Decision problems

September 1, 2020

Uncertainty

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- We cannot know for sure what happened in the past.
- How can we quantify this uncertainty?
- Probabilities!

Axioms of probability

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2. The probability of the impossible event is $P(\emptyset) = 0$
3. The probability of any event $A \in \Sigma$ is $0 \leq P(A) \leq 1$.
4. If A, B are disjoint, i.e. $A \cap B = \emptyset$, meaning that they cannot happen at the same time, then

$$P(A \cup B) = P(A) + P(B)$$

Definition 1 (Conditional probability)

The probability of A happening if we know that B has happened is defined to be:

$$P(A \mid B) \triangleq \frac{P(A \cap B)}{P(B)}.$$

Conditional probabilities obey the same rules as probabilities.

Bayes's theorem

For $P(A_1 \cup A_2) = 1$, $A_1 \cap A_2 = \emptyset$,

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Example 2 (probability of rain)

What is the probability of rain given a forecast x_1 or x_2?

ω_1: rain	$P(\omega_1) = 80\%$
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Table : Prior probability of rain tomorrow

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x_2: dry	$P(x_2 \mid \omega_2) = 50\%$

Table : Probability the forecast is correct

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Table : Probability the forecast is correct

$$P(\omega_1 \mid x_1) = 87.8\%$$
$$P(\omega_1 \mid x_2) = 44.4\%$$

Table : Probability that it will rain given the forecast

Classification in terms of conditional probabilities

- Features $x_t \in \mathcal{X}$.
- Class label $y_t \in \mathcal{Y}$.
- Probability model $P_\mu(x_t \mid y_t)$.
- Prior class probability $P_\mu(y_t = c)$.

$$P_\mu(y_t = c \mid x_t) = \frac{P_\mu(x_t \mid y_t = c) P_\mu(y_t = c)}{\sum_{c' \in \mathcal{Y}} P_\mu(x_t \mid y_t = c') P_\mu(y_t = c')}$$

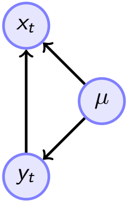
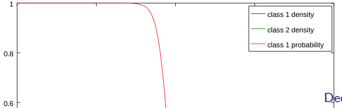
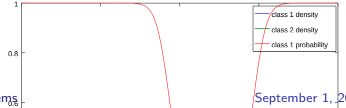


Figure : A generative classification model. μ identifies the model (paramter). x_t are the features and y_t the class label of the t-th example.





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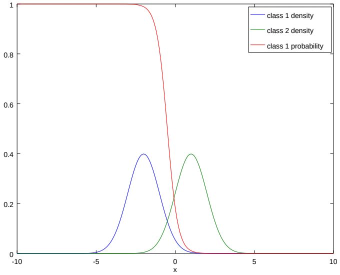
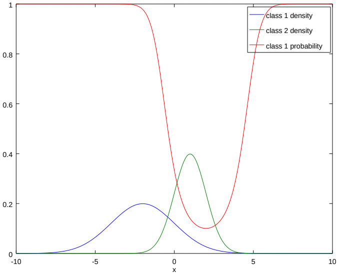
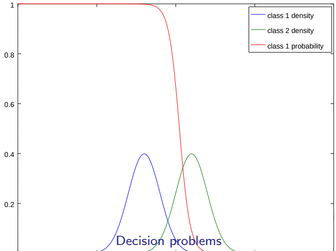


(a) Equal prior and variance



(b) Unequal variance



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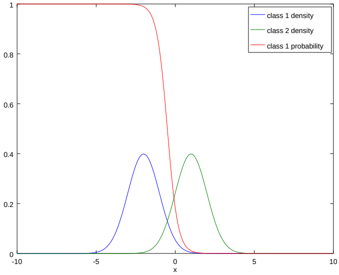

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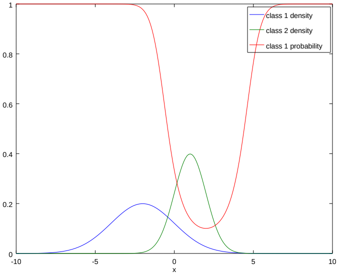

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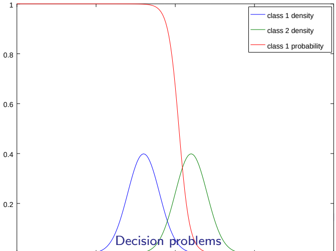

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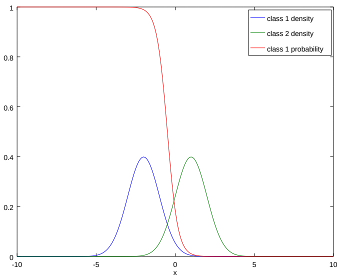

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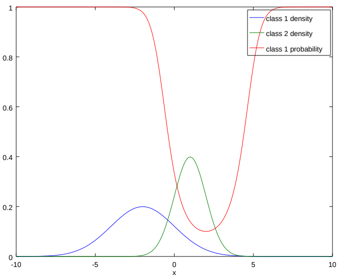

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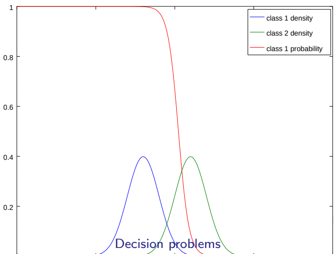

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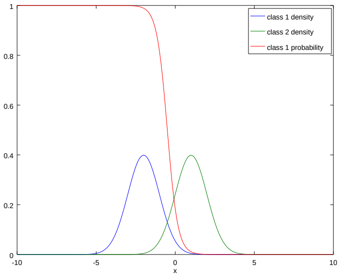

