Seesaw rotational dynamics

1. Newton’s Second Law for Rotation

For any rigid body rotating about a pivot, Newton’s second law takes the form

$$I\,\ddot{\theta} = \sum \tau,$$

where

- I is the total moment of inertia,
- θ is the rotation angle, and
- $\sum \tau$ is the sum of the torques acting on the system.
  Express each term of the torque sum and then divide by I to isolate $\ddot{\theta}$.

2. Gravitational Torque via the 2D Cross Product

a. **The Setup**

Assume two point masses m_1 and m_2 are attached at the ends of a beam (or seesaw) of length L. If the pivot is at the beam’s center, each mass is located at a distance $r = L/2$ from the pivot.

b. **Using the Cross Product**

The gravitational force on each mass is $\mathbf{F}_g$ directed downward. The position vectors from the pivot to each mass are, for example, $\mathbf{r}_1$ and $\mathbf{r}_2$

$$\mathbf{F}_g = \begin{pmatrix} 0 \\ -m_i\,g \end{pmatrix} \quad , \quad \mathbf{r}_1 = \frac{L}{2}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad , \quad \mathbf{r}_2 = -\frac{L}{2}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

The choice of sign is made for convenience so that the gravitational term "supports" or "resists" depending on the difference in masses.
In two dimensions, the torque for each mass is given by the scalar (the “z-component” of the cross product):

$$\tau = \mathbf{r}\times\mathbf{F} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \times \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ F_x & F_y & 0 \end{vmatrix} = \mathbf{i}\cdot\begin{vmatrix} y & 0 \\ F_y & 0 \end{vmatrix} - \mathbf{j}\cdot\begin{vmatrix} x & 0 \\ F_x & 0 \end{vmatrix} + \mathbf{k}\cdot\begin{vmatrix} x & y \\ F_x & F_y \end{vmatrix} = xF_y - yF_x$$

For mass m_1 (located at $-\frac{L}{2}$ on one side)

$$\tau_1 = \mathbf{r}_1 \times \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix} = \left(\frac{L}{2}\cos\theta\right)(-m_1\, g) - \left(\frac{L}{2}\sin\theta\right)(0) = -\frac{L}{2}\, m_1\, g\, \cos\theta.$$

For mass m_2 (located on the opposite side)

$$\tau_2 = \mathbf{r}_2 \times \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix} = \left(-\frac{L}{2}\cos\theta\right)(-m_2\, g) - \left(-\frac{L}{2}\sin\theta\right)(0) = \frac{L}{2}\, m_2\, g\, \cos\theta.$$

c. **Net Gravitational Torque**

The net gravitational torque is then

$$\tau_g = \tau_1 + \tau_2 = \frac{(m_2 - m_1) g\, L}{2} \cos\theta,$$

Thus, when we divide by I (to write the angular acceleration), the gravitational term becomes

$$\tau_g = \gamma\, \cos\theta \quad , \quad \gamma = \frac{(m_2 - m_1) g\, L}{2\, I}.$$

3. Friction Torque

a. **Viscous Friction**

Viscous friction is modeled as being proportional to the angular velocity, where b is the viscous damping coefficient:

$$\tau_{\text{viscous}} = -b\, \dot{\theta},$$

b. **Coulomb Friction**

The Coulomb (dry) friction torque is given by

$$\tau_{\text{Coulomb}} = -\mu_c\, N\, \frac{L}{2}\, \operatorname{sgn}(\dot{\theta}),$$

where

- μ_c is the Coulomb friction coefficient,
- N is the normal force (which we approximate using g in our gravitational context), and
- $L/2$ is the effective lever arm.
  Because the sign function $\operatorname{sgn}(\dot{\theta})$ is discontinuous at zero, we use a smooth approximation:

  $$\operatorname{sgn}(\dot{\theta}) \approx \tanh\left(\frac{\dot{\theta}}{\epsilon}\right),$$

  This approximation facilitates numerical simulation (continuity and derivability), especially useful if it is then linearized. With ϵ as a small positive constant.

$$\tau_{\text{Coulomb}} \approx -\mu_c \frac{g\,L}{2} \tanh\left(\frac{\dot{\theta}}{\epsilon}\right).$$

c. **Total Friction Torque**

$$\tau_{\text{friction}} = \tau_{\text{viscous}} + \tau_{\text{Coulomb}} = -b\,\dot{\theta} - \mu_c \frac{g\,L}{2} \tanh\left(\frac{\dot{\theta}}{\epsilon}\right)$$

When divided by the total moment of inertia I, the friction terms in the angular acceleration become

$$\tau_{\text{friction}} = -\alpha\,\dot{\theta} - \beta \tanh\left(\frac{\dot{\theta}}{\epsilon}\right) \quad , \quad \alpha = \frac{b}{I}, \quad \beta = \frac{\mu_c\,g\,L}{2\,I}$$

