1. Newton’s Second Law for Rotation

For any rigid body rotating about a pivot, Newton’s second law takes the form

$$I\,\ddot{\theta} = \sum \tau,$$

where

- I is the total moment of inertia,
- θ is the rotation angle, and
- $\sum \tau$ is the sum of the torques acting on the system.
  Express each term of the torque sum and then divide by I to isolate $\ddot{\theta}$.

2. Gravitational Torque via the 2D Cross Product

a. The Setup

Assume two point masses m_1 and m_2 are attached at the ends of a beam (or seesaw) of length L. If the pivot is at the beam’s center, each mass is located at a distance $r = L/2$ from the pivot.

b. Using the Cross Product

The gravitational force on each mass is $\mathbf{F}_g$ directed downward. The position vectors from the pivot to each mass are, for example, $\mathbf{r}_1$ and $\mathbf{r}_2$

$$\mathbf{F}_g = \begin{pmatrix} 0 \\ -m_i\, g \end{pmatrix} \quad , \quad \mathbf{r}_1 = \begin{pmatrix} -\frac{L}{2}\cos\theta \\ -\frac{L}{2}\sin\theta \end{pmatrix} \quad , \quad \mathbf{r}_2 = \begin{pmatrix} \frac{L}{2}\cos\theta \\ \frac{L}{2}\sin\theta \end{pmatrix}$$

The choice of sign is made for convenience so that the gravitational term "supports" or "resists" depending on the difference in masses.
In two dimensions, the torque for each mass is given by the scalar (the “z-component” of the cross product):

$$\tau = \mathbf{r} \times \mathbf{F} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \times \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ F_x & F_y & 0 \end{vmatrix} = \mathbf{i} \cdot \begin{vmatrix} y & 0 \\ F_y & 0 \end{vmatrix} - \mathbf{j} \cdot \begin{vmatrix} x & 0 \\ F_x & 0 \end{vmatrix} + \mathbf{k} \cdot \begin{vmatrix} x & y \\ F_x & F_y \end{vmatrix} = xF_y - yF_x$$

For mass m_1 (located at $-\frac{L}{2}$ on one side)

$$\tau_1 = \mathbf{r}_1 \times \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix} = \left(-\frac{L}{2}\cos\theta\right)(-m_1\, g) + \left(\frac{L}{2}\sin\theta\right)(0) = \frac{L}{2}\, m_1\, g\, \cos\theta.$$

For mass m_2 (located on the opposite side)

$$\tau_2 = \mathbf{r}_2 \times \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix} = \left(\frac{L}{2}\cos\theta\right)(-m_2\, g) - \left(\frac{L}{2}\sin\theta\right)(0) = -\frac{L}{2}\, m_2\, g\, \cos\theta.$$

c. Net Gravitational Torque

The net gravitational torque is then

$$\tau_g = \tau_1 + \tau_2 = \frac{(m_1 - m_2) g\, L}{2} \cos\theta,$$

Thus, when we divide by I (to write the angular acceleration), the gravitational term becomes

$$\tau_g = \gamma\, \cos\theta \quad , \quad \gamma = \frac{(m_1 - m_2) g\, L}{2\, I}.$$

3. Friction Torque:

a. Viscous Friction

Viscous friction is modeled as being proportional to the angular velocity, where b is the viscous damping coefficient:

$$\tau_{\text{viscous}} = -b\, \dot{\theta},$$

b. Coulomb Friction

The Coulomb (dry) friction torque is given by

$$\tau_{\text{Coulomb}} = -\mu_c\, N\, \frac{L}{2}\, \text{sgn}(\dot{\theta}),$$

where

- μ_c is the Coulomb friction coefficient,
- N is the normal force (which we approximate using g in our gravitational context), and
- $L/2$ is the effective lever arm.
  Because the sign function $\text{sgn}(\dot{\theta})$ is discontinuous at zero, we use a smooth approximation:

  $$\text{sgn}(\dot{\theta}) \approx \tanh\left(\frac{\dot{\theta}}{\epsilon}\right),$$

  This approximation facilitates numerical simulation (continuity and derivability), especially useful if it is then linearized. With ϵ as a small positive constant.

  $$\tau_{\text{Coulomb}} \approx -\mu_c\, \frac{g\, L}{2}\, \tanh\left(\frac{\dot{\theta}}{\epsilon}\right).$$

c. Total Friction Torque

$$\tau_{\text{friction}} = \tau_{\text{viscous}} + \tau_{\text{Coulomb}} = -b\, \dot{\theta} - \mu_c\, \frac{g\, L}{2}\, \tanh\left(\frac{\dot{\theta}}{\epsilon}\right)$$

When divided by the total moment of inertia I, the friction terms in the angular acceleration become

$$\tau_{\text{friction}} = -\alpha\,\dot{\theta} - \beta\,\tanh\left(\frac{\dot{\theta}}{\epsilon}\right) \quad , \quad \alpha = \frac{b}{I}, \quad \beta = \frac{\mu_c\, g\, L}{2\, I}$$

