boldsymbol{\beta}_p)]\} \\ &= E\{\hat{\boldsymbol{\xi}}^{p'}_{\mathbf{Y}_j|\mathbf{X}_j}\boldsymbol{c}_1'\boldsymbol{W}[\boldsymbol{\beta}_p(\Theta_j) - \boldsymbol{\beta}_p - \boldsymbol{Z}_{pj}(\widehat{\boldsymbol{\beta}}_{pj} - \boldsymbol{\beta}_p)]\} \\ &= E\left\{Tr\left(\hat{\boldsymbol{\xi}}^{p'}_{\mathbf{Y}_j|\mathbf{X}_j}\boldsymbol{c}_1'\boldsymbol{W}[\boldsymbol{\beta}_p(\Theta_j) - \boldsymbol{\beta}_p - \boldsymbol{Z}_{pj}(\widehat{\boldsymbol{\beta}}_{pj} - \boldsymbol{\beta}_p)]\right)\right\} \\ &= Tr\left\{\boldsymbol{c}_1'\boldsymbol{W}\left(E[\boldsymbol{\beta}_p(\Theta_j) - \boldsymbol{\beta}_p - \boldsymbol{Z}_{pj}(\widehat{\boldsymbol{\beta}}_{pj} - \boldsymbol{\beta}_p)]\hat{\boldsymbol{\xi}}^p_{\mathbf{Y}_j|\mathbf{X}_j}\right)\right\}. \end{aligned} \tag{3.30}$$

We have

$$\begin{aligned} & E\{[\boldsymbol{\beta}_p(\Theta_j) - \boldsymbol{\beta}_p - \boldsymbol{Z}_{pj}(\widehat{\boldsymbol{\beta}}_{pj} - \boldsymbol{\beta}_p)]\hat{\boldsymbol{\xi}}^p_{\mathbf{Y}_j|\mathbf{X}_j}\} \\ &= \mathrm{Cov}[\boldsymbol{\beta}_p(\Theta_j), \hat{\boldsymbol{\xi}}^p_{\mathbf{Y}_j|\mathbf{X}_j}] - \boldsymbol{Z}_{pj}\mathrm{Cov}[\widehat{\boldsymbol{\beta}}_{pj}, \hat{\boldsymbol{\xi}}^p_{\mathbf{Y}_j|\mathbf{X}_j}] \\ &= \mathrm{Cov}[\boldsymbol{\beta}_p(\Theta_j)]\boldsymbol{X}_j' - \boldsymbol{Z}_{pj}\Big(\mathrm{Cov}[\boldsymbol{\beta}_p(\Theta_j)]\boldsymbol{X}_j' \\ &\quad + E[\sigma^2_{\xi^p_{\mathbf{Y}_j|\mathbf{X}_j}}(\Theta_j)(\boldsymbol{X}_j'\boldsymbol{X}_j)^{-1}]\boldsymbol{X}_j'\Big) \\ &= 0, \end{aligned} \tag{3.31}$$

by inserting (3.25) into (3.31).

4. Credible regression conditional tail expectation (*CrRCTE*)

In the following we present the conditional tail expectation (*CTE*) risk measures related to quantile regression and the way that they can be embedded into credibility theory.

4.1. Conditional tail expectation (CTE)

Recently, the *CTE* has become an important and popular risk measure due to its simplicity and its coherence. *CTE* is the expected loss, given that the loss is at least as large as some given *VaR*.

Definition 4.1. Let Y denote a loss random variable. The *CTE* of Y at the $100p\%$ confidence level, denoted as $CTE^p(Y)$, is the expected loss given that the loss exceeds the $100p$ percentile (or quantile) of the distribution of Y

$$CTE^p(Y) = E[Y|Y > VaR^p(Y)], \tag{4.1}$$

where $VaR^p(Y)$ is the p-quantile (value at risk) as defined in Section 3.1.

An estimator of $CTE^p(Y)$ is given by

$$\widehat{CTE}^p(Y) = \sum_{i=[np]}^{n}\frac{Y_{n(i)}}{n - [np]}.$$

Furman and Landsman (2006) in order to solve problems that appeared for the estimation of *VaR* and *CTE*, including classical risk ordering process and conditional Chebyshey's inequality, proposed a new risk measure referred as tail variance (*TV*) measure. While the *CTE* risk measure provides information about the average of the tail, the *TV* risk measure estimates the variability along the right tail i.e.,

$$TV^p(Y) = E\{(Y - CTE^p(Y))^2|Y > \xi^p]\}. \tag{4.2}$$

4.1.1. Regression conditional tail expectation (CTE)

Definition 4.2. The regression conditional tail expectation (*CTE*) risk measure is defined as

$$\begin{aligned} CTE^p_{\mathbf{Y}|\mathbf{X}} &= E[Y|Y > VaR^p(Y|\boldsymbol{X} = \boldsymbol{x})] \\ &= \frac{1}{1-p}\int_{\boldsymbol{x}_i'\boldsymbol{\beta}_p}^{\infty} yf_Y(y|x)ds, \end{aligned} \tag{4.3}$$

where $VaR^p(Y|\boldsymbol{X} = \boldsymbol{x})$ is the p-quantile (value at risk) regression defined in (3.1).

Remark 2. The *CTE* reflects only the *np* losses (observations) that are below $VaR^p(Y|\boldsymbol{X} = \boldsymbol{x}) = \boldsymbol{X}\boldsymbol{\beta}_p$, and consequently lacks incentive for mitigating losses below the *VaR*. Moreover, *CTE* does not properly adjust for extreme low-frequency and high-severity losses, since it only accounts for the shortfall.

Remark 3. *CTE* is sensitive to model assumptions of heaviness of tail distributions, or due to appearance of outliers in the data. Here, the robustness of *CTE* which is a coherent measure is questionable.

Definition 4.3. The tail variance measure measures the variability of the right tail and is defined as

$$\begin{aligned} TV^p_{\mathbf{Y}|\mathbf{X}=\boldsymbol{x}} &= TV^p(Y|\boldsymbol{X} = \boldsymbol{x}) \\ &= VaR[Y|Y > VaR^p(Y|\boldsymbol{X} = \boldsymbol{x})] \\ &= E[(Y - CTE^p(Y|\boldsymbol{X} = \boldsymbol{x}))^2|Y > VaR^p(Y|\boldsymbol{X} = \boldsymbol{x})]. \end{aligned}$$

4.2. Empirical model of the credible regression conditional tail expectation (CrRCTE)

In this section we present a generalization of the credible conditional tail expectation (*CrCTE*) as was presented in Pitselis (2016) in the sense that the estimation of *CTE* in based on a regression setting as in Section 3.3. Here, the dependent variable, Y_{ij} depends on some vector of financial components $\boldsymbol{x}_{ij}$ (independent variables) and the conditional tail expectation on a contract $j = 1, \ldots, K$ is as follows:

$$\begin{aligned} CTE^p_{Y_{ij}|\boldsymbol{x}_{ij}}(\Theta_j) &= E[Y_{ij}|Y_{ij} > \boldsymbol{x}'_{ij}\boldsymbol{\beta}_p(\Theta_j), \Theta_j] \\ &= \boldsymbol{x}'_{ij}\boldsymbol{\beta}^{p,n}(\Theta_j), \end{aligned} \tag{4.4}$$

with $j = 1, \ldots, K$ and $i = 1, \ldots, n$, and the regression conditional variance as

$$\begin{aligned} TV^p_{Y_{ij}|\boldsymbol{x}_{ij}} &= VaR[Y_{ij}|Y_{ij} > \boldsymbol{x}'_{ij}\boldsymbol{\beta}_p(\Theta_j), \Theta_j] \\ &= \upsilon^{p,n}(\Theta_j)d_{ii}, \end{aligned} \tag{4.5}$$

where

$$d_{iij} = d_{ij} = \begin{cases} 1 & \text{if } Y_{ij} \geq \boldsymbol{x}'_{ij}\boldsymbol{\beta}_{pj} \\ 0 & \text{otherwise, } i = 1, 2, \ldots, n. \end{cases} \tag{4.6}$$

