Week 1 Number Bases Lecture Note

Notebook:	Computational Mathematics		
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Cornell Notes	**Topic:** Number Bases	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: April 21, 2020

Essential Question:

What is a number base and what are the different number systems available for use in computing?

Questions/Cues:

- What does digit mean?
- What is Base 10?
- What is Base 2?
- What is Base 16 (hexadecimal)?
- How do we convert a number in a generic base b to decimal?
- How do we convert a decimal number to another base?

Notes

- Digit = finger in Latin
- Base 10 = uses digits 0-9 and when you count to 9, you go back to 0 and a digit to the left starting with 1 to denote the continuation of the counting
    - Largest number with single digit in Base 10 is 9, with two digits is 99 and lastly three digits 999
- Base 2 = uses digits 0 and 1, where when you reach 1 and wish to continue, you "roll" the rightmost digit(s) to 0 and place a "1" digit in the leftmost space to continue counting, eg. 1, 2 = 10, 3 = 11, 4 = 100
    - Powers of 10 are rep'ed by multiples of 2 when viewing binary to decimal. Ex. 2 = 10, 4 = 100, 8 = 1000, 16 = 1000

decimal	binary
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111
16	10000

Notice that $10 = 2, 100 = 4, 1000 = 8, 10000 = 16$

*** **For base n, use digits 0, 1,... n-1**

Number bases: Examples

Base 10 digits 0,1,…9

$127_{10} = 1\times10^2 + 2\times10^1 + 7\times10^0 = 100+20+7$

Base 2 digits 0,1

$1001_2 = 1\times2^3 + 0\times2^2 + 0\times2^1 + 1\times2^0 =$
$= 1\times8 + 0\times4 + 0\times2 + 1\times1 = 9_{10}$

Number bases: Examples (cont'd)

Base 2 to base 10: convert 100101011_2

2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
256	128	64	32	16	8	4	2	1
1	0	0	1	0	1	0	1	1

$256 + 0 + 0 + 32 + 16 + 8 + 4 + 2 + 1$
$= 299_{10}$

- Base 16 (hexadecimal) = uses 16 digits from 0-9 and A-F (denoting numbers 10 to 15). When counting in base 16, just like in other number systems, when you run of digits, you place a 0 in the rightmost digit(s) and place "1" digit to continue counting.

Base 16 hexadecimal

16 digits
0,1,…..9,A,B,C,D,E,F

Counting 0,1,…9,A,…F, 10,..,19,1A,…
..1E,1F, 20,21…1FE,1FF,200..

Examples 1F= $1 \times 16^1 + 15 \times 16^0 = 31_{10}$

Generic base *b*, conversion to decimal

$a_n\ a_{n-1}\ a_{n-2} \ldots a_0$

In decimal units corresponds to

$a_n \times b^n + a_{n-1} \times b^{n-1} + \ldots + a_0 \times b^0$

- To convert a decimal number to another base, we use the method of repeated division, where we divide the decimal number by the desired base until the decimal number is reduced to zero and then when listing our answer in the new base form we read the remainders bottom to top; placing each digit of the remainders left to right

Method: repeated division

General example: convert 58_{10} in base 2

58/2 =29 r 0
29/2 =14 r 1
14/2 = 7 r 0
7/2 = 3 r 1
3/2 = 1 r 1
1/2 = 0 r 1

read the remainders bottom up

$58_{10} = 111010_2$

Summary

In this week, we learned about the different number systems and how to convert between them.

Week 1 Number Bases Reading Notes

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Cornell Notes	**Topic:** Number Bases	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: April 22, 2020

Essential Question:

What is a number base and what are the different number systems available for use in computing?

Questions/Cues:

- What are the decimal numbers based upon?
- What is binary number system?
- What is octal number system?
- What is hexadecimal number system?

Notes

- Decimal numbers = based on 10 or powers of 10 and uses base 10, in which there are 10 digits 0-9 with placeholder value like "hundreds, tens and units (ones)" denoted by subscript 10

$$
\begin{aligned}
5192_{10} &= 5000 + 100 + 90 + 2 \\
&= 5(1000) + 1(100) + 9(10) + 2(1) \\
&= 5(10^3) + 1(10^2) + 9(10^1) + 2(10^0)
\end{aligned}
$$

- Binary number system = uses base 2, has only two digits 0 and 1. Numbers in binary are called binary digits or bits. Binary number system is based on powers of 2.

Converting from binary to decimal

Consider the binary number 110101_2. As the base is 2 this means that powers of 2 essentially replace powers of 10:

$$\begin{aligned} 110101_2 &= 1(2^5) + 1(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0) \\ &= 1(32) + 1(16) + 0(8) + 1(4) + 0(2) + 1(1) \\ &= 32 + 16 + 4 + 1 \\ &= 53_{10} \end{aligned}$$

Hence 110101_2 and 53_{10} are equivalent.

WORKED EXAMPLE

14.1 Convert (a) 1111_2, (b) 101010_2 to decimal.

Solution (a)

$$\begin{aligned} 1111_2 &= 1(2^3) + 1(2^2) + 1(2^1) + 1(2^0) \\ &= 1(8) + 1(4) + 1(2) + 1(1) \\ &= 8 + 4 + 2 + 1 \\ &= 15_{10} \end{aligned}$$

(b)

$$\begin{aligned} 101010_2 &= 1(2^5) + 0(2^4) + 1(2^3) + 0(2^2) + 1(2^1) + 0(2^0) \\ &= 1(32) + 0 + 1(8) + 0 + 1(2) + 0 \\ &= 32 + 8 + 2 \\ &= 42_{10} \end{aligned}$$

Table 14.1
Powers of 2

2^0	1	2^4	16	2^8	256
2^1	2	2^5	32	2^9	512
2^2	4	2^6	64	2^{10}	1024
2^3	8	2^7	128		

14.2 Convert 83_{10} to a binary number.

Solution We need to express 83_{10} as the sum of a set of numbers each of which is a power of 2. From Table 14.1 we see that 64 is the highest number in the table that does not exceed the given number of 83. We write

$$83 = 64 + 19$$

We now focus on the 19. From Table 14.1, 16 is the highest number that does not exceed 19. So we write

$$19 = 16 + 3$$

giving

$$83 = 64 + 16 + 3$$

We now focus on the 3 and again using Table 14.1 we may write

$$\begin{aligned} 83 &= 64 + 16 + 2 + 1 \\ &= 2^6 + 2^4 + 2^1 + 2^0 \\ &= 1(2^6) + 0(2^5) + 1(2^4) + 0(2^3) + 0(2^2) + 1(2^1) + 1(2^0) \\ &= 1010011_2 \end{aligned}$$

- Octal number system = use 8 as a base and there are eight digits 0-7 in use. Octal numbers use powers of 8

WORKED EXAMPLES

14.5 Convert 325_8 to a decimal number.

Solution

$$\begin{aligned} 325_8 &= 3(8^2) + 2(8^1) + 5(8^0) \\ &= 3(64) + 2(8) + 5(1) \\ &= 192 + 16 + 5 \\ &= 213_{10} \end{aligned}$$

14.6 Convert 7046_8 to a decimal number.

Solution

$$\begin{aligned} 7046_8 &= 7(8^3) + 0(8^2) + 4(8^1) + 6(8^0) \\ &= 7(512) + 0 + 4(8) + 6(1) \\ &= 3622_{10} \end{aligned}$$

Table 14.2
Powers of 8

8^0	1
8^1	8
8^2	64
8^3	512
8^4	4096
8^5	32768

WORKED EXAMPLES

14.7 Convert 1001 to an octal number.

Solution From Table 14.2 we note that the highest number that does not exceed 1001 is 512. So we write

$$1001 = 512 + 489$$

Looking at the 489, we see that 64 is the highest number that does not exceed 489. We note that

$$489 = 7(64) + 41$$

Finally, looking at 41, we note that 8 is the highest number in Table 14.2 that does not exceed 41. We note that

$$41 = 5(8) + 1$$

so we may write

$$\begin{aligned} 1001 &= 512 + 489 \\ &= 512 + 7(64) + 41 \\ &= 512 + 7(64) + 5(8) + 1 \\ &= 1(8^3) + 7(8^2) + 5(8^1) + 1(8^0) \\ &= 1751_8 \end{aligned}$$

As an alternative we can divide repeatedly by 8, noting the remainder:

	Remainder
$1001 \div 8 = 125$ r 1	1
$125 \div 8 = 15$ r 5	5
$15 \div 8 = 1$ r 7	7
$1 \div 8 = 0$ r 1	1

Reading up the remainder column gives the required octal number: 1751_8.

- Hexadecimal number system = uses 16 as a base, uses 16 digits 0-9, A-F(10 to 15). Hexadecimal numbers are based on powers of 16.

Table 14.3
Hexadecimal digits

Decimal	Hexadecimal
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
10	A
11	B
12	C
13	D
14	E
15	F

WORKED EXAMPLE

14.9 Convert the following hexadecimal numbers to decimal numbers: (a) 93A, (b) F9B3.

Solution (a) Noting that hexadecimal numbers use base 16 we have

$$\begin{aligned} 93A_{16} &= 9(16^2) + 3(16^1) + A(16^0) \\ &= 9(256) + 3(16) + 10(1) \\ &= 2362_{10} \end{aligned}$$

$$\begin{aligned} \text{(b)}\quad F9B3_{16} &= F(16^3) + 9(16^2) + B(16^1) + 3(16^0) \\ &= 15(4096) + 9(256) + 11(16) + 3(1) \\ &= 63923_{10} \end{aligned}$$

Table 14.4
Powers of 16

16^0	1
16^1	16
16^2	256
16^3	4096
16^4	65536

WORKED EXAMPLES

14.10 Convert 14397 to a hexadecimal number.

Solution From Table 14.4 the highest number that does not exceed 14397 is 4096. We write

$$14397 = 3(4096) + 2109$$

We now focus on the 2109. From Table 14.4 the highest number that does not exceed 2109 is 256:

$$2109 = 8(256) + 61$$

Finally, $61 = 3(16) + 13$. So we have

$$\begin{aligned} 14397 &= 3(4096) + 8(256) + 3(16) + 13 \\ &= 3(16^3) + 8(16^2) + 3(16^1) + 13 \end{aligned}$$

From Table 14.3 we see that 13_{10} is D in hexadecimal so we have

$$14397_{10} = 383D_{16}$$

As with other number bases that we have studied, we can convert decimal numbers by repeated division and noting the remainder. The previous example is reworked to illustrate this.

Summary

In this week, we learned about the different number systems and how to convert between them.

Week 2 Number Bases Continued Lecture Note

Notebook:	Computational Mathematics		
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Cornell Notes	**Topic:** Number Bases Continued	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: April 23, 2020

Essential Question:

How do convert non-integer numbers and what are arithmetic operations possible with binary numbers?

Questions/Cues:

- How do we convert a decimal fractional number to binary?
- How do we perform addition in binary?
- How do we perform subtraction in binary?
- How do we perform multiplication in binary?
- How do we perform division in binary?

Notes

- Decimal (Fractional) to Binary = Separate whole integer part of number and convert to binary as normal. For fractional part, multiple the fractional decimal number by 2 writing the answer as a sum of a integer and fractional decimal remainder. eg. 0.375 x 2 = 0.75 = 0 + 0.75. Continue this step until you obtain an answer that a sum of a integer part plus 0 and the last multiplication should resulting in 0 and this is where you stop the process and bring together your answer. Your answer for the fractional decimal part is the integer parts in the conversion read top to bottom. Finally you join your whole integer part from the beginning with the fractional part you just converted to produce a final binary number value

Non-integer numbers: decimal to binary

Example: $17.375_{10} = 1\times10^1+7\times10^0+3\times10^{-1}+7\times10^{-2}+5\times10^{-3}$

$= 10 + 7 + 3/10 + 7/100 + 5/1000$

(Positional value of digits)

How to convert to binary? $17_{10}=10001_2$

Separate integer part 17 and fractional part 0.375

$0.375\times2=0.75=0+0.75$
$0.75\times2=1.5=1+0.5$
$0.5\times2=1.0=1+0$
$0 \rightarrow$ STOP

$0.375_{10}=0.011_2$

$\Rightarrow 17.375_{10}=10001.011_2$

Non-integer numbers: binary to decimal

Example: $1101.101_2 = 1\times2^3+1\times2^2+0\times2^1+1\times2^0+$
$+1\times2^{-1}+0\times2^{-2}+1\times2^{-3}$
$= 8+4+1+1/2+1/8=13.625_{10}$

(Positional value of digits)

in general $a_n\, a_{n-1}\, a_{n-2}\ldots a_0.\, c_{-1}\, c_{-2}\ldots c_{-k}$ in base b

In decimal units corresponds to

$a_n \times b^n + a_{n-1} \times b^{n-1} + \ldots + a_0 \times b^0 +$
$+ c_{-1} \times b^{-1} + c_{-2} \times b^{-2} + \ldots + c_{-k} \times b^{-k}$

- Binary addition = Similar to decimal addition, you add each digit in each column and carrying over "1" when adding together two ones or three ones. In the case of two ones, it is zero carry over 1 and in the case of three ones it is one carry over one.

```
  1          1          1 1
1 0 1+     1 0 1 +    1 0 1 +
1 1 1=     1 1 1 =    1 1 1 =
    0      0 0 0     1 1 0 0
```

$101_2+111_2=1100_2$ *in decimal* $5+7=12$

Operations with binary numbers

Other example: 1010011+1110110=?

```
        1          1   1 1        1 1   1 1
1 0 1 0 0 1 1+   1 0 1 0 0 1 1+   1 0 1 0 0 1 1+
1 1 1 0 1 1 0=   1 1 1 0 1 1 0=   1 1 1 0 1 1 0=
        0 1        0 0 1 0 0 1   1 1 0 0 1 0 0 1
```

1010011+1110110=11001001

In decimal 1010011=83 1110110=118 118+83=201

11001001=1+8+64+128=201

- Binary subtraction = Similar to decimal subtraction, subtraction in binary is quite straightforward; where 1-1 = 0, 1-0 = 1. In the case where subtraction is not immediately possible like 0 - 1, you must borrow one unit from the digit to the left of the zero and the zero becomes a 2 and if the left digit was initially a one it becomes a zero to complete the subtraction operation because it a base 2 number instead of base 10 like decimal. If you borrow a unit from the left digit neighboring a zero that is already promoted to a 2 it becomes a one.

```
      2
1 ~~1~~ 0-
1 0 1=
0 0 1
```

110_2-101_2=001_2 *in decimal 6-5=1*

Other example subtraction: 1110011-1010011

```
    2 ~~2~~1  2
1 ~~1~~ ~~1~~ 0  0 1 1-
1 0 1 0 1 1 0=
0 0 1 1 1 0 1
```

1110011=1+2+16+32+64=115

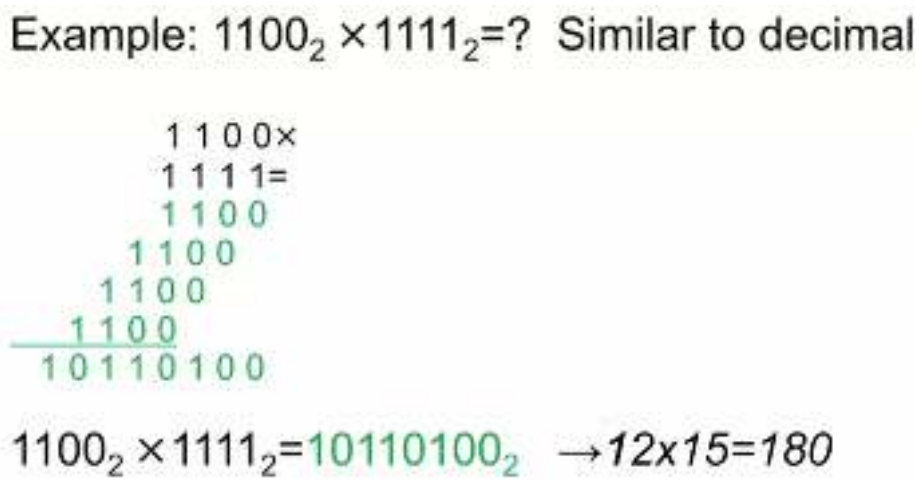
1010110=2+4+16+64=86

115-86=29

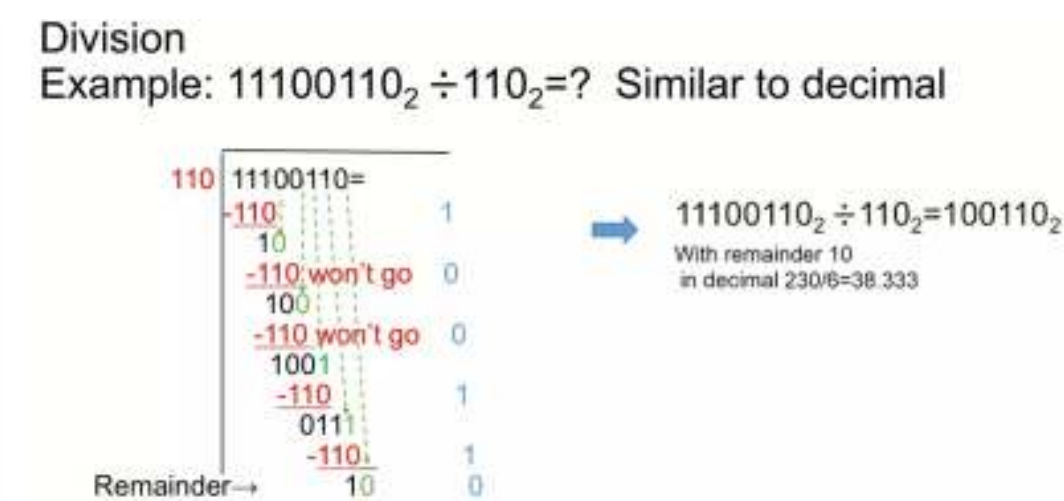
0011101=1+4+8+16=29

- Binary multiplication = Much like decimal multiplication, you line up numbers to multiplied into columns, then you multiplying by selecting the rightmost digit of the multiplier (bottom number) and multiplying that digit throughout the whole of the top number, noting the answer below the number, then you choose the second rightmost digit of the multiplier and multiply it throughout the top number but when noting the answer of the multiplication this second time you write it below the first answer and

moved one digit to the left. Continue to multiply all the digits in the multiplier with the top number, noting the answer of the multiplication each time below previous answers, also moving each lower tier answer one digit to the left. In order to produce the final answer to the given problem, you must sum all the multiplication answers using binary addition and lining each of the answers from the previous multiplication step into columns; adding zeros in empty spaces and adding the resulting sum and writing the final answer in the end at the very bottom once the addition is complete.



- Binary division = Similar to decimal, you arrange the divisor on the outside and dividend on the inside of a long division brace. Then from the dividend you take a number that is larger than the divisor and from that number you subtract the divisor, noting in the quotient at the top how many times the divisor "go into" the dividend. After subtraction of the divisor from the dividend you get what is called the remainder and you bring the next digit from the dividend, here you ask the same question of how many times the divisor "goes into" this number and if the divisor is larger than the number then you add a 0 to the quotient at the top and bring the next digit from the dividend. If after this, the number is larger than the divisor you add a 1 to quotient and the divisor from number and this continue until you reach a remainder of 0 or until you have no more digits to bring from the dividend. Then your final answer is the quotient produced at the top and any remainder leftover from the subtraction.



Summary

In this week, we learned about how to convert fractional decimal numbers to binary and the different arithmetic operation that we can perform on binary numbers.

Week 3 Sequences and Series Lecture Note

Notebook:	Computational Mathematics		
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Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Sequences and Series	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: May 01, 2020

Essential Question:

What are sequences?

Questions/Cues:

- What is a geometric sequence?
- What is an arithmetic sequence?
- What is the formal definition of a sequence?
- What is the Fibonacci Sequence?

Notes

- Geometric Sequence = has the property that the ratio between a value in the sequence and its previous value is constant, it can any number q with a being the initial value of the sequence. The general form of a geometric sequence is as followed:

Geometric sequence

$$a, a\times q, a\times q^2, \ldots a\times q^n ..$$

- Arithmetic sequence = has the property where the nth term in the sequence is the sum of the previous term before it and a constant number q. In other words, the difference between each element of the sequence and its previous is constant.

What is a sequence?

- Formal definition: given a set X a sequence is a function a: $\mathbb{N} \to X$
  i.e. a set $a(0), a(1) \ldots a(n) \ldots$ denoted with $\{a_n\}_{n \in \mathbb{N}}$

- Can be defined explicitly $a_n = f(n)$
  example: $a_n = 2n+1 \to 1,3,5,7,9 \ldots$

- or by recursion i.e. $a_n = f(a_{n-1}, \ldots a_{n-k})$
  example 1: *Arithmetic sequence* $a_n = q + a_{n-1}$
  $\to a_0, a_0+q, a_0+2q, \ldots \to a_n = n \times q + a_0$
  example 2: *Geometric sequence* $a_n = q \times a_{n-1}$
  $\to a_0, a_0 q, a_0 q^2, a_0 q^3 \ldots a_0 q^n \ldots \to a_n = a_0 \times q^n$

- A sequence is said to be convergent, when upon increasing n it approaches a finite constant value

$$\lim_{n \to \infty} a_n = L < \infty$$

example: geometric seq. $q<1 \to q=1/2$
1/2,1/4,1/8……1/64,…. converges to 0

- A sequence is said to be divergent when increasing n it never reaches a constant finite value (either goes to ∞ or oscillates)
  example: geometric seq. $q>1 \to q=2$
  2,4,8,16,32,….256,…

Examples:Fibonacci Sequence

- Definition by recursion: $a_0=0$, $a_1=1$, $a_n = a_{n-1} + a_{n-2}$
- 0,1,1,2,3,5,8,13,21,34,55,89,144….
- $\frac{a_n}{a_{n-1}} \to \varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ Golden Ratio

Summary

In this week, we learned about what a sequence is and the two different kinds of sequences, Geometric and Arithmetic. Finally, in the end we look at a famous example of a sequence

called the Fibonacci Sequence.

Week 3 Sequences & Series Reading Note

Notebook:	Computational Mathematics		
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Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Sequences and Series	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: April 28, 2020

Essential Question:

What are sequences and series and how we define each?

Questions/Cues:

- What is a Sequence?
- What is an Arithmetic Sequence/Progression?
- What is the formula for calculating the nth term of an Arithmetic Sequence/Progression?
- What is a Geometric Sequence/Progression?
- What is the formula for calculating the nth term of a Geometric Sequence/Progression?
- What is a Infinite Sequence?
- What is a Series?
- What is Sigma Notation?
- What is an Arithmetic Series?
- What is a Geometric Series?
- What is a Infinite Geometric Series?

Notes

- Sequence = set of numbers written down in a specific order
  - Each number in the sequence is called a term
  - Sometimes we use the symbol "..." to indicate that the sequence continues
    - Sequences with a finite number of terms are called finite sequences. e.g. 1, 2, 3, ... , 20
    - Sequences that go on forever and are indicate with "..." on the end are called infinite sequences
    - We use a subscript notation to refer to different terms in a sequence. For example, if a sequence is denoted by x then the first term is labeled x_1 if you start counting from 1 or x_0 if you start counting from zero

WORKED EXAMPLES

12.1 The terms of a sequence x are given by $x_k = 2k + 3$. Write down the terms x_1, x_2 and x_3.

Solution If $x_k = 2k + 3$ then replacing k by 1 gives $x_1 = 2 \times 1 + 3 = 5$. Similarly $x_2 = 7$ and $x_3 = 9$.

12.2 The terms of a sequence x are given by $x_k = 4k^2$. Write down the terms x_0, x_1, x_2 and x_3.

Solution If $x_k = 4k^2$ then replacing k by 0 gives $x_0 = 4 \times 0^2 = 0$. Similarly $x_1 = 4$, $x_2 = 16$ and $x_3 = 36$.

- Arithmetic sequence/progression = a sequence formed by adding a fixed amount to the previous term
  - The fixed amount added each time is a constant called the common difference

WORKED EXAMPLES

12.3 Write down four terms of the arithmetic progression that has first term 10 and common difference 3.

Solution The second term is found by adding the common difference 3 to the first term 10. This gives a second term of 13. Similarly to find the third term we add on another 3, and so on. This gives the sequence 10, 13, 16, 19 ....

12.4 Write down six terms of the arithmetic progression that has first term 5 and common difference −2.

Solution Note that in this example the common difference is negative. To find each new term we must add −2. In other words we must subtract 2. This results in

$$5, 3, 1, -1, -3, -5$$

An arithmetic progression can be written

$$a, a + d, a + 2d, a + 3d \ldots$$

a is the **first term**, d is the **common difference**.

The nth term of an arithmetic progression is given by $a + (n - 1)d$.

