IN-STK 5000: Introductory assignment

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The purpose of this assignment is to evaluate the background knowledge of the students in the course. Please provide as precise and concise answers as possible.

1 Probability theory

In this section we consider probability as a measure, i.e. as a function from sets to $[0, 1]$. All events are subsets of the universal set Ω, so that $P(\Omega) = 1$, $P(\emptyset) = 0$.

EXERCISE 1. If A, B are mutually exclusive events i.e. $A \cap B = \emptyset$, then

$$P(A \cup B) = P(A) + P(B)$$

EXERCISE 2 (Union bound). If A, B are not exclusive events, i.e. $A \cap B \neq \emptyset$, then

$$P(A \cup B) \leq P(A) + P(B)$$

EXERCISE 3 (Conditional probability). If A, B are two events, with $P(B) > 0$, then conditional probability is defined as

$$P(A \mid B) \triangleq \frac{P(A \cap B)}{P(B)}$$

EXERCISE 4 (Marginal probability). Let $A_1, \ldots, A_n$ be mutually exclusive events so that $\bigcup_{i=1}^n A_i = \Omega$ and $B \subset \Omega$ an arbitrary other event. Then:

$$P(B) = \sum_{A_i} P(A_i \cap B) = \sum_{A_i} P(B \mid A_i) P(A_i)$$

2 Random variables and statistics

EXERCISE 5. A real-valued random variable x is simply a mapping $x : \Omega \to \mathbb{R}$. Write the definition of the expectation of x drawn from P, where P is a probability measure on (Ω, Σ) and Σ is the σ-algebra generated by Ω.

$$\mathbb{E}(x) = \sum_{\omega \in \Omega} x(\omega) P(\omega)$$

EXERCISE 6. The sample mean μ_n of n i.i.d random variables $x_1, \ldots, x_n$ is defined as

$$\mu_n \triangleq \frac{1}{n} \sum_{i=1}^n x_i$$

EXERCISE 7. Write the expectation of the sample mean μ_n in relation to $x_1, \ldots, x_n$. Since x_i are i.i.d, there is some $\bar{x}$ so that $\mathbb{E}\, x_i = \bar{x}$ for all i. Then

$$\mathbb{E}\, \mu_n = \mathbb{E}\, \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\, x_i = \bar{x}$$

EXERCISE 8. A null hypothesis test at significance level p is constructed by using a test statistic $\pi : \mathcal{X} \to [0, 1)$ mapping from the space of possible data to the interval $[0, 1)$, so that the test rejects the null hypothesis whenever $\pi(x) < p$. Does this mean that:

1. The probability that the test will falsely reject the null hypothesis is p.
2. The probability that the test will falsely accept the null hypothesis is p.
3. The probability that the test will falsely reject the alternative hypothesis is p.
4. The probability that the test will falsely accept the alternative hypothesis is p.
5. Given the data x, the probability that the null hypothesis is true is $\pi(x)$.
6. Given the data x, the probability that the null hypothesis is false is $\pi(x)$.
7. Given the data x, the probability that the alternative hypothesis is true is $\pi(x)$.
8. Given the data x, the probability that the alternative hypothesis is false is $\pi(x)$.

Null hypothesis tests that have a fixed significance level p are designed so that, if the data comes from a given null hypothesis, then the probability that the test statistic $\pi(x) < p$ is exactly equal to p. The probability of falsely accepting the null hypothesis, however, depends on the unknown alternative hypothesis and so cannot be computed. Consequently the correct answers are 1 and 4 (Since the decision rule either accepts or rejects the null hypothesis, 4 is correct too).

3 Linear algebra

EXERCISE 9. If $\boldsymbol{x} = x_1, \ldots, x_n$, $\boldsymbol{y} = y_1, \ldots, y_n$ are two column vectors in $\mathbb{R}^n$, what is their inner product:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^\top \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$$

EXERCISE 10. The matrix

$$\boldsymbol{A}^+ \triangleq (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top.$$

is the left-pseudoinverse of $\boldsymbol{A}$. Complete the following:

$$\boldsymbol{A}^+ \boldsymbol{A} = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{A} = \boldsymbol{I}$$

4 Calculus

EXERCISE 11. If $f : \mathcal{X} \to \mathbb{R}$ is a twice-differentiable function, what are *sufficient* conditions for x_0 to be a *local maximum* of the function, i.e. there exists $\epsilon > 0$ so that $f(x_0) \geq f(x)$ for all $x : |x - x_0| < \epsilon$? If $df(x_0)/dx = 0$ then x_0 is either a saddle point, a maximum or a minimum. If in addition $d^2 f(x_0)/dx^2 < 0$, then x_0 is a maximum.

EXERCISE 12. Solve the following integral, for $T > 0$

$$\int_1^T \frac{1}{x} \, \mathrm{d}x = \ln T$$