4. Total Moment of Inertia

a. **Beam's Moment of Inertia**

For a uniform bar (beam) of mass m_b and length L rotating about its center, the moment of inertia is obtained by integrating:

1. **Density**: $\lambda = m_b/L$
2. **Elemental mass**: $dm = \lambda\,dx$
3. **Moment contribution**: $dI = x^2\,dm = x^2\,\lambda\,dx$
4. **Integrate**:

$$I_b = \int_{-L/2}^{L/2} x^2\,\lambda\,dx = \lambda \int_{-L/2}^{L/2} x^2\,dx = \frac{m_b}{L}\frac{L^3}{12} = \frac{1}{12}m_b L^2$$

b. **Point Masses' Moments of Inertia**

Each point mass m_i located at a distance $r = L/2$ from the pivot contributes

$$I_i = m_i\left(\frac{L}{2}\right)^2 = \frac{1}{4}m_i L^2.$$

Thus, for both masses:

$$I_{\text{points}} = \frac{1}{4}\,(m_1 + m_2)\,L^2.$$

c. **Total Moment of Inertia**

Adding the beam's and the point masses' moments of inertia gives

$$I = I_b + I_{\text{points}} = \frac{1}{12} m_b L^2 + \frac{1}{4}(m_1 + m_2)L^2.$$

5. Adding the Control Force Torque

To include the contribution of a **control force** F generated by a **drone motor fixed vertically** on one end of the beam (e.g., at $x = \frac{L}{2}$), we must analyze the **torque generated** by that force around the pivot.
We start from the general expression in scalar 2D form $\tau = xF_y - yF_x$. Alternatively, torque magnitude in terms of the angle between the vectors is $\tau = r \cdot F \cdot \sin(\theta_{\text{rel}})$

where:

- $r = \|\mathbf{r}\| = \frac{L}{2}$ is the distance from the pivot to the point where the force is applied,
- $F = \|\mathbf{F}\|$,
- θ_{rel} is the angle between $\mathbf{r}$ and $\mathbf{F}$.

In our system:

- The motor is mounted **perpendicular to the beam** in the **vertical direction**, and thus, $\mathbf{F} = (0, F)$ in the beam's reference frame.
- The point of application is at $\mathbf{r} = (-\frac{L}{2}, 0)$, i.e., the left end of the beam.
- The angle between $\mathbf{r}$ and $\mathbf{F}$ is exactly 90°, so $\sin(\theta_{\text{rel}}) = 1$.
  Then:

  $$\tau_{\text{control}} = \frac{L}{2} \cdot F \cdot \sin(90^\circ) = \frac{L}{2} \cdot F$$

  Dividing both sides by I, we define $\delta = L/(2I)$:

  $$\tau_{\text{control}} = \delta F$$

6. Assembling the Equation

Writing Newton's second law with all the torques:

$$I\,\ddot{\theta} = -b\,\dot{\theta} - \mu_c\,\frac{g\,L}{2}\,\tanh\!\left(\frac{\dot{\theta}}{\epsilon}\right) + \frac{(m_1 - m_2)g\,L}{2}\,\cos\theta + \frac{L}{2}\,F.$$

Dividing through by I gives

$$\ddot{\theta} = -\frac{b}{I}\,\dot{\theta} - \frac{\mu_c\,g\,L}{2I}\,\tanh\!\left(\frac{\dot{\theta}}{\epsilon}\right) + \frac{(m_1 - m_2)g\,L}{2I}\,\cos\theta + \delta\,F.$$

With the constant definitions

$$\alpha = \frac{b}{I}, \quad \beta = \frac{\mu_c\, g\, L}{2I}, \quad \gamma = \frac{(m_1 - m_2) g\, L}{2I}, \quad \delta = \frac{L}{2I},$$

this becomes

$$\ddot{\theta} + \alpha\,\dot{\theta} + \beta\,\tanh\left(\frac{\dot{\theta}}{\epsilon}\right) = \gamma\,\cos\theta + \delta\,F.$$

7. State-Space Representation

To express the model in state-space form, define the state vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}.$$

Then, the state equations are

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\alpha\,x_2 - \beta\,\tanh\left(\frac{x_2}{\epsilon}\right) + \gamma\,\cos x_1 + \delta\,F. \end{aligned}}$$

As $\epsilon \to 0$, $\tanh(x_2/\epsilon) \to \mathrm{sign}(x_2)$ but this function is undefined at zero

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\alpha\,x_2 - \beta\,\mathrm{sign}\,(x_2) + \gamma\,\cos x_1 + \delta\,F. \end{aligned}}$$

with the constants defined as:

- $\alpha = \frac{b}{I}$,
- $\beta = \frac{\mu_c\, g\, L}{2I}$,
- $\gamma = \frac{(m_1 - m_2) g\, L}{2I}$,
- $\delta = \frac{L}{2I}$.