Based on the regression CTE^p, we develop the $CrCTE^p$ risk measures, which is the weighted average of the individual regression conditional tail expectation and the collective (industry) regression conditional tail expectation, with the following assumptions:

(i) The contracts are independent [i.e. the vectors $(\Theta_j, \boldsymbol{Y}_j)$ for $j = 1, \ldots, K$] and the variables Θ_j are identically distributed,
(ii) $\mathbf{CTE}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}(\Theta_j) = E[\boldsymbol{Y}_j|\boldsymbol{Y}_j > \boldsymbol{X}_j\boldsymbol{\beta}_p(\Theta_j), \Theta_j] = \boldsymbol{X}_j\boldsymbol{\beta}^{p,n}(\Theta_j)$,
(iii) $\mathbf{TV}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}(\Theta_j) = VaR[\boldsymbol{Y}_j|\boldsymbol{Y}_j > \boldsymbol{X}_j\boldsymbol{\beta}_p(\Theta_j), \Theta_j] = \upsilon^{p,n}(\Theta_j)\boldsymbol{D}_j$, where $\boldsymbol{D}_j$ is a diagonal $n \times n$ matrix with elements d_{ii}, as defined in (4.6).

Here the structural parameters are defined as follows

$$\Xi^p_{CTE} = E[\mathbf{CTE}^p_{\boldsymbol{Y}_j|\boldsymbol{X}_j}(\Theta_j)] = \boldsymbol{X}_j\boldsymbol{\beta}^{p,n}, \tag{4.7}$$

$$\Upsilon^{p,n} = E[\upsilon^{p,n}(\Theta_j)], \tag{4.8}$$

$$\boldsymbol{A}^{p,n} = \mathrm{Cov}(\boldsymbol{\beta}^{p,n}(\Theta_j)). \tag{4.9}$$

4.2.1. Results on the regression conditional tail expectation

Let $\hat{\boldsymbol{\beta}}_{pj}$ be the regression quantile coefficient defined by the minimization (3.14) and let $\boldsymbol{\beta}^{p,n}_j, j = 1, \ldots, K$ is the conditional tail least squares estimator (*LSE*), which is calculated as the ordinary *LSE* after trimmed-off Y_{ij} with $d_{iij} = 0, i = 1, 2, \ldots, n$, i.e.,

$$\widehat{\boldsymbol{\beta}}^{p,n}_j = (\boldsymbol{X}'_j\boldsymbol{D}^{-1}_j\boldsymbol{X}_j)^{-1}\boldsymbol{X}'_j\boldsymbol{D}^{-1}_j\boldsymbol{Y}_j, \tag{4.10}$$

where $\boldsymbol{Y}_j = (Y_{1j}, \ldots, Y_{nj})'$ and $\hat{\boldsymbol{\beta}}^{p,n} = (\hat{\beta}^{p,n}_0, \hat{\beta}^{p,n}_1, \ldots, \hat{\beta}^{p,n}_k)'$. Then given Θ_j, the variance–covariance of $\widehat{\boldsymbol{\beta}}^{p,n}_j$ is

$$\mathrm{Cov}(\widehat{\boldsymbol{\beta}}^{p,n}_j|\Theta_j) = \upsilon^{p,n}(\Theta_j)(\boldsymbol{X}'_j\boldsymbol{D}^{-1}_j\boldsymbol{X}_j)^{-1}.$$

Remark 4. The conditional tail *LSE* can be considered as a special case of the trimmed *LSE* as was presented in Jurečková (1984) [see also Ruppert and Carroll, 1980].

4.3. Empirical credibility estimation for (RCTE)

We consider a vector of random variable $\boldsymbol{H} = (H_1, \ldots, H_L)'$ that has to be estimated optimally by an inhomogeneous linear combination of the risks contained in the vectors $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_K$, with $\boldsymbol{Y}_j = (Y_{1j}, \ldots, Y_{nj})'$, i.e., the lth element of $\boldsymbol{H}$ is of the form

$$H_l = z_{0l} + \sum_{j=1}^{K}\sum_{i=1}^{n} z_{ijl}Y_{ij},$$

for fixed coefficients z_{0l} and z_{ijl}. The vector $\boldsymbol{H}^* = (H^*_1, \ldots, H^*_L)'$ for which the mean square error

$$MSE = E[(\boldsymbol{H} - \boldsymbol{H}^*)'\boldsymbol{Q}(\boldsymbol{H} - \boldsymbol{H}^*)] \tag{4.11}$$

attains its minimum is given by the following theorem, where $\boldsymbol{Q}$ is symmetric and positive semi-definite.

Lemma 4.1 (De Vylder, 1976)**.** *The optimal linear credibility estimator of* $\boldsymbol{H}$ *is equal to*

$$\boldsymbol{H}^* = E[\boldsymbol{H}] + \sum_{j=1}^{K}\mathrm{Cov}(\boldsymbol{H}, \boldsymbol{Y}_j)[\mathrm{Var}(\boldsymbol{Y}_j)]^{-1}(\boldsymbol{Y}_j - E(\boldsymbol{Y}_j))$$

and $\boldsymbol{H}^*$ *does not depend on the weighted matrix* $\boldsymbol{Q}$ *as defined in* (4.11).

Theorem 4.1. *Under the assumptions of Section* 4.2*, the best estimate of the posterior parameter vector* $E[\boldsymbol{\beta}^{p,n}(\Theta_j)|\boldsymbol{Y}_j I_{[Y_{ij} > \xi^p_{Y_{ij}|\boldsymbol{x}_{ij}}]}, \boldsymbol{X}_j]$ ($\boldsymbol{X}_j$ *is fixed and known)*

$$\boldsymbol{M}^{p,n}_j = \boldsymbol{Z}^{p,n}_j\widehat{\boldsymbol{\beta}}^{p,n}_j + (\boldsymbol{I} - \boldsymbol{Z}^{p,n}_j)\boldsymbol{\beta}^{p,n}, \tag{4.12}$$

where

$$\boldsymbol{Z}^{p,n}_j = \boldsymbol{A}^{p,n}\left(\boldsymbol{A}^{p,n} + E[\upsilon^{p,n}(\boldsymbol{X}'_j\boldsymbol{D}^{-1}_j\boldsymbol{X}_j)^{-1}]\right)^{-1}, \tag{4.13}$$

with $E(\widehat{\boldsymbol{\beta}}^{p,n}_j|\Theta_j) = \boldsymbol{\beta}^{p,n}(\Theta_j)$, $E[\boldsymbol{\beta}^{p,n}(\Theta_j)] = \boldsymbol{\beta}^{p,n}$ *and* $\widehat{\boldsymbol{\beta}}^{p,n}_j$ *as in* (4.10).