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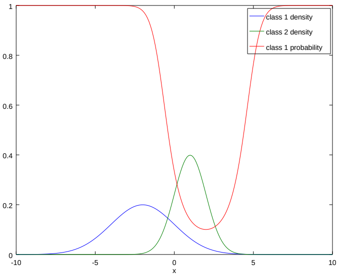

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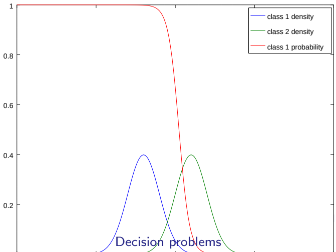
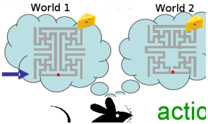


Subjective probability

Subjective probability measure ξ

- If we think event A is more likely than B, then $\xi(A) > \xi(B)$.
- Usual rules of probability apply:
  1. $\xi(A) \in [0, 1]$.
  2. $\xi(\emptyset) = 0$.
  3. If $A \cap B = \emptyset$, then $\xi(A \cup B) = \xi(A) + \xi(B)$.

Bayesian inference illustration



Use a subjective belief $\xi(\mu)$ on $\mathcal{M}$

- Prior belief $\xi(\mu)$ represents our initial uncertainty.

Bayesian inference illustration

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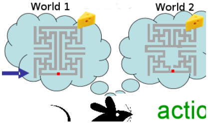
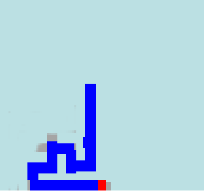


prior



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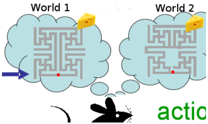
Bayesian inference illustration



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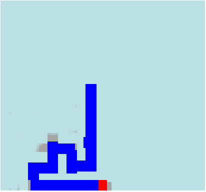

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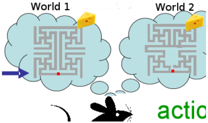
- Prior belief $\xi(\mu)$ represents our initial uncertainty.
- We observe history h.
- Each possible μ assigns a probability $P_\mu(h)$ to h.
- We can use this to update our belief via Bayes' theorem to obtain the posterior belief:

$$\xi(\mu \mid h) \propto P_\mu(h)\xi(\mu) \quad \text{(conclusion = evidence} \times \text{prior)}$$



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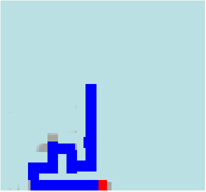

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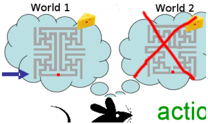

conclusion

Some examples

Example 4

John claims to be a medium. He throws a coin n times and predicts its value always correctly. Should we believe that he is a medium?

- μ_1: John is a medium.
- μ_0: John is not a medium.

The answer depends on what we expect a medium to be able to do, and how likely we thought he'd be a medium in the first place.

Bayesian inference

Family of models $\mathcal{M} = \{\mu_1, \ldots, \mu_k\}$

Defines a family of probabilities for any data x:

$$\{P_\mu | \mu \in \mathcal{M}\}, \qquad P_\mu(x) \equiv \mathbb{P}(x \mid \mu).$$

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Interpretation

- $\mathcal{M}$: Set of all possible models that could describe the data.
- $P_\mu(x)$: Probability of x under model μ.
- Alternative notation $\mathbb{P}(x \mid \mu)$: Probability of x given that model μ is correct.
- $\xi(\mu)$: Our belief, before seeing the data, that μ is correct.
- $\xi(\mu \mid x)$: Our belief, aftering seeing the data, that μ is correct.

Exercise 1 (Continued example for medium)

$$P_\mu(x) = \prod_{t=1}^{n} P_\mu(x_t). \qquad \text{(independence property)}$$

If a classmate correctly predicts 4 coin tosses, what is your belief they are a medium?

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$$P_{\mu_0}(x_t = 1) = 1/2, \qquad P_{\mu_0}(x_t = 0) = 1/2. \qquad \text{(non-medium model)}$$

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$$\xi(\mu_1 \mid x) = \frac{P_{\mu_1}(x)\xi(\mu_1)}{\mathbb{P}_\xi(x)} \qquad \text{(posterior belief)}$$

$$\mathbb{P}_\xi(x) \triangleq P_{\mu_1}(x)\xi(\mu_1) + P_{\mu_0}(x)\xi(\mu_0). \qquad \text{(marginal distribution)}$$

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Sequential update of beliefs

	M	T	W	T	F	S	S
CNN	0.5	0.6	0.7	0.9	0.5	0.3	0.1
SMHI	0.3	0.7	0.8	0.9	0.5	0.2	0.1
YR	0.6	0.9	0.8	0.5	0.4	0.1	0.1
Rain?	Y	Y	Y	N	Y	N	N

Table : Predictions by three different entities for the probability of rain on a particular day, along with whether or not it actually rained.

Exercise 2

- n meteorological stations $\{\mu_i \mid i = 1, \ldots, n\}$
- The i-th station predicts rain $P_{\mu_i}(x_t \mid x_1, \ldots, x_{t-1})$.
- Let $\xi_t(\mu)$ be our belief at time t. Derive the next-step belief $\xi_{t+1}(\mu) \triangleq \xi_t(\mu|y_t)$ in terms of the current belief ξ_t.
- Write a python function that computes this posterior

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Bayesian inference for Bernoulli distributions

Estimating a coin's bias

A fair coin comes heads 50% of the time. We want to test an unknown coin, which we think may not be completely fair.