Linear Approximation of Dynamical Systems with Inputs and Outputs

System Description

Consider a dynamical system with **state** $x \in \mathbb{R}^n$, **input** $u \in \mathbb{R}^m$, and **output** $y \in \mathbb{R}^p$:

$$\dot{x} = f(x, u) \quad , \quad y = g(x, u, t),$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$ are smooth functions.

Equilibrium Points

An equilibrium point $(x_{\text{eq}}, u_{\text{eq}})$ satisfies:

$$f(x_{\text{eq}}, u_{\text{eq}}) = 0.$$

At equilibrium, the state remains stationary if undisturbed, and u_{eq} is the steady-state input.

Multivariable Taylor Series Linearization

State Dynamics Linearization: Expand $f(x, u)$ around $(x_{\text{eq}}, u_{\text{eq}})$ using a first-order Taylor series:

$$f(x, u) \approx f(x_{\text{eq}}, u_{\text{eq}}) + \left.\frac{\partial f}{\partial x}\right|_{\text{eq}} (x - x_{\text{eq}}) + \left.\frac{\partial f}{\partial u}\right|_{\text{eq}} (u - u_{\text{eq}}).$$

Since $f(x_{\text{eq}}, u_{\text{eq}}) = 0$, the linearized dynamics become:

$$\dot{x} \approx \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\text{eq}}}_{A} (x - x_{\text{eq}}) + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{\text{eq}}}_{B} (u - u_{\text{eq}}).$$

Output Equation Linearization: Similarly, linearize $y = g(x, u, t)$ around $(x_{\text{eq}}, u_{\text{eq}})$:

$$y \approx g(x_{\text{eq}}, u_{\text{eq}}, t) + \underbrace{\left.\frac{\partial g}{\partial x}\right|_{\text{eq}}}_{C} (x - x_{\text{eq}}) + \underbrace{\left.\frac{\partial g}{\partial u}\right|_{\text{eq}}}_{D} (u - u_{\text{eq}}).$$

Linearized System Representation

Perturbation Variables: Define deviations from equilibrium, where $y_{\text{eq}} = g(x_{\text{eq}}, u_{\text{eq}}, t)$ is the equilibrium output:

$$\begin{aligned} z &= x - x_{\text{eq}}, \\ v &= u - u_{\text{eq}}, \\ \Delta y &= y - y_{\text{eq}}, \end{aligned}$$

Linear State-Space Model: The linearized system becomes:

$$\begin{aligned} \dot{z} &= Az + Bv, \\ \Delta y &= Cz + Dv. \end{aligned}$$

Stability Analysis

- **Stability** depends on the eigenvalues of $A = \left.\frac{\partial f}{\partial x}\right|_{\text{eq}}$:
  - **Stable**: All eigenvalues of A have $\mathrm{Re}(\lambda_i) < 0$.
  - **Unstable**: Any eigenvalue has $\mathrm{Re}(\lambda_i) > 0$.
  - **Marginal**: Eigenvalues with $\mathrm{Re}(\lambda_i) = 0$ (nonlinear terms dominate).

Solution of the Linearized System

1. **The initial condition** for the linearized system z_0 is defined as the deviation of the original system's initial state x_0 at t_0 from equilibrium x_{eq}:

$$z_0 = x_0 - x_{\text{eq}},$$

2. **State Trajectory**:

$$z(t) = e^{At} z(0) + \int_0^t e^{A(t-\tau)} B v(\tau) d\tau \quad , \quad x(t) = x_{\text{eq}} + z(t)$$

3. **Output Response**:

$$\Delta y(t) = Cz(t) + Dv(t) \quad , \quad y(t) = y_{\text{eq}} + \Delta y(t)$$

Equilibrium and Linearization Analysis

1. System Dynamics

The state-space model is:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha\, x_2 - \beta\, \tanh\left(\frac{x_2}{\epsilon}\right) + \gamma\, \cos x_1 + \delta\, F \end{aligned}$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$, and the constants are:

- $\alpha = b/I$ (viscous damping)
- $\beta = \mu_c g L/(2I)$ (Coulomb friction magnitude)
- $\gamma = (m_2 - m_1) g L/(2I)$ (gravitational torque factor)
- $\delta = L/(2I)$ (control input gain)

2. Equilibrium Points

Equilibrium points ($\mathbf{x}_{eq}$) occur when $\dot{\mathbf{x}} = \mathbf{0}$.