Proof. Now the lth element of $\boldsymbol{H}^*$ in Lemma 4.1 takes the form

$$H^{p,n}_l = z_{0l} + \sum_{j=1}^{K}\sum_{i=1}^{n} z_{ijl}Y_{ij}I_{[Y_{ij} > \xi^p_{Y_{ij}|\boldsymbol{x}_{ij}}]},$$

for $l = 1, \ldots, L$, $i = 1, 2, \ldots, n$, $j = 1, \ldots, K$ and $\boldsymbol{H}$ is equal to the vector $\boldsymbol{\beta}^{p,n}(\Theta_j)$ in the regression conditional tail expectation, then based on the assumptions and results of Section 4.2 we obtain

$$\begin{aligned} \boldsymbol{M}^{p,n}_j &= E[\boldsymbol{\beta}^{p,n}(\Theta_j)] + \sum_{j=1}^{K}\mathrm{Cov}\left(\boldsymbol{\beta}^{p,n}(\Theta_j), \boldsymbol{Y}_j I_{[Y_{ij} > \xi^p_{Y_{ij}|\boldsymbol{x}_{ij}}]}\right) \\ &\quad \times \left(\mathrm{Var}[\boldsymbol{Y}_j I_{[Y_{ij} > \xi^p_{Y_{ij}|\boldsymbol{x}_{ij}}]}]\right)^{-1}\left(\boldsymbol{Y}_j I_{[Y_{ij} > \xi^p_{Y_{ij}|\boldsymbol{x}_{ij}}]} - \boldsymbol{X}_j\boldsymbol{\beta}^{p,n}\right) \\ &= \boldsymbol{\beta}^{p,n} + \boldsymbol{A}^{p,n}\boldsymbol{X}'_j\left(\boldsymbol{X}_j\boldsymbol{A}^{p,n}\boldsymbol{X}'_j + E[\upsilon^{p,n}(\Theta_j)]\boldsymbol{D}_j\right)^{-1} \\ &\quad \times \left(\boldsymbol{Y}_j I_{[Y_{ij} > \xi^p_{Y_{ij}|\boldsymbol{x}_{ij}}]} - \boldsymbol{X}_j\boldsymbol{\beta}^{p,n}\right). \end{aligned}$$

After some calculations based on well known techniques we obtain

$$\begin{aligned} \boldsymbol{M}^{p,n}_j &= \boldsymbol{\beta}^{p,n} + \boldsymbol{A}^{p,n}\left(\boldsymbol{A}^{p,n} + \Upsilon^{p,n}(\boldsymbol{X}'_j\boldsymbol{D}^{-1}_j\boldsymbol{X}_j)^{-1}\right)^{-1} \\ &\quad \times (\boldsymbol{X}'_j\boldsymbol{D}^{-1}_j\boldsymbol{X}_j)^{-1}\boldsymbol{X}'_j\boldsymbol{D}^{-1}_j\left(\boldsymbol{Y}_j I_{[Y_{ij} > \xi^p_{Y_{ij}|\boldsymbol{x}_{ij}}]} - \boldsymbol{X}_j\boldsymbol{\beta}^{p,n}\right) \\ &= \boldsymbol{\beta}^{p,n} + \boldsymbol{Z}^{p,n}_j\widehat{\boldsymbol{\beta}}^{p,n}_j + \boldsymbol{Z}^{p,n}_j\boldsymbol{\beta}^{p,n} \end{aligned}$$

that implies (4.12).

4.3.1. Parameter estimation

An estimator of the parameter $\boldsymbol{\beta}^{p,n}$ is given by

$$\widehat{\boldsymbol{\beta}}^{p,n} = \frac{1}{K}\sum_{j=1}^{K}\widehat{\boldsymbol{\beta}}_j^{p,n},$$

with $\widehat{\boldsymbol{\beta}}_j^{p,n}$ as in (4.10). An estimator $\widehat{\Upsilon}^{p,n}$ is given by

$$\widehat{\Upsilon}^{p,n} = \frac{1}{K}\sum_{j=1}^{K}\widehat{\upsilon}^{p,n}(\Theta_j),$$

with

$$\widehat{\upsilon}^{p,n}(\Theta_j) = \frac{1}{(n-[np])}(\boldsymbol{Y}_j - \boldsymbol{X}_j\widehat{\boldsymbol{\beta}}_j^{p,n})'\boldsymbol{D}_j^{-1}(\boldsymbol{Y}_j - \boldsymbol{X}_j\widehat{\boldsymbol{\beta}}_j^{p,n}),$$

and an estimator of $\boldsymbol{A}^{p,n}$ by

$$\widehat{\boldsymbol{A}}^{p,n} = \frac{1}{K-1}\sum_{j=1}^{K}(\widehat{\boldsymbol{\beta}}_j^{p,n} - \widehat{\boldsymbol{\beta}}^{p,n})(\widehat{\boldsymbol{\beta}}_j^{p,n} - \widehat{\boldsymbol{\beta}}^{p,n})' - \frac{1}{K}\sum_{j=1}^{K}\widehat{\upsilon}^{p,n}(\Theta_j)(\boldsymbol{X}_j'\boldsymbol{D}_j^{-1}\boldsymbol{X}_j)^{-1}.$$

5. Credible regression tail conditional median (*CrRTCM*)

This section provides the *CrRTCM* similarly as the *CrRVaR* previously presented in Section 3.

5.1. Tail conditional median (TCM)

Ogryczak and Zawadzki (2002) provided a formal definition of the tail conditional median solution concept. Heyde et al. (2007) proposed the tail conditional median (*TCM*) and compared it with the tail conditional expectation. They provided numerical results to illustrate the robustness of the proposed *TCM*. They have shown numerically that the *TCM* is more robust than the *CTE* in the sense that it is less sensitive to the tail behavior of the underlying distribution.

It is well known that *VaR* does not satisfy the subadditivity property of coherent measure, so as the *TCM*. However, if the *TCM* is estimated in the same level p as the *VaR* is estimated, or if *VaR* is calculated at a higher level, the problem of subadditivity of *VaR* is easily solved. Artzner et al. (1999) in one of their examples showed that the 10% *VaR* does not satisfy subadditivity for Y_1 and Y_2. However, the 10% *TCM* that corresponds to 5% *VaR*, satisfies subadditivity.

Definition 5.1. The *TCM* at level p is defined as

$$TCM^p(Y) = median[Y|Y > VaR^p(Y)].$$

In other words $TCM^p(Y)$ is the conditional median of Y given that $Y \geq VaR^p(Y)$. If Y is continuous then

$$TCM^p(Y) = VaR^{\frac{p+1}{2}}(Y).$$

For example, if we want to measure the loss beyond $p = 95\%$ level, we can use *VaR* at $\frac{p+1}{2} = 97.5\%$, which is the tail conditional median at 95% level.

5.1.1. Regression tail conditional median (TCM)

Definition 5.2. If Y is a random variable then the regression *TCM* is defined as

$$\begin{aligned} TCM^p(Y|\boldsymbol{X}=\boldsymbol{x}) &= median[Y|Y > VaR^p(Y|\boldsymbol{X}=\boldsymbol{x})] \\ &= VaR^{\frac{p+1}{2}}(Y|\boldsymbol{X}=\boldsymbol{x}). \end{aligned} \tag{5.1}$$

5.2. The empirical model of credible regression tail conditional median (CrRTCM)

Based on the definition of the regression $TCM^p(Y|\boldsymbol{X} = \boldsymbol{x})$, we develop the *CrRTCM* risk measures, which are the weighted average of the individual regression tail conditional median and the industry regression tail conditional median. The assumptions are similar as in Section 3.3,

(i) The contracts are independent [i.e. the vectors $(\Theta_j, \boldsymbol{Y}_j)$ for $j = 1, \ldots, K$] and the variables Θ_j are identically distributed,

(ii) $E(\widehat{\xi}^{\frac{p+1}{2}}_{Y_{ij}|\boldsymbol{x}_{ij}}|\Theta_j) = TCM^p_{Y_{ij}|\boldsymbol{x}_{ij}}(\Theta_j) = median[Y_{ij}|Y_{ij} > \boldsymbol{x}'_{ij}\boldsymbol{\beta}_p(\Theta_j), \Theta_j]$, for $i = 1, \ldots, n$, $j = 1, \ldots, K$, where $\boldsymbol{x}_{ij}$ is the ith row of a $(n \times k)$ fixed design matrix $\boldsymbol{X}_j$ of full rank $k(< n)$ and $\boldsymbol{\beta}_p(\Theta_j)$ an unknown $k \times 1$ vector of regression parameters associated with the $p^\star$th ($p^\star = \frac{p+1}{2}$) quantile. The above in a matrix form can be written as

$$E(\widehat{\boldsymbol{\xi}}^{p^\star}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \boldsymbol{X}_j\boldsymbol{\beta}_{p^\star}(\Theta_j).$$