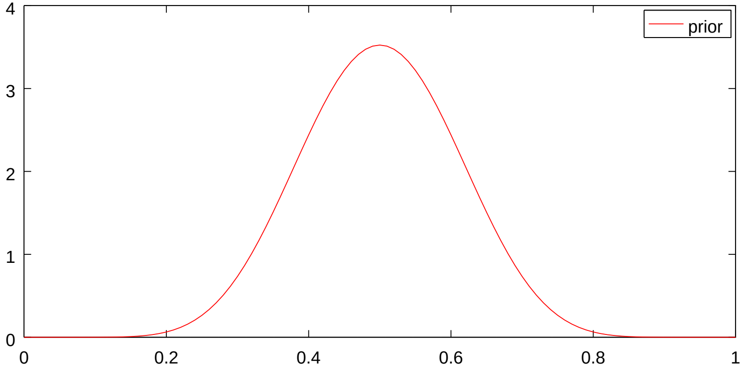

Figure : Prior belief ξ about the coin bias θ.

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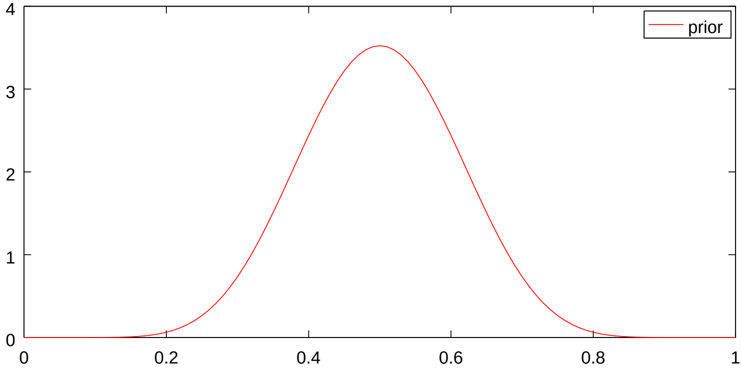

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For a sequence of throws $x_t \in \{0, 1\}$,

$$P_\theta(x) \propto \prod_t \theta^{x_t} (1-\theta)^{1-x_t} = \theta^{\#\text{Heads}} (1-\theta)^{\#\text{Tails}}$$

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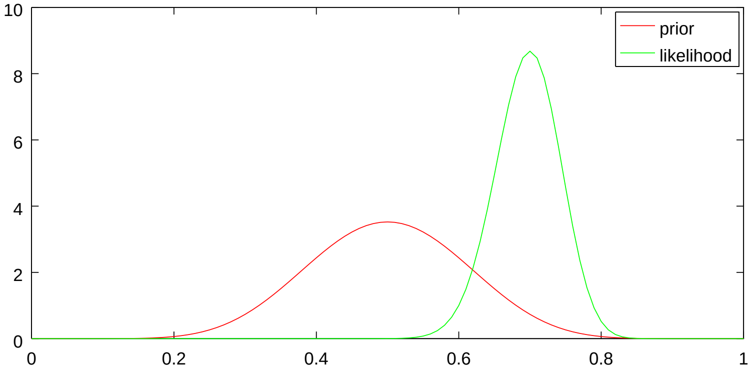

Figure : Prior belief ξ about the coin bias θ and likelihood of θ for the data.

Say we throw the coin 100 times and obtain 70 heads. Then we plot the likelihood $P_\theta(x)$ of different models.

Bayesian inference for Bernoulli distributions

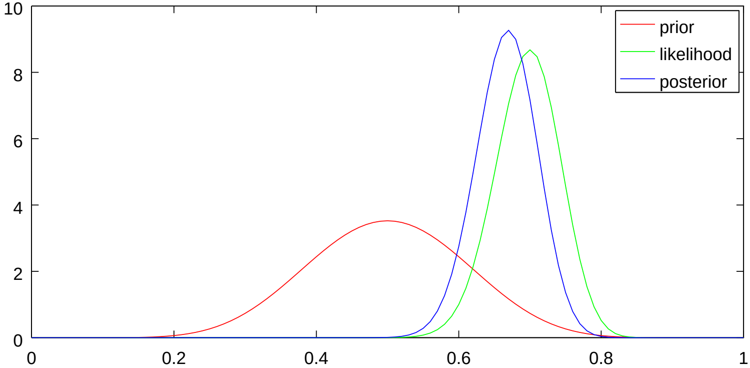

Figure : Prior belief $\xi(\theta)$ about the coin bias θ, likelihood of θ for the data, and posterior belief $\xi(\theta \mid x)$

From these, we calculate a posterior distribution over the correct models. This represents our conclusion given our prior and the data.

Learning outcomes

Understanding

- The axioms of probability, marginals and conditional distributions.
- The philosophical underpinnings of Bayesianism.
- The simple conjugate model for Bernoulli distributions.

Skills

- Be able to calculate with probabilities using the marginal and conditional definitions and Bayes rule.
- Being able to implement a simple Bayesian inference algorithm in Python.

Reflection

- How useful is the Bayesian representation of uncertainty?
- How restrictive is the need to select a prior distribution?
- Can you think of another way to explicitly represent uncertainty in a way that can incorporate new evidence?