1. $\dot{x}_1 = x_2 = 0 \implies \dot{\theta}_{eq} = 0$.
2. Substituting $x_2 = 0$ into $\dot{x}_2$:
   $0 = -0 - \beta \tanh(0) + \gamma \cos x_1 + \delta F$

$$\gamma \cos \theta_{eq} + \delta F = 0 \quad \implies \quad \cos \theta_{eq} = -\frac{\delta F}{\gamma}$$

- **Case 1: No Control ($F = 0$)**
  $\cos\theta_{eq} = 0 \implies \theta_{eq} = \frac{\pi}{2} + n\pi$, for any integer n.
  The distinct equilibrium points are $(\theta_{eq}, \dot{\theta}_{eq}) = (\pm\frac{\pi}{2}, 0)$. These correspond to the vertical positions.
- **Case 2: With Control ($F \neq 0$)**
  Equilibrium points exist if $|\frac{\delta F}{\gamma}| \leq 1$.
  If so, $\theta_{eq} = \pm\arccos(-\frac{\delta F}{\gamma}) + 2k\pi$.

3. Linearization Analysis

1. Non-linear System Equations

Let the state vector be $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$.
The system can be written as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, F)$, where

$$\mathbf{f}(\mathbf{x}, F) = \begin{pmatrix} x_2 \\ -\alpha\, x_2 - \beta\, \tanh\left(\frac{x_2}{\epsilon}\right) + \gamma\, \cos x_1 + \delta\, F \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{x}, F) \\ f_2(\mathbf{x}, F) \end{pmatrix}$$

Note on $\tanh$: Using $\tanh(x_2/\epsilon)$ instead of $\operatorname{sgn}(x_2)$ makes the function differentiable at $x_2 = 0$, allowing for straightforward Jacobian calculation at equilibrium.

3. Compute Jacobians

We linearize $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, F)$ around an operating point $(\mathbf{x}_{op}, F_{op})$

Matrix A (Jacobian):

$$A = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

The partial derivatives are:

- $\frac{\partial f_1}{\partial x_1} = \frac{\partial(x_2)}{\partial x_1} = 0$
- $\frac{\partial f_1}{\partial x_2} = \frac{\partial(x_2)}{\partial x_2} = 1$
- $\frac{\partial f_2}{\partial x_1} = \frac{\partial}{\partial x_1}(-\alpha x_2 - \beta\tanh(\frac{x_2}{\epsilon}) + \gamma\cos x_1 + \delta F) = -\gamma\sin x_1$
- $\frac{\partial f_2}{\partial x_2} = \frac{\partial}{\partial x_2}(-\alpha x_2 - \beta\tanh(\frac{x_2}{\epsilon}) + \gamma\cos x_1 + \delta F) = -\alpha - \beta\frac{1}{\epsilon}\operatorname{sech}^2(\frac{x_2}{\epsilon})$

Evaluate at the equilibrium point $(\mathbf{x}_{eq}, F_{eq}) = (\theta_{eq}, 0, F_{eq})$:

- $\left.\frac{\partial f_1}{\partial x_1}\right|_{eq} = 0$
- $\left.\frac{\partial f_1}{\partial x_2}\right|_{eq} = 1$

- $\left.\frac{\partial f_2}{\partial x_1}\right|_{eq} = -\gamma \sin\theta_{eq}$
- $\left.\frac{\partial f_2}{\partial x_2}\right|_{eq} = -\alpha - \frac{\beta}{\epsilon}\text{sech}^2(0) = -\alpha - \frac{\beta}{\epsilon}$

So, the Jacobian matrix at equilibrium is:

$$A = \begin{pmatrix} 0 & 1 \\ -\gamma \sin\theta_{eq} & -\alpha - \beta/\epsilon \end{pmatrix}$$

Matrix B (Input Matrix):

$$B = \frac{\partial \mathbf{f}}{\partial F} = \begin{pmatrix} \frac{\partial f_1}{\partial F} \\ \frac{\partial f_2}{\partial F} \end{pmatrix}$$

- $\frac{\partial f_1}{\partial F} = \frac{\partial (x_2)}{\partial F} = 0$
- $\frac{\partial f_2}{\partial F} = \frac{\partial}{\partial F}(-\alpha x_2 - \beta \tanh(\frac{x_2}{\epsilon}) + \gamma \cos x_1 + \delta F) = \delta$

Evaluating at equilibrium (or anywhere, as it's constant):

$$B = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

The Linearized System

The initial conditions for the linearized system are $\mathbf{z}(0)$ and $\mathbf{x}(0)$ are the initial conditions of the non linear system

$$\mathbf{z}(0) = \mathbf{x}(0) - \mathbf{x}_{eq}$$

The real input is $\Delta F(t) = F(t) - F_{\text{eq}}$

$$\boxed{\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\gamma \sin\theta_{eq} & -\alpha - \beta/\epsilon \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \delta \end{pmatrix} \Delta F}$$

The restore the original coordinates use

$$\mathbf{x}(t) = \mathbf{x}_{eq} + \mathbf{z}(t)$$

This represents the linearized dynamics of the deviations ($z_1 = \Delta\theta$, $z_2 = \Delta\dot{\theta}$) from the equilibrium point $(\theta_{eq}, 0)$ due to input deviations ΔF.