(iii) The variance of the tail conditional median is written as

$$\mathrm{Var}(\widehat{\boldsymbol{\xi}}^{p^*}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \sigma^2_{\xi^{p^*}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)\boldsymbol{I}_{n\times n}.$$

The structural parameters are defined as follows ($p^* = \frac{p+1}{2}$)

$$\begin{aligned} &\boldsymbol{\beta}_{p^\star} = E[\boldsymbol{\beta}_{p^\star}(\Theta_j)], \qquad \boldsymbol{A}_{p^\star} = \mathrm{Cov}[\boldsymbol{\beta}_{p^\star}(\Theta_j)], \\ &s^2_{\xi^{p^\star}_{\boldsymbol{Y}|\boldsymbol{X}}} = E[\sigma^2_{\xi^{p^\star}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}}(\Theta_j)]. \end{aligned} \tag{5.2}$$

Then we proceed similarly as in Section 3, since the tail conditional median can be also defined by letting $\frac{p+1}{2}$ instead of p in Section 3.

6. Credible quantile regression tail expectation (*CrQRTE*)

Although, the regression tail conditional median is more robust compared with the conditional tail expectation, it does not provide details on the behavior of the tail of the distribution.

In the following we introduce the credible quantile regression tail expectation, which takes into account, changes over time across the tail of the distribution. While in the CTE^p we average the observations that exceed or equal a certain level p, in the quantile tail expectation we average quantiles that exceed or are equal to this certain level p. This means that from a unique sequence of regression quantiles $\{\boldsymbol{\xi}^{\tau_1}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}, \ldots, \boldsymbol{\xi}^{\tau_m}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}\}$, with $0 < \tau_1 < \tau_2 < \cdots < \tau_p < \tau_{p+1} < \cdots < \tau_m < 1$, we average the $m-p-1$ tail quantiles, $\boldsymbol{\xi}^{\tau_p}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}, \boldsymbol{\xi}^{\tau_{p+1}}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}, \ldots, \boldsymbol{\xi}^{\tau_m}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}$, i.e.,

$$\bar{\boldsymbol{\xi}}^{p,m}_{\boldsymbol{Y}_j|\boldsymbol{X}_j} = \frac{1}{m-p-1}\sum_{l=p}^{m}\boldsymbol{\xi}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}, \tag{6.1}$$

where $\boldsymbol{\xi}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j} = (\xi^{\tau_l}_{Y_{1j}|\boldsymbol{x}_{1j}}, \ldots, \xi^{\tau_l}_{Y_{nj}|\boldsymbol{x}_{nj}})'$ with $\xi^{\tau_l}_{Y_{ij}|\boldsymbol{x}_{ij}} = VaR^{\tau_l}(Y_{ij}|\boldsymbol{x}_{ij}) = \boldsymbol{x}'_{ij}\boldsymbol{\beta}^{\tau_l}_j$, $l = p, p+1, \ldots, m$.

Remark 5. The number of quantiles that might be averaged in the tail of distribution depends on the sample size and is the one that can provide minimum variance (see Mosteller, 1946).

6.1. Empirical CrQRTE

The quantile tail expectation within the framework of credibility techniques provides the credible quantile regression tail expectation ($CrQRTE^p$), which is more robust compared with the credible conditional tail expectation. For this model the assumptions are:

(i) The contracts are independent [i.e. the vectors $(\Theta_j, \boldsymbol{Y}_j)$ for $j = 1, \ldots, K$] and the variables Θ_j are identically distributed,

(ii) $E[\widehat{\xi}^{\tau_p}_{Y_{ij}|\boldsymbol{x}_{ij}}(\Theta_j)] = VaR^{\tau_p}(Y_{ij}|\boldsymbol{x}_{ij}, \Theta_j) = \boldsymbol{x}'_{ij}\boldsymbol{\beta}^{\tau_p}(\Theta_j)$, for $i = 1, \ldots, n$, $j = 1, \ldots, K$, where $\boldsymbol{x}_{ij}$ is the ith row of a $(n \times k)$ fixed design matrix $\boldsymbol{X}_j$ of full rank $k(< n)$ and $\boldsymbol{\beta}^{\tau_p}(\Theta_j)$ an unknown $k \times 1$ vector of quantile regression parameters associated with τ_p. In a matrix form it can be written as

$$\boldsymbol{\Xi}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}(\Theta_j) = E(\bar{\boldsymbol{\xi}}^{p,m}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \boldsymbol{X}_j\boldsymbol{\beta}^{p,m}_j(\Theta_j),$$

where $\boldsymbol{\beta}^{p,m}_j(\Theta_j)$ is the mean vector of quantile regression coefficients from level p to level m,

(iii) $\mathrm{Var}(\bar{\boldsymbol{\xi}}^{p,m}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \upsilon^{p,m}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}(\Theta_j)\boldsymbol{I}_n$.

The structural parameters are defined as follows:

$$\boldsymbol{\Xi}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}} = E[\boldsymbol{\Xi}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}(\Theta_j)], \qquad \Upsilon_{\xi^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}} = E[\upsilon^{p,m}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}(\Theta_j)],$$
$$\boldsymbol{\Psi}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}} = \mathrm{Var}[\boldsymbol{\Xi}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}(\Theta_j)]. \tag{6.2}$$

Theorem 6.1. *Under the above assumptions, the best estimate of the posterior parameter vector* $E[\boldsymbol{\beta}^{p,m}(\Theta_j)|\xi^{\tau_p}_{Y_{ij}|\boldsymbol{x}_{ij}}, \xi^{\tau_{p+1}}_{Y_{ij}|\boldsymbol{x}_{ij}}, \ldots, \xi^{\tau_m}_{Y_{ij}|\boldsymbol{x}_{ij}}]$ *is*

$$\boldsymbol{C}^{p,m}_j = \boldsymbol{Z}^{p,m}_j\widehat{\boldsymbol{\beta}}^{p,m}_j + (\boldsymbol{I} - \boldsymbol{Z}^{p,m}_j)\boldsymbol{\beta}^{p,m}, \tag{6.3}$$

where

$$\boldsymbol{Z}^{p,m}_j = \boldsymbol{\Psi}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}\left(\boldsymbol{\Psi}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}} + E[\upsilon^{p,m}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}(\Theta_j)(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}]\right)^{-1}, \tag{6.4}$$

with $E(\widehat{\boldsymbol{\beta}}^{p,m}_j|\Theta_j) = \boldsymbol{\beta}^{p,m}(\Theta_j)$ *and* $E[\boldsymbol{\beta}^{p,m}(\Theta_j)] = \boldsymbol{\beta}^{p,m}$.