Preferences

Example 5

Food

A McDonald's cheeseburger

B Surstromming

C Oatmeal

Money

A 10,000,000 SEK

B 10,000,000 USD

C 10,000,000 BTC

Entertainment

A Ticket to Liseberg

B Ticket to Rebstar

C Ticket to Nutcracker

Rewards and utilities

- Each choice is called a reward $r \in \mathcal{R}$.
- There is a utility function $U : \mathcal{R} \to \mathbb{R}$, assigning values to reward.
- We (weakly) prefer A to B iff $U(A) \geq U(B)$.

Exercise 3

From your individual preferences, derive a common utility function that reflects everybody's preferences in the class for each of the three examples. Is there a simple algorithm for deciding this? Would you consider the outcome fair?

Preferences among random outcomes

Example 6

Would you rather ...

- A Have 100 EUR now?
- B Flip a coin, and get 200 EUR if it comes heads?

Risk and monetary rewards

Preferences among random outcomes

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The expected utility hypothesis

Rational decision makers prefer choice A to B if

$$\mathbb{E}(U|A) \geq \mathbb{E}(U|B),$$

where the expected utility is

$$\mathbb{E}(U|A) = \sum_r U(r)\,\mathbb{P}(r|A).$$

In the above example, $r \in \{0, 100, 200\}$ and $U(r)$ is increasing, and the coin is fair.

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Risk and monetary rewards

- If U is convex, we are risk-seeking.

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Risk and monetary rewards

- If U is convex, we are risk-seeking.
- If U is concave, we are risk-averse.

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Risk and monetary rewards

- If U is convex, we are risk-seeking.
- If U is linear, we are risk neutral.
- If U is concave, we are risk-averse.

Uncertain rewards

- Decisions $a \in \mathcal{A}$
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Example 7

You are going to work, and it might rain. What do you do?

- a_1: Take the umbrella.
- a_2: Risk it!
- ω_1: rain
- ω_2: dry

$\rho(\omega, a)$	a_1	a_2
ω_1	dry, carrying umbrella	wet
ω_2	dry, carrying umbrella	dry

$U[\rho(\omega, a)]$	a_1	a_2
ω_1	0	-10
ω_2	0	1

Table : Rewards and utilities.

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- $\max_a \min_\omega U = 0$

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Expected utility

$$\mathbb{E}(U \mid a) = \sum_r U[\rho(\omega, a)] \, \mathbb{P}(\omega \mid a)$$

Example 8

You are going to work, and it might rain. The forecast said that the probability of rain (ω_1) was 20%. What do you do?

- a_1: Take the umbrella.
- a_2: Risk it!

$\rho(\omega, a)$	a_1	a_2
ω_1	dry, carrying umbrella	wet
ω_2	dry, carrying umbrella	dry
$U[\rho(\omega, a)]$	a_1	a_2
ω_1	0	-10
ω_2	0	1
$\mathbb{E}_P(U \mid a)$	0	-1.2

Table : Rewards, utilities, expected utility for 20% probability of rain.

Bayes decision rules

Consider the case where outcomes are independent of decisions:

$$U(\xi, a) \triangleq \sum_{\mu} U(\mu, a)\xi(\mu)$$

This corresponds e.g. to the case where $\xi(\mu)$ is the belief about an unknown world.

Definition 9 (Bayes utility)

The maximising decision for ξ has an expected utility equal to:

$$U^*(\xi) \triangleq \max_{a \in \mathcal{A}} U(\xi, a). \tag{2.1}$$

The n-meteorologists problem

Exercise 4

- Meteorological models $\mathcal{M} = \{\mu_1, \ldots, \mu_n\}$
- Rain predictions at time t: $p_{t,\mu} \triangleq P_\mu(x_t = \mathrm{rain})$.
- Prior probability $\xi(\mu) = 1/n$ for each model.
- Should we take the umbrella?

	M	T	W	T	F	S	S
CNN	0.5	0.6	0.7	0.9	0.5	0.3	0.1
SMHI	0.3	0.7	0.8	0.9	0.5	0.2	0.1
YR	0.6	0.9	0.8	0.5	0.4	0.1	0.1
Rain?	Y	Y	Y	N	Y	N	N

Table : Predictions by three different entities for the probability of rain on a particular day, along with whether or not it actually rained.

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1. What is your belief about the quality of each meteorologist after each day?
2. What is your belief about the probability of rain each day?

$$P_\xi(x_t = \mathrm{rain} \mid x_1, x_2, \dots x_{t-1}) = \sum_{\mu \in \mathcal{M}} P_\mu(x_t = \mathrm{rain} \mid x_1, x_2, \dots x_{t-1}) \xi(\mu \mid x_1, x_2, \dots x_{t-1})$$

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3. Assume you can decide whether or not to go running each day. If you go running and it does not rain, your utility is 1. If it rains, it's -10. If you don't go running, your utility is 0. What is the decision maximising utility in expectation (with respect to the posterior) each day?

Deciding a class given a model

- Features $x_t \in \mathcal{X}$.
- Label $y_t \in \mathcal{Y}$.
- Decisions $a_t \in \mathcal{A}$.
- Decision rule $\pi(a_t \mid x_t)$ assigns probabilities to actions.

Standard classification problem

$$\mathcal{A} = \mathcal{Y}, \qquad U(a, y) = \mathbb{I}\{a = y\}$$

Exercise 5

If we have a model $P_\mu(y_t \mid x_t)$, and a suitable U, what is the optimal decision to make?

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For standard classification,

$$a_t \in \arg\max_{a \in \mathcal{A}} P_\mu(y_t = a \mid x_t)$$

Deciding the class given a model family

- Training data $D_T = \{(x_i, y_i) \mid i = 1, \ldots, T\}$
- Models $\{P_\mu \mid \mu \in \mathcal{M}\}$.
- Prior ξ on $\mathcal{M}$.