Analysis at Specific Points (assuming $F = 0$)

- **Around $\theta = 0$ (Horizontal):**
  - This is **not an equilibrium point** unless $\gamma = 0$ ($m_1 = m_2$).

- Linearizing here yields $A = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha - \beta/\epsilon \end{pmatrix}$.
  - Linearized equation: $\Delta\ddot{\theta} + (\alpha + \beta/\epsilon)\Delta\dot{\theta} = \delta\Delta F$.
  - **Issue:** The restoring force term dependent on $\Delta\theta$ (from gravity) vanishes ($-\gamma\sin(0) = 0$). The linearization **loses the essential angular dynamics** and doesn't capture the tendency to fall away from horizontal if $m_1 \neq m_2$.
- **Around $\theta_{eq} = \pi/2$ (Vertical):**
  - $A = \begin{pmatrix} 0 & 1 \\ -\gamma & -\alpha - \beta/\epsilon \end{pmatrix}$.
  - Linearized equation: $\Delta\ddot{\theta} + (\alpha + \beta/\epsilon)\Delta\dot{\theta} + \gamma\Delta\theta = \delta\Delta F$.
  - **Usefulness:** This is a standard 2nd order system. It captures the local dynamics. If $\gamma > 0$ (i.e., $m_2 > m_1$), the $+\gamma\Delta\theta$ term represents a **restoring torque**, making this equilibrium **stable** (assuming damping $\alpha, \beta > 0$).
- **Around $\theta_{eq} = -\pi/2$ (Vertical):**
  - $A = \begin{pmatrix} 0 & 1 \\ \gamma & -\alpha - \beta/\epsilon \end{pmatrix}$.
  - Linearized equation: $\Delta\ddot{\theta} + (\alpha + \beta/\epsilon)\Delta\dot{\theta} - \gamma\Delta\theta = \delta\Delta F$.
  - **Instability:** If $\gamma > 0$ ($m_2 > m_1$), the $-\gamma\Delta\theta$ term represents an **anti-restoring torque**, pushing the system away from equilibrium. The linearized system has a positive eigenvalue, indicating **instability**, matching simulation observations.

4. Stability vs. n and γ

The equilibrium angles are $\theta_{eq} = \frac{\pi}{2} + n\pi$. Stability depends on n (even/odd) and the sign of $\gamma = (m_2 - m_1)gL/(2I)$.

- **If n is even** (e.g., -2, 0, 2,...): $\theta_{eq} \equiv \pi/2$.
  - Linearized term: $+\gamma\Delta\theta$.
  - **Stable** if $\gamma > 0$ ($m_2 > m_1$, heavier mass m_2 is down).
  - **Unstable** if $\gamma < 0$ ($m_1 > m_2$, heavier mass m_1 is up).
- **If n is odd** (e.g., -1, 1, 3,...): $\theta_{eq} \equiv -\pi/2$.
  - Linearized term: $-\gamma\Delta\theta$.
  - **Stable** if $\gamma < 0$ ($m_1 > m_2$, heavier mass m_1 is down).
  - **Unstable** if $\gamma > 0$ ($m_2 > m_1$, heavier mass m_2 is up).

Conclusion on negative n:
The stability for $\theta_{eq} = \pi/2 + n\pi$ with *negative* n depends on whether n is even or odd:

- If n is negative **odd** ($\equiv -\pi/2$): Unstable if $\gamma > 0$, Stable if $\gamma < 0$.
- If n is negative **even** ($\equiv \pi/2$): Stable if $\gamma > 0$, Unstable if $\gamma < 0$.

Swapping the masses ($m_1 \leftrightarrow m_2$) flips the sign of γ, thus **reversing the stability** of both vertical equilibrium configurations ($\pi/2$ and $-\pi/2$).