4. Total Moment of Inertia

a. Beam's Moment of Inertia

For a uniform bar (beam) of mass m_b and length L rotating about its center, the moment of inertia is obtained by integrating:

1. **Density**: $\lambda = m_b/L$
2. **Elemental mass**: $dm = \lambda\, dx$
3. **Moment contribution**: $dI = x^2\, dm = x^2\, \lambda\, dx$
4. **Integrate**:

$$I_b = \int_{-L/2}^{L/2} x^2\, \lambda\, dx = \lambda \int_{-L/2}^{L/2} x^2\, dx = \frac{m_b}{L}\,\frac{L^3}{12} = \frac{1}{12} m_b L^2$$

b. Point Masses' Moments of Inertia

Each point mass m_i located at a distance $r = L/2$ from the pivot contributes

$$I_i = m_i \left(\frac{L}{2}\right)^2 = \frac{1}{4} m_i L^2.$$

Thus, for both masses:

$$I_{\text{points}} = \frac{1}{4}\,(m_1 + m_2)\,L^2.$$

c. Total Moment of Inertia

Adding the beam's and the point masses' moments of inertia gives

$$I = I_b + I_{\text{points}} = \frac{1}{12} m_b L^2 + \frac{1}{4}(m_1 + m_2) L^2.$$

5. Adding the Control Force Torque

To include the contribution of a **control force** F generated by a **drone motor fixed vertically** on one end of the beam (e.g., at $x = \frac{L}{2}$), we must analyze the **torque generated** by that force around the pivot.
We start from the general expression in scalar 2D form $\tau = xF_y - yF_x$. Alternatively, torque magnitude in terms of the angle between the vectors is $\tau = r \cdot F \cdot \sin(\theta_{\text{rel}})$

where:

- $r = \|\mathbf{r}\| = \frac{L}{2}$ is the distance from the pivot to the point where the force is applied,
- $F = \|\mathbf{F}\|$,
- θ_{rel} is the angle between $\mathbf{r}$ and $\mathbf{F}$.

In our system:

- The motor is mounted **perpendicular to the beam** in the **vertical direction**, and thus, $\mathbf{F} = (0, F)$ in the beam's reference frame.
- The point of application is at $\mathbf{r} = (-\frac{L}{2}, 0)$, i.e., the left end of the beam.
- The angle between $\mathbf{r}$ and $\mathbf{F}$ is exactly 90°, so $\sin(\theta_{\text{rel}}) = 1$.
  Then:

$$\tau_{\text{control}} = \frac{L}{2} \cdot F \cdot \sin(90^\circ) = \frac{L}{2} \cdot F$$

  Dividing both sides by I, we define $\delta = L/(2I)$:

$$\tau_{\text{control}} = \delta F$$

6. Assembling the Equation

Writing Newton's second law with all the torques:

$$I\,\ddot{\theta} = -b\,\dot{\theta} - \mu_c\,\frac{g\,L}{2}\,\tanh\!\left(\frac{\dot{\theta}}{\epsilon}\right) + \frac{(m_1 - m_2)g\,L}{2}\,\cos\theta + \frac{L}{2}\,F.$$

Dividing through by I gives

$$\ddot{\theta} = -\frac{b}{I}\,\dot{\theta} - \frac{\mu_c\,g\,L}{2I}\,\tanh\!\left(\frac{\dot{\theta}}{\epsilon}\right) + \frac{(m_1 - m_2)g\,L}{2I}\,\cos\theta + \delta\,F.$$

With the constant definitions

$$\alpha = \frac{b}{I}, \quad \beta = \frac{\mu_c\,g\,L}{2I}, \quad \gamma = \frac{(m_1 - m_2)g\,L}{2I}, \quad \delta = \frac{L}{2I},$$

this becomes

$$\ddot{\theta} + \alpha\,\dot{\theta} + \beta\,\tanh\!\left(\frac{\dot{\theta}}{\epsilon}\right) = \gamma\,\cos\theta + \delta\,F.$$

7. State-Space Representation

To express the model in state-space form, define the state vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}.$$

Then, the state equations are

$$\boxed{\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\alpha\, x_2 - \beta\, \tanh\!\left(\frac{x_2}{\epsilon}\right) + \gamma\, \cos x_1 + \delta\, F.\end{aligned}}$$

with the constants defined as:

- $\alpha = \frac{b}{I}$,
- $\beta = \frac{\mu_c\, g\, L}{2I}$,
- $\gamma = \frac{(m_1 - m_2) g\, L}{2I}$,
- $\delta = \frac{L}{2I}$.

Linearization

Consider a nonlinear system represented by:

$$\dot{x} = f(x, u, t), \quad y(t) = g(x, u, t)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. To design controllers or analyze stability, we often linearize the system around an **equilibrium point** x_0, u_0.