WORKED EXAMPLE

12.5 Use the formula for the nth term to find the 10th term of an arithmetic progression with first term 3 and common difference 5.

Solution Using the formula nth term $= a + (n - 1)d$ with $a = 3$, $d = 5$ and $n = 10$ we find

$$10\text{th term} = 3 + (10 - 1) \times 5 = 3 + 9 \times 5 = 48$$

The 10th term is therefore 48. Using the formula allows us to state the 10th term without having to calculate all the previous terms.

- Geometric Sequence/Progression = a sequence formed by multiplying the previous term by a fixed amount
  - The fixed by which each term is multiplied is called the common ratio
    - If you're given the first term of a sequence & and the common ratio it is easy to generate subsequent terms.

WORKED EXAMPLES

12.6 Find the first six terms of the geometric sequence with first term 3 and common ratio 2.

Solution The first term is 3. The second term is found by multiplying the first by the common ratio 2. So, the second term is 6. The third term is found by multiplying the second by 2, to give 12. We continue in this fashion to generate the sequence 3, 6, 12, 24, 48, 96.

12.7 Write down the first five terms of the geometric sequence with first term 5 and common ratio $\frac{2}{3}$.

Solution The second term is found by multiplying the first by $\frac{2}{3}$. So the second term is $5 \times \frac{2}{3} = \frac{10}{3}$. The third term is found by multiplying this by $\frac{2}{3}$. Continuing in this way we generate the sequence

$$5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27}, \frac{80}{81}$$

12.8 A geometric progression is given by $1, \frac{1}{2}, \frac{1}{4}, \ldots$. What is its common ratio?

Solution The first term is 1. We must ask by what number must the first term be multiplied to give us the second. The answer is clearly $\frac{1}{2}$. The common ratio is $\frac{1}{2}$. This can be checked by verifying that the third term is indeed $\frac{1}{4}$. More generally, the common ratio of a geometric sequence can be found by dividing any term by the previous term. Observe this in the earlier examples.

12.9 Write down the first six terms of the geometric sequence with first term 4 and common ratio -1.

Solution Note that in this example the common ratio is negative. This will have an interesting effect, as you will see. Starting with the first term and repeatedly multiplying by -1 gives the sequence $4, -4, 4, -4, 4, -4$. Notice that because the common ratio is negative, the terms of the sequence alternate in sign.

A geometric progression can be written

$$a, ar, ar^2, ar^3 \ldots$$

a is the **first term**, r is the **common ratio**.

Where:

the first term is a
the second term is ar
the third term is ar^2
the fourth term is ar^3

The nth term of a geometric progression is given by ar^{n-1}.

WORKED EXAMPLE

12.10 Write down the seventh term of the geometric progression that has first term 2 and common ratio 3.

Solution The nth term is ar^{n-1}. Here $a = 2$ and $r = 3$. The seventh term is $(2)(3)^{7-1} = 2 \times 3^6 = 2 \times 729 = 1458$.

- Infinite Sequence = a sequence that continues indefinitely and we use the ... to indicate this.
  - It can sometimes be the case that as we move along the sequence, the terms get closer and closer to a fixed value. For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$, terms are getting smaller and smaller if we continue forever these terms approach the

value 0, sequence can also be written as $x_k = \frac{1}{k}, \text{for } k = 1,2,3\ldots$. As k gets larger and larger and approaches infinity, terms in the sequence get closer and closer to 0; we say that "$\frac{1}{k}$ tends to 0 as k tends to infinity" or "as k tends to infinity, the limit of the sequence is 0"

- $\text{Written as } \lim_{k \to \infty} \frac{1}{k} = 0, \text{where 'lim' stands for limit}$
  - $\lim_k \to \infty$ means we must examine the behaviour of the sequence as k gets larger and larger
  - $*$ Very important point to note here, when a sequence possesses a limit it is said to **converge**. However, not all sequences possess a limit or a value they approach, thus are said to **diverge**. For example, the sequence defined by $x_k = 3k - 2, \text{which is } 1, 4, 7, 10\ldots$ is such an example. In this example, as k gets larger and larger, so too do the terms of the sequence; tending to no such value L as in the previous examples above.

WORKED EXAMPLE

12.11 (a) Write down the first four terms of the sequence $x_k = 3 + \frac{1}{k^2}$, $k = 1, 2, 3 \ldots$.

(b) Find, if possible, the limit of this sequence as k tends to infinity.

Solution (a) The first four terms are given by

$$x_1 = 3 + \frac{1}{1} = 4, \quad x_2 = 3 + \frac{1}{4} = 3\tfrac{1}{4}, \quad x_3 = 3 + \frac{1}{9} = 3\tfrac{1}{9},$$
$$x_4 = 3 + \frac{1}{16} = 3\tfrac{1}{16}$$

(b) As more terms are included we see that x_k approaches 3 because the quantity $\frac{1}{k^2}$ becomes smaller and smaller. We can write

$$\lim_{k \to \infty} \left(3 + \frac{1}{k^2}\right) = 3$$

This sequence converges to the limit 3.

- Series = when the terms of a sequence are added together, the result is known as a series. Eg. 1 + 2 + 3 + 4 + 5
  - A series is a sum. If the series contains a finite number of terms we can add them all up and obtain the sum of the series
    - If the series contains an infinite number of terms it is more complicated because an infinite series may have a finite sum in which case it is said to converge or alternatively it may not have a finite sum in which case it said to diverge.
- Sigma notation($\sum$) = a concise and convenient way of writing long sums.
  - For example, $1 + 2 + 3 + 4 + 5 + \ldots + 10 + 11 + 12$ can be written concisely using sigma notation as: $\sum_{k=1}^{k=12} k$, which stands for a sum of all the values of k as k ranges through all whole numbers from 1 to 12.
    - The lowermost and uppermost values of k are written at the bottom and top of the sigma sign respectively
      - The lowermost value of k is commonly $k = 1$ or $k = 0$ but values are possible. When working with sigma notation, always check the lowermost value of k
      - Sometimes the sigma notation is itself is used in shorthand, meaning the $'k=\,'$ part written at the bottom can be omitted.

For example, our previous example can be written as : $\sum_{k=1}^{12} k$ or $\sum_{1}^{12} k$

WORKED EXAMPLES

12.12 Write out explicitly what is meant by

$$\sum_{k=1}^{k=5} k^3$$

Solution We must let k range from 1 to 5, cube each value of k, and add the results:

$$\sum_{k=1}^{k=5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$$

12.13 Express $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ concisely using sigma notation.

Solution Each term takes the form $\frac{1}{k}$, where k varies from 1 to 4. In sigma notation we could write this as

$$\sum_{k=1}^{k=4} \frac{1}{k}$$

12.14 Write the sum

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_{19} + x_{20}$$

using sigma notation.

Solution The sum may be written as

$$\sum_{k=1}^{k=20} x_k$$

- $*$ In general, $(-1)^k$ takes on values of 1 when k is even, and - 1 when k is odd. A factor of $(-1)^k$ in the terms of a series means that the terms alternate in sign

WORKED EXAMPLES

12.15 Write out fully what is meant by $\sum_{k=1}^{4}(-1)^k 2^k$.

Solution When $k = 1$, $(-1)^k 2^k = (-1)^1 2^1 = -2$.

When $k = 2$, $(-1)^k 2^k = (-1)^2 2^2 = 4$.

When $k = 3$, $(-1)^k 2^k = (-1)^3 2^3 = -8$.

When $k = 4$, $(-1)^k 2^k = (-1)^4 2^4 = 16$.

So

$$\sum_{k=1}^{4} (-1)^k 2^k = -2 + 4 - 8 + 16$$

Note that the signs are alternating owing to the presence of $(-1)^k$ in the sum.

12.16 Write out fully what is meant by

$$\sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i+1}$$

Solution Note that the lowermost value of i is $i = 0$.

When $i = 0$, $\frac{(-1)^{i+1}}{2i+1} = \frac{(-1)^1}{1} = -1$

When $i = 1$, $\frac{(-1)^{i+1}}{2i+1} = \frac{(-1)^2}{3} = \frac{1}{3}$

Substituting in values for $i = 2$, $i = 3$, $i = 4$ and $i = 5$ produces the sum

$$\sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11}$$

- Arithmetic Series = When the terms of an arithmetic sequence are added, the result is known as an arithmetic series
  - If the arithmetic series has a large number of terms then finding its sum by directly adding all the terms is ill-advised. Fortunately, the following formula makes it easier to find the sum of an arithmetic series:

The sum of the first n terms of an arithmetic series with first term a and common difference d is denoted by S_n and given by

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

WORKED EXAMPLE

12.17 Find the sum of the first 10 terms of the arithmetic series with first term 3 and common difference 4.

Solution Use the formula $S_n = \frac{n}{2}(2a + (n-1)d)$ with $n = 10$, $a = 3$ and $d = 4$:

$$S_{10} = \frac{10}{2}(2 \times 3 + (10-1) \times 4) = 5(6 + 36) = 210$$

- Geometric Series = When the terms of a geometric sequence are added, the result is known as a geometric series. The following is the formula that allows us to find the sum of a geometric series:

The sum of the first n terms of a geometric series with first term a and common ratio r is denoted by S_n and given by

$$S_n = \frac{a(1-r^n)}{1-r} \qquad \text{provided } r \text{ is not equal to } 1$$

  - The formula excludes the case where $r = 1$ because the denominator becomes zero and division by zero is never allowed

WORKED EXAMPLE

12.18 Find the sum of the first five terms of the geometric series with first term 2 and common ratio 3.

Solution Use the formula $S_n = \frac{a(1-r^n)}{1-r}$ with $n = 5$, $a = 2$ and $r = 3$:

$$S_5 = \frac{2(1-3^5)}{1-3} = \frac{2(1-243)}{-2} = 242$$

- Infinite Geometric Series = When terms of an infinite geometric sequence are added, the result is an infinite geometric series, in the case that sum of the terms is finite and can be found.
    - Consider the special of an infinite geometric series in which the common ratio r lies between -1 and 1. In this case the sum always exists and its value can be found using the following formula:

The sum of an infinite number of terms of a geometric series is denoted by S_∞ and is given by

$$S_\infty = \frac{a}{1-r} \qquad \text{provided } -1 < r < 1$$

- Note in this formula if the common ratio is larger than 1 or less than -1, $r > 1 \text{or } r < -1$, then the series doesn't converge and the sum of an infinite geometric series can't be found

WORKED EXAMPLE

12.19 Find the sum of the infinite geometric series with first term 2 and common ratio $\frac{1}{3}$.

Solution Using the formula $S_\infty = \frac{a}{1-r}$ with $a = 2$ and $r = \frac{1}{3}$,

$$S_\infty = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$$

Notice that we can only make use of the formula because the value of r lies between -1 and 1.

Summary

In this week, we learned about what a sequence and what is series is. Alongside this, we explored the difference between an arithmetic and geometric sequence/series. Secondly, we looked at how arithmetic/geometric series from their respective sequences. Finally, we touched on the notion of the infinite and infinity with infinite sequences and a infinite geometric series.

Week 4 Sequences and Series Continued Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-01 5:39 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Sequences and Series Continued	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: May 01, 2020

Essential Question:

What is a series?

Questions/Cues:

- What is a series?

Notes

Series

- Consider again a generic sequence $\{a_n\}$ with n=0,1,2....
  we can look at the sum of its elements
  $a_0,\ a_0+a_1,\ a_0+a_1+a_2,\ a_0+a_1+a_2+a_3,\ldots$
- These sums define another sequence $\{s_n\}$ n=0,1,2...
  with $s_n=a_0+a_1+\ldots a_n \rightarrow s_0=a_0,\ s_1=a_0+a_1,\ s_2=a_0+a_1+a_2,\ldots$
- $\{s_n\}$ is called a series, it is a type of sequence that is obtained as sum of the elements of another sequence
- A series is also indicated with $\sum \rightarrow s_n=\sum_{i=0}^{n} a_i$

Summary

In this week, we learned about what a series and how it necessary to have an original sequence in order to construct a series.

Week 5 Modular Arithmetic Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-05 5:27 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Modular Arithmetic	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: May 05, 2020

Essential Question:

What are the operations performed with congruent numbers and their application to the field of computer science?

Questions/Cues:

- What is modular arithmetic?
- What is general way of writing congruence between two numbers?
- What is the more formal definition of modular arithmetic?
- How do you perform modular arithmetic on negative numbers?

Notes

- Modular Arithmetic = A way to classify integers, in a sense it is an arithmetic over integers; first introduced by mathematician and physicist Carl Friedrich Gauss
    - used in comp sci when dealing with long numbers, simplifying operation with long numbers
    - At the core of modular arithmetic is the idea of congruence between integers
    - Two numbers a & b are congruent, for example "mod 2" or modulus 2 if when they are divided by 2, they have the same remainder
- Congruency (general) =

$$a \overset{congruent}{\equiv} b \pmod{k} \Leftrightarrow a = nk + R, b = mk + R$$

    - Where R is the remainder and a, b, k are generic integers

- Example $3 \equiv 5 (mod 2)$
  since $3/2=1$ with $R=1$ $5/2=2$ with $R=1$

- Familiar example: the clock,

  if now it is 8 AM after 7 hs it will be 15 or 3 PM (it is "*mod 12*")

  $\rightarrow$ $15 \equiv 3 (mod\ 12)$

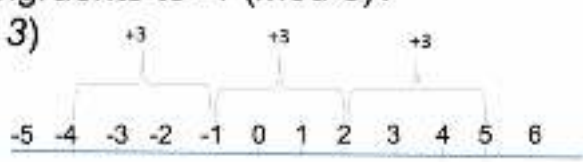
  since $15/12=1$ with $R=3$ $3/12=0$ with $R=3$

  - Modular Arithmetic (formal) = $\bmod k$ is mapping by congruence all integers to the subset of non-negative integers smaller than k to which all other integers can be shown to be congruent to that is: $Min_k = \{0,1,2,\ldots k-1\}$
    - The subset is referred to as the minimal subset
      - This subset is special because it is made of integers that when they are divided by the modulus k they coincide with the remainder
    - For example, modulus 12 or $\bmod 12$, the minimal subset is $Min_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$ and no other integers can be shown to be congruent to the elements of this minimal subset. So if you have a positive integer, you just divide by k, and take the remainder & that remainder will give you the element of the minimal subset to which that number is congruent to

- With negative integers when we divide by *k* we need to get a <u>non-negative remainder</u>
  Ex: -17 *mod 12*
  you cannot do $-17/12= -1$ with $R=-5$ negative (wrong)
  but $-17/12 = -2$ with $R=7$ positive
  $\rightarrow -17 \equiv 7 (mod\ 12)$
- Other way: -17 reverse sign $\rightarrow 17 \equiv 5 (mod\ 12)$ reverse sign to 5 and add $k(=12)$ $-5+12=7$ $\rightarrow -17 \equiv 7 (mod\ 12)$

- Other example $-12 (mod\ 5) \rightarrow 12 \equiv 2 (mod\ 5)$ $\rightarrow -2+5=3$
  $\rightarrow -12 \equiv 3 (mod\ 5)$

- What are the integers congruents to -1 (*mod 3*)?
  $\ldots \equiv -4 \equiv -1 \equiv 2 \equiv 5 \equiv \ldots (mod\ 3)$



since $a \equiv b (mod\ 3) \Leftrightarrow a=n3+R$ and $b=m3+R$
$\rightarrow a-b= (n-m)3$

Examples

- 23 what is the smallest congruent number *mod 5*?
  23/5 =4 with R=3 → 23 ≡3 (*mod 5*)

- 1101024 what is the smallest congruent number *mod 5*?
  1101024=1101020+4 →R=4 → 1101024 ≡4 (*mod 5*)

- -2367(*mod 5*) ? → 2367/5=473 R=2 -2+5=3
  → -2367≡3(*mod 5*)

Summary

In this week, we learned about what modular arithmetic is and how it can be performed on both positive and negative numbers.

Week 5 Modular Arithmetic Reading Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-07 4:10 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Modular Arithmetic	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: May 06, 2020

Essential Question:

What is divisibility and congruence among two numbers?

Questions/Cues:

- What do we mean when we say a divides b?
- What is meant by a trivial divisor?
- What are some basic properties of divisibility?
- What is the division algorithm/theorem?
- What is a consequence observed when applying the division algorithm?
- What is a prime number/integer?
- What is "a mod n" and how does this relate to the remainder of a divides n?
- What do mean when it is said that "a is congruent to b modulus n"?
- What is a residue of a modulo n?
- What properties does the congruence relation have in common with equality relation?
- What is a residue class and its importance in the congruence of two numbers?
- What are some arithmetic properties of congruences?

Notes

Definition 1.2.1. Let a and b be integers with $a \neq 0$. We say a divides b, denoted by $a \mid b$, if there exists an integer c such that $b = ac$. When a divides b, we say that a is a *divisor* (or *factor*) of b, and b is a *multiple* of a. If a does not divide b, we write $a \nmid b$. If $a \mid b$ and $0 < a < b$, then a is called a *proper divisor* of b.

Remark 1.2.1. We never use 0 as the left member of the pair of integers in $a \mid b$, however, 0 may occur as the right member of the pair, thus $a \mid 0$ for every integer a not zero. Under this restriction, for $a \mid b$, we may say that b is divisible by a, which is equivalent to say that a is a divisor of b. The notation $a^\alpha \mid\mid b$ is sometimes used to indicate that $a^\alpha \mid b$ but $a^{\alpha+1} \nmid b$.

Example 1.2.1. The integer 200 has the following positive divisors (note that, as usual, we shall be only concerned with positive divisors, not negative divisors, of an integer):

$$1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200.$$

Thus, for example, we can write

$$8 \mid 200, \ 50 \mid 200, \ 7 \nmid 200, \ 35 \nmid 200.$$

Definition 1.2.2. A divisor of n is called a *trivial divisor* of n if it is either 1 or n itself. A divisor of n is called a *nontrivial divisor* if it is a divisor of n, but is neither 1, nor n.

Example 1.2.2. For the integer 18, 1 and 18 are the trivial divisors, whereas 2, 3, 6 and 9 are the nontrivial divisors. The integer 191 has only two trivial divisors and does not have any nontrivial divisors.

Some basic properties of divisibility are given in the following theorem:

Theorem 1.2.1. Let a, b and c be integers. Then

(1) if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.
(2) if $a \mid b$, then $a \mid bc$, for any integer c.
(3) if $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof.

(1) Since $a \mid b$ and $a \mid c$, we have

$$b = ma, \ c = na, \quad m, n \in \mathbb{Z}.$$

Thus $b + c = (m + n)a$. Hence, $a \mid (m + n)a$ since $m + n$ is an integer. The result follows.

(2) Since $a \mid b$ we have

$$b = ma, \quad m \in \mathbb{Z}.$$

Multiplying both sides of this equality by c gives

$$bc = (mc)a$$

which gives $a \mid bc$, for all integers c (whether or not $c = 0$).

(3) Since $a \mid b$ and $b \mid c$, there exists integers m and n such that

$$b = ma, \quad c = nb.$$

Thus, $c = (mn)a$. Since mn is an integer the result follows.

Theorem 1.2.2 (Division algorithm). For any integer a and any positive integer b, there exist unique integers q and r such that

$$a = bq + r, \quad 0 \leq r < b, \tag{1.41}$$

where a is called the *dividend*, q the *quotient*, and r the *remainder*. If $b \nmid a$, then r satisfies the stronger inequalities $0 < r < a$.

Example 1.2.3. Let $b = 15$. Then

(1) when $a = 255$, $a = b \cdot 17 + 0$, so $q = 17$ and $r = 0 < 15$.

(2) when $a = 177$, $a = b \cdot 11 + 12$, so $q = 11$ and $r = 12 < 15$.

(3) when $a = -783$, $a = b \cdot (-52) + 3$, so $q = -52$ and $r = 3 < 15$.

Definition 1.2.3. Consider the following equation

$$a = 2q + r, \quad a, q, r \in \mathbb{Z},\ 0 \leq r < q.$$

Then if $r = 0$, then a is *even*, whereas if $r = 1$, then a is *odd*.

Definition 1.2.4. A positive integer n greater than 1 is called *prime* if its only divisors are n and 1. A positive integer n that is greater than 1 and is not prime is called *composite*.

Example 1.2.4. The integer 23 is prime since its only divisors are 1 and 23, whereas 22 is composite since it is divisible by 2 and 11.

Definition 1.6.1. Let a be an integer and n a positive integer greater than 1. We define "a mod n" to be the remainder r when a is divided by n, that is

$$r = a \bmod n = a - \lfloor a/n \rfloor n. \tag{1.219}$$

We may also say that "r is equal to a reduced modulo n".

Remark 1.6.1. It follows from the above definition that a mod n is the integer r such that $a = \lfloor a/n \rfloor n + r$ and $0 \leq r < n$, which was known to the ancient Greeks and Chinese some 2000 years ago.

Definition 1.6.2. Let a and b be integers and n a positive integer. We say that "a is *congruent* to b modulo n", denoted by

$$a \equiv b \pmod{n} \tag{1.220}$$

if n is a divisor of $a - b$, or equivalently, if $n \mid (a - b)$. Similarly, we write

$$a \not\equiv b \pmod{n} \tag{1.221}$$

if a is not congruent (or incongruent) to b modulo n, or equivalently, if $n \nmid (a - b)$. Clearly, for $a \equiv b \pmod{n}$ (resp. $a \not\equiv b \pmod{n}$), we can write $a = kn - b$ (resp. $a \neq kn - b$) for some integer k. The integer n is called the *modulus*.

Clearly,

$$\begin{aligned} a \equiv b \pmod{n} &\iff n \mid (a - b) \\ &\iff a = kn + b, \quad k \in \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} a \not\equiv b \pmod{n} &\iff n \nmid (a - b) \\ &\iff a \neq kn + b, \quad k \in \mathbb{Z} \end{aligned}$$

Definition 1.6.3. If $a \equiv b \pmod{n}$, then b is called a *residue* of a modulo n. If $0 \leq b \leq m-1$, b is called the *least nonnegative residue* of a modulo n.

Remark 1.6.2. It is common, particularly in computer programs, to denote the least nonnegative residue of a modulo n by $a \bmod n$. Thus, $a \equiv b \pmod{n}$ if and only if $a \bmod n = b \bmod n$, and, of course, $a \not\equiv b \pmod{n}$ if and only if $a \bmod n \neq b \bmod n$.

Example 1.6.2. The following are some examples of congruences or incongruences.

$$\begin{aligned} 35 &\equiv 11 \pmod{12} && \text{since} && 12 \mid (35-11) \\ &\not\equiv 12 \pmod{11} && \text{since} && 11 \nmid (35-12) \\ &\equiv 2 \pmod{11} && \text{since} && 11 \mid (35-2) \end{aligned}$$

The congruence relation has many properties in common with the equality relation. For example, we know from high-school mathematics that equality is

(1) reflexive: $a = a$, $\forall a \in \mathbb{Z}$;
(2) symmetric: if $a = b$, then $b = a$, $\forall a, b \in \mathbb{Z}$;
(3) transitive: if $a = b$ and $b = c$, then $a = c$, $\forall a, b, c \in \mathbb{Z}$.

We shall see that congruence modulo n has the same properties:

Theorem 1.6.1. Let n be a positive integer. Then the congruence modulo n is

(1) reflexive: $a \equiv a \pmod{n}$, $\forall a \in \mathbb{Z}$;
(2) symmetric: if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$, $\forall a, b \in \mathbb{Z}$;
(3) transitive: if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$, $\forall a, b, c \in \mathbb{Z}$.

Proof.

(1) For any integer a, we have $a = 0 \cdot n + a$, hence $a \equiv a \pmod{n}$.
(2) For any integers a and b, if $a \equiv b \pmod{n}$, then $a = kn + b$ for some integer k. Hence $b = a - kn = (-k)n + a$, which implies $b \equiv a \pmod{n}$, since $-k$ is an integer.
(3) If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a = k_1 n + b$ and $b = k_2 n + c$. Thus, we can get

$$a = k_1 n + k_2 n + c = (k_1 + k_2)n + c = k' n + c$$

which implies $a \equiv c \pmod{n}$, since k' is an integer. □

Theorem 1.6.1 shows that the congruence modulo n is an equivalence relation on the set of integers $\mathbb{Z}$. But note that the divisibility relation $a \mid b$ is reflexive, and transitive but not symmetric; in fact if $a \mid b$ and $b \mid a$ then $a = b$, so it is not an equivalence relation. The congruence relation modulo n partitions $\mathbb{Z}$ into n *equivalence classes*. In number theory, we call these classes *congruence classes*, or *residue classes*. More formally, we have:

Definition 1.6.4. If $x \equiv a \pmod{n}$, then a is called a *residue* of x modulo n. The *residue class* of a modulo n, denoted by $[a]_n$ (or just $[a]$ if no confusion will be caused), is the set of all those integers that are congruent to a modulo n. That is,

$$\begin{aligned}[a]_n &= \{x : x \in \mathbb{Z} \text{ and } x \equiv a \pmod{n}\} \\ &= \{a + kn : k \in \mathbb{Z}\}. \end{aligned} \tag{1.222}$$

Note that writing $a \in [b]_n$ is the same as writing $a \equiv b \pmod{n}$.