Lyapunov Stability Analysis for Linearized System

1. Linear System Model

The system under consideration is the linearized dynamics of a seesaw around the vertical equilibrium point $\theta_{\text{eq}} = \frac{\pi}{2}$. This equilibrium is assumed to be stable in the absence of damping and friction, requiring $m_2 > m_1$. Equivalent to $\theta_{\text{eq}} = -\frac{\pi}{2}$ for $m_1 > m_2$. The linearized dynamics are described by the linear time-invariant (LTI) system:

$$\dot{\mathbf{z}} = A\mathbf{z},$$

where $\mathbf{z}$ is the state vector representing the deviation from the equilibrium and the angular velocity:

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \theta - \theta_{\text{eq}} \\ \dot{\theta} \end{bmatrix}.$$

The system matrix A is given by:

$$A = \begin{pmatrix} 0 & 1 \\ -\gamma & -\alpha - \frac{\beta}{\epsilon} \end{pmatrix},$$

where the parameters are defined in terms of the physical properties of the seesaw (m_1, m_2 are masses, g is gravity, L is half the seesaw length, I is the moment of inertia, b is viscous damping coefficient, μ_c is Coulomb friction magnitude, and ϵ is a small positive constant used to handle the sign function in the original nonlinear model):

$$\gamma = \frac{(m_2 - m_1)gL}{2I}, \quad \alpha = \frac{b}{I}, \quad \beta = \frac{\mu_c gL}{2I}, \quad \epsilon > 0.$$

For the equilibrium $\theta_{\text{eq}} = \pi/2$ to be a point we can analyze for stability in this linearized model, we typically need $\gamma > 0$, which corresponds to $m_2 > m_1$. We also assume positive viscous damping ($b > 0$, so $\alpha > 0$) and non-zero Coulomb friction effects ($\mu_c > 0$, so $\beta > 0$). The moment of inertia I is always positive.

2. Lyapunov Function Candidate

To analyze the stability of the equilibrium point $\mathbf{z} = \mathbf{0}$ using Lyapunov's Direct Method, we choose a quadratic Lyapunov function candidate:

$$V(\mathbf{z}) = \mathbf{z}^T P\mathbf{z},$$

where P is a symmetric, positive definite matrix ($P = P^T \succ 0$). For a 2×2 system, the matrix P has the form:

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}.$$

For P to be positive definite, its leading principal minors must be positive: $p_{11} > 0$ and $\det(P) = p_{11}p_{22} - p_{12}^2 > 0$. The positive definiteness of P ensures that $V(\mathbf{z}) > 0$ for all $\mathbf{z} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$.

3. Solving the Lyapunov Equation

For a linear system $\dot{\mathbf{z}} = A\mathbf{z}$, asymptotic stability at the origin is proven if there exists a positive definite matrix P such that the time derivative of the Lyapunov function, $\dot{V}(\mathbf{z})$, is negative definite. The derivative of $V(\mathbf{z})$ along the system trajectories is given by:

$$\dot{V}(\mathbf{z}) = \frac{d}{dt}(\mathbf{z}^T P \mathbf{z}) = \dot{\mathbf{z}}^T P \mathbf{z} + \mathbf{z}^T P \dot{\mathbf{z}}.$$

Substituting $\dot{\mathbf{z}} = A\mathbf{z}$:

$$\dot{V}(\mathbf{z}) = (A\mathbf{z})^T P \mathbf{z} + \mathbf{z}^T P (A\mathbf{z}) = \mathbf{z}^T A^T P \mathbf{z} + \mathbf{z}^T P A \mathbf{z} = \mathbf{z}^T (A^T P + PA)\mathbf{z}.$$

To ensure $\dot{V}(\mathbf{z})$ is negative definite, we can choose a positive definite matrix Q and solve the algebraic **Lyapunov equation**:

$$A^T P + PA = -Q.$$

A standard approach is to choose Q as the identity matrix, $Q = I$. Substituting the matrix A, the general form of P, and $Q = I$:

$$\begin{pmatrix} 0 & -\gamma \\ 1 & -\left(\alpha + \frac{\beta}{\epsilon}\right) \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\gamma & -\left(\alpha + \frac{\beta}{\epsilon}\right) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Performing the matrix multiplications:

$$\begin{pmatrix} -\gamma p_{12} & -\gamma p_{22} \\ p_{11} - \left(\alpha + \frac{\beta}{\epsilon}\right) p_{12} & p_{12} - \left(\alpha + \frac{\beta}{\epsilon}\right) p_{22} \end{pmatrix} + \begin{pmatrix} -\gamma p_{12} & p_{11} - \left(\alpha + \frac{\beta}{\epsilon}\right) p_{12} \\ -\gamma p_{22} & p_{12} - \left(\alpha + \frac{\beta}{\epsilon}\right) p_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Adding the matrices on the left-hand side and equating corresponding elements to the matrix on the right-hand side results in the following system of linear equations for the elements of P:

$$\begin{pmatrix} -2\gamma p_{12} & p_{11} - \gamma p_{22} - \left(\alpha + \frac{\beta}{\epsilon}\right) p_{12} \\ p_{11} - \gamma p_{22} - \left(\alpha + \frac{\beta}{\epsilon}\right) p_{12} & 2\left(p_{12} - \left(\alpha + \frac{\beta}{\epsilon}\right) p_{22}\right) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This matrix equation gives us three distinct equations (due to the symmetry of P and $-Q$):