Posterior over classification models

$$\xi(\mu \mid D_T) = \frac{P_\mu(y_1, \ldots, y_T \mid x_1, \ldots, x_T)\xi(\mu)}{\sum_{\mu' \in \mathcal{M}} P_{\mu'}(y_1, \ldots, y_T \mid x_1, \ldots, x_T)\xi(\mu')}$$

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If not dealing with time-series data, we assume independence between x_t:

$$P_\mu(y_1, \ldots, y_T \mid x_1, \ldots, x_T) = \prod_{i=1}^{T} P_\mu(y_i \mid x_i)$$

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The Bayes rule for maximising $\mathbb{E}_\xi(U \mid a, x_t, D_T)$

The decision rule simply chooses the action:

$$a_t \in \arg\max_{a \in \mathcal{A}} \sum_y \sum_{\mu \in \mathcal{M}} P_\mu(y_t = y \mid x_t)\xi(\mu \mid D_T)U(a, y) \tag{3.1}$$

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We can rewrite this by calculating the posterior marginal marginal label probability

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$$= \arg\max_{a \in \mathcal{A}} \sum_y \mathbb{P}_{\xi \mid D_T}(y_t \mid x_t)U(a, y) \tag{3.2}$$

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Approximating the model

Full Bayesian approach for infinite $\mathcal{M}$

Here ξ can be a probability density function and

$$\xi(\mu \mid D_T) = P_\mu(D_T)\xi(\mu) / \mathbb{P}_\xi(D_T), \qquad \mathbb{P}_\xi(D_T) = \int_{\mathcal{M}} P_\mu(D_T)\xi(\mu)\,\mathrm{d},$$

can be hard to calculate.

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Maximum a posteriori model

We only choose a single model through the following optimisation:

$$\mu_{\mathrm{MAP}}(\xi, D_T) = \arg\max_{\mu \in \mathcal{M}} P_\mu(D_T)\xi(\mu)$$

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We only choose a single model through the following optimisation:

$$\mu_{\mathrm{MAP}}(\xi, D_T) = \arg\max_{\mu \in \mathcal{M}} \overbrace{\ln P_\mu(D_T)}^{\text{goodness of fit}} + \underbrace{\ln \xi(\mu)}_{\text{regulariser}} .$$

Learning outcomes

Understanding

- Preferences, utilities and the expected utility principle.
- Hypothesis testing and classification as decision problems.
- How to interpret p-values Bayesian tests.
- The MAP approximation to full Bayesian inference.

Skills

- Being able to implement an optimal decision rule for a given utility and probability.
- Being able to construct a simple null hypothesis test.

Reflection

- When would expected utility maximisation not be a good idea?
- What does a p value represent when you see it in a paper?
- Can we prevent high false discovery rates when using p values?
- When is the MAP approximation good?

Simple hypothesis testing

The simple hypothesis test as a decision problem

- $\mathcal{M} = \{\mu_0, \mu_1\}$
- a_0: Accept model μ_0
- a_1: Accept model μ_1

U	μ_0	μ_1
a_0	1	0
a_1	0	1

Table : Example utility function for simple hypothesis tests.

Example 10 (Continuation of the medium example)

- μ_1: that John is a medium.
- μ_0: that John is not a medium.

$$\mathbb{E}_\xi(U \mid a_0) = 1 \times \xi(\mu_0 \mid \boldsymbol{x}) + 0 \times \xi(\mu_1 \mid \boldsymbol{x}), \qquad \mathbb{E}_\xi(U \mid a_1) = 0 \times \xi(\mu_0 \mid \boldsymbol{x}) + 1 \times \xi(\mu_1 \mid \boldsymbol{x})$$

Null hypothesis test

Many times, there is only one model under consideration, μ_0, the so-called null hypothesis.

The null hypothesis test as a decision problem

- a_0: Accept model μ_0
- a_1: Reject model μ_0

Example 11

Construction of the test for the medium

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- In particular, we can fix a policy that only chooses a_1 when μ_0 is true a proportion δ of the time.
- This can be done by construcing a threshold test from the inverse-CDF.

Using p-values to construct statistical tests

Definition 12 (Null statistical test)

The statistic $f : \mathcal{X} \to [0, 1]$ is designed to have the property:

$$P_{\mu_0}(\{x \mid f(x) \leq \delta\}) = \delta.$$

If our decision rule is:

$$\pi(a \mid x) = \begin{cases} a_0, & f(x) \geq \delta \\ a_1, & f(x) < \delta, \end{cases}$$

the probability of rejecting the null hypothesis when it is true is exactly δ.

The value of the statistic $f(x)$, otherwise known as the p-value, is uninformative.

Issues with p-values

- They only measure quality of fit on the data.
- Not robust to model misspecification.
- They ignore effect sizes.
- They do not consider prior information.
- They do not represent the probability of having made an error.
- The null-rejection error probability is the same irrespective of the amount of data (by design).