Example 1.6.3. Let $n = 5$. Then there are five residue classes, modulo 5, namely the sets:

$$\begin{aligned}
[0]_5 &= \{\cdots, -15, -10, -5, 0, 5, 10, 15, 20, \cdots\}, \\
[1]_5 &= \{\cdots, -14, -9, -4, 1, 6, 11, 16, 21, \cdots\}, \\
[2]_5 &= \{\cdots, -13, -8, -3, 2, 7, 12, 17, 22, \cdots\}, \\
[3]_5 &= \{\cdots, -12, -7, -2, 3, 8, 13, 18, 23, \cdots\}, \\
[4]_5 &= \{\cdots, -11, -6, -1, 4, 9, 14, 19, 24, \cdots\}.
\end{aligned}$$

The first set contains all those integers congruent to 0 modulo 5, the second set contains all those congruent to 1 modulo 5, $\cdots$, and the fifth (i.e., the last) set contains all those congruent to 4 modulo 5. So, for example, the residue class $[2]_5$ can be represented by any one of the elements in the set

$$\{\cdots, -13, -8, -3, 2, 7, 12, 17, 22, \cdots\}.$$

Clearly, there are infinitely many elements in the set $[2]_5$.

Example 1.6.4. In residue classes modulo 2, $[0]_2$ is the set of all even integers, and $[1]_2$ is the set of all odd integers:

$$\begin{aligned}
[0]_2 &= \{\cdots, -6, -4, -2, 0, 2, 4, 6, 8, \cdots\}, \\
[1]_2 &= \{\cdots, -5, -3, -1, 1, 3, 5, 7, 9, \cdots\}.
\end{aligned}$$

Example 1.6.5. In congruence modulo 5, we have

$$\begin{aligned}[9]_5 &= \{9 + 5k : k \in \mathbb{Z}\} = \{9, 9 \pm 5, 9 \pm 10, 9 \pm 15, \cdots\} \\ &= \{\cdots, -11, -6, -1, 4, 9, 14, 19, 24, \cdots\}.\end{aligned}$$

We also have

$$\begin{aligned}[4]_5 &= \{4 + 5k : k \in \mathbb{Z}\} = \{4, 4 \pm 5, 4 \pm 10, 4 \pm 15, \cdots\} \\ &= \{\cdots, -11, -6, -1, 4, 9, 14, 19, 24, \cdots\}.\end{aligned}$$

So, clearly, $[4]_5 = [9]_5$.

Definition 1.6.5. If $x \equiv a \pmod{n}$ and $0 \leq a \leq n - 1$, then a is called the *least (nonnegative) residue* of x modulo n.

Example 1.6.6. Let $n = 7$. There are seven residue classes, modulo 7. In each of these seven residue classes, there is exactly one least residue of x modulo 7. So, the complete set of all least residues x modulo 7 is $\{0, 1, 2, 3, 4, 5, 6\}$.

$\mathbb{Z}/n\mathbb{Z}$	also denoted by $\mathbb{Z}_n$, residue classes modulo n; a ring of integers; a field if n is prime

The finite set $\mathbb{Z}/n\mathbb{Z}$ is closely related to the infinite set $\mathbb{Z}$. So, it is natural to ask if it is possible to define addition and multiplication in $\mathbb{Z}/n\mathbb{Z}$ and do some reasonable kind of arithmetic there. Surprisingly, addition, subtraction and multiplication in $\mathbb{Z}/n\mathbb{Z}$ will be much the same as that in $\mathbb{Z}$. Let us first investigate some elementary arithmetic properties of congruences.

Theorem 1.6.5. For all $a, b, c, d \in \mathbb{Z}$ and $n \in \mathbb{Z}_{>1}$, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

(1) $a \pm b \equiv c \pm d \pmod{n}$,
(2) $a \cdot b \equiv c \cdot d \pmod{n}$,
(3) $a^m \equiv b^m \pmod{n}$, $\forall m \in \mathbb{N}$.

Theorem 1.6.5 is equivalent to the following theorem, since

$$\begin{aligned} a \equiv b \pmod{n} &\iff a \bmod n = b \bmod n, \\ a \bmod n &\iff [a]_n, \\ b \bmod n &\iff [b]_n. \end{aligned}$$

Theorem 1.6.6. For all $a, b, c, d \in \mathbb{Z}$, if $[a]_n = [b]_n$, $[c]_n = [d]_n$, then

(1) $[a \pm b]_n = [c \pm d]_n$,
(2) $[a \cdot b]_n = [c \cdot d]_n$,
(3) $[a^m]_n = [b^m]_n$, $\forall m \in \mathbb{N}$.

The fact that the congruence relation modulo n is stable for addition (subtraction) and multiplication means that we can define binary operations, again called addition (subtraction) and multiplication on the set of $\mathbb{Z}/n\mathbb{Z}$ of equivalence classes modulo n as follows (in case only one n is being discussed, we can simply write $[x]$ for the class $[x]_n$):

$$\begin{aligned} [a]_n + [b]_n &= [a+b]_n \\ [a]_n - [b]_n &= [a-b]_n \\ [a]_n \cdot [b]_n &= [a \cdot b]_n \end{aligned}$$

Example 1.6.11. Let $n = 12$, then

$$\begin{aligned} [7]_{12} +_{12} [8]_{12} &= [7+8]_{12} = [15]_{12} = [3]_{12}, \\ [7]_{12} -_{12} [8]_{12} &= [7-8]_{12} = [-1]_{12} = [11]_{12}, \\ [7]_{12} \cdot_{12} [8]_{12} &= [7 \cdot 8]_{12} = [56]_{12} = [8]_{12}. \end{aligned}$$

In many cases, we may still prefer to write the above operations as follows:

$$\begin{aligned} &7 + 8 = 15 \equiv 3 \pmod{12}, \\ &7 - 8 = -1 \equiv 11 \pmod{12}, \\ &7 \cdot 8 = 56 \equiv 8 \pmod{12} \end{aligned}$$

We summarise the properties of addition and multiplication modulo n in the following two theorems.

Theorem 1.6.7. The set $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n has the following properties with respect to addition:

(1) Closure: $[x] + [y] \in \mathbb{Z}/n\mathbb{Z}$, for all $[x], [y] \in \mathbb{Z}/n\mathbb{Z}$.
(2) Associative: $([x] + [y]) + [z] = [x] + ([y] + [z])$, for all $[x], [y], [z] \in \mathbb{Z}/n\mathbb{Z}$.
(3) Commutative: $[x] + [y] = [y] + [x]$, for all $[x], [y] \in \mathbb{Z}/n\mathbb{Z}$.
(4) Identity, namely, $[0]$.
(5) Additive inverse: $-[x] = [-x]$, for all $[x] \in \mathbb{Z}/n\mathbb{Z}$.

Theorem 1.6.8. The set $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n has the following properties with respect to multiplication:

(1) Closure: $[x] \cdot [y] \in \mathbb{Z}/n\mathbb{Z}$, for all $[x], [y] \in \mathbb{Z}/n\mathbb{Z}$.
(2) Associative: $([x] \cdot [y]) \cdot [z] = [x] \cdot ([y] \cdot [z])$, for all $[x], [y], [z] \in \mathbb{Z}/n\mathbb{Z}$.
(3) Commutative: $[x] \cdot [y] = [y] \cdot [x]$, for all $[x], [y] \in \mathbb{Z}/n\mathbb{Z}$.
(4) Identity, namely, $[1]$.
(5) Distributivity of multiplication over addition: $[x] \cdot ([y] + [z]) = ([x] \cdot [y]) + ([x] \cdot [z])$, for all $[x], [y], [z] \in \mathbb{Z}/n\mathbb{Z}$.

Definition 1.6.10. Two integers x and y are said to be multiplicative inverses if

$$xy \equiv 1 \pmod{n}, \tag{1.228}$$

where n is a positive integer greater than 1.

Summary

In this week, we learned about the divisibility of two numbers and the congruence between two numbers.

Week 6 Modular Arithmetic Continued Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-07 5:45 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Modular Arithmetic Continued	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: May 07, 2020

Essential Question:

What are the operations that we can perform in modular arithmetic?

Questions/Cues:

- How do we perform addition and/or subtraction in modular arithmetic?
- How do we perform multiplication in modular arithmetic?
- How do we perform division in modular arithmetic?

Notes

- Addition/Subtraction(in mod) = map the operands(numbers) being added or subtracted to the minimal subset of the mod term, remembering that the minimal subset is 0 to k-1. Mapping the numbers to the subset simply mean that each number is replaced by its equivalent congruency from minimal subset. Remember that this equivalent congruency is obtained by dividing each number being operated on by the mod term ie. 12 and comparing the remainder to see which number in the minimal subset each number is congruent to and replacing it with such and performing the said operation, if after the operation the resultant number is not in the subset, divide the number by the mod term until the number is one of the ones in the minimal subset

- Addition and subtraction (*mod k*): map numbers to Min_k, then sum or subtract. Adjust if result is not in Min_k

- Examples(*mod12*):mapping to Min_{12}={0,1,2,3,4,5,6,7,8,9,10,11}

$$\overset{\equiv 2}{14} + \overset{\equiv 4}{28} \equiv 2 + 4 = 6 \rightarrow 14+28 \equiv 6 (mod\ 12)$$

$$\overset{\equiv 4}{28} - \overset{\equiv 2}{14} \equiv 4 - 2 = 2 \rightarrow 28-14 \equiv 2 (mod\ 12)$$

$$\overset{\equiv 11}{11} + \overset{\equiv 4}{28} \equiv 11+4 = \underset{\text{remap}}{15} \equiv 3 \rightarrow 11+28 \equiv 3 (mod 12)$$

- Multiplication(in mod) = map the operands(numbers) being multiplied to the minimal subset of the mod term, remembering that the minimal subset is 0 to k-1. Mapping the numbers to the subset simply mean that each number is replaced by its equivalent congruency from minimal subset. Remember that this equivalent congruency is obtained by dividing each number being operated on by the mod term ie. 12 and comparing the remainder to see which number in the minimal subset each number is congruent to and replacing it with such and performing the multiplication, if after the multiplication the resultant number is not in the subset, divide the number by the mod term until the number is one of the ones in the minimal subset

- Multiplication (*mod k*): map numbers to Min_k, then multiply. Adjust if result is not in Min_k

- Examples(*mod 12*): mapping to {0,1,2,3,4,5,6,7,8,9,10,11}

  $\overset{\equiv 2}{14} \times \overset{\equiv 4}{28} \equiv 2 \times 4 = 8 \rightarrow 14 \times 28 \equiv 8 (mod\ 12)$

  $\overset{\equiv 11}{11} \times \overset{\equiv 4}{28} \equiv 11 \times 4 = 44 \equiv 8 \rightarrow 11 \times 28 \equiv 8 (mod 12)$

- Division(in mod) = Division in modular arithmetic can be straightforwardly extended and applied like the previous operations mentioned. When mapping a "divisor/divided" to the minimal subset, you must make sure that the dividend is defined during congruency because division by zero is not defined. To extend division, you must first define the multiplicative inverse in modulus k, the multiplicative inverse m^{-1} of the integer m is such that m times m^{-1} = 1 in modulus k. Then, we can define the division of two numbers in modulus k as $a \times b^{-1} (\bmod k)$

- Division cannot be straightforwardly extended

- Example (*mod6*) 4/12 not defined since 12 ≡0 (*mod 6*)

- Define first <u>multiplicative inverse</u> (*mod k*)

- Multiplicative inverse m^{-1} of integer m : $m \times m^{-1} = 1 (mod\ k)$

- Then define a/b (*mod k*) as $a \times b^{-1}$ adopting multiplication
- Example: find 2^{-1} (*mod 7*)? → Min_7={0,1,2,3,4,5,6}

  Try all integers in Min_7

  2×0≡0 (*mod 7*) 2×1≡2 (*mod 7*) 2×2≡4(*mod 7*) 2×3≡ 6(*mod 7*)
  2×4≡1(*mod 7*) 2×5≡3 (*mod 7*) 2×6≡6 (*mod 7*)

  → 2^{-1}=4(*mod 7*)

- Division Example: $6/2 (\bmod 7) = 6 \times 2^{-1} = 6 \times 4 = 24 \equiv 3 (\bmod 7)$

- *Note: inverse cannot always be found*

  Ex. : inverse of 2(*mod4*) $Min_4=\{0,1,2,3\}$

  $2\times0\equiv0\ (mod\ 4)$ $2\times1=2\equiv2$ $2\times2=4\equiv0$ $2\times3=6\equiv2$

Summary

In this week, we learned about the different operations we can perform in modular arithmetic.

Week 7 Angles, Triangles, Trigonometry Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-14 4:06 PM
Author:	SUKHJIT MANN		

Cornell Notes	Angles, triangles, trigonometry	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: May 14, 2020

Essential Question:

What are angles and what is trigonometry and how are these related to the study of triangles?

Questions/Cues:

- What is an angle?
- What are some special types of angles, how do we measure radians and how do we convert between radians/degrees?
- What are some properties of triangles?
- What are similar triangles?
- What are properties of right triangles?

Notes

What is an angle?

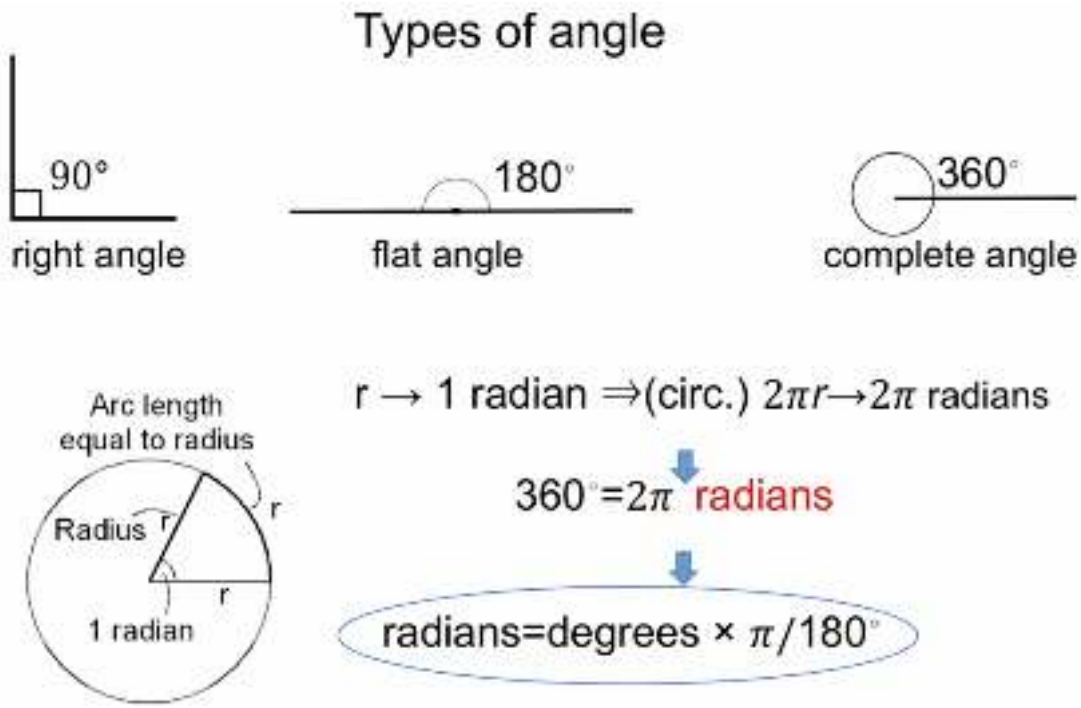
- It is a measure of the separation of two rays emanating from a vertex *v*

- It is measured in degrees (sexagesimal) or radians
- 1 Degree is 1/180 of a flat angle

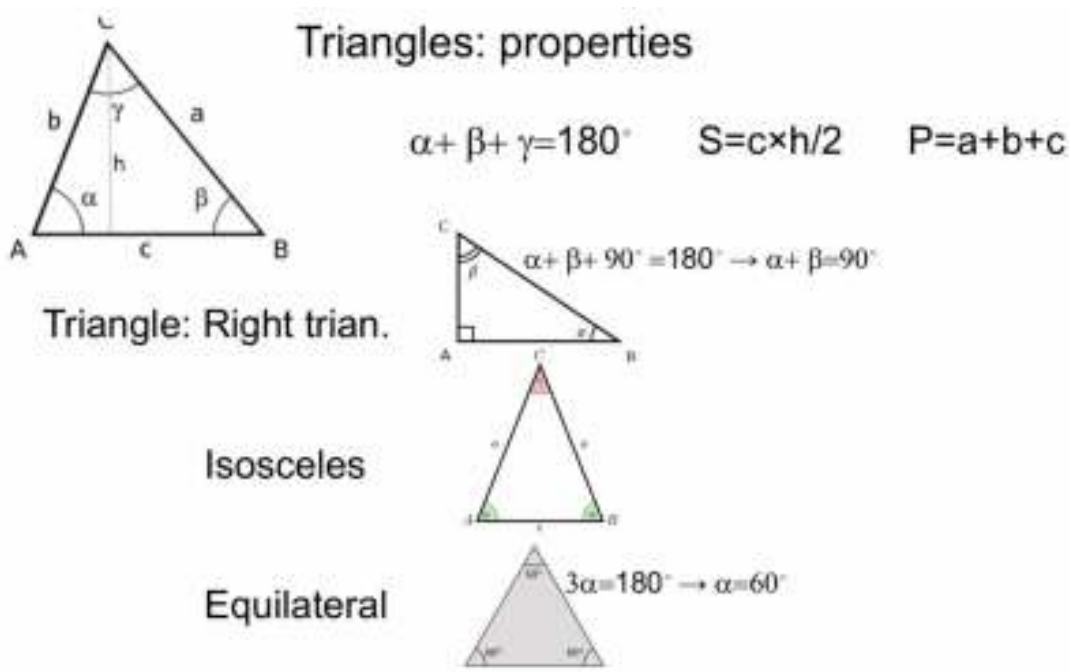
- 1 min is 1/60 of a degree, 1 sec is 1/60 of min
- ex. 35° 23' 12"

    - A flat angle is just the separation between two rays that emanate from the vertex and they depart in opposite direction. A flat angle corresponds to $180°$

- When working with degrees, we are working in base 60



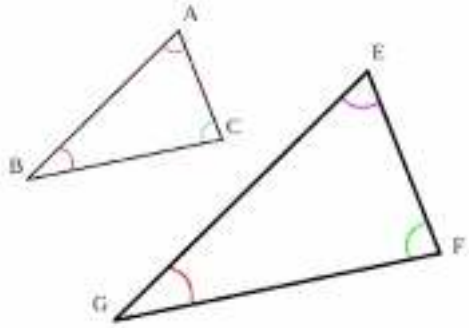
- The complete angle is the angle that is formed by two rays emanating from a vertex when they coincide
- Angles are periodic, in degree they periodical of 360°



- In a isosceles triangle, two sides are same length and the angles adjacent to the non-equal side are equal

- Similar Triangles = have the same angles but they have sides which are rescaled by the same rescaling factor; they have the same angles and proportional sides. This means the ratio between each side of one triangle and the corresponding side of the similar triangle are the same

Similar Triangles

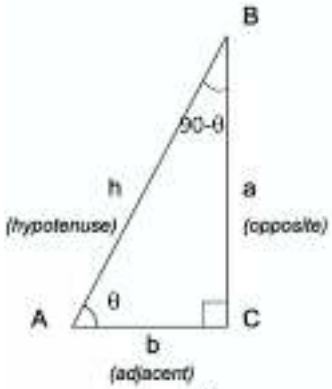


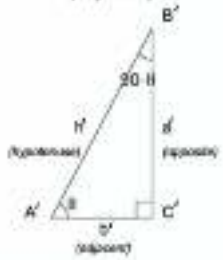
Similar Triangles
rescale one (zoom in or out) and will coincide with the other

Same angles, proportional sides:

→ AB/EG=AC/EF=BC/GF

Right Triangles: properties





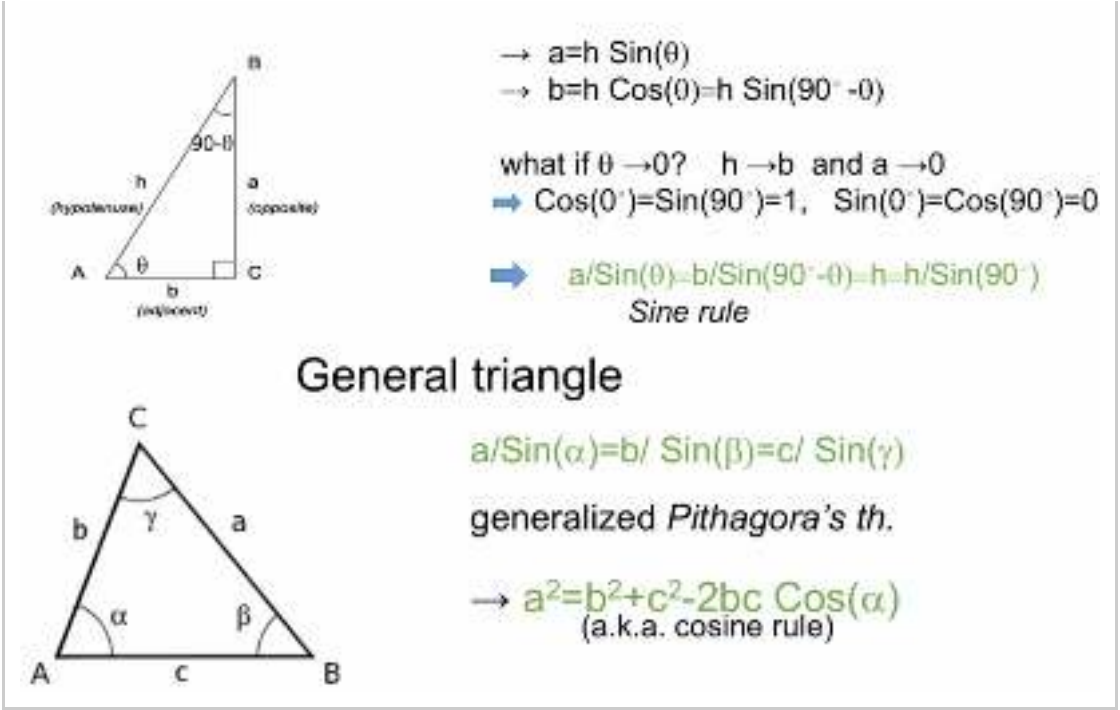
h/h'=b/b' → b/h=b'/h'=Cos(θ)=Sin(90-θ)
h/h'=a/a' → a/h=a'/h'=Cos(90-θ)=Sin(θ)
a/a'=b/b' → a/b=a'/b'=$\frac{\mathrm{Sin}(\theta)}{\mathrm{Cos}(\theta)}$ =Tan(θ)

1)Sin(θ)=a/h=$\frac{opposite}{hypotenuse}$ → a=h Sin(θ)
2)Cos(θ)=b/h=$\frac{adjacent}{hypotenuse}$ → b=h Cos(θ)
3)Tan(θ)=a/b=$\frac{opposite}{adjacent}$ → a=b Tan(θ)

$a^2+b^2=h^2$ *Pithagora's theorem*

From 1) and 2) it follows
$h^2\,\mathrm{Sin}^2(\theta)+h^2\,\mathrm{Cos}^2(\theta)=h^2$

→ $\mathrm{Sin}^2(\theta)+\mathrm{Cos}^2(\theta)=1$



Summary

In this week, we learned about what an angle is. Alongside this, we looked at properties of triangles, Pythagoras theorem, the laws of sines and cosines

Week 7 Angles, Triangles, Trigonometry Reading Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-14 2:45 PM
Author:	SUKHJIT MANN		

Cornell Notes	Angles, triangles, trigonometry	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: May 12, 2020

Essential Question:

What are angles and what is trigonometry and how are these related to the study of triangles?

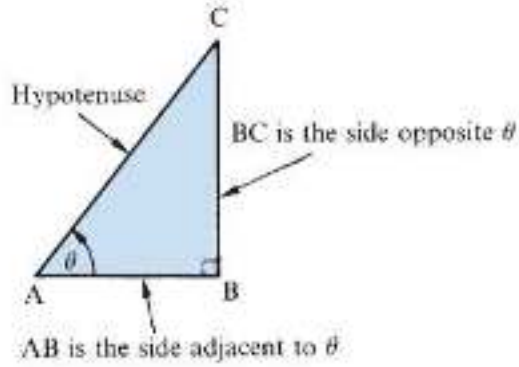
Questions/Cues:

- What is a right angle?
- How are a side and an angle referred to in a triangle?
- What is the hypotenuse?
- What are three basic trigonometric ratios?
- Given one or more of the trigonometric ratio, how do we find theta or how do we denote the inverse of a trig ratio?
- What are Quadrants?
- How do we calculate angles outside the range of 0° to 360°?
- What are the graphs of $y = \sin\theta, y = \cos\theta, \text{and } y = \tan\theta$ like?
- What is an identity?
- What is the most crucial identity in all of trigonometry?
- How do we solve equations involving trigonometric ratios?