Proof. We have to minimize the following square error

$$\begin{aligned} Q &= E\{[\boldsymbol{\beta}^{p,m}(\Theta_j) - \boldsymbol{C}^{p,m}_j]'[\boldsymbol{\beta}^{p,m}(\Theta_j) - \boldsymbol{C}^{p,m}_j]\} \\ &= E\{[\boldsymbol{\beta}^{p,m}(\Theta_j) - \boldsymbol{\beta}^{p,m} - \boldsymbol{Z}^{p,m}_j(\widehat{\boldsymbol{\beta}}^{p,m}_j - \boldsymbol{\beta}^{p,m})]' \\ &\quad \times [\boldsymbol{\beta}^{p,m}(\Theta_j) - \boldsymbol{\beta}^{p,m} - \boldsymbol{Z}^{p,m}_j(\widehat{\boldsymbol{\beta}}^{p,m}_j - \boldsymbol{\beta}^{p,m})]\}. \end{aligned}$$

Using the product rule and differentiating with respect to the matrix $\boldsymbol{Z}^{p,m}_j$ and setting the result equal to zero after some trivial calculations we obtain

$$\begin{aligned} &-2E\left\{\left(\boldsymbol{\beta}^{p,m}(\Theta_j) - \boldsymbol{\beta}^{p,m} - \boldsymbol{Z}^{p,m}_j(\widehat{\boldsymbol{\beta}}^{p,m}_j - \boldsymbol{\beta}^{p,m})\right)\right. \\ &\quad \left.\times (\widehat{\boldsymbol{\beta}}^{p,m}_j - \boldsymbol{\beta}^{p,m})'\right\} = 0, \end{aligned} \tag{6.5}$$

which implies

$$\boldsymbol{Z}^{p,m}_j = \mathrm{Cov}[\boldsymbol{\beta}^{p,n}(\Theta_j)]\{E(\mathrm{Cov}[\widehat{\boldsymbol{\beta}}^{p,m}_j|\Theta_j]) + \mathrm{Cov}(E[\widehat{\boldsymbol{\beta}}^{p,m}_j|\Theta_j])\}^{-1},$$

that yields (6.4).

6.2. Parameter estimation

An individual estimator $\widehat{\boldsymbol{\beta}}^{p,m}_j$, of $\boldsymbol{\beta}^{p,m}_j(\Theta_j)$, which is the average of quantile regression coefficients of contract $j(j = 1, \ldots, K)$ that exceed or equal the pth quantile, is given by

$$\widehat{\boldsymbol{\beta}}^{p,m}_j = \frac{1}{m-p+1}\sum_{l=p}^{m}\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}, \tag{6.6}$$

with

$$\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j} = (\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}\boldsymbol{X}'_j\widehat{\boldsymbol{\xi}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j} \tag{6.7}$$

and an estimator of the parameter $\boldsymbol{\beta}^{p,m}$ is given by

$$\widehat{\boldsymbol{\beta}}^{p,m} = \frac{1}{K}\sum_{j=1}^{K}\widehat{\boldsymbol{\beta}}^{p,m}_j.$$

Given Θ_j, the variance–covariance matrix of $\widehat{\boldsymbol{\beta}}^{p,m}_j$ can be obtained as

$$\begin{aligned} \mathrm{Cov}(\widehat{\boldsymbol{\beta}}^{p,m}_j|\Theta_j) &= \mathrm{Cov}\left(\frac{1}{m-p+1}\sum_{l=p}^{m}\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j\right) \\ &= \left(\frac{1}{m-p+1}\right)^2\left[\sum_{l=p}^{m}\mathrm{Var}(\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)\right. \\ &\quad \left. + 2\sum_{l<l'}^{m}\mathrm{Cov}(\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}, \widehat{\boldsymbol{\beta}}^{\tau_{l'}}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)\right]. \end{aligned}$$

It is known that (see Bassett and Koenker, 1978)

$$\mathrm{Var}(\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \frac{\tau_l(1-\tau_l)}{n[f(\xi^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)]^2}(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}$$

and

$$\mathrm{Cov}(\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}, \widehat{\boldsymbol{\beta}}^{\tau_{l'}}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j) = \frac{\tau_l(1-\tau_{l'})}{nf(\xi^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)f(\xi^{\tau_{l'}}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)}(\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}.$$

Then an estimator $\widehat{\Phi}_{\xi^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}}$, of $E[\mathrm{Cov}(\widehat{\boldsymbol{\beta}}^{p,m}_j|\Theta_j)]$, is obtained as

$$\begin{aligned} \widehat{\boldsymbol{\Phi}}_{\xi^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}} &= \frac{1}{K(m-p+1)^2}\left[\sum_{j=1}^{K}\sum_{l=p}^{m}\widehat{\mathrm{Var}}(\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)\right. \\ &\quad \left. + 2\sum_{j=1}^{K}\sum_{\tau_l<\tau_{l'}}^{m}\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\beta}}^{\tau_l}_{\boldsymbol{Y}_j|X_j}, \widehat{\boldsymbol{\beta}}^{\tau_{l'}}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)\right] \\ &= \frac{1}{Kn(m-p+1)^2}\sum_{j=1}^{K}\left[\sum_{l=p}^{m}\frac{\tau_l(1-\tau_l)}{[\hat{f}(\xi^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)]^2}\right. \\ &\quad \left. + 2\sum_{\tau_l<\tau_{l'}}^{m}\frac{\tau_l(1-\tau_{l'})}{\hat{f}(\xi^{\tau_l}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)\hat{f}(\xi^{\tau_{l'}}_{\boldsymbol{Y}_j|\boldsymbol{X}_j}|\Theta_j)}\right](\boldsymbol{X}'_j\boldsymbol{X}_j)^{-1}. \end{aligned}$$

Finally, we obtain an estimator of $\boldsymbol{\Psi}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}$ as

$$\widehat{\boldsymbol{\Psi}}^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}} = \frac{1}{K-1}\sum_{j=1}^{K}(\widehat{\boldsymbol{\beta}}^{p,m}_j - \widehat{\boldsymbol{\beta}}^{p,m})'(\widehat{\boldsymbol{\beta}}^{p,m}_j - \widehat{\boldsymbol{\beta}}^{p,m}) - \widehat{\boldsymbol{\Phi}}_{\xi^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}}.$$

7. Numerical illustrations

In what follows we illustrate the performance of the proposed credible risk measures with industry financial data based on the results of Sections 3–6.

The data was created by *CMPT-IND-RETS* using the 201407*CRSP* database (see, and Fama and French, 2016). It contains value and equal weighted returns for 10 industry portfolios in relation to the three factor model introduced by Fama and French (1993). The portfolios are constructed with monthly returns from July 1926 to July 2014. In particular, the risk measure for each of these portfolios needs to be forecast for the new period.

In assessing risk measures of these portfolios on the basis of their individual returns, we have to take into account two extreme points of view: (a) The difference in the observed profit/loss (P/L) returns is a result of the random nature of assets returns and no portfolio is better than another. Based on this point of view, the best forecast for the risk measure of an individual portfolio is the average of the industry's risk measure over all 10 portfolios.

Table 1
Descriptive statistics for 10 industry portfolio returns and the F/F 3 research factors.

Portfolio	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	Mkt.RF	SMB	HML
Min	−24.61	−34.82	−29.81	−26.00	−33.85	−21.56	−30.22	−34.08	−32.85	−29.90	29.07	−16.40	−13.03
1st Qu.	−1.39	−2.76	−2.04	−2.31	−2.71	−1.34	−2.10	−1.90	−1.64	−2.13	−2.04	−1.56	−1.29
Median	1.13	1.01	1.39	0.92	1.27	0.95	1.12	1.11	1.07	1.30	1.02	0.05	0.22
Mean	0.98	1.11	1.03	1.07	1.09	0.87	1.01	1.10	0.89	0.91	0.65	0.22	0.40
3rd Qu.	3.67	4.85	4.30	4.55	5.03	3.27	4.09	4.04	3.59	4.23	3.66	1.73	1.75
Max.	34.39	79.87	57.29	33.47	53.54	28.19	42.25	37.13	42.85	58.82	37.93	37.47	33.82
Percentage (%) of negative returns and of negative F/F 3 research factors													
%	37.46	42.48	39.83	41.72	41.81	37.75	39.36	38.88	39.45	40.87	39.92	49.20	45.79

Note: Each variable has 1057 observations, from July 1926 to July 2014.

(b) The difference in the observed returns is not random, but is systematic. They are due to the varying risk profiles across the industry portfolios. Based on this point of view, the best forecast for the risk measure of an individual portfolio is the risk measure of the individual portfolio itself.