1. $-2\gamma p_{12} = -1$
2. $p_{11} - \gamma p_{22} - \left(\alpha + \frac{\beta}{\epsilon}\right)p_{12} = 0$
3. $2\left(p_{12} - \left(\alpha + \frac{\beta}{\epsilon}\right)p_{22}\right) = -1$

From equation (1), assuming $\gamma \neq 0$, we can solve for p_{12}:

$$p_{12} = \frac{1}{2\gamma}.$$

Since we assume $\gamma > 0$ for stability at $\theta_{eq} = \pi/2$, p_{12} is a well-defined positive value.

From equation (3):

$$p_{12} - \left(\alpha + \frac{\beta}{\epsilon}\right)p_{22} = -\frac{1}{2}.$$

Substitute the value of p_{12}:

$$\frac{1}{2\gamma} - \left(\alpha + \frac{\beta}{\epsilon}\right)p_{22} = -\frac{1}{2}.$$

Rearranging and solving for p_{22}:

$$\left(\alpha + \frac{\beta}{\epsilon}\right)p_{22} = \frac{1}{2\gamma} + \frac{1}{2} = \frac{1+\gamma}{2\gamma}.$$

Since $\alpha > 0$, $\beta > 0$, and $\epsilon > 0$, the term $\left(\alpha + \frac{\beta}{\epsilon}\right) > 0$. As $\gamma > 0$, the right-hand side is also positive. Thus, p_{22} is positive:

$$p_{22} = \frac{1+\gamma}{2\gamma\left(\alpha + \frac{\beta}{\epsilon}\right)}.$$

From equation (2), we can solve for p_{11}:

$$p_{11} = \gamma p_{22} + \left(\alpha + \frac{\beta}{\epsilon}\right)p_{12}.$$

Substitute the derived values of p_{12} and p_{22}:

$$p_{11} = \gamma\left(\frac{1+\gamma}{2\gamma\left(\alpha + \frac{\beta}{\epsilon}\right)}\right) + \left(\alpha + \frac{\beta}{\epsilon}\right)\left(\frac{1}{2\gamma}\right).$$

Simplifying the expression for p_{11}:

$$p_{11} = \frac{1+\gamma}{2\left(\alpha+\frac{\beta}{\epsilon}\right)} + \frac{\alpha+\frac{\beta}{\epsilon}}{2\gamma}.$$

Given that $\gamma > 0$, $\alpha > 0$, $\beta > 0$, and $\epsilon > 0$, all terms in the expression for p_{11} are positive, which means $p_{11} > 0$.

Thus, we have found the unique symmetric matrix P that solves the Lyapunov equation $A^T P + PA = -I$:

$$P = \begin{pmatrix} \frac{1+\gamma}{2\left(\alpha+\frac{\beta}{\epsilon}\right)} + \frac{\alpha+\frac{\beta}{\epsilon}}{2\gamma} & \frac{1}{2\gamma} \\ \frac{1}{2\gamma} & \frac{1+\gamma}{2\gamma\left(\alpha+\frac{\beta}{\epsilon}\right)} \end{pmatrix}.$$

4. Positive Definiteness of P

For P to be a valid matrix for constructing a Lyapunov function, it must be positive definite. For a symmetric 2×2 matrix, the conditions for positive definiteness are that the leading principal minors are positive: $p_{11} > 0$ and $\det(P) > 0$.

We have already shown that $p_{11} = \frac{1+\gamma}{2\left(\alpha+\frac{\beta}{\epsilon}\right)} + \frac{\alpha+\frac{\beta}{\epsilon}}{2\gamma}$. Since $\gamma > 0$, $\alpha > 0$, $\beta > 0$, and $\epsilon > 0$, all terms are positive, so $p_{11} > 0$.