Notes

- Right angle = A 90 degree angle, denoted a "square-box" or ∟
- Side (triangle) = A side is made my joining two points together and the resultant side is simply referred as the two uppercase letter that make up the side. ie. AB = the side joining points A and B
- Angle (triangle) = The angle at a particular point is made by joining two sides together. For example, in the figure below, the angle at A, made by the sides AB and AC, is written $\measuredangle BAC$, $\measuredangle A$ or simply as A. In the figure below, this angle is labelled as θ

Figure 22.1
A right-angled triangle ABC



- Hypotenuse = The side opposite the right angle

$$\sin\theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{AB}{AC}$$

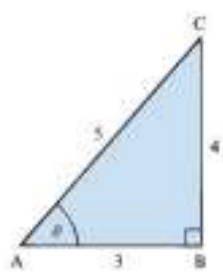
$$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta} = \frac{BC}{AB}$$

  - All trig ratios are defined as the ratio of two lengths and so they themselves are with no units
  - Since the hypotenuse is always the longest side of a triangle, $\sin\theta$ and $\cos\theta$ can never be greater than 1

WORKED EXAMPLES

22.1 Calculate $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $\triangle ABC$ as shown in Figure 22.2.

Figure 22.2
$\triangle ABC$ for Worked Example 22.1



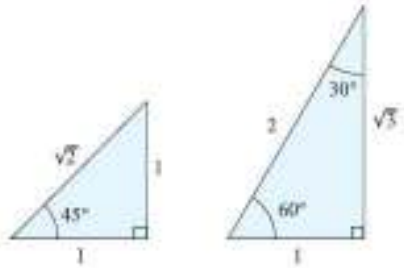
Solution

$$\sin\theta = \frac{BC}{AC} = \frac{4}{5} = 0.8$$

$$\cos\theta = \frac{AB}{AC} = \frac{3}{5} = 0.6$$

$$\tan\theta = \frac{BC}{AB} = \frac{4}{3} = 1.3333$$

Figure 22.3



22.2 Using the triangles shown in Figure 22.3 write down expressions for

(a) sin 45°, cos 45°, tan 45°

(b) sin 60°, cos 60°, tan 60°

(c) sin 30°, cos 30°, tan 30°

Solution (a) $\sin 45^\circ = \frac{1}{\sqrt{2}}$, $\cos 45^\circ = \frac{1}{\sqrt{2}}$, $\tan 45^\circ = \frac{1}{1} = 1$

(b) $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$

(c) $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$

Notice that we have left our solutions in their exact forms using square roots if necessary rather than given their decimal approximations. These, so-called, **surd forms** are frequently used.

****Note that the triangles in the above example are also referred to as special triangles, with one being a 45, 45, 90 triangle and the other being a 30, 60, 90 triangle**
****Also an angle in degrees has the symbol ° Otherwise assume that the angle is measured in radians.**

WORKED EXAMPLES

22.3 Use a scientific calculator to evaluate

(a) sin 65° (b) cos 17° (c) tan 50° (d) sin 1 (e) cos 1.5
(f) tan 0.5

Solution For (a), (b) and (c) ensure your calculator is set to DEGREES, and not to RADIANS or GRADS.

(a) sin 65° = 0.9063 (b) cos 17° = 0.9563 (c) tan 50° = 1.1918

Now, change the MODE of your calculator in order to work in radians. Check that

(d) sin 1 = 0.8415 (e) cos 1.5 = 0.0707 (f) tan 0.5 = 0.5463

22.4 Prove

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

Solution From Figure 22.1, we have

$$\sin\theta = \frac{BC}{AC} \qquad \cos\theta = \frac{AB}{AC}$$

and so

$$\frac{\sin\theta}{\cos\theta} = \frac{BC/AC}{AB/AC} = \frac{BC}{AC} \times \frac{AC}{AB} = \frac{BC}{AB}$$

But

$$\tan\theta = \frac{BC}{AB}$$

and so

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

- Finding angle given trig ratios = If $\sin\theta = x$, we write $\theta = \sin^{-1}x$
  - This is the same as saying that θ is the angle whose sine is x
  - The superscript -1 does not denote a power. It's a notation for the inverse of the trigonometrical ratios

WORKED EXAMPLES

22.5 Use a scientific calculator to evaluate

(a) $\sin^{-1} 0.5$ (b) $\cos^{-1} 0.3$ (c) $\tan^{-1} 2$

Solution (a) Using the $\sin^{-1}$ button we have

$$\sin^{-1} 0.5 = 30^\circ$$

This is another way of saying that $\sin 30^\circ = 0.5$.

(b) Using a scientific calculator we see

$$\cos^{-1} 0.3 = 72.5^\circ$$

that is,

$$\cos 72.5^\circ = 0.3$$

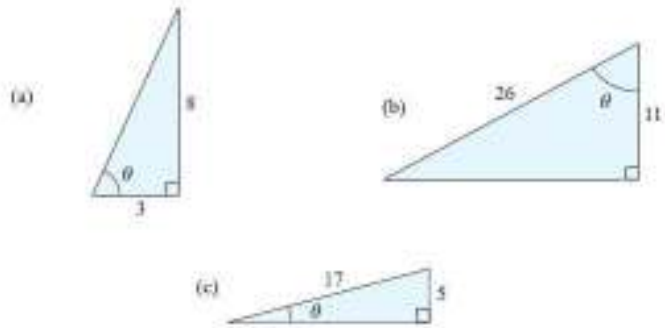
(c) Using a calculator we see $\tan^{-1} 2 = 63.4^\circ$ and hence $\tan 63.4^\circ = 2$.

22.6 Calculate the angles θ in Figures 22.4(a), (b) and (c).

Solution (a) In Figure 22.4(a) we are given the length of the side opposite θ and the length of the adjacent side. Hence we can make use of the tangent ratio. We write

$$\tan\theta = \frac{8}{3} = 2.6667$$

Figure 22.4



Therefore θ is the angle whose tangent is 2.6667, written

$$\theta = \tan^{-1} 2.6667$$

Using a calculator we can find the angle θ in either degrees or radians. If your calculator is set to degree mode, verify that $\theta = 69.44°$. Check also that you can obtain the equivalent answer in radians, that is 1.212.

(b) In Figure 22.4(b) we are given the length of the side adjacent to θ and the length of the hypotenuse. Hence we use the cosine ratio:

$$\cos\theta = \frac{11}{26} = 0.4231$$

Therefore

$$\theta = \cos^{-1} 0.4231$$

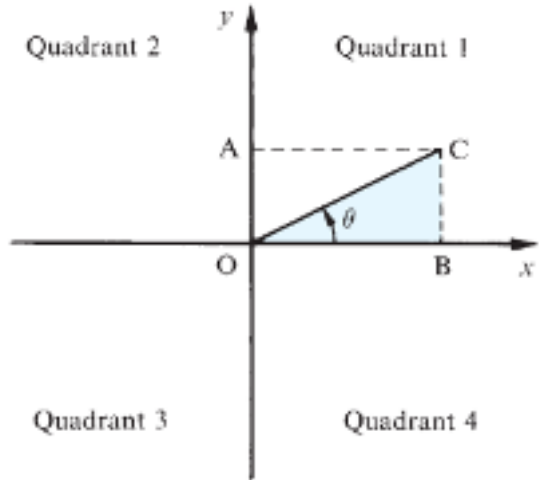
Using a calculator check that $\theta = 64.97°$.

(c) In Figure 22.4(c) $\sin\theta = \frac{5}{17}$. Hence

$$\theta = \sin^{-1}\frac{5}{17} = 17.10°$$

- Quadrants (xy plane) = the xy plane can be divided into in four quadrants as shown in the figure below. The origin is O. The arm formed by OC can rotate into any of the quadrants. We measure the anticlockwise angle from the positive x axis to the arm and call this angle θ
  - When arm is in quadrant 1, $0° \leq \theta \leq 90°$
  - When arm is in quadrant 2, $90° \leq \theta \leq 180°$
  - When arm is in quadrant 3, $180° \leq \theta \leq 270°$
  - When arm is in quadrant 4, $270° \leq \theta \leq 360°$
    - On occasion, angles are measured in a clockwise direction from the positive x axis. In this case, these are conventionally taken to be negative. For example, angles of $-60°$ and $-120°$ correspond to same position as $300° \text{ and } 240°$ each respectively as shown below

Figure 23.1
The plane is divided into four quadrants



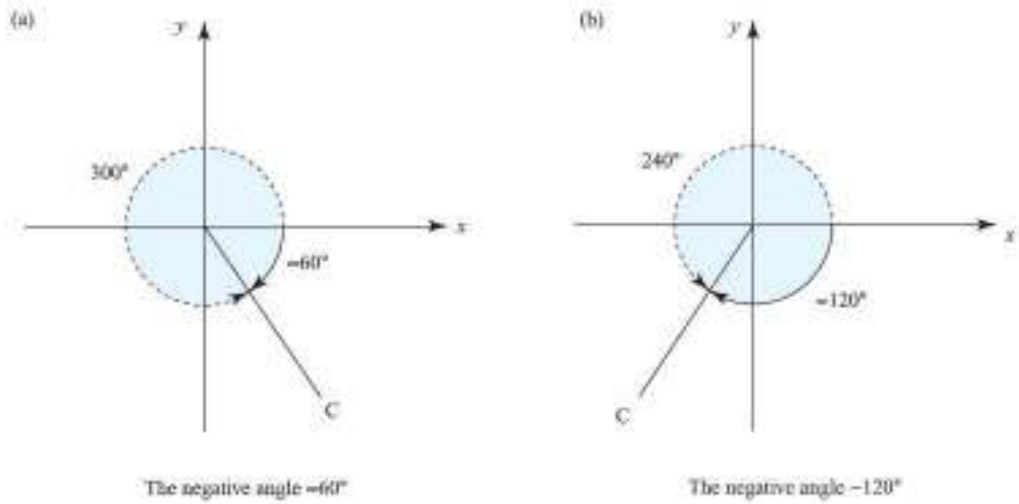
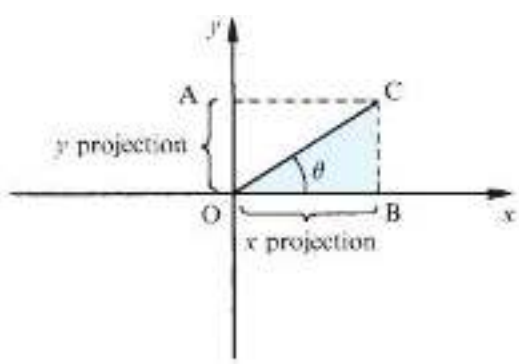


Figure 23.2
The negative angles of −60° and −120°

- x-y projections = length of the side of the triangle formed by the arm OC. OB is the x-projection and OA is the y-projection. OA and OB can be negative depending on the position of the arm OC, but the length of the arm is always positive

Figure 23.3
The x projection of OC is OB; the y projection of OC is OA



$$\sin\theta = \frac{y \text{ projection of arm OC}}{\text{OC}} = \frac{\text{OA}}{\text{OC}}$$

$$\cos\theta = \frac{x \text{ projection of arm OC}}{\text{OC}} = \frac{\text{OB}}{\text{OC}}$$

$$\tan\theta = \frac{y \text{ projection of arm OC}}{x \text{ projection of arm OC}} = \frac{\text{OA}}{\text{OB}}$$

Note that the extended definition is in terms of projections and so θ is not limited to a maximum value of 90°.

Projections in the first quadrant

Consider the arm OC in the first quadrant, as shown in Figure 23.3. From the right-angled triangle OCB we have

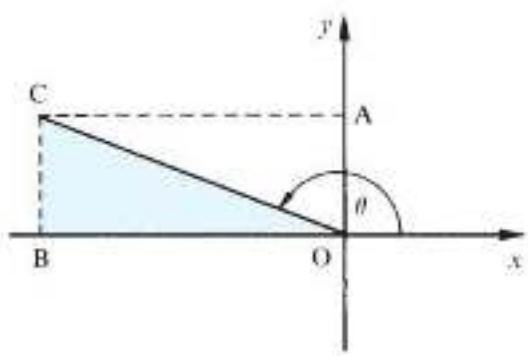
$$\sin\theta = \frac{\text{BC}}{\text{OC}} \qquad \cos\theta = \frac{\text{OB}}{\text{OC}} \qquad \tan\theta = \frac{\text{BC}}{\text{OB}}$$

Alternatively, using the extended definition we could also write

$$\sin\theta = \frac{\text{OA}}{\text{OC}} \qquad \cos\theta = \frac{\text{OB}}{\text{OC}} \qquad \tan\theta = \frac{\text{OA}}{\text{OB}}$$

Noting that OA = BC, we see that the two definitions are in agreement when $0° \leqslant \theta \leqslant 90°$.

Figure 23.4
When OC is in the second quadrant the x projection is negative



Projections in the second quadrant

We now consider the arm in the second quadrant, as shown in Figure 23.4. The x projection of OC is onto the negative part of the x axis; the y projection of OC is onto the positive part of the y axis. Hence $\sin\theta$ is positive, whereas $\cos\theta$ and $\tan\theta$ are negative.

Projections in the third quadrant

When the arm is in the third quadrant, as shown in Figure 23.5, the x and y projections are both negative. Hence for $180° < \theta < 270°$, $\sin\theta$ and $\cos\theta$ are negative and $\tan\theta$ is positive.

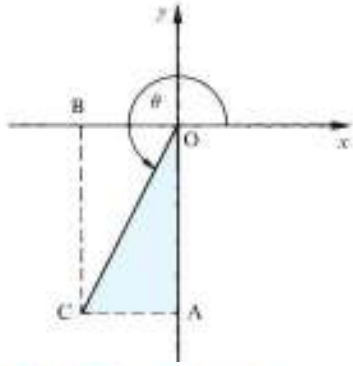


Figure 23.5
Both x and y projections are negative

Projections in the fourth quadrant

Finally the arm OC is rotated into the fourth quadrant, as shown in Figure 23.6. The x projection is positive, the y projection is negative, and so $\sin\theta$ and $\tan\theta$ are negative and $\cos\theta$ is positive.

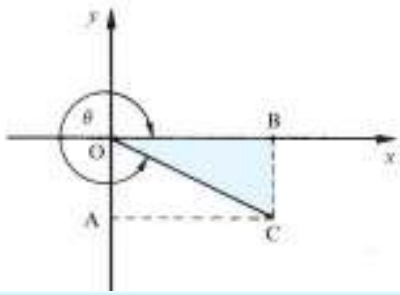


Figure 23.6
The x projection is positive and the y projection is negative

	Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
$\sin\theta$	positive	positive	negative	negative
$\cos\theta$	positive	negative	negative	positive
$\tan\theta$	positive	negative	positive	negative

- Angles (outside the degree range) = This is done by adding or subtracting $360°$ from an angle. By adding/subtracting $360°$ is equivalent to rotating the arm through a complete revolution. This leaves the position unchanged and thus leaves the trigonometrical ration unaltered. This is mathematically stated by the following:

$$\sin\theta = \sin(\theta + 360°) = \sin(\theta - 360°)$$

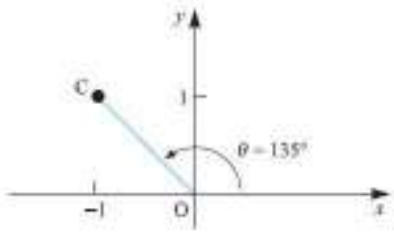
$$\cos\theta = \cos(\theta + 360°) = \cos(\theta - 360°)$$

$$\tan\theta = \tan(\theta + 360°) = \tan(\theta - 360°)$$

WORKED EXAMPLES

23.1 Find $\tan\theta$ in Figure 23.7.

Figure 23.7



Solution Angle θ is in the second quadrant. Hence its tangent will be negative. We can find its value from

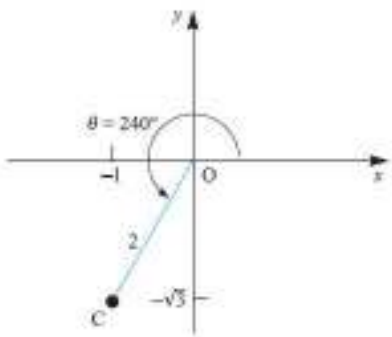
$$\tan\theta = \frac{y \text{ projection}}{x \text{ projection}}$$

$$= \frac{1}{-1}$$

$$= -1$$

23.2 Find $\sin\theta$ in Figure 23.8.

Figure 23.8



Solution Angle θ is in the third quadrant. Hence its sine will be negative. We can find its value from

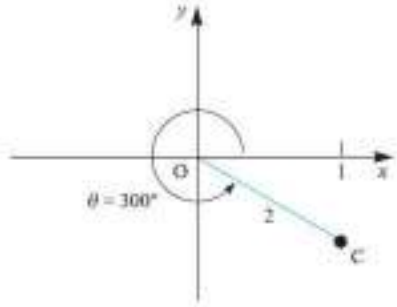
$$\sin\theta = \frac{x \text{ projection}}{OC}$$

$$= \frac{-\sqrt{3}}{2}$$

$$= -\frac{\sqrt{3}}{2}$$

23.3 Find $\cos\theta$ in Figure 23.9.

Figure 23.9



Solution Angle θ is in the fourth quadrant. Hence its cosine will be positive. We can find its value from

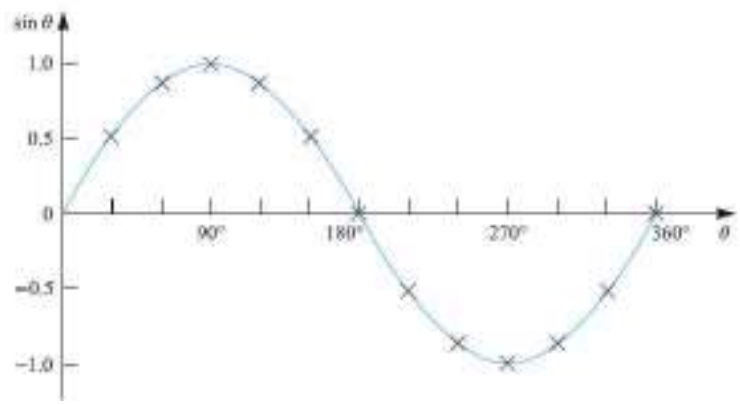
$$\cos\theta = \frac{x \text{ projection}}{OC}$$

$$= \frac{1}{2}$$

Table 23.1

θ	0°	30°	60°	90°	120°	150°	180°
$\sin\theta$	0	0.5	0.8660	1	0.8660	0.5	0
θ	210°	240°	270°	300°	330°	360°	
$\sin\theta$	−0.5	−0.8660	−1	−0.8660	−0.5	0	

Figure 23.11
The function $y = \sin\theta$



- If we extend values of θ to below 0° and above 360°, the values of $\sin\theta$ are simply repeated. The pattern established by graph of $\sin\theta$ is repeated every 360°

Figure 23.12
The values of $\sin\theta$ repeat every 360°

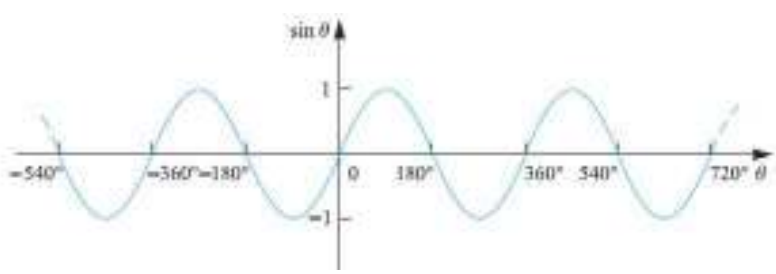


Table 23.2

θ	0°	30°	60°	90°	120°	150°	180°
$\cos\theta$	1	0.8660	0.5	0	−0.5	−0.8660	−1

θ	210°	240°	270°	300°	330°	360°
$\cos\theta$	−0.8660	−0.5	0	0.5	0.8660	1

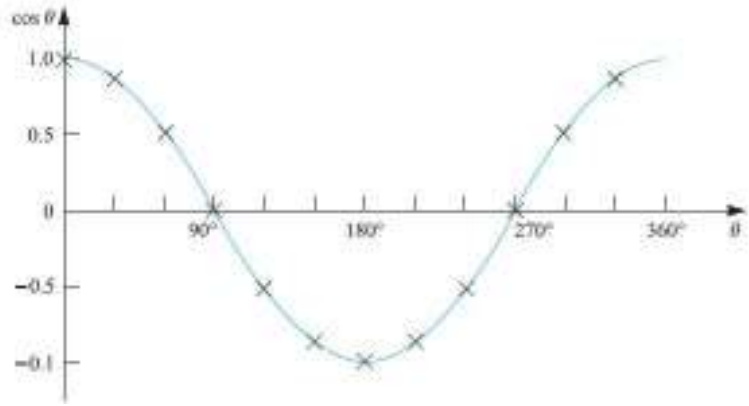


Figure 23.13
The function $y = \cos\theta$

- If we extend values of θ to below 0° and above 360°, the values of $\cos\theta$ are simply repeated. The pattern established by the graph of $\cos\theta$ is repeated every 360°

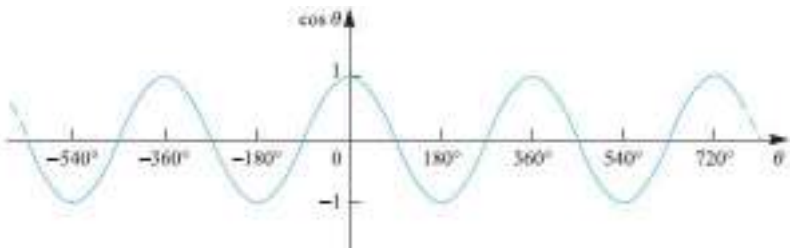


Figure 23.14
The values of $\cos\theta$ repeat every 360°

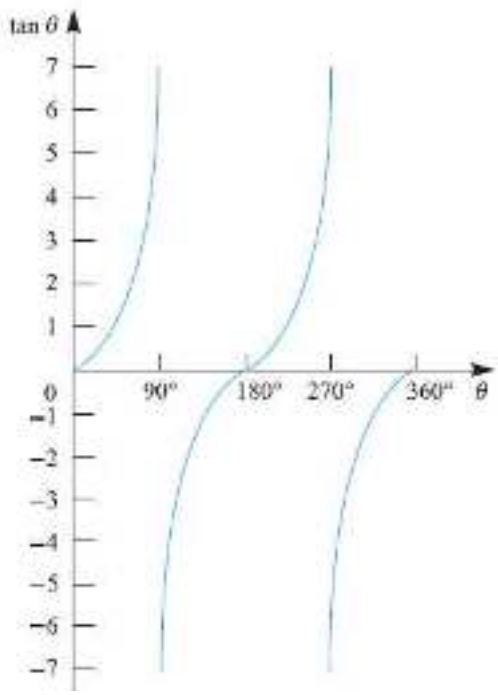
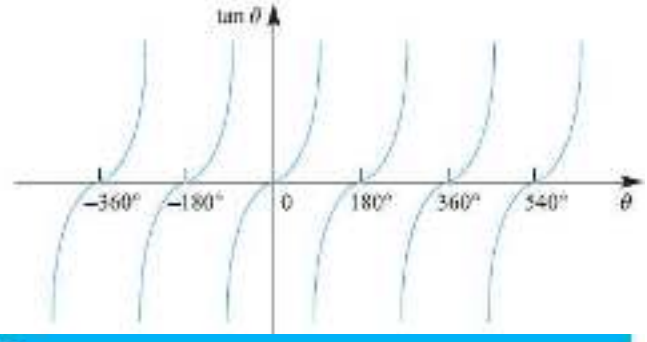


Figure 23.15
The function $y = \tan\theta$

- The pattern established by $\tan\theta$ is repeated every 180°. The values of $\tan\theta$ extend from minus infinity to plus infinity

Figure 23.16
The values of $\tan\theta$ repeat every 180°



WORKED EXAMPLES

23.4 Draw $y = \sin 2\theta$ for $0° \leqslant \theta \leqslant 180°$.

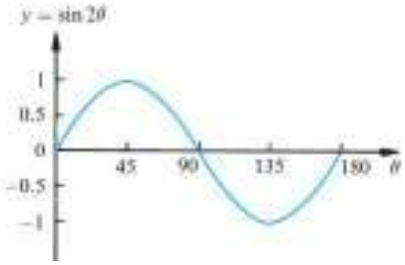
Solution Values of θ and $\sin 2\theta$ are given in Table 23.3. A graph of $y = \sin 2\theta$ is shown in Figure 23.17.