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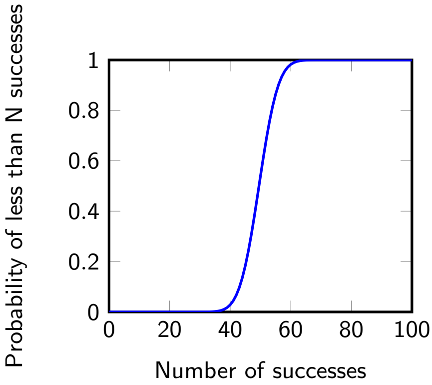

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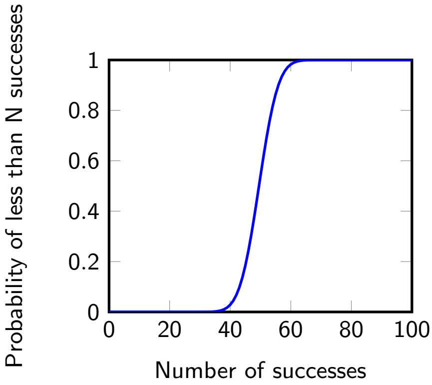
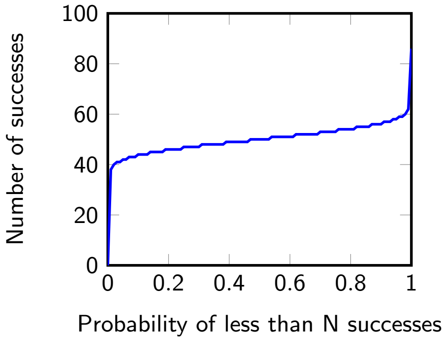
- CDF: $P_{\mu_0}(N \leq n \mid K = 100)$
- ICDF: the number of successes that will happen with probability at least δ





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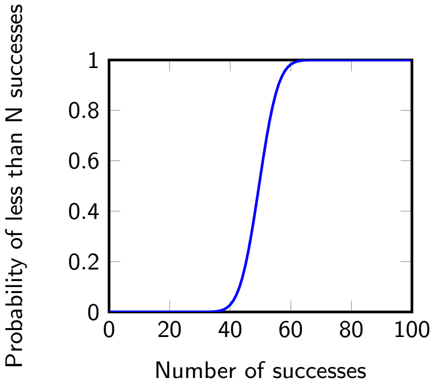
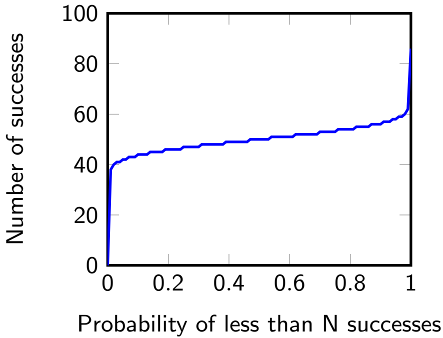
- CDF: $P_{\mu_0}(N \leq n \mid K = 100)$
- ICDF: the number of successes that will happen with probability at least δ
- e.g. we'll get at most 50 successes a proportion $\delta = 1/2$ of the time.





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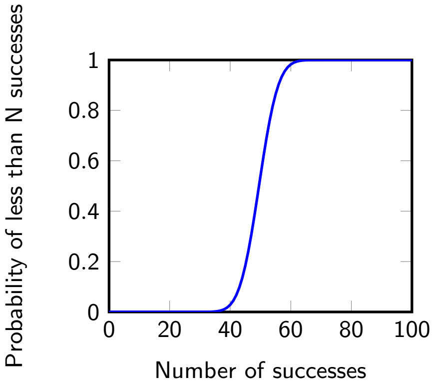
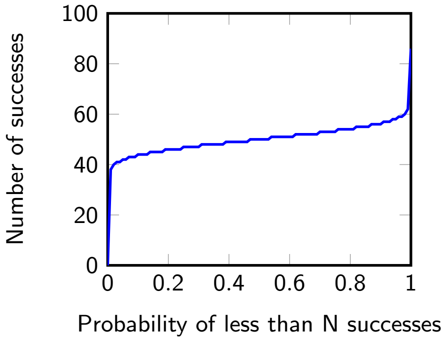
- e.g. we'll get at most 50 successes a proportion $\delta = 1/2$ of the time.
- Using the (inverse) CDF we can construct a policy π that selects a_1 when μ_0 is true only a δ portion of the time, for any choice of δ.





Building a test

The test statistic

We want the test to reflect that we don't have a significant number of failures.

$$f(x) = 1 - \mathrm{binocdf}(\sum_{t=1}^{n} x_t, n, 0.5)$$

What $f(x)$ is and is not

- It is a **statistic** which is $\leq \delta$ a δ portion of the time when μ_0 is true.
- It is **not** the probability of observing x under μ_0.
- It is **not** the probability of μ_0 given x.

Exercise 6

- Let us throw a coin 8 times, and try and predict the outcome.

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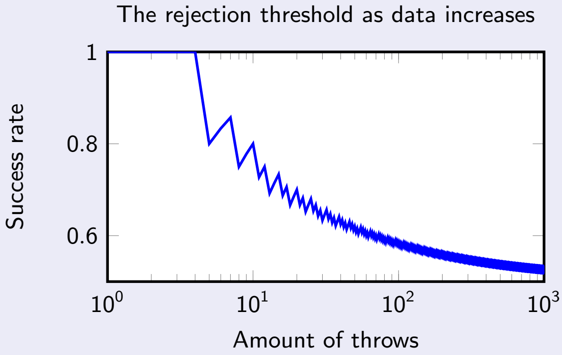

Figure : Here we see how the rejection threshold, in terms of the success rate, changes with the number of throws to achieve an error rate of $\delta = 0.05$.