Now, we compute the determinant of P:

$$\det(P) = p_{11}p_{22} - p_{12}^2.$$

Substituting the derived values of p_{11}, p_{22}, and p_{12}:

$$\det(P) = \left(\frac{1+\gamma}{2\left(\alpha+\frac{\beta}{\epsilon}\right)} + \frac{\alpha+\frac{\beta}{\epsilon}}{2\gamma}\right)\left(\frac{1+\gamma}{2\gamma\left(\alpha+\frac{\beta}{\epsilon}\right)}\right) - \left(\frac{1}{2\gamma}\right)^2.$$

Expanding the first term:

$$\det(P) = \frac{(1+\gamma)^2}{4\gamma\left(\alpha+\frac{\beta}{\epsilon}\right)^2} + \frac{\left(\alpha+\frac{\beta}{\epsilon}\right)(1+\gamma)}{4\gamma^2\left(\alpha+\frac{\beta}{\epsilon}\right)} - \frac{1}{4\gamma^2}.$$

Simplifying the second term:

$$\det(P) = \frac{(1+\gamma)^2}{4\gamma\left(\alpha+\frac{\beta}{\epsilon}\right)^2} + \frac{1+\gamma}{4\gamma^2} - \frac{1}{4\gamma^2}.$$

$$\det(P) = \frac{(1+\gamma)^2}{4\gamma\left(\alpha + \frac{\beta}{\epsilon}\right)^2} + \frac{1+\gamma-1}{4\gamma^2}.$$

$$\det(P) = \frac{(1+\gamma)^2}{4\gamma\left(\alpha + \frac{\beta}{\epsilon}\right)^2} + \frac{\gamma}{4\gamma^2}.$$

Since $\gamma \neq 0$, we can simplify the second term:

$$\det(P) = \frac{(1+\gamma)^2}{4\gamma\left(\alpha + \frac{\beta}{\epsilon}\right)^2} + \frac{1}{4\gamma}.$$

Given that $\gamma > 0$, $\alpha > 0$, $\beta > 0$, and $\epsilon > 0$, both terms in the expression for $\det(P)$ are positive. Therefore, $\det(P) > 0$.

Since $p_{11} > 0$ and $\det(P) > 0$, the matrix P obtained by solving the Lyapunov equation with $Q = I$ is indeed positive definite.

5. Time Derivative of the Lyapunov Function

By solving the Lyapunov equation $A^T P + PA = -Q$ with $Q = I$, the time derivative of the Lyapunov function $V(\mathbf{z})$ along the trajectories of the system $\dot{\mathbf{z}} = A\mathbf{z}$ is directly given by:

$$\dot{V}(\mathbf{z}) = \mathbf{z}^T(A^T P + PA)\mathbf{z} = \mathbf{z}^T(-Q)\mathbf{z}.$$

With $Q = I$, we have:

$$\dot{V}(\mathbf{z}) = -\mathbf{z}^T I\mathbf{z} = -\|\mathbf{z}\|^2.$$

The term $\|\mathbf{z}\|^2$ represents the squared Euclidean norm of the state vector $\mathbf{z}$. It is positive for any $\mathbf{z} \neq \mathbf{0}$ and zero only when $\mathbf{z} = \mathbf{0}$. Therefore, $\dot{V}(\mathbf{z})$ is negative definite: $\dot{V}(\mathbf{z}) < 0$ for all $\mathbf{z} \neq \mathbf{0}$ and $\dot{V}(\mathbf{0}) = 0$.

6. Conclusion

We have successfully applied Lyapunov's Direct Method to analyze the stability of the equilibrium point $\mathbf{z} = \mathbf{0}$ for the linearized seesaw system $\dot{\mathbf{z}} = A\mathbf{z}$. We found a quadratic Lyapunov function candidate $V(\mathbf{z}) = \mathbf{z}^T P\mathbf{z}$ by solving the Lyapunov equation $A^T P + PA = -I$ for the matrix P.

The matrix P was found to be positive definite under the physical assumptions that lead to $\gamma > 0$, $\alpha > 0$, and $\beta > 0$ (i.e., $m_2 > m_1$, positive viscous damping, and non-zero Coulomb friction effects). This ensures that $V(\mathbf{z})$ is positive definite ($V(\mathbf{z}) > 0$ for $\mathbf{z} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$). Furthermore, quadratic forms with a positive definite matrix P are radially unbounded, meaning $V(\mathbf{z}) \to \infty$ as $\|\mathbf{z}\| \to \infty$.

The time derivative of $V(\mathbf{z})$ along the system trajectories was found to be $\dot{V}(\mathbf{z}) = -\|\mathbf{z}\|^2$, which is negative definite ($\dot{V}(\mathbf{z}) < 0$ for $\mathbf{z} \neq \mathbf{0}$ and $\dot{V}(\mathbf{0}) = 0$).

According to Lyapunov's Direct Method, if there exists a positive definite, radially unbounded Lyapunov function whose derivative is negative definite, then the equilibrium point is globally asymptotically stable. In this case, the equilibrium $\mathbf{z} = \mathbf{0}$ of the linearized system is globally asymptotically stable.

The global asymptotic stability of the linearized system at $\mathbf{z} = \mathbf{0}$ implies that the corresponding equilibrium point of the original nonlinear system, $\theta_{\text{eq}} = \frac{\pi}{2}$, is locally asymptotically stable.