Table 23.3
Values of θ and corresponding values of $\sin 2\theta$

θ	0	15	30	45	60	75	90	105
2θ	0	30	60	90	120	150	180	210
$\sin 2\theta$	0	0.5000	0.8660	1	0.8660	0.5000	0	−0.5000

θ	120	135	150	165	180
2θ	240	270	300	330	360
$\sin 2\theta$	−0.8660	−1	−0.8660	−0.5000	0

Figure 23.17
A graph of $y = \sin 2\theta$



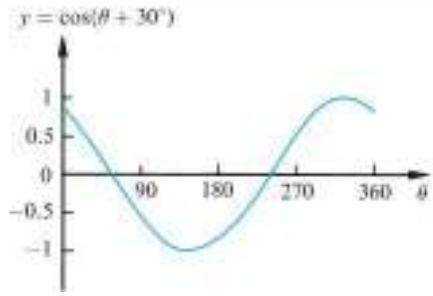
23.5 Draw a graph of $y = \cos(\theta + 30°)$ for $0° \leqslant \theta \leqslant 360°$.

Solution A table of values is drawn up in Table 23.4. Figure 23.18 shows a graph of $y = \cos(\theta + 30°)$.

Table 23.4
Values of θ and corresponding values of $\cos(\theta + 30°)$

θ	0	30	60	90	120	150	180	210
$\theta + 30$	30	60	90	120	150	180	210	240
$\cos(\theta + 30)$	0.8660	0.5000	0	−0.5000	−0.8660	−1	−0.8660	−0.5000
θ	240	270	300	330	360			
$\theta + 30$	270	300	330	360	390			
$\cos(\theta + 30)$	0	0.5000	0.8660	1	0.8660			

Figure 23.18
Graph of $y = \cos(\theta + 30°)$



- Identity = looks like an equation, crucial difference is that the left-hand side and right-hand side of an identity are equal for all values of the variable involved. On the other hand, an equation contains one or more unknown quantities, which must be found before the left-hand and right-hand sides are equal
  - $\frac{\sin\theta}{\cos\theta} = \tan\theta$ is an identity since $\frac{\sin\theta}{\cos\theta}$ and $\tan\theta$ have the same value whatever the value of θ
- The fundamental identity of trigonometry = $\sin^2 A + \cos^2 A = 1$
  - the symbols/variables A and B stand for any angle we choose
  - ****Note that** $(\cos A)^2$ is usually written as $\cos^2 A$ and $(\sin A)^2$ is written as $\sin^2 A$
  - The fundamental identity shows that sin A and cos A are closely related. Knowing sin A the identity can be used to calculate cos A and vice versa
  - By manipulating this identity we are able to derive a list of common trigonometric identities

Table 24.1
Common trigonometrical identities

$$\frac{\sin A}{\cos A} = \tan A$$

$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$

$$\sin(A-B) = \sin A \cos B - \sin B \cos A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos 2A = (\cos A)^2 - (\sin A)^2 = \cos^2 A - \sin^2 A$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)$$

$$\sin A = -\sin(-A)$$

$$\cos A = \cos(-A)$$

$$\tan A = -\tan(-A)$$

$$\sin \theta = \sin(180^\circ - \theta)$$

$$\cos \theta = -\cos(180^\circ - \theta)$$

$$\tan \theta = -\tan(180^\circ - \theta)$$

$$\sin \theta = -\sin(\theta - 180^\circ)$$

$$\cos \theta = -\cos(\theta - 180^\circ)$$

$$\tan \theta = \tan(\theta - 180^\circ)$$

$$\sin \theta = -\sin(360^\circ - \theta)$$

$$\cos \theta = \cos(360^\circ - \theta)$$

$$\tan \theta = -\tan(360^\circ - \theta)$$

WORKED EXAMPLE

24.1 (a) Show that $\sin\theta = \sin(180^\circ - \theta)$.

(b) Show that $\cos\theta = -\cos(180^\circ - \theta)$.

(c) From parts (a) and (b) deduce that $\tan\theta = -\tan(180^\circ - \theta)$.

Solution (a) Consider $\sin(180^\circ - \theta)$. Because the right-hand side is the sine of the difference of two angles we use the identity for $\sin(A - B)$:

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

Letting $A = 180^\circ$ and $B = \theta$ we obtain

$$\begin{aligned}\sin(180^\circ - \theta) &= \sin 180^\circ \cos\theta - \sin\theta \cos 180^\circ \\ &= 0.\cos\theta - \sin\theta(-1) \\ &= \sin\theta\end{aligned}$$

Hence $\sin\theta = \sin(180^\circ - \theta)$.

(b) Consider $\cos(180^\circ - \theta)$. Here we are dealing with the cosine of the difference of two angles. We use the identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Letting $A = 180^\circ$ and $B = \theta$ we obtain

$$\begin{aligned}\cos(180^\circ - \theta) &= \cos 180^\circ \cos\theta + \sin 180^\circ \sin\theta \\ &= -1(\cos\theta) + 0(\sin\theta) \\ &= -\cos\theta\end{aligned}$$

Hence $\cos\theta = -\cos(180^\circ - \theta)$.

(c) We note from Table 24.1 that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and so, using the results of parts (a) and (b),

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta} \\ &= \frac{\sin(180^\circ - \theta)}{-\cos(180^\circ - \theta)} \\ &= -\tan(180^\circ - \theta)\end{aligned}$$

WORKED EXAMPLE

24.2 Use a calculator to evaluate the following trigonometrical ratios and hence verify the identities obtained in Worked Example 24.1.

(a) $\sin 30°$ and $\sin 150°$

(b) $\cos 70°$ and $\cos 110°$

(c) $\tan 30°$ and $\tan 150°$

Solution (a) Using a calculator verify that $\sin 30° = 0.5$. Similarly, $\sin 150° = 0.5$. We see that

$$\sin 30° = \sin 150° = \sin(180° - 30°)$$

This verifies the first of the identities for the case when $\theta = 30°$.

(b) $\cos 70° = 0.3420$ and $\cos 110° = -0.3420$. Hence

$$\cos 70° = -\cos 110° = -\cos(180° - 70°)$$

This verifies the second of the identities for the case when $\theta = 70°$.

(c) $\tan 30° = 0.5774$ and $\tan 150° = -0.5774$. So

$$\tan 30° = -\tan 150° = -\tan(180° - 30°)$$

This verifies the third of the identities for the case when $\theta = 30°$.

WORKED EXAMPLES

24.3 Evaluate the following trigonometrical ratios using a calculator and hence verify the results in the previous Key point.

(a) $\sin 30°$ and $\sin 210°$

(b) $\cos 40°$ and $\cos 220°$

(c) $\tan 50°$ and $\tan 230°$

Solution (a) $\sin 30° = 0.5$ and $\sin 210° = -0.5$. Note that $\sin 210° = -\sin 30°$.

(b) $\cos 40° = 0.7660$ and $\cos 220° = -0.7660$. Note that $\cos 220° = -\cos 40°$.

(c) $\tan 50° = 1.1918$ and $\tan 230° = 1.1918$. Note that $\tan 50° = \tan 230°$.

24.4 Evaluate the following trigonometrical ratios using a calculator and hence verify the results in the previous Key point.

(a) $\sin 70°$ and $\sin 290°$

(b) $\cos 40°$ and $\cos 320°$

(c) $\tan 20°$ and $\tan 340°$

Solution (a) $\sin 70° = 0.9397$, $\sin 290° = -0.9397$ and so $\sin 290° = -\sin 70°$.

(b) $\cos 40° = 0.7660$, $\cos 320° = 0.7660$ and so $\cos 40° = \cos 320°$.

(c) $\tan 20° = 0.3640$, $\tan 340° = -0.3640$ and so $\tan 340° = -\tan 20°$.

WORKED EXAMPLES

24.5 Simplify

$$1 - \sin A \cos A \tan A$$

Solution

$$1 - \sin A \cos A \tan A = 1 - \sin A \cos A \left(\frac{\sin A}{\cos A}\right)$$

$$= 1 - \sin^2 A$$

$$= \cos^2 A \qquad \text{since } \cos^2 A + \sin^2 A = 1$$

24.6 Show that

$$\cos^4 A - \sin^4 A = \cos 2A$$

Solution We note that

$$\cos^4 A - \sin^4 A = (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$

$$= 1(\cos^2 A - \sin^2 A) \qquad \text{using } \cos^2 A + \sin^2 A = 1$$

$$= \cos 2A \qquad \text{using Table 24.1}$$

24.7 Show that

$$\frac{\sin 6\theta - \sin 4\theta}{\sin \theta} = 2 \cos 5\theta$$

Solution We note the identity

$$\sin A - \sin B = 2 \sin\left(\frac{A - B}{2}\right) \cos\left(\frac{A + B}{2}\right)$$

Letting $A = 6\theta$, $B = 4\theta$ we obtain

$$\sin 6\theta - \sin 4\theta = 2 \sin\theta \cos 5\theta$$

and so

$$\frac{\sin 6\theta - \sin 4\theta}{\sin \theta} = 2 \cos 5\theta$$

- Trigonometric Equations = make use of the inverse trigonometric functions $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$
  - For example, since $\sin 30^\circ = 0.5$ we write $30^\circ = \sin^{-1}0.5$ and say 30° is the angle who sine is 0.5. It is also true that $\sin 150^\circ = 0.5$ and $150^\circ = \sin^{-1}0.5$. So the inverse sine, cosine, or tangent of a number can yield more than one answer
    - Your calculator will only give one value, the other values must be deduced from knowledge of the trig functions $y = \sin\theta, y = \cos^{-1}\theta, \text{and } \tan\theta$
    - Make sure to pay attention to the range that is stated in the question to know how many values are required for your answer

WORKED EXAMPLES

24.8 Find all values of $\sin^{-1} 0.4$ in the range $0°$ to $360°$.

Solution Let $\theta = \sin^{-1} 0.4$, that is $\sin\theta = 0.4$. Since $\sin\theta$ is positive then one value is in the first quadrant and another value is in the second quadrant. Using a scientific calculator we see

$$\theta = \sin^{-1} 0.4 = 23.6°$$

The formula $\sin\theta = \sin(180° - \theta)$ was derived on page 294

This is the value in the first quadrant. The value in the second quadrant is found using $\sin\theta = \sin(180° - \theta)$. Hence the value in the second quadrant is $180° - 23.6° = 156.4°$.

24.9 Find all values of θ in the range $0°$ to $360°$ such that

(a) $\theta = \cos^{-1}(-0.5)$ (b) $\theta = \tan^{-1} 1$

Solution (a) We have $\theta = \cos^{-1}(-0.5)$, that is $\cos\theta = -0.5$. As $\cos\theta$ is negative then θ must be in the second and third quadrants. Using a calculator we find $\cos^{-1}(-0.5) = 120°$. This is the value of θ in the second quadrant. We now seek the value of θ in the third quadrant.

We have, from the Key point on page 295,

$$\cos(\theta - 180°) = -\cos\theta$$

We are given $\cos\theta = -0.5$. So

$$\cos(\theta - 180°) = 0.5$$

Now, since θ is in the third quadrant, $\theta - 180°$ must be an acute angle whose cosine equals 0.5. That is,

$$\theta - 180° = 60°$$

$$\theta = 240°$$

This is the value of θ in the third quadrant. The required solutions are thus $\theta = 120°, 240°$.

(b) We are given $\theta = \tan^{-1} 1$, that is $\tan\theta = 1$. Since $\tan\theta$ is positive, there is a value of θ in the first and third quadrants. Using a calculator we find $\tan^{-1} 1 = 45°$. This is the value of θ in the first quadrant.

We have, from the Key point on page 295,

$$\tan(\theta - 180°) = \tan\theta$$

We are given $\tan\theta = 1$. So

$$\tan(\theta - 180°) = 1$$

Now $\theta - 180°$ must be an acute angle with tangent equal to 1. That is,

$$\theta - 180° = 45°$$

$$\theta = 225°$$

The required values of θ are $45°$ and $225°$.

24.10 Find all values of θ in the range 0° to 360° such that $\theta = \sin^{-1}(-0.5)$.

Solution We know $\sin\theta = -0.5$ and so θ must be in the third and fourth quadrants. Using a calculator we find

$$\sin^{-1}(-0.5) = -30^\circ$$

We require solutions in the range 0° to 360°. Recall that adding 360° to an angle leaves the values of the trigonometrical ratios unaltered. So

$$\sin^{-1}(-0.5) = -30^\circ + 360^\circ = 330^\circ$$

We have found the value of θ in the fourth quadrant.

We now seek the value in the third quadrant. We have, from the Key point on page 295,

$$\sin(\theta - 180^\circ) = -\sin\theta$$

We are given $\sin\theta = -0.5$. Therefore

$$\sin(\theta - 180^\circ) = 0.5$$

Now $\theta - 180^\circ$ must be an acute angle with sine equal to 0.5. That is,

$$\theta - 180^\circ = 30^\circ$$

$$\theta = 210^\circ$$

The required values of θ are 210° and 330°.

24.11 Find all values of θ in the range 0° to 360° such that $\sin 2\theta = 0.5$.

Solution We make the substitution $z = 2\theta$. As θ varies from 0° to 360° then z varies from 0° to 720°. Hence the problem as given is equivalent to finding all values of z in the range 0° to 720° such that $\sin z = 0.5$.

$$\sin z = 0.5$$

$$z = \sin^{-1} 0.5$$

$$= 30^\circ$$

Also

$$\sin(180^\circ - 30^\circ) = \sin 30^\circ$$

$$\sin 150^\circ = \sin 30^\circ = 0.5$$

$$\sin^{-1} 0.5 = 150^\circ$$

The values of z in the range 0° to 360° are 30° and 150°. Recalling that adding 360° to an angle leaves its trigonometrical ratios unaltered, we see that $30^\circ + 360^\circ = 390^\circ$ and $150^\circ + 360^\circ = 510^\circ$ are values of z in the range 360° to 720°. The solutions for z are thus

$$z = 30^\circ, 150^\circ, 390^\circ, 510^\circ$$

and hence the required values of θ are given by

$$\theta = \frac{z}{2} = 15^\circ, 75^\circ, 195^\circ, 255^\circ$$

24.12 Solve $\cos 3\theta = -0.5$ for $0^\circ \leqslant \theta \leqslant 360^\circ$.

Solution We substitute $z = 3\theta$. As θ varies from 0° to 360° then z varies from 0° to 1080°. Hence the problem is to find values of z in the range 0° to 1080° such that $\cos z = -0.5$.

We begin by finding values of z in the range 0° to 360°. Using Worked Example 24.9(a) we see that

$$z = 120^\circ, 240^\circ$$

To find values of z in the range 360° to 720° we add 360° to each of these solutions. This gives

$$z = 120^\circ + 360^\circ = 480^\circ \qquad \text{and} \qquad z = 240^\circ + 360^\circ = 600^\circ$$

To find values of z in the range 720° to 1080° we add 360° to these solutions. This gives

$$z = 840^\circ, 960^\circ$$

Hence

$$z = 120^\circ, 240^\circ, 480^\circ, 600^\circ, 840^\circ, 960^\circ$$

Finally, using $\theta = \frac{z}{3}$ we obtain values of θ:

$$\theta = \frac{z}{3} = 40^\circ, 80^\circ, 160^\circ, 200^\circ, 280^\circ, 320^\circ$$

24.13 Solve

$$\tan(2\theta + 20^\circ) = 0.3 \qquad 0^\circ \leqslant \theta \leqslant 360^\circ$$

Solution We substitute $z = 2\theta + 20^\circ$. As θ varies from 0° to 360° then z varies from 20° to 740°.

First we solve

$$\tan z = 0.3 \qquad 0^\circ \leqslant z \leqslant 360^\circ$$

This leads to $z = 16.7^\circ, 196.7^\circ$. Values of z in the range 360° to 720° are

$$z = 16.7^\circ + 360^\circ, 196.7^\circ + 360^\circ$$
$$= 376.7^\circ, 556.7^\circ$$

By adding a further 360° values of z in the range 720° to 1080° are found. These are

$$z = 736.7^\circ, 916.7^\circ$$

Hence values of z in the range 0° to 1080° are

$$z = 16.7^\circ, 196.7^\circ, 376.7^\circ, 556.7^\circ, 736.7^\circ, 916.7^\circ$$

Values of z in the range 20° to 740° are thus

$$z = 196.7^\circ, 376.7^\circ, 556.7^\circ, 736.7^\circ$$

The values of θ in the range 0° to 360° are found using $\theta = (z - 20^\circ)/2$:

$$\theta = \frac{z - 20^\circ}{2} = 88.35^\circ, 178.35^\circ, 268.35^\circ, 358.35^\circ$$

Summary

In this week, we learned the basic trigonometric ratios that govern the study of triangles. Also we examined special triangles and corresponding trig ratios that are generated through these triangles. We also looked at the extended definition of the three trig ratio, the inverse trig functions, the fundamental trig identity, the various derived trig identities and lastly trigonometric equations.

Week 8 Angles, triangles, and trigonometry Continued Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-14 4:31 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Sequences and Series Continued	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: May 14, 2020

Essential Question:

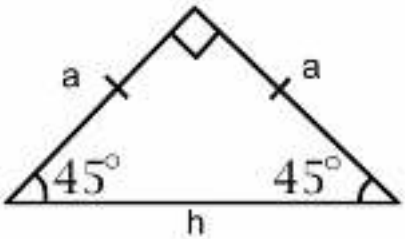
What are angles and what is trigonometry and how are these related to the study of triangles?

Questions/Cues:

- What is an example of Pythagoras theorem applied?
- What is an example of the Law of Sines applied?
- What is an example of the Law of Cosines applied?

Notes

Examples: Triangle Rectangle Isosceles



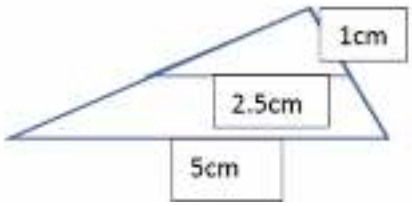
$a^2+b^2=2a^2=h^2$ Pithagora's theorem

→ $a=h/\sqrt{2}$

$a/h=1/\sqrt{2}=\sin(45°)$

→ $\sin(45°)=1/\sqrt{2}=\sqrt{2}/2$

What is the length of the righthand side of the larger triangle?



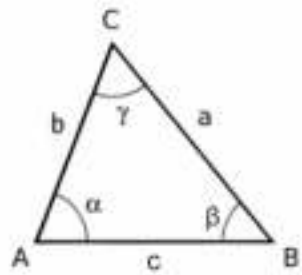
Similar triangles

→ 1cm/X=2.5cm/5cm

→ X=2cm

Examples: Generic Triangle

a=8 cm b=3 cm $\alpha=58^\circ$ c? Note: $\sin(58^\circ)=0.848$



Use sine ratio: $a/\sin(\alpha)=b/\sin(\beta)=c/\sin(\gamma)$

$\rightarrow \sin(\beta)=b\sin(\alpha)/a \rightarrow \sin(\beta)=3\times0.848/8=0.318$

$\rightarrow \beta=\sin^{-1}(0.318)=18.54^\circ$

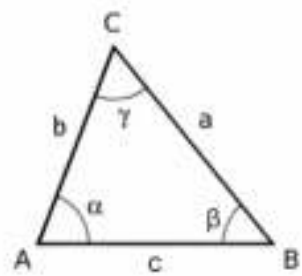
$\rightarrow \gamma=180^\circ-18.54^\circ-58^\circ=103.46^\circ$

but $c/\sin(\gamma)=a/\sin(\alpha)$

$\rightarrow c=a\sin(\gamma)/\sin(\alpha)=8\times0.97/0.848=9.15\text{cm}$

Examples: Generic Triangle

a=6 cm b=4 cm c=3cm $\alpha=?$



Use cosine rule: $a^2=b^2+c^2-2bc\cos(\alpha)$

$\rightarrow \cos(\alpha)=\frac{b^2+c^2-a^2}{2bc}=\frac{16+9-36}{24}=-0.458$

$\rightarrow \alpha=\cos^{-1}(-0.458)=117.3^\circ$

Summary

In this week, we learned how apply Pythagoras theorem, Sine law and Cosine law to a triangle.

Week 9 Graph Sketching & Kinematics Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-22 5:27 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Graph Sketching & Kinematics	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: May 22, 2020

Essential Question:

What is a function and what are its applications to kinematics (simple motion)?

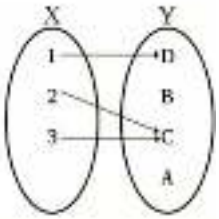
Questions/Cues:

- What is the definition of a function?
- What are surjective, injective and bijective functions?
- What are Cartesian Coordinates?
- What is the Distance Formula used to find the distance between two points P and Q?

Notes

What is a function?

A function $f(x)$ links elements x,y of two sets X and Y



$f(x)$ tells you what to do with an input x:
ex. $f(x)=2x+4$ multiply by 2 and add 4
$f(3)=2\times3+4=10$ $f(-1)=2\times(-1)+4=2$ $f(13)=2\times13+4=143$

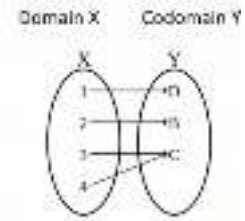
Domain of a function: elements of X on which f is defined
ex. $-4<x<4$ or $(-4,4)$: all values between -4 and 4 excluding -4,4
$-4\leq x\leq 4$ or $[-4,4]$ includes -4,4
$-4<x\leq 4$ or $(-4,4]$ includes 4 not -4
$-4\leq x<4 \cup 6<x<8$ or $[-4,4) \cup (6,8)$ etc...

Codomain of a function: elements of Y linked by f to X

(codomain also called range or image)

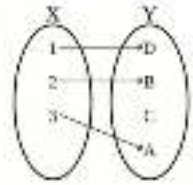
- the function f(x) = 2x+4 maps a real number to a real number; the domain is the set of all real number
- The domain and range can also be restricted and written in interval notation by inequalities or braces like above

What is a function?