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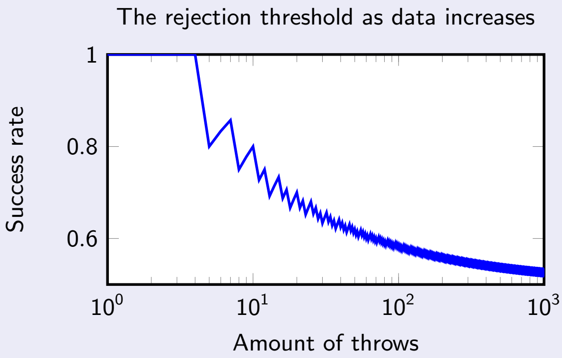
- Select a p-value threshold so that $\delta = 0.05$. For 8 throws, this corresponds to > 6 successes or $\geq 87.5\%$ success rate.
- Let's calculate the p-value for each one of you



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- Let's calculate the p-value for each one of you
- What is the rejection performance of the test?

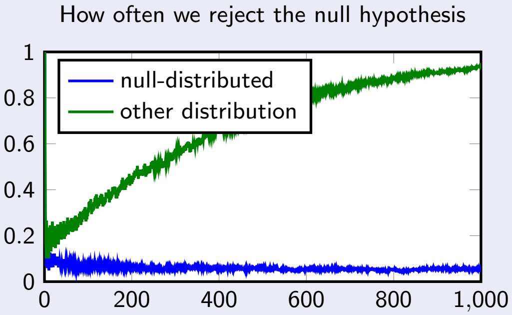

Figure : Here we see the rejection rate of the null hypothesis (μ_0) for two cases. Firstly, for the case when μ_0 is true. Secondly, when the data is generated from $\mathcal{Bernoulli}(0.55)$.

Statistical power and false discovery.

Beyond not rejecting the null when it's true, we also want:

- High power: Rejecting the null when it is false.
- Low false discovery rate: Accepting the null when it is true.

Power

The power depends on what hypothesis we use as an alternative.

False discovery rate

False discovery depends on how likely it is a priori that the null is false.

The Bayesian version of the test

Example 13

1. Set $U(a_i, \mu_j) = \mathbb{I}\{i = j\}$.
2. Set $\xi(\mu_i) = 1/2$.
3. μ_0: $\mathcal{B}ernoulli(1/2)$.
4. μ_1: $\mathcal{B}ernoulli(\theta)$, $\theta \sim \mathcal{U}nif([0, 1])$.
5. Calculate $\xi(\mu \mid x)$.
6. Choose a_i, where $i = \arg\max_j \xi(\mu_j \mid x)$.

Bayesian model averaging for the alternative model μ_1

$$P_{\mu_1}(x) = \int_{\Theta} B_\theta(x) \, \mathrm{d}\beta(\theta) \tag{3.3}$$

$$\xi(\mu_0 \mid x) = \frac{P_{\mu_0}(x)\xi(\mu_0)}{P_{\mu_0}(x)\xi(\mu_0) + P_{\mu_1}(x)\xi(\mu_1)} \tag{3.4}$$

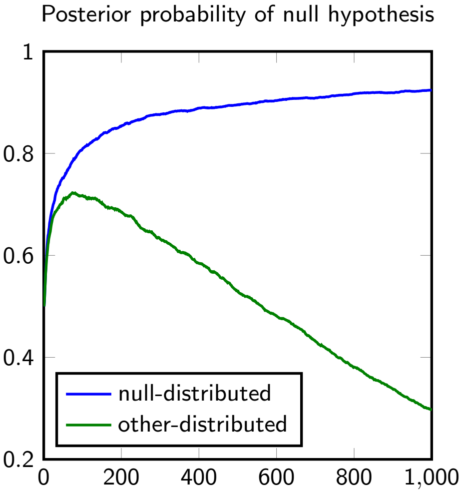

Figure : Here we see the convergence of the posterior probability.

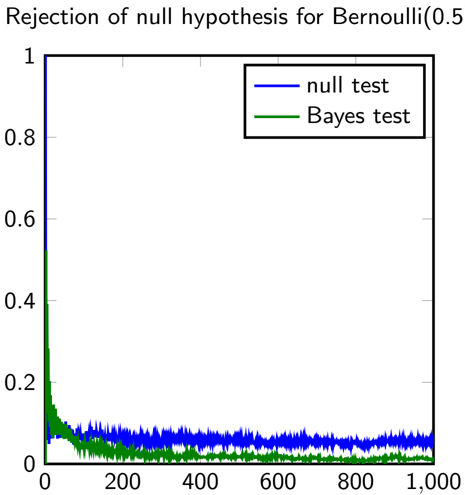

Figure : Comparison of the rejection probability for the null and the Bayesian test when μ_0 is true.

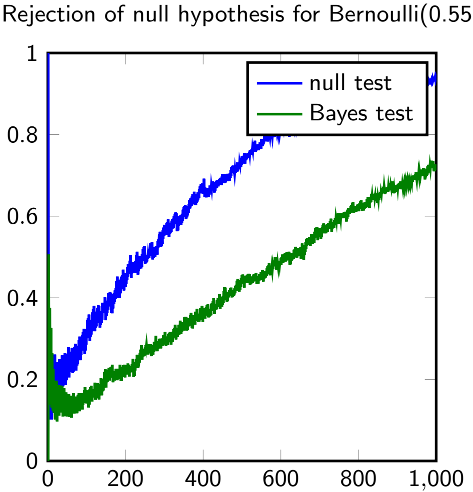

Figure : Comparison of the rejection probability for the null and the Bayesian test when μ_1 is true.