The credibility regression approach to this problem is to take a weighted average of these two extreme choices and the methodology of Sections 3–6 may be adapted with some simple modification. As a P/L we consider a random variable Y, where as a loss outcome would be $Y < 0$ and we observe the lower tail of the distribution.

In the usual *CAPM*, and the Fama–French three factor model, volatility is reduced by diversification, i.e., we add more assets to a portfolio. In our credibility model there is no need for diversification of each individual portfolio, because the model itself takes into account the volatility of each individual portfolio as well as the market volatility that is composed from these individual portfolios. The 10 industry portfolios (dependent variables), are the following:

(1) **NoDur**: Consumer NonDurables—Food, Tobacco, Textiles, Apparel, Leather, Toys.
(2) **Durbl**: Consumer Durables—Cars, TV's, Furniture, Household Appliances.
(3) **Manuf**: Manufacturing—Machinery, Trucks, Planes, Chemicals, Off Furn, Paper.
(4) **Enrgy**: Oil, Gas, and Coal Extraction and Products.
(5) **HiTec**: Business Equipment—Computers, Software, and Electronic Equipment.
(6) **Telcm**: Telephone and Television Transmission.
(7) **Shops**: Wholesale, Retail, and Some Services (Laundries, Repair Shops).
(8) **Hlth**: Healthcare, Medical Equipment, and Drugs.
(9) **Utils**: Utilities.
(10) **Other**: Other—Mines, Construction, Building Material, Transportation, Hotels, Bus Service, Entertainment, Finance.

For each of our 10 regression models we consider the same design matrix with independent variables Fama/French 3 Research Factors. The Fama/French 3 Research Factors are constructed using the 6 value–weight portfolios formed on size and book-to-market. See Fama and French (1993) for the description of the 6 size/book-to-market portfolios.

The *SMB* Factor that stands for Small Minus Big, is the average return on the three small portfolios minus the average return on the three big portfolios. It is designed to measure the additional return (often referred to as "size premium") investors have historically received by investing in stocks of companies with relatively small market capitalization.

The *HML* Factor that stands for High Minus Low is the average return on the two value portfolios minus the average return on the two growth portfolios. It has been constructed to measure the "value premium" provided to investors for investing in companies with high book-to-market values.

The Factor *RM-RF*, is the excess return on the market. *RM* is the return on the value weighted portfolio of the stocks in the six size book-to-market equity (*BE*/*ME*) portfolios, plus the negative BE stocks excluded from the portfolios. RF is the one-month bill rate.

The coefficients of each of our 10 regression models are the same as betas in the Capital Asset Pricing Model (*CAPM*), measuring the exposure an asset has to market risk (*RM-RF*), the level of exposure to size risk (*SMB*) and the level of exposure to value risk (*HML*). The *SMB* and *HML* factors are the most commonly used, because they have the greatest predictive power of any two additional factors, often yielding an R^2 value close to 0.95.

Table 1 provides some descriptive statistics of the (P/L) monthly returns (dependent variables) for the 10 industry portfolios and for the Fama/French 3 Research Factors. Extreme values indicate that there is a substantial variability within each portfolio. A percentage (%) of negative monthly returns and negative monthly Fama/French 3 Research Factors are presented at the end of Table 1. The number of observations in each regression is $n = 1057$.

Remark 6. Individual quantile regression was tested and all models provided better fit with no intercept (no adding value to the portfolio) in the three risk factors equation.

Tables 2–4 illustrate the results of credibility techniques embedded with regression *VaR*, *CTM*, *CTE* and quantile tail expectation (QTE). Here, as a dependent variable we consider the monthly returns of each of the 10 industry portfolios in relation to the 3 research factors.

The first three lines of Table 2(a), show the values of coefficients of quantile regression (*VaR*) for each of the 10 quantile regression models determining the threshold of the possible 5% losses and in the following lines the values of the corresponding credible quantile regression coefficients. The values of the overall measure of the variability in the tail of distribution $s^2_{\xi^{0.05}_{\boldsymbol{Y}|\boldsymbol{X}}}$, the variance of the mean $\boldsymbol{A}_{0.05}$ and the credible risk factor $\boldsymbol{Z}_{0.05j}$, are also presented in the same table. We observe that the coefficients of *CrRVaR* only slightly deviate from the individual coefficients of *VaR* and this is due to the fact that the mean variance is small in comparison to the variance of the mean.

Similar results we obtain when our estimation is based on the median of the tail of the distribution, *CrRCTM* (Table 2(b)), *CrRCTE* (Table 3(a)) and *CrQRTE* (Table 4(a)). This is mainly a consequence of the fact that each of the mean variance of $\widehat{S}^2_{\xi^{0.025}_{\boldsymbol{Y}|\boldsymbol{X}}}$, $\widehat{\Upsilon}^{p,n}$ and $\widehat{\Upsilon}_{\xi^{p,m}_{\boldsymbol{Y}|\boldsymbol{X}}}$ is small in comparison to each of the corresponding variance of the mean, which means that there is no much variability within each portfolio of assets. For the application of *CrRCTE* model (Table 3(b)), for each portfolio, five quantiles have been estimated (0.95, 0.975, 0.98, 0.99, 0.995).

Table 2
10 industry portfolios.

Monthly returns from July 1926 to July 2014

a: Credible regression value at risk (*CrRVaR*) $p = 0.05$

Individual quantile (*VaR*) regression coefficients

Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{\beta}_{0.05j,Mkt.RF}$	0.714	1.091	1.052	0.856	1.170	0.594	0.594	0.798	0.680	1.018
$\widehat{\beta}_{0.05j,SMB}$	−0.004	0.023	−0.005	−0.280	0.078	−0.172	−0.172	−0.098	−0.269	0.087
$\widehat{\beta}_{0.05j,HML}$	−0.103	0.065	0.004	0.110	−0.471	−0.150	−0.150	−0.375	0.161	0.281

Credible quantile (*VaR*) regression coefficients

Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{B}^{Cred}_{0.05j,Mkt.RF}$	0.712	1.090	1.052	0.856	1.170	0.594	0.594	0.800	0.680	1.018
$\widehat{B}^{Cred}_{0.05j,SMB}$	−0.006	0.023	−0.005	−0.277	0.079	−0.172	−0.172	−0.100	−0.269	0.085
$\widehat{B}^{Cred}_{0.05j,HML}$	−0.103	0.065	0.004	0.110	−0.470	−0.150	−0.150	−0.374	0.160	0.280

$\widehat{\boldsymbol{\beta}}_{0.05}$	$\widehat{\boldsymbol{A}}_{0.05}$			$\widehat{\boldsymbol{Z}}_{0.05j}$			$\widehat{s}^2_{\xi^{0.05}_{\boldsymbol{Y}\mid\boldsymbol{X}}}$
0.857	0.409451	0.187263	0.001023	0.996955	0.004730	0.000660	
−0.081	0.187263	0.167039	−0.047084	0.187263	0.167039	−0.047084	9.584
−0.063	0.001023	−0.047084	0.498908	0.000676	−0.000920	0.998358	

b: Credible regression conditional tail median (*CrRCTM*) $p = 0.025$

Individual regression tail median coefficients

Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{\beta}_{0.025j,Mkt.RF}$	0.714	1.082	1.048	0.852	1.159	0.5833	0.583	0.798	0.679	1.016
$\widehat{\beta}_{0.025j,SMB}$	−0.004	0.010	−0.003	−0.278	0.086	−0.178	−0.1778	−0.099	−0.279	0.088
$\widehat{\beta}_{0.025j,HML}$	−0.103	0.056	0.003	0.112	−0.474	−0.150	−0.150	−0.376	0.156	0.281