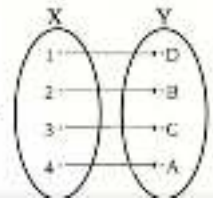


<u>Surjective function</u>: to each y∈Y →at least one x∈X

<u>Injective function</u>:
to each x∈X → only one distinct y∈Y



<u>Bijective function</u>: Injective+Surjective



- A bijective function has one-to-one correspondence, it's a one-to-one function

Cartesian Coordinates

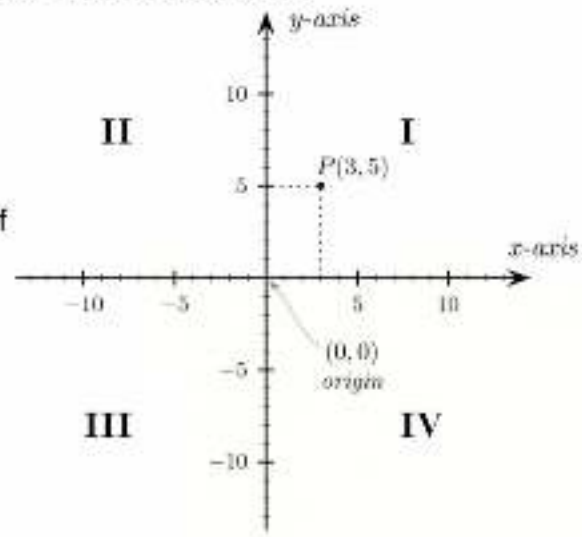
System of two perpendicular axes, *x*,*y*, to map and label points on the plane:

each point P is labeled by a pair of numbers (x,y)

x is the length of the projection of the point on the *x*-axis

and y is the length of the projection of the point on *y*-axis

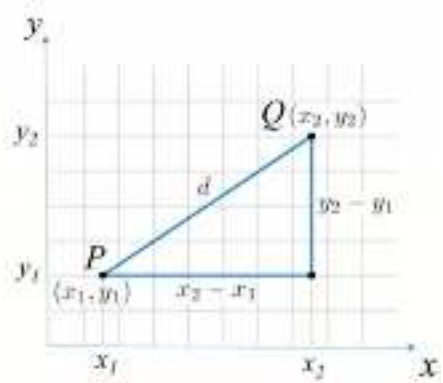
Generic point on *y*-axis P(0,y)
Generic point on *x*-axis P(x,0)



Distance between P and Q:

Using Pythagoras theorem

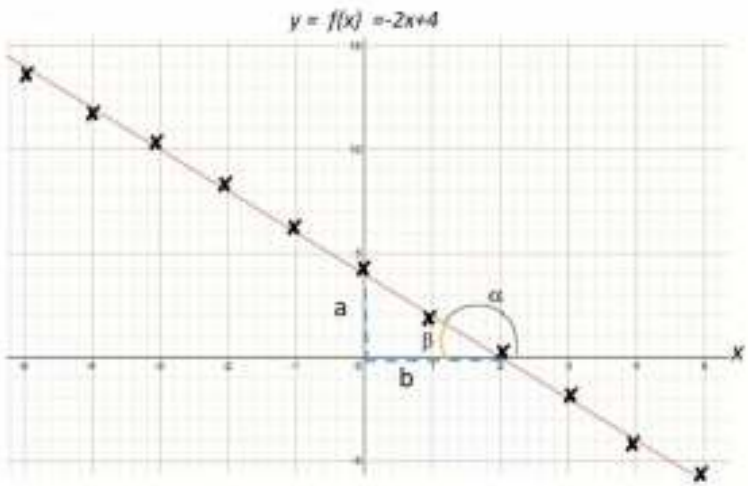
$$d_{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Examples: f(x)= -2x+4 Domain R

x	-5	-4	-3	-2	-1	0
f(x)	=-2(-5)+4=14	=12	=10	=8	=6	=4
coordinates	(-5, 14)	(-4,12)	(-3,10)	(-2,8)	(-1,6)	(0,4)

x	1	2	3	4	5
f(x)	=-2(1)+4=2	=0	=-2	=-4	
coordinates	(1,2)	(2,0)	(3,-2)	(4,-4)	



Note: $\beta=180-\alpha$
$\tan(\beta)=a/b=4/2=2$

In general for a straight line
$y=mx+n$

with $m=\tan(\alpha)$

In our case n=4
$m=\tan(\alpha)=-\tan(\beta)=-2$

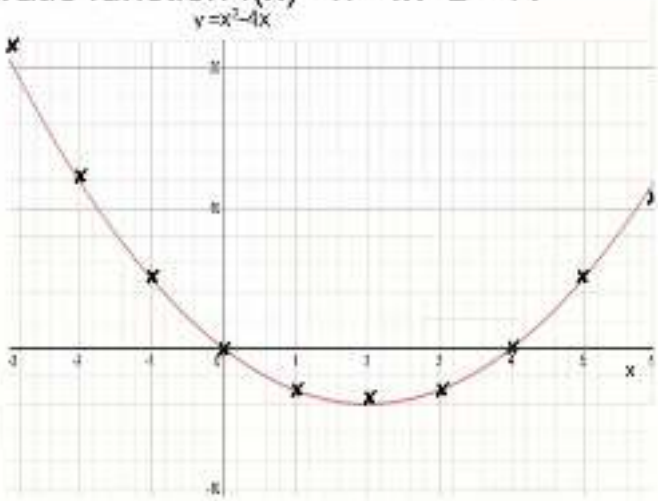
Intersection with y-axis $\rightarrow x=0 \rightarrow y_0=f(0)=-2(0)+4=4$
Intersection with x-axis $\rightarrow y=0 \rightarrow f(x_0)=0$
Solve $-2x_0+4=0 \rightarrow 2x_0=4 \rightarrow x_0=2$

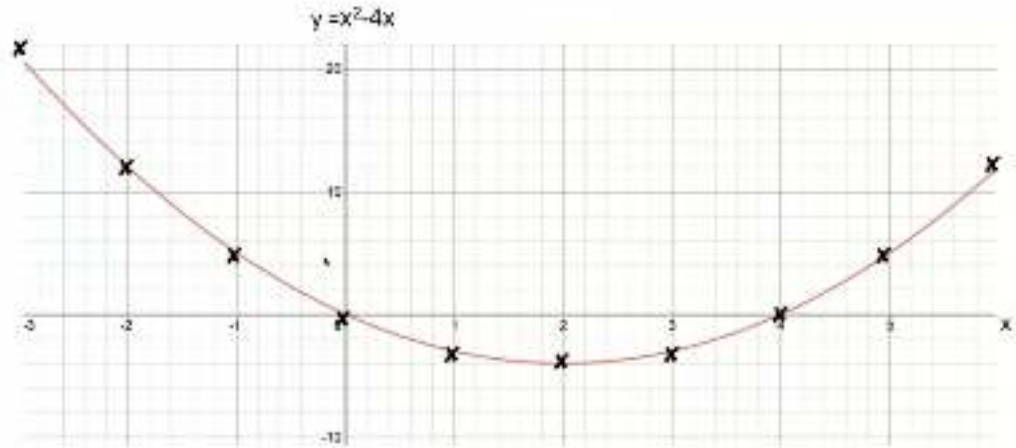
Examples: quadratic function $f(x)= x^2-4x$ D= R

x	-3	-2	-1
f(x)	$=(-3)^2-12=21$	=12	=5
coordinates	(-3,21)	(-2,12)	(-1,5)

x	0	1	2
f(x)	=0	=-3	=-4
coordinates	(0,0)	(1,-3)	(2,-4)

x	3	4	5	6
f(x)	$=(3)^2-12=-3$	=0	=5	=12
coordinates	(1,-3)	(4,0)	(5,5)	(6,12)





Intersection with y-axis $\rightarrow x=0 \rightarrow y_0=f(0)=(0)^2-4(0)=0$
Intersection with x-axis $\rightarrow y=0 \rightarrow f(x_0)=x_0^2-4x_0=0$

Solve $x_0^2-4x_0=x_0(x_0-4)=0 \rightarrow x_0=0, x_0=4$

Generic quadratic function $f(x)= ax^2+bx+c$

Intersection with y-axis $\rightarrow x=0 \rightarrow y_0=f(0)=a(0)^2+b(0)+c=c$
Intersection with x-axis $\rightarrow y=0 \rightarrow f(x_0)=ax_0^2+bx_0+c=0$

Solve $ax_0^2+bx_0+c=0 \rightarrow x_0=(-b\pm\sqrt{(b^2-4ac)})/(2a)$

Cubic function $f(x)= ax^3+bx^2+cx+d$ Ex: x^3-4x D=R

x	-4	-3	-2	-1	0	1
f(x)	$=(-4)^3-4(-4)=-48$	$=(-3)^3-4(-3)=-15$	$=(-2)^3-4(-2)=0$	$=3$	$=0$	$=-3$
coordinates	(-4,-48)	(-3,-15)	(-2,0)	(-1,3)	(0,0)	(1,-3)

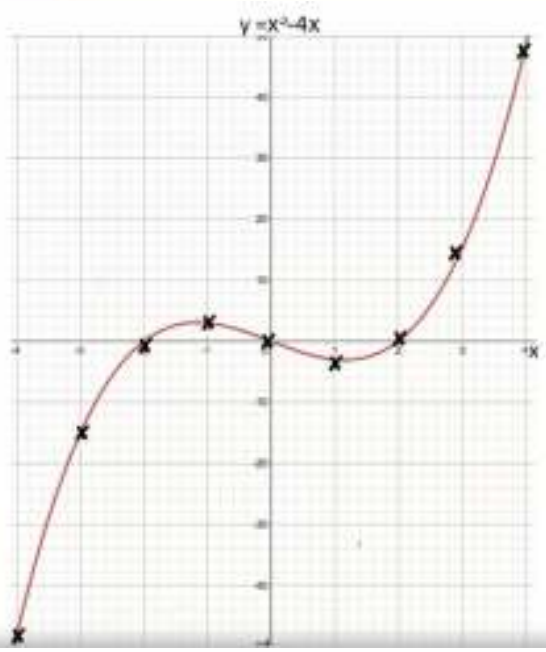
x	2	3	4
f(x)	$=(2)^3-4(2)=0$	$=(3)^3-4(3)=15$	$=48$
coordinates	(2,0)	(3,15)	(4,48)

Intersection with y-axis $\rightarrow x=0$
$\rightarrow y_0=f(0)=(0)^3-4(0)=0$

Intersection with x-axis $\rightarrow y=0$
$\rightarrow f(x_0)=x_0^3-4x_0=0$

Solve $x_0^3-4x_0=x_0(x_0^2-4)=0$
$\rightarrow x_0=0, x_0=\pm 2$

Note: a vertical line intersects the curve in <u>only one point</u>
$\rightarrow$ single-valued functions



Summary

In this week, we learned about what a function, surjective/injective functions, the Cartesian coordinate system and distance formula.

Week 9 Graph Sketching & Kinematics Reading Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-18 3:02 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Graph Sketching & Kinematics Reading	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: May 17, 2020

Essential Question:

What is a function and what are its applications to kinematics (simple motion)?

Questions/Cues:

- What are axes?
- What is an interval?
- What is the notation used to concisely describe intervals?
- What are the different types of intervals?
- What is domain and range?

Notes

- Axes = horizontal & vertical lines that divide the x-y plane in four quadrants. These axes intersect at a point O called the origin. The horizontal axis is used to represent the independent variable, commonly x and the vertical axis is used to represent the dependent variable, commonly y.
    - A scale is drawn on both axes in such a way that at the origin x = 0 and y = 0. Positive x values lie to the right and negative x values to the left. Positive y values are above the origin, negative y values are below the origin.
        - It's not essential that the scales on both axes are the same
    - Each point in the plane corresponds to a specific value of x and y. We usually call the x value the x coordinate and call the y value the y coordinate. To refer to a specific point we give both coordinates in brackets in form (x, y), always giving the x coordinate first

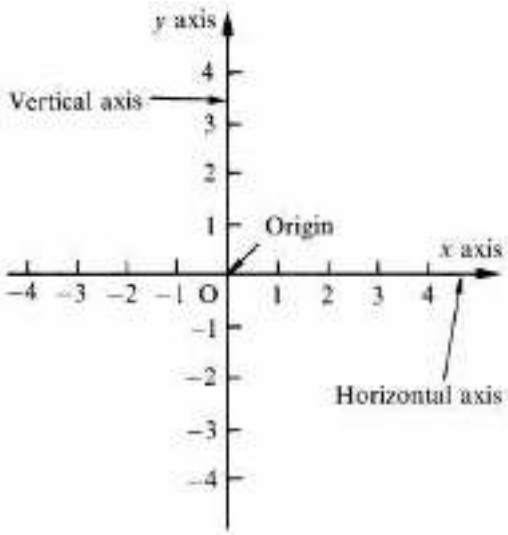


Figure 17.1
The $x-y$ plane

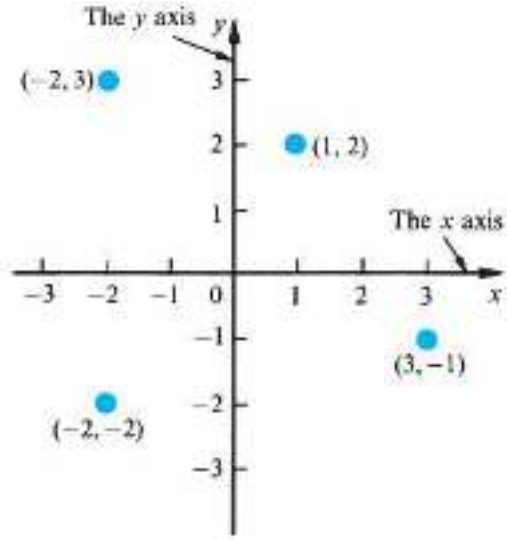


Figure 17.2
Several points in the $x-y$ plane

WORKED EXAMPLE

17.1 Draw the $x-y$ plane and on it mark the points whose coordinates are (1, 2), (3, −1), (−2, 3), (−2, −2).

Solution The plane is drawn in Figure 17.2, and the given points are indicated. Note that the first coordinate in each bracket is the x coordinate.

- Interval = part of an axis
- Greater than = $>$ means greater than. ex. 6 > 5. If $x_1 > x_2$ then x_1 is to the right of x_2 on the x axis.
    - The symbol $\geq$ means 'greater than or equal to', ie. $10 \geq 8$ and $8 \geq 8$ are both true.
- Less than = $<$ means less than. ie. 4 < 5. If $x_1 < x_2$ then x_1 is to the left of x_2 on the x axis
    - The symbol $\leq$ means 'less than or equal to'. ie. $6 \leq 9 \text{ and } 6 \leq 6$

** The symbols $>, \geq, <, \leq$ are known as inequalities

    - Adding and subtracting the same quantity from both sides of an inequality leaves the inequality unchanged. For any quantity k if:

$$x > y$$

then

$$x + k > y + k \text{ and } x - k > y - k.$$

Similarly, if $x < y$ then

$$x + k < y + k \text{ and } x - k < y - k.$$

Multiplying and dividing an inequality needs greater care. If an inequality is multiplied or divided by a **positive** quantity then the inequality remains unchanged. If $x > y$ and k is positive, then

$$kx > ky \text{ and } \frac{x}{k} > \frac{y}{k}$$

Similarly, if $x < y$ and k is positive then

$$kx < ky \text{ and } \frac{x}{k} < \frac{y}{k}$$

If the inequality is multiplied or divided by a **negative** quantity then the inequality is reversed, that is, 'greater than' must be replaced by 'less than' and vice versa. If $x > y$ and k is negative, then

$$kx < ky \text{ and } \frac{x}{k} < \frac{y}{k}$$

Note that the 'greater than' has been replaced by a 'less than'. Similarly, if $x < y$ and k is negative then

$$kx > ky \text{ and } \frac{x}{k} > \frac{y}{k}$$

Note that the 'less than' has been replaced by 'greater than'.

Intervals

$\mathbb{R}$ is called the set of real numbers.

We often need to represent intervals on the number line. To help us do this we introduce the set $\mathbb{R}$. $\mathbb{R}$ is the symbol we use to denote all numbers from minus infinity to plus infinity. All numbers, including integers, fractions and decimals, belong to $\mathbb{R}$. To show that a number, x, is in this set, we write $x \in \mathbb{R}$ where $\in$ means 'belongs to'.

There are three different kinds of interval:

(a) *The closed interval* An interval that includes its end-points is called a **closed interval**. All the numbers from 1 to 3, including both 1 and 3, comprise a closed interval, and this is denoted using square brackets, $[1, 3]$. Any number in this closed interval must be greater than or equal to 1, and also less than or equal to 3. Thus if x is any number in the interval, then $x \geqslant 1$ and also $x \leqslant 3$. We write this compactly as $1 \leqslant x \leqslant 3$. Finally we need to show explicitly that x is any number on the x axis, and not just an integer for example. So we write $x \in \mathbb{R}$. Hence the interval $[1, 3]$ can be expressed as

$$\{x : x \in \mathbb{R},\ 1 \leqslant x \leqslant 3\}$$

This means the set contains all the numbers x with $x \geqslant 1$ and $x \leqslant 3$.

(b) *The open interval* Any interval that does not include its end-points is called an **open interval**. For example, all the numbers from 1 to 3, but excluding 1 and 3, comprise an open interval. Such an interval is denoted using round brackets, $(1, 3)$. The interval may be written using set notation as

$$\{x : x \in \mathbb{R},\ 1 < x < 3\}$$

We say that x is **strictly greater** than 1, and **strictly less** than 3, so that the values of 1 and 3 are excluded from the interval.

(c) *The semi-open or semi-closed interval* An interval may be open at one end and closed at the other. Such an interval is called **semi-open** or, as some authors say, **semi-closed**. The interval $(1, 3]$ is a semi-open interval. The square bracket next to the 3 shows that 3 is included in the interval; the round bracket next to the 1 shows that 1 is not included in the interval. Using set notation we would write

$$\{x : x \in \mathbb{R},\ 1 < x \leqslant 3\}$$

When marking intervals on the x axis there is a notation to show whether or not the end-point is included. We use ● to show that an end-point is included (i.e. closed) whereas ○ is used to denote an end-point that is not included (i.e. open).

WORKED EXAMPLES

17.2 Describe the interval $[-3, 4]$ using set notation and illustrate it on the x axis.

Solution The interval $[-3, 4]$ is given in set notation by

$$\{x : x \in \mathbb{R}, -3 \leqslant x \leqslant 4\}$$

Figure 17.3 illustrates the interval.

Figure 17.3
The interval $[-3, 4]$

17.3 Describe the interval $(1, 4)$ using set notation and illustrate it on the x axis.

Solution The interval $(1, 4)$ is expressed as

$$\{x : x \in \mathbb{R}, 1 < x < 4\}$$

Figure 17.4 illustrates the interval on the x axis.

Figure 17.4
The interval $(1, 4)$

WORKED EXAMPLE

17.4 Plot a graph of $y = 2x - 1$ for $-3 \leqslant x \leqslant 3$.

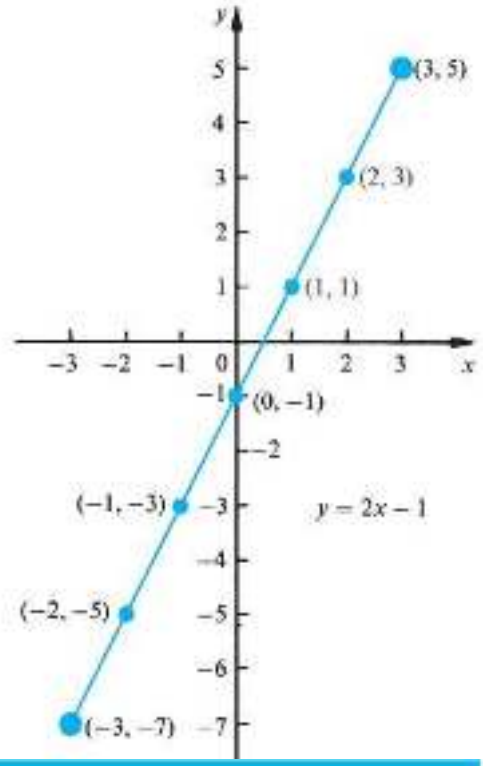
Solution We first calculate the value of y for several values of x. A table of x values and corresponding y values is drawn up as shown in Table 17.1.

Table 17.1
Values of x and y when $y = 2x - 1$

x	−3	−2	−1	0	1	2	3
y	−7	−5	−3	−1	1	3	5

The independent variable x varies from -3 to 3; the dependent variable y varies from -7 to 5. The x axis must accommodate values from -3 to 3; the y axis must accommodate values from -7 to 5. Each pair of values of x and y is represented by a unique point in the x–y plane, with coordinates (x, y). Each pair of values in the table is plotted as a point and then the points are joined to form the graph (Figure 17.5). By joining the points we see that, in this example, the graph is a straight line. We were asked to plot the graph for $-3 \leqslant x \leqslant 3$. Therefore, we have indicated each end-point of the graph by a ● to show that it is included. This follows the convention used for labelling closed intervals given in §17.2.

Figure 17.5
A graph of $y = 2x - 1$



WORKED EXAMPLE

17.5 (a) Plot a graph of $y = x^2 - 2x + 2$ for $-2 \leqslant x \leqslant 3$.

(b) Use your graph to determine which of the following points lie on the graph: (1, 2), (0, 2), and (1, 1).

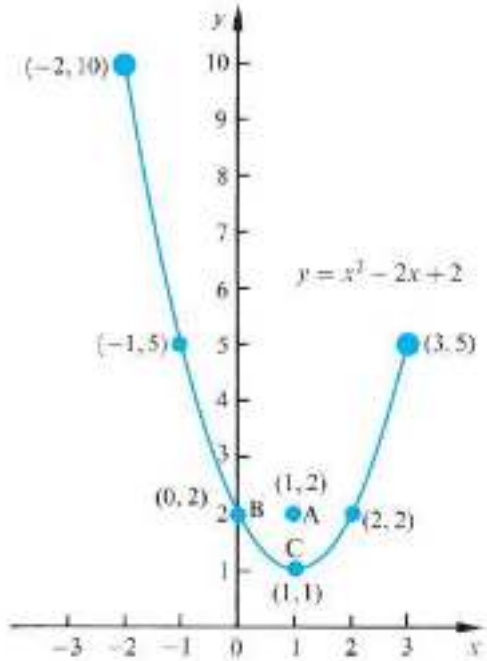
Solution (a) Table 17.2 gives values of x and the corresponding values of y. The independent variable is x and so the x axis is horizontal. The calculated points are plotted and joined. Figure 17.6 shows the graph. In this example the graph is not a straight line but rather a curve.

(b) The point (1, 2) has an x coordinate of 1 and a y coordinate of 2. The point is labelled by A on Figure 17.6. Points (0, 2) and (1, 1) are plotted and labelled by B and C. From the figure we can see that (0, 2) and (1, 1) lie on the graph.

Table 17.2
Values of x and y when $y = x^2 - 2x + 2$

x	−2	−1	0	1	2	3
y	10	5	2	1	2	5

Figure 17.6
A graph of
$y = x^2 - 2x + 2$



- Domain = the set of values we allow the independent variable to take is called the domain of the function. If the domain is not actually specified in any particular case, it's taken to be the largest set possible
- Range = The set of values we allow the dependent variable to take is called the range/output of the function

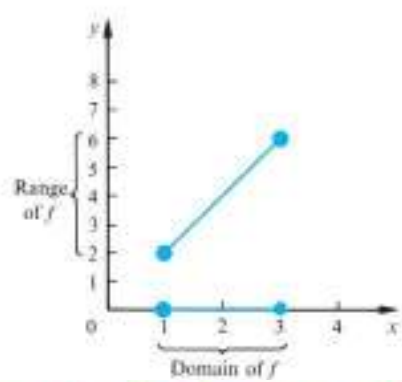
WORKED EXAMPLES

17.6 The function f is given by $y = f(x) = 2x$, for $1 \leqslant x \leqslant 3$.

(a) State the independent variable.
(b) State the dependent variable.
(c) State the domain of the function.
(d) Plot a graph of the function.
(e) State the range of the function.

Solution (a) The independent variable is x.
(b) The dependent variable is y.
(c) Since we are given $1 \leqslant x \leqslant 3$ the domain of the function is the interval $[1, 3]$, that is all values from 1 to 3 inclusive.
(d) The graph of $y = f(x) = 2x$ is shown in Figure 17.7.

Figure 17.7
Graph of $y = f(x) = 2x$



(e) The range is the set of values taken by the output, y. From the graph we see that as x varies from 1 to 3 then y varies from 2 to 6. Hence the range of the function is $[2, 6]$.

17.7 Consider the function $y = f(t) = 4t$.

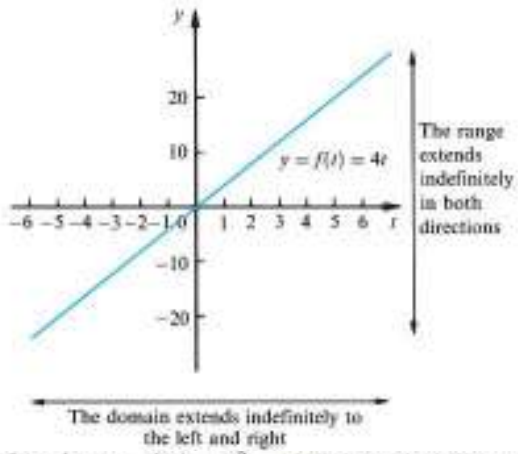
(a) State which is the independent variable and which is the dependent variable.

(b) Plot a graph of the function and give its domain and range.

Solution (a) The independent variable is t and the dependent variable is y.

(b) No domain is specified so it is taken to be the largest set possible. The domain is $\mathbb{R}$, the set of all (real) numbers. However, it would be impractical to draw a graph whose domain was the whole extent of the t axis, and so a selected portion is shown in Figure 17.8. Similarly, only a restricted portion of the range can be drawn, although in this example the range is also $\mathbb{R}$.

Figure 17.8
A graph of $f(t) = 4t$



17.8 Consider the function $y = f(x) = x^2 - 1$: (a) state the independent variable; (b) state the dependent variable; (c) state the domain; (d) plot a graph of f and determine the range.

Solution (a) The independent variable is x.

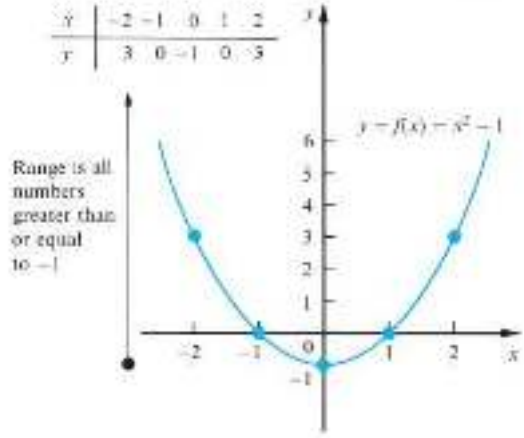
(b) The dependent variable is y.

(c) No domain is specified so it is taken to be the largest set possible. The domain is $\mathbb{R}$.

