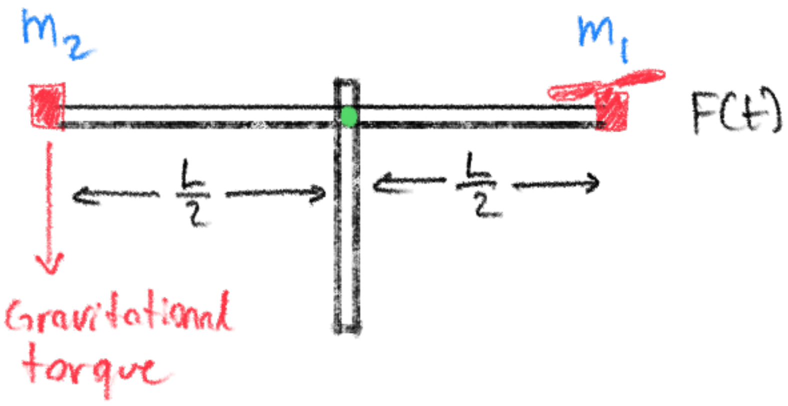
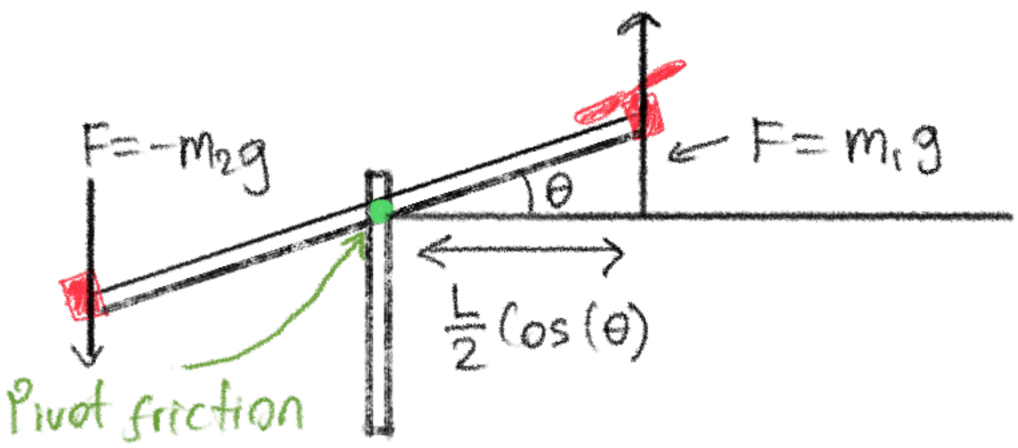
Seesaw rotational dynamics





Gravitational torque

$$\tau_1 = r \times F = [x, y, 0] \times [F_x, F_y, 0]$$
$$= [0, 0, xF_y - yF_x] = \frac{L}{2}\cos(\theta) m_1 g$$

$$\tau_2 = \frac{L}{2}\cos(\theta)(-m_2 g)$$

Total gravitational torque:

$$\boxed{\tau_g = (m_1 - m_2) g \frac{L}{2}\cos(\theta)}$$

Pivot friction

Coulomb friction

$$\tau = \mu_c \cdot F_{normal} \cdot \mathrm{Sign}(V)$$

$$\tau_{f_1} = \mu_c (m_1 + m_2) g \cdot \mathrm{Sign}(\dot{\theta})$$
$$\approx \mu_c (m_1 + m_2) g \tanh\left(\frac{\dot{\theta}}{\varepsilon}\right), \quad \varepsilon > 0$$

$$\tau = bV$$ Viscous friction

$$\tau_{f_2} = b\dot{\theta}$$

$$\tau_f = \tau_{f_1} + \tau_{f_2}$$ Total pivot friction

$$\boxed{\tau_f = -\mu_c (m_1 + m_2) g \tanh\left(\frac{\dot{\theta}}{\varepsilon}\right) - b\dot{\theta}}$$

Motor torque: $\tau_F = \frac{L}{2} F(t)$

Total moment of inertia

$$I = m_1\left(\frac{L}{2}\right)^2 + m_2\left(\frac{L}{2}\right)^2 = \frac{L^2}{4}(m_1 + m_2)$$

Newtons second Law for rotation

$$I\ddot{\theta} = \sum_i \tau_i$$

$$I\ddot{\theta} = \tau_g + \tau_f + \tau_F$$

$$\frac{L^2}{4}(m_1 + m_2)\ddot{\theta} = \frac{L}{2}\left[(m_1 - m_2) g \cos(\theta) + F(t)\right] - \mu_c (m_1 + m_2) g \tanh\left(\frac{\dot{\theta}}{\varepsilon}\right) - b\dot{\theta}$$

$$\ddot{\theta} = -\underbrace{\frac{4b}{L^2(m_1 + m_2)}}_{\alpha}\dot{\theta} - \underbrace{\frac{4\mu_c g}{L}}_{\beta}\tanh\left(\frac{\dot{\theta}}{\varepsilon}\right) + \underbrace{\frac{2(m_1 - m_2) g}{L(m_1 + m_2)}}_{\gamma}\cos(\theta) + \underbrace{\frac{2}{L(m_1 + m_2)}}_{P} F(t)$$

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha x_2 - \beta \tanh\left(\frac{x_2}{\varepsilon}\right) + \gamma \cos(x_1) + P F(t) \end{aligned}}$$

Linearization

Linearization

Point $x_1 = 0,\ x_2 = 0 \qquad F = 0$

$$\dot{x} = f(x, u) \qquad A = \left.\frac{\partial f}{\partial x}\right|_{(x_0, u_0)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_0, u_0)}$$

$$y = h(x, u) \qquad C = \left.\frac{\partial h}{\partial x}\right|_{(x_0, u_0)} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_0, u_0)}$$

$$\delta\dot{x} = A\,\delta x + B\,\delta u$$

$$\delta y = C\,\delta x + D\,\delta u$$

$$\delta x = x - x_0$$

$$\delta u = u - u_0$$

$$f_1(x_1, x_2) = x_2$$

$$f_2(x_1, x_2, F) = -\alpha x_2 - \beta \tanh\left(\frac{x_2}{\varepsilon}\right) + \gamma \cos(x_1) + \rho F(t)$$

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{x_1=0} = 0 \qquad \left.\frac{\partial f_1}{\partial x_2}\right|_{x_2=0} = 1 \qquad \left.\frac{\partial f_2}{\partial F}\right|_{F=0} = \rho$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{x_1=0} = -\gamma\,\mathrm{sen}(x_1)\Big|_{x_1=0} = 0 \qquad \left.\frac{\partial f_2}{\partial x_2}\right|_{x_2=0} = -\alpha - \left.\frac{\mathrm{sech}^2(x_2)}{\varepsilon}\right|_{x_2=0} = -\alpha - \frac{1}{\varepsilon}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha - \frac{1}{\varepsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} F(t) \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$