1. Find the Equilibrium Point

Solve for x_0, u_0 such that:

$$f(x_0, u_0) = 0$$

This ensures the system is in steady state $\dot{x} = 0$ at the equilibrium.

2. Compute Jacobian Matrices

Approximate the system with **Jacobian matrices** evaluated at (x_0, u_0):

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c|c} \frac{\partial f}{\partial x} & nbsp; & \frac{\partial f}{\partial u} \\ \hline \frac{\partial g}{\partial x} & \frac{\partial g}{\partial u} & nbsp; \end{array}\right]_{(x_0, u_0)}$$

3. Form the Linearized System

The linearized state-space model is:

$$\begin{aligned} \delta\dot{x} &= A\delta x + B\delta u \\ \delta y &= C\delta x + D\delta u \end{aligned}$$

where:

- $\delta x = x - x_0$ (perturbation in state),
- $\delta u = u - u_0$ (perturbation in input),
- $\delta y = y - g(x_0, u_0)$ (perturbation in output).

1. Equilibrium Points

An equilibrium point of the system is defined as a state where the derivatives of the state variables vanish. That is, when

$$\dot{x}_1 = 0 \quad \text{and} \quad \dot{x}_2 = 0.$$

From the state-space equations, this implies:
For the first state:

$$\dot{x}_1 = x_2 = 0.$$

For the second state:

$$\dot{x}_2 = -\alpha\, x_2 - \beta\, \tanh\Big(\frac{x_2}{\epsilon}\Big) + \gamma\, \cos x_1 + \delta\, F = 0.$$

If no external control is applied, i.e., $F = 0$, and with $x_2 = 0$, the second equation reduces to

$$\gamma\, \cos x_1 = 0.$$

Thus, the equilibrium condition for x_1 becomes

$$\cos x_1 = 0 \quad \Longrightarrow \quad x_1 = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

The bar in vertical orientation corresponds to $(x_1, x_2) = (\pi/2, 0)$.

Alternatively, if a constant control input $F = F_e$ is used, the equilibrium condition for x_1 becomes

$$\gamma\, \cos x_1 + \delta\, F_e = 0 \quad \Longrightarrow \quad x_1 = \cos^{-1}(\delta F_e/)$$

By appropriately choosing F_e, one may shift the equilibrium to a desired configuration (for instance, at $(0, 0)$).

2. First-Order Perturbation and Linearization

To analyze the dynamics around an equilibrium point $(x_{1,e}, x_{2,e}, F_e)$, we define small perturbations around this point. We write

$$\begin{aligned} x_1 &= x_{1,e} + \Delta x_1, \\ x_2 &= x_{2,e} + \Delta x_2, \\ Fnbsp; &= F_e + \Delta F. \end{aligned}$$

Inserting these into the original nonlinear state equations and expanding in a Taylor series, we keep only the first-order (linear) terms in the perturbations. This process is known as *first-order linearization*.

The nonlinear state equations are given by

$$\mathbf{f}(x_1, x_2, F) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2, F) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\alpha\, x_2 - \beta\, \tanh\big(\frac{x_2}{\epsilon}\big) + \gamma\, \cos x_1 + \delta\, F \end{pmatrix}.$$

Performing a Taylor series expansion around the equilibrium yields

$$\Delta \dot{x} \approx \underbrace{\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(x_{1,e},x_{2,e})}}_{A} \Delta x + \underbrace{\left.\frac{\partial \mathbf{f}}{\partial F}\right|_{F_e}}_{B} \Delta F,$$

$$\boxed{A = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(x_{1,e},x_{2,e})} = \begin{pmatrix} 0 & 1 \\ -\gamma \cos x_{1,e} & -\alpha - \dfrac{\beta}{\epsilon}\operatorname{sech}^2\!\left(\dfrac{x_{2,e}}{\epsilon}\right) \end{pmatrix} \quad , \quad B = \left.\frac{\partial \mathbf{f}}{\partial F}\right|_{F_e} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}}$$

3. Shift of Initial Conditions

Since the linearization is performed about an equilibrium point $(x_{1,e}, x_{2,e}, F_e)$, the original initial conditions of the nonlinear system, are expressed as

$$\Delta x(0) = \begin{pmatrix} x_1(0) - x_{1,e} \\ x_2(0) - x_{2,e} \end{pmatrix}.$$

Summary

- The equilibrium points are determined by setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, leading to conditions such as $\cos x_1 = 0$ (or a modified version when an external control is present).
- The system is linearized by introducing perturbations around the equilibrium values, i.e., $x_1 = x_{1,e} + \Delta x_1$, $x_2 = x_{2,e} + \Delta x_2$, and $F = F_e + \Delta F$.
- A first-order Taylor expansion provides the linear approximation $\Delta \dot{x} \approx A\,\Delta x + B\,\Delta F$, where the Jacobian matrices A and B are obtained from the partial derivatives of the system's dynamics.
- The initial conditions of the original system are shifted by subtracting the equilibrium point, yielding the perturbations that act as the initial conditions for the linearized system.