(d) A table of values and graph of $f(x) = x^2 - 1$ are shown in Figure 17.9. From the graph we see that the smallest value of y is -1, which occurs when $x = 0$. Whether x increases or decreases from 0, the value of y increases. The range is thus all values greater than or equal to -1.

Figure 17.9
A graph of $f(x) = x^2 - 1$

x	−2	−1	0	1	2
y	3	0	−1	0	3



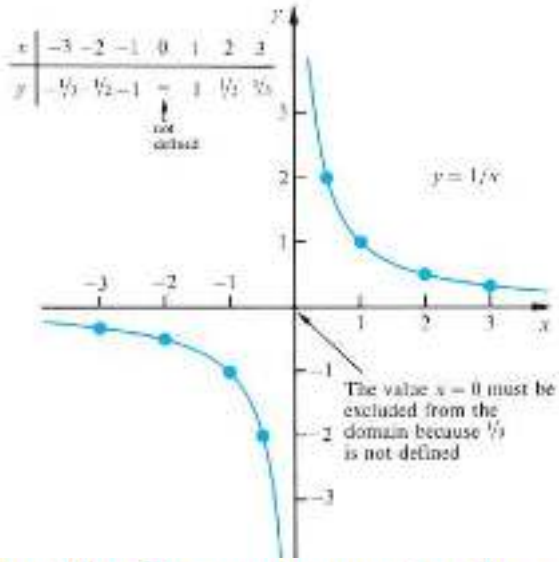
We can write the range using the set notation introduced in §17.2 as

$$\text{range} = \{y : y \in \mathbb{R},\ y \geqslant -1\}$$

17.9 Given the function $y = 1/x$, state the domain and range.

Solution The domain is the largest set possible. All values of x are permissible except $x = 0$ since $\frac{1}{0}$ is not defined. It is never possible to divide by 0. Thus the domain comprises all real numbers except 0. A table of values and graph are shown in Figure 17.10. As x varies, y can take on any value except 0. In this example we see that the range is thus the same as the domain. We note that the graph is split at $x = 0$. For small, positive

Figure 17.10
A graph of $y = 1/x$



values of x, y is large and positive. The graph approaches the y axis for very small, positive values of x. For small, negative values of x, y is large and negative. Again, the graph approaches the y axis for very small, negative values of x. We say that the y axis is an **asymptote**. In general, if the graph of a function approaches a straight line we call that line an asymptote.

WORKED EXAMPLE

17.10 Consider again the piecewise function in Worked Example 16.10:

$$y(x) = \begin{cases} x^2 + 1 & -1 \leqslant x \leqslant 2 \\ 3x & 2 < x \leqslant 6 \\ 2x + 1 & x > 6 \end{cases}$$

(a) Plot a graph of the function.

(b) State the domain of the function.

(c) State the range of the function.

Solution (a) Figure 17.11 shows a graph of $y(x)$.

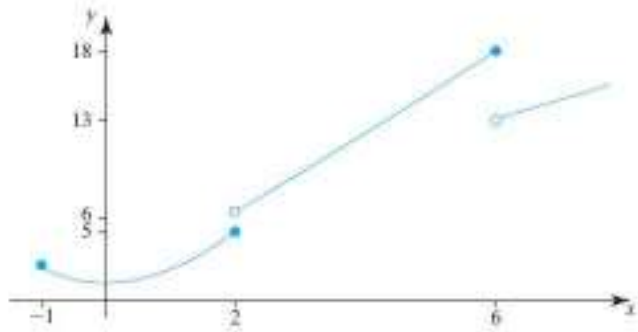


Figure 17.11
A graph of $y(x)$ for Worked Example 17.10

The different rules that make up the definition of y determine the three different pieces of the graph. Note the use of ● and ○ to denote the closed and open end-points.

(b) The domain is $[-1, \infty)$, that is

$$\text{domain} = \{x \in \mathbb{R},\ x \geqslant -1\}$$

(c) The smallest value of y is $y = 1$, which occurs when $x = 0$. From the graph we see that the range is $[1, 5]$ and $(6, \infty)$, that is

$$\text{range} = \{y \in \mathbb{R},\ 1 \leqslant y \leqslant 5 \text{ and } y > 6\}$$

WORKED EXAMPLES

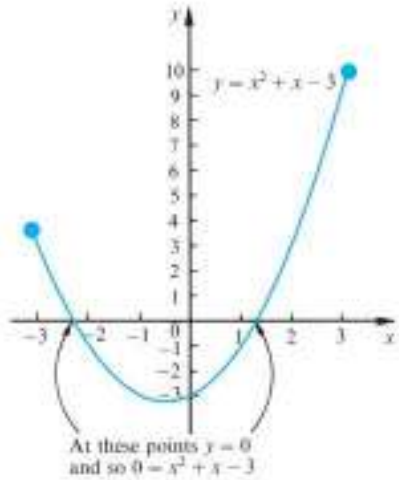
17.11 Find graphically all solutions in the interval $[-3, 3]$ of the equation $x^2 + x - 3 = 0$.

Solution We draw a graph of $y(x) = x^2 + x - 3$. Table 17.3 gives x and y values and Figure 17.12 shows a graph of the function. We have plotted $y = x^2 + x - 3$ and wish to solve $0 = x^2 + x - 3$. Thus we read from the graph the coordinates of the points at which $y = 0$. Such points must be on the x axis. From the graph the points are $(1.3, 0)$ and $(-2.3, 0)$. So the solutions of

Table 17.3
Values of x and y when $y = x^2 + x - 3$

x	-3	-2	-1	0	1	2	3
y	3	-1	-3	-3	-1	3	9

Figure 17.12
A graph of $y = x^2 + x - 3$



$x^2 + x - 3 = 0$ are $x = 1.3$ and $x = -2.3$. Note that these are approximate solutions, being dependent upon the accuracy of the graph and the x and y scales used. Increased accuracy can be achieved by drawing an enlargement of the graph around the values $x = 1.3$ and $x = -2.3$. So, for example, we could draw $y = x^2 + x - 3$ for $1.2 \leqslant x \leqslant 1.5$ and $y = x^2 + x - 3$ for $-2.4 \leqslant x \leqslant -2.2$.

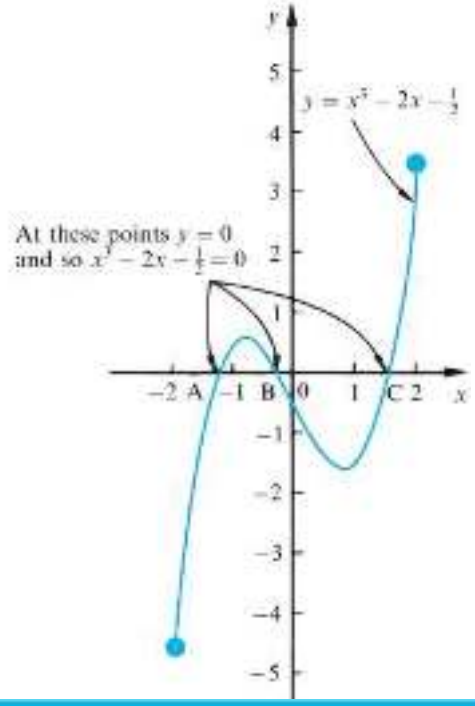
17.12 Find solutions in the interval $[-2, 2]$ of the equation $x^3 = 2x + \frac{1}{2}$ using a graphical method.

Solution The problem of solving $x^3 = 2x + \frac{1}{2}$ is identical to that of solving $x^3 - 2x - \frac{1}{2} = 0$. So we plot a graph of $y = x^3 - 2x - \frac{1}{2}$ for $-2 \leqslant x \leqslant 2$ and then locate the points where the curve cuts the x axis, that is, where the y coordinate is 0. Table 17.4 gives x and y values and Figure 17.13 shows a graph of the function. We now consider points on the graph where the y coordinate is zero. These points are marked A, B and C. Their x coordinates are -1.27, -0.26 and 1.53. Hence the solutions of $x^3 = 2x + \frac{1}{2}$ are approximately $x = -1.27$, $x = -0.26$ and $x = 1.53$.

Table 17.4
Values of x and y when $y = x^3 - 2x - \frac{1}{2}$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	-4.5	-0.875	0.5	0.375	-0.5	-1.375	-1.5	-0.125	3.5

Figure 17.13
A graph of
$y = x^3 - 2x - \frac{1}{2}$



WORKED EXAMPLES

17.13 Solve graphically

$$3x - y = 9 \qquad (17.1)$$

$$x + 2y = -4 \qquad (17.2)$$

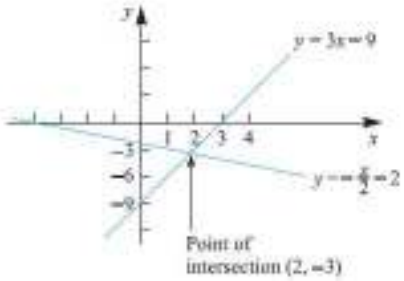
Solution Both equations are rearranged so that y is the subject. This gives

$$y = 3x - 9 \qquad (17.3)$$

$$y = -\frac{x}{2} - 2 \qquad (17.4)$$

Since Equations 17.3 and 17.4 are simple rearrangements of Equations 17.1 and 17.2 then they have the same solution. Equations 17.3 and 17.4 are drawn; Figure 17.14 illustrates this.

Figure 17.14
Points of intersection give the solution to simultaneous equations



We seek values of x and y that fit both equations. For a point to lie on both graphs, the point must be at the intersection of the graphs. In other words, solutions to simultaneous equations are given by the points of intersection. Reading from the graph, the point of intersection is at $x = 2$, $y = -3$. Hence $x = 2$, $y = -3$ is a solution of the given simultaneous equations.

17.14 Solve graphically

$$4x - y = 0$$
$$3x + y = 7$$

Solution We write the equations with y as subject:

$$y = 4x$$
$$y = -3x + 7$$

These are now plotted as shown in Figure 17.15.

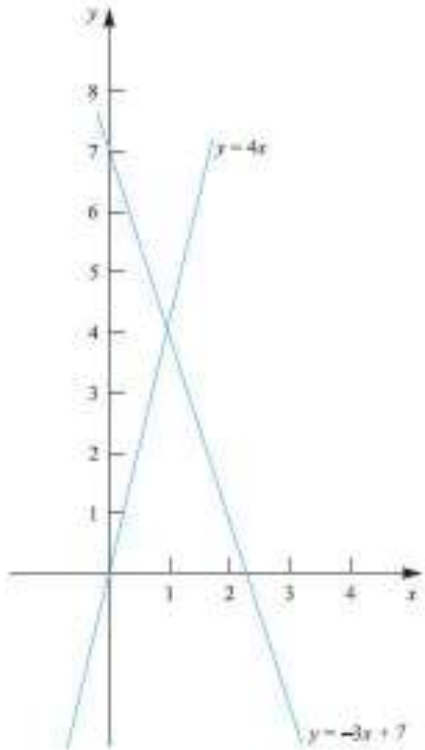


Figure 17.15
The graphs intersect at $x = 1$, $y = 4$

The point of intersection is $x = 1$, $y = 4$ and so this is the solution of the given simultaneous equations.

17.15 Solve graphically

$$x^2 - y = -1$$
$$2x - y = -3$$

Solution Writing the equations with y as subject gives

$$y = x^2 + 1$$
$$y = 2x + 3$$

These are plotted as shown in Figure 17.16.

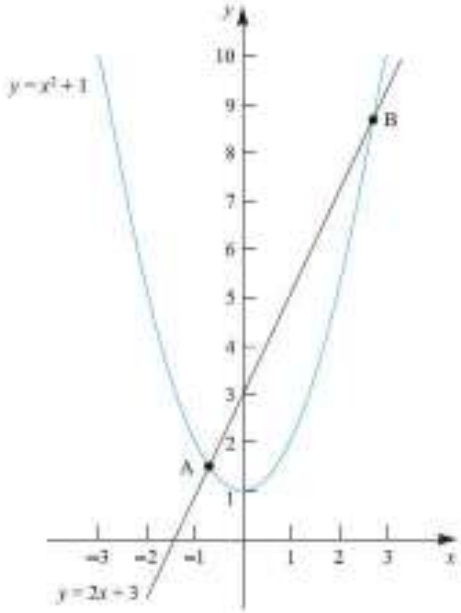


Figure 17.16
The graphs have two points of intersection and hence there are two solutions

There are two points of intersection, A and B. From the graph it is difficult to extract an accurate estimate. However, by using a graphics calculator or package, greater accuracy can be achieved. The coordinates of A are $x = -0.73$, $y = 1.54$; the coordinates of B are $x = 2.73$, $y = 8.46$. Hence there are two solutions to the given equations: $x = -0.73$, $y = 1.54$ and $x = 2.73$, $y = 8.46$.

17.16 (a) On the same axes draw graphs of $y = x^3$ and $y = 2x + \frac{1}{2}$ for $-2 \leqslant x \leqslant 2$.

(b) Note the x coordinates of the points where the two graphs intersect. By referring to Worked Example 17.12 what do you conclude? Can you explain your findings?

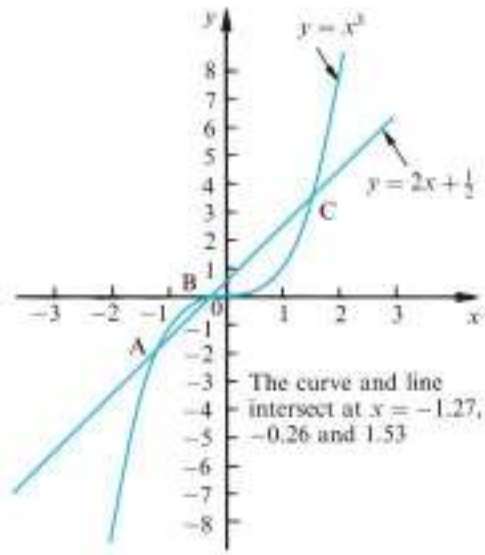
Solution (a) Table 17.5 gives x values and the corresponding values of x^3 and $2x + \frac{1}{2}$. Figure 17.17 shows a graph of $y = x^3$ together with a graph of $y = 2x + \frac{1}{2}$.

(b) The two graphs intersect at A, B and C. The x coordinates of these points are -1.27, -0.26 and 1.53. We note from Worked Example 17.12 that these values are the solutions of $x^3 = 2x + \frac{1}{2}$. We explain this as

Table 17.5
Values of x and y when $y = x^3$ and $y = 2x + \frac{1}{2}$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^3	-8	-3.375	-1	-0.125	0	0.125	1	3.375	8
$2x + \frac{1}{2}$	-3.5	-2.5	-1.5	-0.5	0.5	1.5	2.5	3.5	4.5

Figure 17.17
Graphs of $y = x^3$ and $y = 2x + \frac{1}{2}$



follows. Where $y = x^3$ and $y = 2x + \frac{1}{2}$ intersect, their y values are identical and so at these points

$$x^3 = 2x + \frac{1}{2}$$

By reading the x coordinates at these points of intersection we are finding those x values for which $x^3 = 2x + \frac{1}{2}$.

Summary

In this week, we learned about intervals, inequalities, solving a system of equations graphically and finding solutions to a function graphically within a given interval.

Week 9 Graph Sketching & Kinematics Reading Note 2

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-20 7:26 PM
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Cornell Notes	**Topic:** Graph Sketching & Kinematics Reading	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: May 18, 2020

Essential Question:

What is a function and what are its applications to kinematics (simple motion)?

Questions/Cues:

- What is the Cartesian Plane?
- What is the Pythagorean Theorem and the Distance formula?
- What is the Midpoint formula?
- What is an equation and its graph?
- What is the point-plotting method?
- What are intercepts?
- What is symmetry and what are the graphical and/or algebraic tests for symmetry?
- What is a circle?
- What is a linear equation?
- How do we calculate the slope of a line given two points?
- What is point-slope form?
- What are parallel and perpendicular lines?
- What is the definition of a function?
- What are the characteristics of a function from Set A to Set B?
- What are the four ways to represent a function?
- How is the domain of a function described?
- What is the graph of a function f?
- What is the vertical line test for functions?
- What are the zeros of a function?
- What are increasing/decreasing functions?
- What is relative/local maximum or minimum?
- What is average rate of change?
- What are even/odd functions?
- What are the characteristics of the linear function?
- What is the graph of the squaring function $f(x) = x^2$ like?
- What are the graphs of the cubic, square root and reciprocal functions like?
- What are step and piecewise-defined functions?
- What are the parent functions most commonly found throughout algebra?
- What are vertical and horizontal shifts?
- What is a reflection in terms of the graph of a function?
- What are nonrigid transformations?

- How can we arithmetically combine two functions together?
- What is the composition of functions?

Notes

- Cartesian Plane = representing ordered pairs of real numbers by points in the plane called the rectangular coordinate system or Cartesian plane
    - Two real number lines intersecting at right angles form the Cartesian plane. The horizontal real number line is usually called the x-axis, and the vertical real number line is usually called the y-axis. The point of intersection of these two axes is the origin and the two axes divide the plane into four quadrants.
    - Each point in the plane corresponds to an ordered pair (x, y) of real numbers x and y, called coordinates of the point. The x-coordinate represents the directed distance from the y-axis to the point and the y-coordinate represents the directed distance from the x-axis to the point

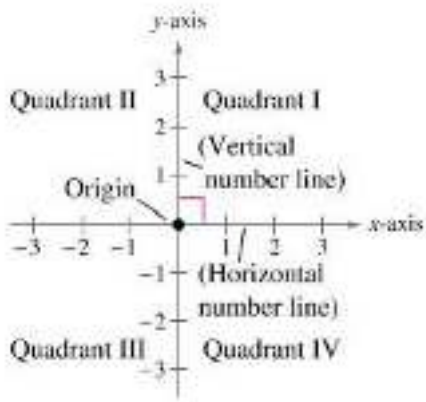


Figure 1.1

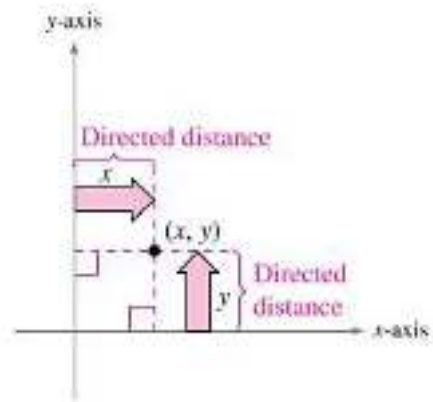


Figure 1.2

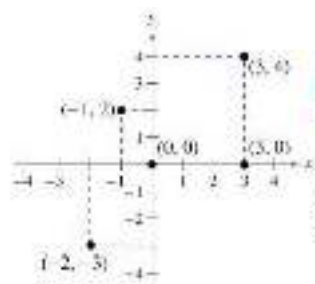


Figure 1.3

EXAMPLE 1 **Plotting Points in the Cartesian Plane**

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

Solution To plot the point $(-1, 2)$, imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. Plot the other four points in a similar way, as shown in Figure 1.3.

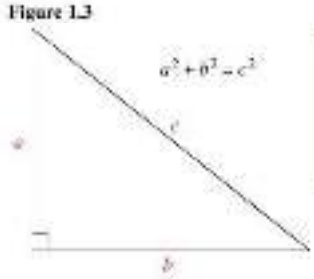


Figure 1.4

Pythagorean Theorem

For a right triangle with hypotenuse length c and sides lengths a and b, you have $a^2 + b^2 = c^2$, as shown in Figure 1.4. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

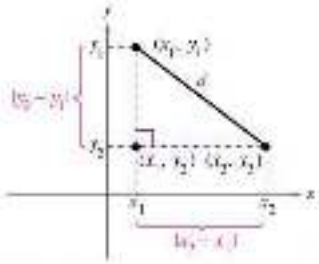


Figure 1.5

Using the points (x_1, y_1) and (x_2, y_2), you can form a right triangle, as shown in Figure 1.5. The length of the hypotenuse of the right triangle is the distance d between the two points. The length of the vertical side of the triangle is $|y_2 - y_1|$ and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem,

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This result is the **Distance Formula.**

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 3 **Finding a Distance**

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Algebraic Solution

Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then apply the Distance Formula.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= \sqrt{[3 - (-2)]^2 + (4 - 1)^2}$	Substitute for x_1, x_2, y_1 and y_2.
$= \sqrt{(5)^2 + (3)^2}$	Simplify.
$= \sqrt{34}$	Simplify.
≈ 5.83	Use a calculator.

So, the distance between the points is about 5.83 units.

Check

$d^2 \stackrel{?}{=} 5^2 + 3^2$	Pythagorean Theorem
$(\sqrt{34})^2 \stackrel{?}{=} 5^2 + 3^2$	Substitute for d.
$34 = 34$	Distance checks. ✓

Graphical Solution

Use centimeter graph paper to plot the points $A(-2, 1)$ and $B(3, 4)$. Carefully sketch the line segment from A to B. Then use a centimeter ruler to measure the length of the segment.

The line segment measures about 5.8 centimeters. So, the distance between the points is about 5.8 units.

EXAMPLE 4 **Verifying a Right Triangle**

Show that the points

$$(2, 1), \quad (4, 0), \quad \text{and} \quad (5, 7)$$

are vertices of a right triangle.

Solution The three points are plotted in Figure 1.6. Using the Distance Formula, the lengths of the three sides are

$$d_1 = \sqrt{(5 - 2)^2 + (7 - 1)^2} = \sqrt{9 + 36} = \sqrt{45},$$
$$d_2 = \sqrt{(4 - 2)^2 + (0 - 1)^2} = \sqrt{4 + 1} = \sqrt{5}, \text{ and}$$
$$d_3 = \sqrt{(5 - 4)^2 + (7 - 0)^2} = \sqrt{1 + 49} = \sqrt{50}.$$

Because $(d_1)^2 + (d_2)^2 = 45 + 5 = 50 = (d_3)^2$, you can conclude by the converse of the Pythagorean Theorem that the triangle is a right triangle.

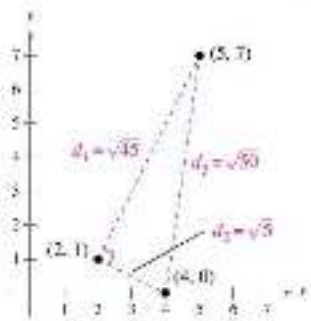


Figure 1.6

The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, find the average values of the respective coordinates of the two endpoints using the **Midpoint Formula.**

The Midpoint Formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

EXAMPLE 5 Finding the Midpoint of a Line Segment

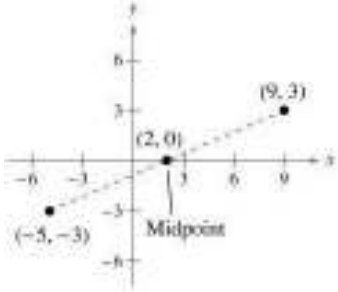


Figure 1.7

Find the midpoint of the line segment joining the points

$$(-5, -3) \quad \text{and} \quad (9, 3).$$

Solution Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \text{Midpoint Formula}$$

$$= \left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2}\right) \qquad \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2.$$

$$= (2, 0) \qquad \text{Simplify.}$$

The midpoint of the line segment is $(2, 0)$, as shown in Figure 1.7.

EXAMPLE 6 Finding the Length of a Pass

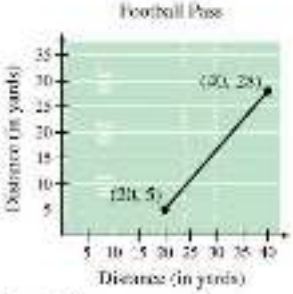


Figure 1.8

A football quarterback throws a pass from the 28-yard line, 40 yards from the sideline. A wide receiver catches the pass on the 5-yard line, 20 yards from the same sideline, as shown in Figure 1.8. How long is the pass?

Solution The length of the pass is the distance between the points $(40, 28)$ and $(20, 5)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{Distance Formula}$$

$$= \sqrt{(40 - 20)^2 + (28 - 5)^2} \qquad \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2.$$

$$= \sqrt{20^2 + 23^2} \qquad \text{Simplify.}$$

$$= \sqrt{400 + 529} \qquad \text{Simplify.}$$

$$= \sqrt{929} \qquad \text{Simplify.}$$

$$\approx 30 \qquad \text{Use a calculator.}$$

So, the pass is about 30 yards long.