Credible regression tail median coefficients

Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{B}^{Cred}_{0.025j,Mkt.RF}$	0.715	1.082	1.047	0.851	1.159	0.584	0.584	0.798	0.679	1.017
$\widehat{B}^{Cred}_{0.025j,SMB}$	−0.006	0.010	−0.003	−0.275	0.087	−0.178	−0.178	−0.099	−0.277	0.086
$\widehat{B}^{Cred}_{0.025j,HML}$	−0.103	0.056	0.002	0.112	−0.473	−0.150	−0.150	−0.375	0.155	0.281

$\widehat{\boldsymbol{\beta}}_{0.025}$	$\widehat{\boldsymbol{A}}_{0.025}$			$\widehat{\boldsymbol{Z}}_{0.025j}$			$\widehat{s}^2_{\xi^{0.025}_{\boldsymbol{Y}\mid\boldsymbol{X}}}$
0.851	0.408279	0.191923	0.002381	0.996825	0.004855	0.000739	
−0.083	0.191923	0.172607	−0.052872	0.006946	0.986229	−0.001393	9.640
−0.064	0.002381	−0.052872	0.497798	0.000770	0.001076	0.998315	

7.0.1. *Effect of a single outlier to credible risk measures*

In the following, the sensitivity of our credible risk measures is examined in the presence of outlier events. In order to implement this, we replace the last observation of the 10th asset, $Y_{1057,10} = -2.14$, with an artificial outlier, $Y_{1057,10} = -100.00$.

More specifically, with the appearance of these artificial outliers we have the following:

(a) The values of *CrRVaR* and *CrRTCM* remain unchanged being simply the thresholds of possible 0.05%, 0.025% losses, respectively and is indifferent of how serious the losses beyond that threshold actually are.

(b) The values of the individual regression *CTE* coefficients, especially the coefficient of the *SBM* of the 10th portfolio (Other, Table 3(b), right) for the modified data with one outlier $Y_{1057,10} = -100.00$, are different from the original regression *CTE* coefficients (Table 3(a), right) with no outlier ($Y_{1057,10} = -2.14$). This implies changes to the values of the credible regression *CTE* coefficients. Here we have a lack of robustness and an unsatisfactory behavior of credible conditional tail expectation, in the presence of a single outlier. This happens because of the sensitivity of the between-contracts covariance $\widehat{\boldsymbol{A}}^{p,n}$ and the expected within-contract variance $\widehat{\Upsilon}^{p,n}$ estimators in the presence of an outlier event, as are shown in Table 3(b), which leads to different value of the credible risk factor $\widehat{\boldsymbol{Z}}^{p,n}$. An outlier does not only affect the credible measure of the contaminated contract, but it also affects other contracts too.

(c) Finally, the values of *CrQRTE*, $\widehat{\boldsymbol{C}}^{p,m}_j$ in Table 4(b) remain almost unchanged (slightly deviate from the individual values of $\widehat{\boldsymbol{\beta}}^{p,m}_j$), with the presence of the outlier $Y_{(1057,10)} = -100.00$, in comparison with the values of *CrQRTE*, $\widehat{\boldsymbol{C}}^{p,m}_j$ in Table 4(a) with the original data. This means that in the presence of outlier events, *CrQRTE* is more robust than the other credible risk measures showing the superiority of *CrQRTE* among the other credible risk measures.

8. Concluding remarks

In the insurance and financial market, especially today because of the current crisis, there is a need to investigate the relation between the industry risk and individuals' institutions risk in the tail of the loss distribution. Our main goal was the construction of techniques for obtaining credible risk measures in a regression setting. These new measures connect individual risk information of similar insurance contracts or financial portfolios with collective (e.g. several similar but non identical to each other financial components) risk information.

Our models implemented with Fama and French (2016) data, where the dependent variables are the monthly returns of 10 industry portfolios and the independent variables are the 3 risk factors, obtaining the values of risk measures for each individual portfolio of returns. This can be used for evaluating portfolios and examining changes at different points of the distribution.

Table 3
10 industry Portfolios.

Monthly returns from July 1926 to July 2014

a: Credible regression conditional tail expectation (CrRCTE)

Individual regression *CTE* coefficients

Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{\beta}^{p,n}_{j_{Mkt.RF}}$	0.845	1.300	1.095	1.038	1.355	0.843	1.109	1.071	1.132	1.009
$\widehat{\beta}^{p,n}_{j_{SMB}}$	−0.024	0.308	0.020	−0.111	−0.270	−0.263	−0.035	−0.513	−0.390	−0.041
$\widehat{\beta}^{p,n}_{j_{HML}}$	−0.030	0.103	0.105	0.173	−0.420	−0.270	−0.175	−0.048	0.073	0.412

Credible regression *CTE* coefficients

Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{M}^{p,n}_{j_{Mkt.RF}}$	0.851	1.303	1.097	1.038	1.350	0.846	1.111	1.066	1.128	1.108
$\widehat{M}^{p,n}_{j_{SMB}}$	−0.041	0.286	0.012	−0.110	−0.258	−0.270	−0.049	−0.486	−0.366	−0.037
$\widehat{M}^{p,n}_{j_{HML}}$	−0.029	0.103	0.102	0.168	−0.408	−0.263	−0.169	−0.050	0.069	0.401

$\widehat{\beta}^{p,n}$	$\widehat{A}^{p,n}$			$\widehat{Z}_j^{p,n}$			$\widehat{\Upsilon}^{p,n}$
1.090	0.237771	0.066361	−0.014452	0.983733	0.013358	−0.002754	
−0.132	0.066361	0.465676	0.173083	0.040628	0.929530	0.022110	16.082
−0.008	−0.014452	0.173083	0.495680	0.001341	0.006684	0.970959	

b: CrRCTE with the presence of 1 artificial outlier $Y_{1057,13} = -100$

Individual regression *CTE* coefficients

Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{\beta}^{p,n}_{j_{Mkt.RF}}$	0.845	1.300	1.095	1.038	1.355	0.843	1.109	1.071	1.132	1.008
$\widehat{\beta}^{p,n}_{j_{SMB}}$	−0.024	0.308	0.020	−0.111	−0.270	−0.263	−0.035	−0.513	−0.390	0.592
$\widehat{\beta}^{p,n}_{j_{HML}}$	−0.030	0.103	0.105	0.173	−0.420	−0.270	−0.175	−0.048	0.073	0.453

Credible regression *CTE* coefficients

Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{M}^{p,n}_{j_{Mkt.RF}}$	0.853	1.298	1.094	1.035	1.348	0.850	1.111	1.063	1.123	1.016
$\widehat{M}^{p,n}_{j_{HML}}$	−0.048	0.292	0.021	−0.092	−0.267	−0.283	−0.052	−0.469	−0.343	0.557
$\widehat{M}^{p,n}_{j_{HML}}$	−0.022	0.105	0.098	0.156	−0.393	−0.251	−0.158	−0.061	0.052	0.439

$\widehat{\beta}^{p,n}$	$\widehat{A}^{p,n}$			$\widehat{Z}_j^{p,n}$			$\widehat{\Upsilon}^{p,n}$
1.079	0.239474	0.018070	−0.058046	0.968973	0.018773	−0.014846	
−0.068	0.018070	0.908366	0.463966	0.071544	0.890164	0.095508	34.736
−0.003	−0.058046	0.463966	0.515512	−0.012685	0.039397	0.910693	

The application of our results to Fama/French data set is very important to finance, because it shows that credibility theory can be applied to financial data incorporating the individual risk of a financial component (stock, bond, etc.) and a collective risk. It is well known that credibility has been developed in many areas of insurance applications, but there are not many applications of credibility theory in finance. Of course, our results could be valid for other data sets for casualty, life insurance or financial risk management. Implementing our credible risk measures with Fama/French data, we hope to convey the aspect of credibility to the rich area of finance, a powerful tool that takes into consideration individual as well collective information.