Concentration inequalities and confidence intervals

Further reading

Points of significance (Nature Methods)

- Importance of being uncertain `https://www.nature.com/articles/nmeth.2613`
- Error bars `https://www.nature.com/articles/nmeth.2659`
- P values and the search for significance `https://www.nature.com/articles/nmeth.4120`
- Bayes' theorem `https://www.nature.com/articles/nmeth.3335`
- Sampling distributions and the bootstrap `https://www.nature.com/articles/nmeth.3414`

Classification as an optimisation problem.

The μ-optimal classifier

$$\max_{\theta \in \Theta} f(\pi_\theta, \mu, U), \qquad f(\pi_\theta, \mu, U) \triangleq \mathbb{E}_\mu^{\pi_\theta}(U) \tag{4.1}$$

$$f(\pi_\theta, \mu, U) = \sum_{x,y,a} U(a, y) \pi_\theta(a \mid x) P_\mu(y \mid x) P_\mu(x) \tag{4.2}$$

$$\approx \sum_{t=1}^{T} \sum_{a_t} U(a_t, y_t) \pi_\theta(a_t \mid x_t), \qquad (x_t, y_t)_{t=1}^{T} \sim P_\mu. \tag{4.3}$$

Stochastic gradient methdos

Gradient ascent

$$\theta_{i+1} = \theta_i + \alpha \nabla_\theta g(\theta_i).$$

Stochastic gradient ascent

$$g(\theta) = \int_{\mathcal{M}} f(\theta, \mu) \,\mathrm{d}\xi(\mu)$$

$$\theta_{i+1} = \theta_i + \alpha \nabla_\theta f(\theta_i, \mu_i), \qquad \mu_i \sim \xi.$$

Two views of neural networks

Neural network classification model $P_\theta(\boldsymbol{y} \mid \boldsymbol{x}_t)$

$$\begin{tikzcd} \boldsymbol{x}_t \arrow[r, bend right] & \boldsymbol{y}_t \end{tikzcd}$$

Objective: Find the best model for D_T.

Neural network classification policy $\pi(a_t \mid \boldsymbol{x}_t)$

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Difference between the two views

- We can use standard probabilistic methods for P.
- Finding the optimal π is an optimisation problem.

Linear networks and the perceptron algorithm

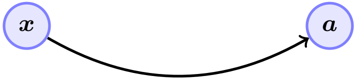


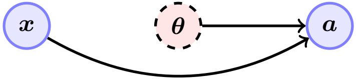
Figure : Abstract graphical model for a neural network

Definition 14 (Linear classifier)

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\theta}_1 & \cdots & \boldsymbol{\theta}_C \end{bmatrix} = \begin{bmatrix} \theta_{1,1} & \cdots & \theta_{1,C} \\ \vdots & \ddots & \vdots \\ \theta_N & \cdots & \theta_{N,C} \end{bmatrix}$$

$$\pi_{\Theta}(a \mid \boldsymbol{x}) = \exp\left(\boldsymbol{\theta}_a^\top \boldsymbol{x}\right) / \sum_{a'} \exp\left(\boldsymbol{\theta}_{a'}^\top \boldsymbol{x}\right)$$

Linear networks and the perceptron algorithm



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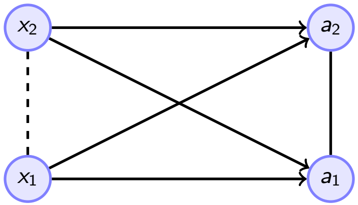

Figure : Graphical model for a linear neural network.

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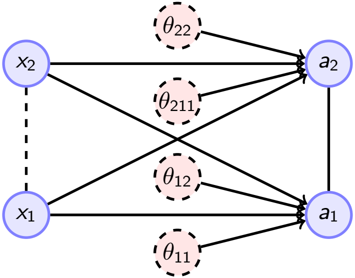

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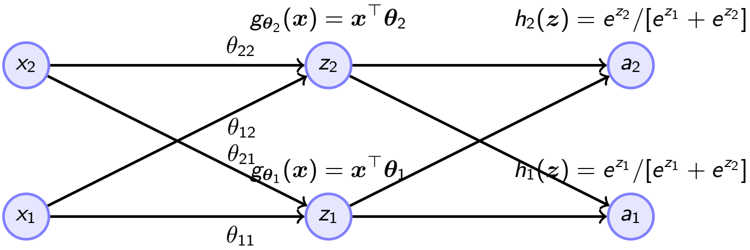

Figure : Architectural view of a linear neural network.

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Gradient ascent for a matrix U

$$\max_\theta \sum_{t=1}^{T} \sum_{a_t} U(a_t, y_t) \pi_\theta(a_t \mid x_t) \qquad \text{(objective)}$$

$$\nabla_\theta \sum_{t=1}^{T} \sum_{a_t} U(a_t, y_t) \pi_\theta(a_t \mid x_t) \qquad \text{(gradient)}$$

$$= \sum_{t=1}^{T} \sum_{a_t} U(a_t, y_t) \nabla_\theta \pi_\theta(a_t \mid x_t) \qquad (4.4)$$

Chain Rule of Differentiation

$$f(z), z = g(x), \qquad \frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} \qquad \text{(scalar version)}$$

$$\nabla_\theta \pi = \nabla_g \pi \nabla_\theta g \qquad \text{(vector version)}$$

Learning outcomes

Understanding

- Classification as an optimisation problem.
- (Stochastic) gradient methods and the chain rule.
- Neural networks as probability models or classification policies.
- Linear neural netwoks.
- Nonlinear network architectures.

Skills

- Using a standard NN class in python.

Reflection

- How useful is the ability to have multiple non-linear layers in a neural network.
- How rich is the representational power of neural networks?
- Is there anything special about neural networks other than their allusions to biology?