In the sequel, credible regression risk measures were obtained within these ten industry portfolios of assets and the effect of a single outlier to credible risk measures was investigated. Similarly, as in classical credibility regression estimation, we have a lack of robustness and an unsatisfactory behavior with credible risk measure estimation *CrRCTE* and *CrRTCM*, in the presence of a single outliers. This happens because of the sensitivity of the between-assets covariance and the expected within-asset variance estimators as shown in Tables 2–4. In cases where we have more than one outlier that affects the *VaR* of an individual portfolio (contract), the *CrRVaR* provides an unsatisfactory behavior similar to that of the *CrRTCM*.

On the other hand, *CrQRTE* provides satisfactory results in the presence of very large claims, provides quite well adjusted estimates keeping the quantile regression credibility estimation equitable, i.e. there is equity between portfolios of assets in risk measure sharing.

Consequently, all the above show the superiority of the *CrQRTE* estimation, which from a regulatory perspective, is preferable to other above mentioned credible risk measures. The main reason is that it takes into account changes over time across the tail of the distribution and at the same time it is robust to outlier events in the tail of the distribution. The CrRTCM is not very robust (although the individual RTCM are robust) to outlier events in the tail of the distribution, but it is more appropriate when we are concerned with margin requirements in financial trading, insurance premiums and technical provisions, because it reports only a conditional quantile and ignores outcomes beyond that (similar to the conditional *VaR*). From a regulatory perspective the *CrRCTE* is also a preferable credible risk measure, because it reflects the credible mean size of losses exceeding a p-quantile (VaR^p), but is not robust and is sensitive to model assumptions and outliers. From shareholders or management perspective, the credible quantile regression (*VaR*) is a meaningful risk-measure since the default event itself is of primary concern and the size of shortfall is only secondary.

While the quantile regression estimators are robust against long-tailedness of the errors e, they may be unreliable under departures from the model assumptions, in particular when the x has leverage points. Because of this Rousseeuw and Hubert (1999) introduced the notion of depth in a regression setting. They considered depth-based regression quantiles that estimate

Table 4
10 industry portfolios.

Monthly returns from July 1926 to July 2014										
a: Credible quantile regression tail expectation (CrQRTE)										
Individual quantile regression tail expectation coefficients										
Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{\beta}^{p,m}_{j_{Mkt.RF}}$	0.720	1.094	1.052	0.855	1.170	0.599	0.800	0.804	0.687	1.019
$\widehat{\beta}^{p,m}_{j_{SMB}}$	−0.002	0.008	0.002	−0.273	0.083	−0.179	0.128	−0.101	−0.270	0.090
$\widehat{\beta}^{p,m}_{j_{SMB}}$	−0.091	0.069	0.013	0.121	−0.461	−0.147	−0.226	−0.362	0.179	0.285
Credible quantile regression tail expectation coefficients										
Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{C}^{p,m}_{j_{Mkt.RF}}$	0.720	1.093	1.052	0.855	1.170	0.599	0.908	0.804	0.687	1.019
$\widehat{C}^{p,m}_{j_{Mkt.RF}}$	−0.003	0.008	0.002	−0.271	0.084	−0.180	0.126	−0.101	−0.269	0.088
$\widehat{C}^{p,m}_{j_{Mkt.RF}}$	−0.091	0.069	0.012	0.121	−0.461	−0.147	−0.226	−0.362	0.178	0.285
$\widehat{\beta}^{p,m}_{0.05}$	$\widehat{\Psi}^{p,m}_{Y\|x}$			$\widehat{Z}^{p,m}_j$			$\widehat{\Upsilon}^{p,m}_{\xi_{Y\|X}}$			
0.891	0.326161	0.164758	−0.031043	0.997079	0.003771	0.000682				
−0.052	0.164758	0.193338	−0.091076	0.005693	0.989596	−0.001391	9.376			
−0.062	−0.031043	−0.091076	0.518362	0.000721	−0.001119	0.998383				
Quantiles (0.80, 0.90, 0.95, 0.975, 0.99)										
b: CrQRTE with the presence of 1 artificial outlier $Y_{1057,13} = -100$										
Individual quantile regression tail expectation coefficients										
Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{\beta}^{p,m}_{j_{Mkt.RF}}$	0.720	1.094	1.052	0.855	1.170	0.599	0.8	0.804	0.687	1.018
$\widehat{\beta}^{p,m}_{j_{SMB}}$	−0.002	0.008	0.002	−0.273	0.083	−0.179	0.128	−0.101	−0.270	0.093
$\widehat{\beta}^{p,m}_{j_{SMB}}$	−0.091	0.069	0.013	0.121	−0.461	−0.147	−0.226	−0.362	0.179	0.284
Credible quantile regression tail expectation coefficients										
Coefficients	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
$\widehat{C}^{p,m}_{j_{Mkt.RF}}$	0.720	1.093	1.052	0.855	1.169	0.599	0.908	0.804	0.687	1.019
$\widehat{C}^{p,m}_{j_{Mkt.RF}}$	−0.003	0.008	0.002	−0.271	0.084	−0.180	0.126	−0.101	−0.269	0.092
$\widehat{C}^{p,m}_{j_{Mkt.RF}}$	−0.091	0.069	0.013	0.121	−0.461	−0.147	−0.226	−0.362	0.178	0.283
$\widehat{\beta}^{p,m}$	$\widehat{\Psi}^{p,m}_{Y\|x}$			$\widehat{Z}^{p,m}_j$			$\widehat{\Upsilon}^{p,m}_{\xi_{Y\|X}}$			
0.891	0.326013	0.165251	−0.031354	0.996810	0.004093	0.000731				
−0.051	0.165251	0.194447	−0.089795	0.006202	0.988713	−0.001477	10.278			
−0.062	−0.031354	−0.089795	0.517389	0.000775	−0.001193	0.998233				
Quantiles (0.80, 0.90, 0.95, 0.975, 0.99)										

the conditional quantile of y given x, as do customary L_1-based regression quantiles of Koenker and Bassett (1978), but with the conditional advantage of being robust to leverage points. Adrover et al. (2004) defined robust regression quantile estimates that are robust when the predictors contain leverage points and attain a maximum breakdown point. Neykov et al. (2012) considered a robust estimation in the framework of quantile regression, the least trimmed quantile regression, which is based on trimming in order to reduce the influence of the outliers in the explanatory variables. These robust regression quantile estimates that are also robust to outliers in the design matrix space can be incorporated with credibility estimation and produce some more robust risk measures, an idea for our next project.

When the insurance claim (or financial component) distribution is known, credibility techniques can be extended to the problem of forecasting the distribution of individual risk, based upon a collective statistics and individual experience data. If it is assumed that prior statistics are available from a collective statistics, of somewhat heterogeneous insurance contracts (or financial components), then a credibility approach similar to the credibility distribution approach introduced by Jewell (1976) can be applied, which is our project in progress.

Credible risk measures may also be calculated based on credibility confidence limits, which means that the level of available capital cannot be lower than the left confidence limit. Similarly, risk margin can be calculated by considering credibility confidence levels approaches. Credible risk measures enjoy the advantages and disadvantages of credibility theory as well as of the coherent risk measures, although there is a need for more research in that direction. However, a simple application of credibility regression techniques is not always a case for the creation of risk measures. Most of the time the application of a specific credible risk measure depends on financial (or insurance) data we want to apply and the outcome we want to produce.

In the context of linear empirical Bayes estimation of quantiles, the weighted case (that includes unequal number of observations on each contract) in a credibility model of quantiles, as well as the asymptotic optimality of credibility estimators can be determined similarly as in Norberg (1980). All the above related to the empirical Bayes quantile premium estimation including performance of the estimators are ideas for our next projects.

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