EXAMPLE 7 Estimating Annual Sales

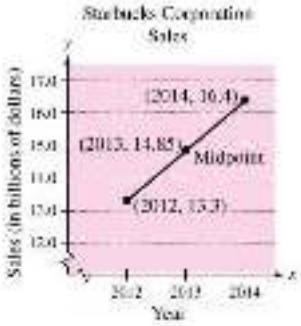


Figure 1.9

Starbucks Corporation had annual sales of approximately $13.3 billion in 2012 and $16.4 billion in 2014. Without knowing any additional information, what would you estimate the 2013 sales to have been? *(Source: Starbucks Corporation)*

Solution Assuming that sales followed a linear pattern, you can estimate the 2013 sales by finding the midpoint of the line segment connecting the points $(2012, 13.3)$ and $(2014, 16.4)$.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \text{Midpoint Formula}$$

$$= \left(\frac{2012 + 2014}{2}, \frac{13.3 + 16.4}{2}\right) \qquad \text{Substitute for } x_1, x_2, y_1, \text{ and } y_2.$$

$$= (2013, 14.85) \qquad \text{Simplify.}$$

So, you would estimate the 2013 sales to have been about $14.85 billion, as shown in Figure 1.9. (The actual 2013 sales were about $14.89 billion.)

EXAMPLE 8 Translating Points in the Plane

The triangle in Figure 1.10 has vertices at the points $(-1, 2)$, $(1, -2)$, and $(2, 3)$. Shift the triangle three units to the right and two units up and find the coordinates of the vertices of the shifted triangle shown in Figure 1.11.

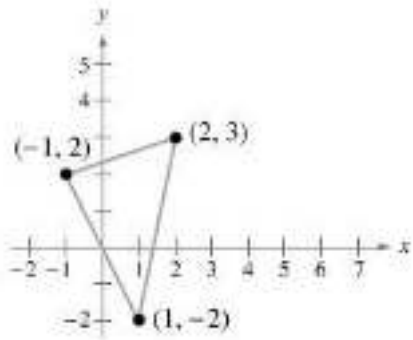


Figure 1.10

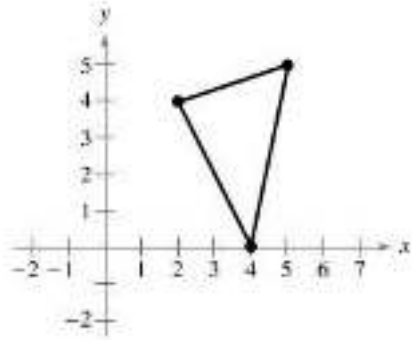


Figure 1.11

Solution To shift the vertices three units to the right, add 3 to each of the x-coordinates. To shift the vertices two units up, add 2 to each of the y-coordinates.

Original Point	Translated Point
$(-1, 2)$	$(-1 + 3, 2 + 2) = (2, 4)$
$(1, -2)$	$(1 + 3, -2 + 2) = (4, 0)$
$(2, 3)$	$(2 + 3, 3 + 2) = (5, 5)$

- Equation = a relationship between two quantities is expressed as an equation in two variables. For example, y = 7 - 3x is an equation in x and y when the substitutions x = a and y = b result in a true statement. For example, (1, 4) is a solution of y = 7 - 3x because 4 = 7 - 3(1) is a true statement
    - The graph of an equation is the set of all points that are solutions of the equation

EXAMPLE 1 Determining Solution Points

Determine whether (a) $(2, 13)$ and (b) $(-1, -3)$ lie on the graph of $y = 10x - 7$.

Solution

a. $y = 10x - 7$ — Write original equation.

$13 \stackrel{?}{=} 10(2) - 7$ — Substitute 2 for x and 13 for y.

$13 = 13$ — $(2, 13)$ is a solution. ✓

The point $(2, 13)$ *does* lie on the graph of $y = 10x - 7$ because it is a solution point of the equation.

b. $y = 10x - 7$ — Write original equation.

$-3 \stackrel{?}{=} 10(-1) - 7$ — Substitute -1 for x and -3 for y.

$-3 \neq -17$ — $(-1, -3)$ is not a solution.

The point $(-1, -3)$ *does not* lie on the graph of $y = 10x - 7$ because it is *not* a solution point of the equation.

The basic technique used for sketching the graph of an equation is the **point-plotting method.**

The Point-Plotting Method of Graphing

1. When possible, isolate one of the variables.
2. Construct a table of values showing several solution points.
3. Plot these points in a rectangular coordinate system.
4. Connect the points with a smooth curve or line.

It is important to use negative values, zero, and positive values for x (if possible) when constructing a table.

EXAMPLE 2 Sketching the Graph of an Equation

Sketch the graph of

$$3x + y = 7.$$

Solution

First, isolate the variable y.

$$y = -3x + 7 \qquad \text{Solve equation for } y.$$

Next, construct a table of values that consists of several solution points of the equation. For example, when $x = -3$,

$$y = -3(-3) + 7 = 16$$

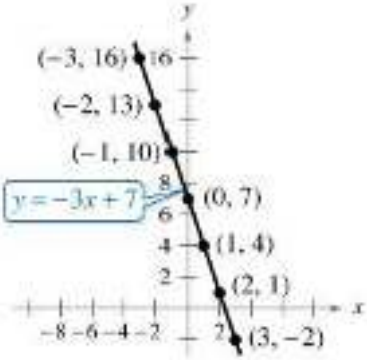
which implies that $(-3, 16)$ is a solution point of the equation.

x	$y = -3x + 7$	(x, y)
-3	16	$(-3, 16)$
-2	13	$(-2, 13)$
-1	10	$(-1, 10)$
0	7	$(0, 7)$
1	4	$(1, 4)$
2	1	$(2, 1)$
3	-2	$(3, -2)$

From the table, it follows that

$$(-3, 16),\ (-2, 13),\ (-1, 10),\ (0, 7),\ (1, 4),\ (2, 1),\ \text{and}\ (3, -2)$$

are solution points of the equation. Plot these points and connect them with a line, as shown below.



EXAMPLE 3 **Sketching the Graph of an Equation**

Sketch the graph of

$$y = x^2 - 2.$$

Solution

The equation is already solved for y, so begin by constructing a table of values.

x	-2	-1	0	1	2	3
$y = x^2 - 2$	2	-1	-2	-1	2	7
(x, y)	$(-2, 2)$	$(-1, -1)$	$(0, -2)$	$(1, -1)$	$(2, 2)$	$(3, 7)$

Next, plot the points given in the table, as shown in Figure 1.12. Finally, connect the points with a smooth curve, as shown in Figure 1.13.

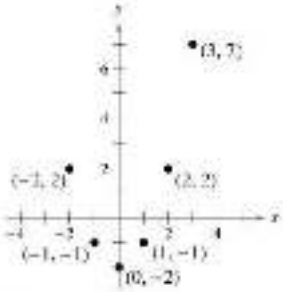


Figure 1.12

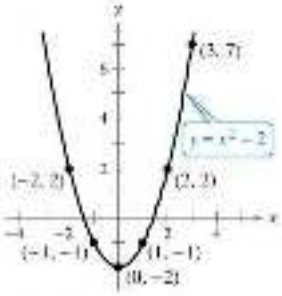


Figure 1.13

REMARK One of your goals in this course is to learn to classify the basic shape of a graph from its equation. For instance, you will learn that the *linear equation* in Example 2 can be written in the form

$$y = mx + b$$

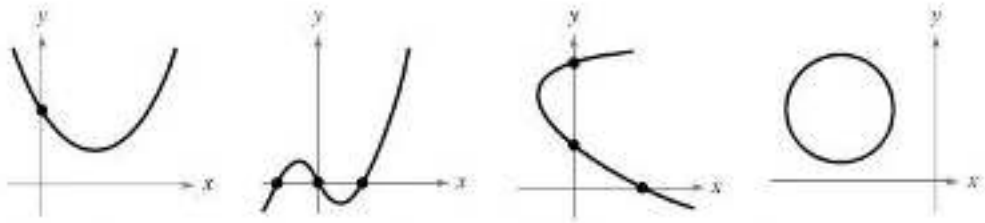
and its graph is a line. Similarly, the *quadratic equation* in Example 3 has the form

$$y = ax^2 + bx + c$$

and its graph is a parabola. ▷

Intercepts of a Graph

Solution points of an equation that have zero as either the x-coordinate or the y-coordinate are called **intercepts.** They are the points at which the graph intersects or touches the x- or y-axis. It is possible for a graph to have no intercepts, one intercept, or several intercepts, as shown in the graphs below.



No x-intercepts
One y-intercept

Three x-intercepts
One y-intercept

One x-intercept
Two y-intercepts

No intercepts

Note that an x-intercept can be written as the ordered pair $(a, 0)$ and a y-intercept can be written as the ordered pair $(0, b)$. Sometimes it is convenient to denote the x-intercept as the x-coordinate a of the point $(a, 0)$ or the y-intercept as the y-coordinate b of the point $(0, b)$. Unless it is necessary to make a distinction, the term *intercept* will refer to either the point or the coordinate.

Finding Intercepts

1. To find x-intercepts, let y be zero and solve the equation for x.
2. To find y-intercepts, let x be zero and solve the equation for y.

EXAMPLE 4 Finding x- and y-Intercepts

Find the x- and y-intercepts of the graph of

$$y = x^3 - 4x.$$

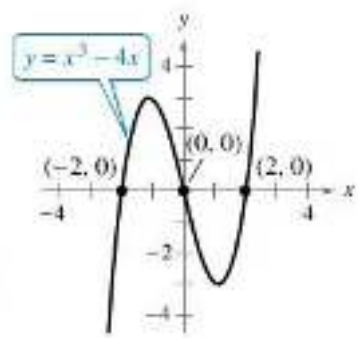
Solution

To find the x-intercepts of the graph of $y = x^3 - 4x$, let $y = 0$. Then

$$0 = x^3 - 4x$$
$$= x(x^2 - 4)$$

has the solutions $x = 0$ and $x = \pm 2$.

x-intercepts: $(0, 0)$, $(2, 0)$, $(-2, 0)$ See figure.



To find the y-intercept of the graph of $y = x^3 - 4x$, let $x = 0$. Then

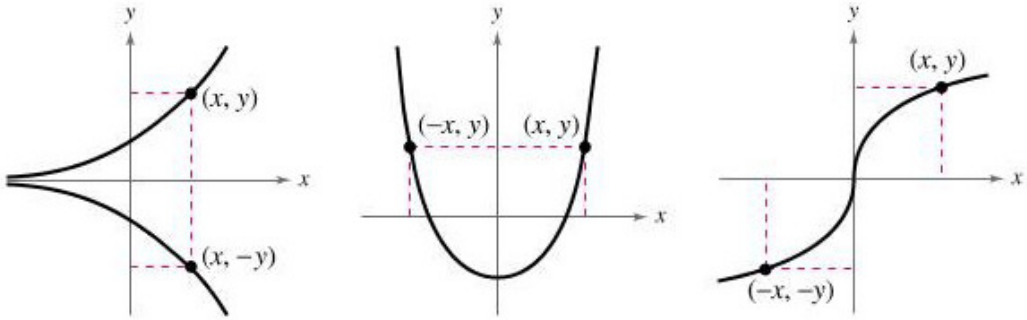
$$y = (0)^3 - 4(0)$$

has one solution, $y = 0$.

y-intercept: $(0, 0)$ See figure.

Symmetry

Graphs of equations can have **symmetry** with respect to one of the coordinate axes or with respect to the origin. Symmetry with respect to the x-axis means that when you fold the Cartesian plane along the x-axis, the portion of the graph above the x-axis coincides with the portion below the x-axis. Symmetry with respect to the y-axis or the origin can be described in a similar manner. The graphs below show these three types of symmetry.

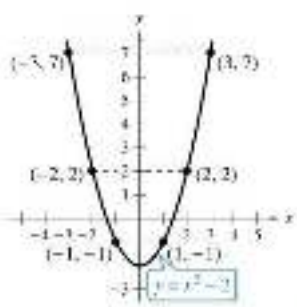


x-Axis symmetry y-Axis symmetry Origin symmetry

Knowing the symmetry of a graph *before* attempting to sketch it is helpful, because then you need only half as many solution points to sketch the graph. Graphical and algebraic tests for these three basic types of symmetry are described below.

Graphical Tests for Symmetry

1. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is on the graph, $(x, -y)$ is also on the graph.
2. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is on the graph, $(-x, y)$ is also on the graph.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is on the graph, $(-x, -y)$ is also on the graph.



y-Axis symmetry
Figure 1.14

For example, the graph of $y = x^2 - 2$ is symmetric with respect to the y-axis because (x, y) and $(-x, y)$ are on the graph of $y = x^2 - 2$. (See the table below and Figure 1.14.)

x	-3	-2	-1	1	2	3
y	7	2	-1	-1	2	7
(x, y)	$(-3, 7)$	$(-2, 2)$	$(-1, -1)$	$(1, -1)$	$(2, 2)$	$(3, 7)$

Algebraic Tests for Symmetry

1. The graph of an equation is symmetric with respect to the x-axis when replacing y with $-y$ yields an equivalent equation.
2. The graph of an equation is symmetric with respect to the y-axis when replacing x with $-x$ yields an equivalent equation.
3. The graph of an equation is symmetric with respect to the origin when replacing x with $-x$ and y with $-y$ yields an equivalent equation.

EXAMPLE 5 Testing for Symmetry

Test $y = 2x^3$ for symmetry with respect to both axes and the origin.

Solution

x-Axis:	$y = 2x^3$	Write original equation.
	$-y = 2x^3$	Replace y with $-y$. Result is not an equivalent equation.
y-Axis:	$y = 2x^3$	Write original equation.
	$y = 2(-x)^3$	Replace x with $-x$.
	$y = -2x^3$	Simplify. Result is not an equivalent equation.
Origin:	$y = 2x^3$	Write original equation.
	$-y = 2(-x)^3$	Replace y with $-y$ and x with $-x$.
	$-y = -2x^3$	Simplify.
	$y = 2x^3$	Simplify. Result is an equivalent equation.

Of the three tests for symmetry, the test for origin symmetry is the only one satisfied. So, the graph of $y = 2x^3$ is symmetric with respect to the origin (see Figure 1.15).

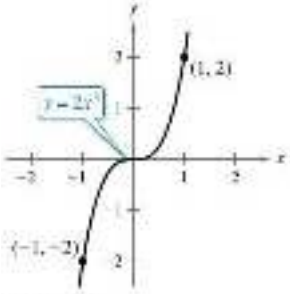


Figure 1.15

EXAMPLE 6 Using Symmetry as a Sketching Aid

Use symmetry to sketch the graph of $x - y^2 = 1$.

Solution Of the three tests for symmetry, the test for x-axis symmetry is the only one satisfied, because $x - (-y)^2 = 1$ is equivalent to $x - y^2 = 1$. So, the graph is symmetric with respect to the x-axis. Find solution points above (or below) the x-axis and then use symmetry to obtain the graph, as shown in Figure 1.16.

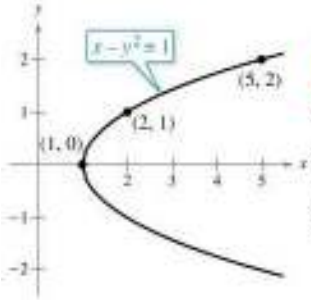


Figure 1.16

EXAMPLE 7 Sketching the Graph of an Equation

Sketch the graph of $y = |x - 1|$.

Solution This equation fails all three tests for symmetry, so its graph is not symmetric with respect to either axis or to the origin. The absolute value bars tell you that y is always nonnegative. Construct a table of values. Then plot and connect the points, as shown in Figure 1.17. Notice from the table that $x = 0$ when $y = 1$. So, the y-intercept is $(0, 1)$. Similarly, $y = 0$ when $x = 1$. So, the x-intercept is $(1, 0)$.

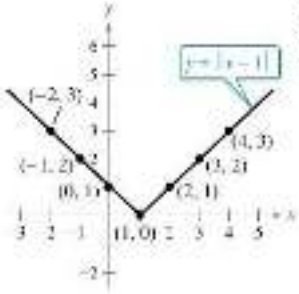


Figure 1.17

x	-2	-1	0	1	2	3	4
$y = \lvert x - 1 \rvert$	3	2	1	0	1	2	3
(x, y)	$(-2, 3)$	$(-1, 2)$	$(0, 1)$	$(1, 0)$	$(2, 1)$	$(3, 2)$	$(4, 3)$

Circles

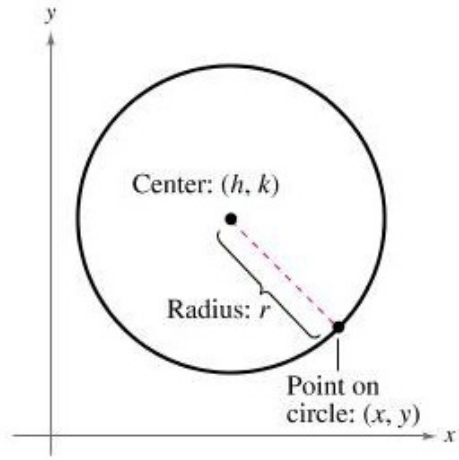
A **circle** is a set of points (x, y) in a plane that are the same distance r from a point called the center, (h, k), as shown at the right. By the Distance Formula,

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

By squaring each side of this equation, you obtain the **standard form of the equation of a circle.** For example, for a circle with its center at $(h, k) = (1, 3)$ and radius $r = 4$,

$$\sqrt{(x-1)^2 + (y-3)^2} = 4 \qquad \text{Substitute for } h, k, \text{ and } r.$$

$$(x-1)^2 + (y-3)^2 = 16. \qquad \text{Square each side.}$$



Standard Form of the Equation of a Circle

A point (x, y) lies on the circle of **radius** r and **center** (h, k) if and only if

$$(x-h)^2 + (y-k)^2 = r^2.$$

From this result, the standard form of the equation of a circle with radius r *and center at the origin,* $(h, k) = (0, 0)$, is

$$x^2 + y^2 = r^2. \qquad \text{Circle with radius } r \text{ and center at origin}$$

EXAMPLE 8 **Writing the Equation of a Circle**

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure 1.18. Write the standard form of the equation of this circle.

Solution

The radius of the circle is the distance between $(-1, 2)$ and $(3, 4)$.

$$\begin{aligned} r &= \sqrt{(x-h)^2 + (y-k)^2} && \text{Distance Formula} \\ &= \sqrt{[3-(-1)]^2 + (4-2)^2} && \text{Substitute for } x, y, h, \text{ and } k. \\ &= \sqrt{4^2 + 2^2} && \text{Simplify.} \\ &= \sqrt{16 + 4} && \text{Simplify.} \\ &= \sqrt{20} && \text{Radius} \end{aligned}$$

Using $(h, k) = (-1, 2)$ and $r = \sqrt{20}$, the equation of the circle is

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 && \text{Equation of circle} \\ [x-(-1)]^2 + (y-2)^2 &= (\sqrt{20})^2 && \text{Substitute for } h, k, \text{ and } r. \\ (x+1)^2 + (y-2)^2 &= 20. && \text{Standard form} \end{aligned}$$

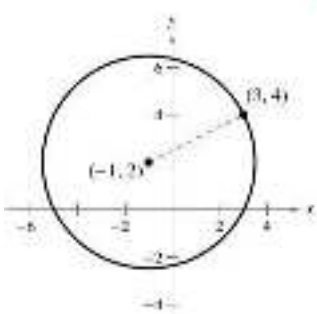


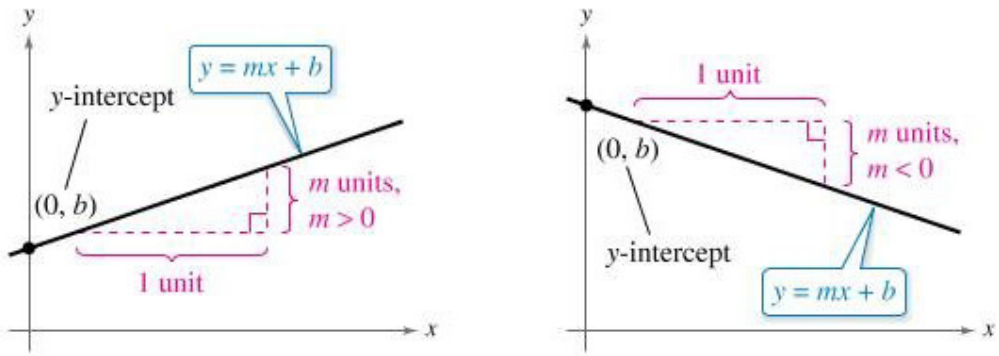
Figure 1.18

- Linear Equation = for relating two variables is the linear equation in two variables $y = mx + b$
  - The equation is linear because its graph is a straight line
  - By letting x = 0, you obtain
    - $y = m(0) + b = b$, so the line crosses the y-axis at y = b. In other words, the y-intercept is (0, b). The steepness, or slope of the line is m.

$$y = mx + b$$

Slope ↑ ↑ y-Intercept

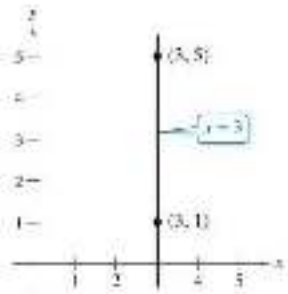
- The slope of a non-vertical line is the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right



Positive slope, line rises

Negative slope, line falls

- A linear equation written in slope-intercept form has the form $y = mx + b$



Slope is undefined.
Figure 1.20

The Slope-Intercept Form of the Equation of a Line

The graph of the equation

$$y = mx + b$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

Once you determine the slope and the y-intercept of a line, it is relatively simple to sketch its graph. In the next example, note that none of the lines is vertical. A vertical line has an equation of the form

$$x = a. \qquad \text{Vertical line}$$

The equation of a vertical line cannot be written in the form $y = mx + b$ because the slope of a vertical line is undefined (see Figure 1.20).

EXAMPLE 1 Graphing Linear Equations

Sketch the graph of each linear equation.

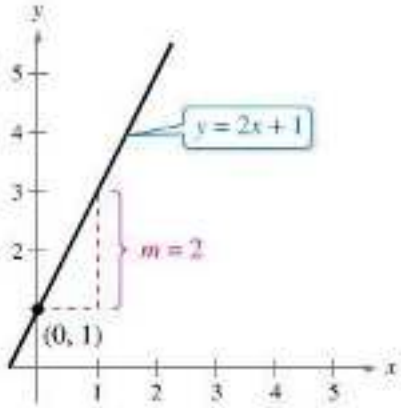
a. $y = 2x + 1$

b. $y = 2$

c. $x + y = 2$

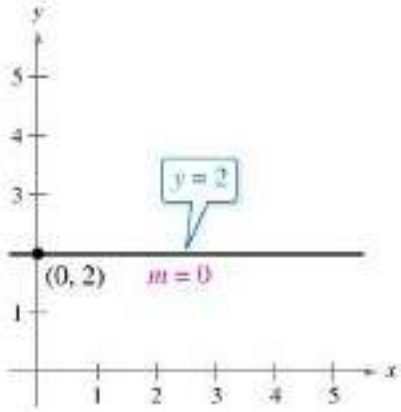
Solution

a. Because $b = 1$, the y-intercept is $(0, 1)$. Moreover, the slope is $m = 2$, so the line *rises* two units for each unit the line moves to the right (see figure).



When m is positive, the line rises.

b. By writing this equation in the form $y = (0)x + 2$, you find that the y-intercept is $(0, 2)$ and the slope is $m = 0$. A slope of 0 implies that the line is horizontal—that is, it does not rise *or* fall (see figure).

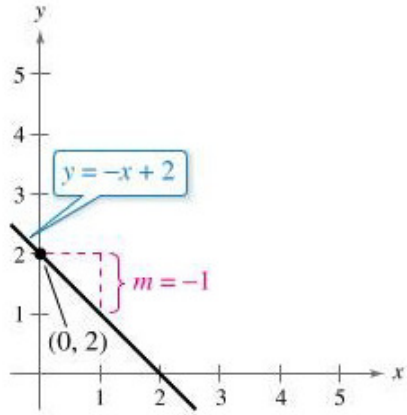


When m is 0, the line is horizontal.

c. By writing this equation in slope-intercept form

$x + y = 2$	Write original equation.
$y = -x + 2$	Subtract x from each side.
$y = (-1)x + 2$	Write in slope-intercept form.

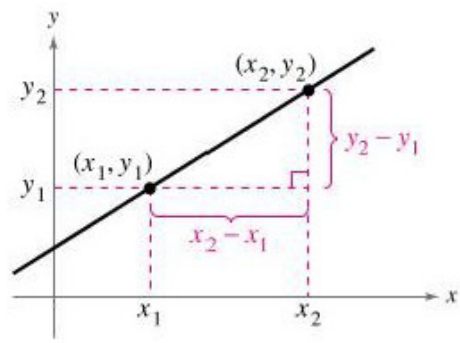
you find that the y-intercept is $(0, 2)$. Moreover, the slope is $m = -1$, so the line *falls* one unit for each unit the line moves to the right (see figure).



When m is negative, the line falls.

Finding the Slope of a Line

Given an equation of a line, you can find its slope by writing the equation in slope-intercept form. When you are not given an equation, you can still find the slope by using two points on the line. For example, consider the line passing through the points (x_1, y_1) and (x_2, y_2) in the figure below.



As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction.

$$y_2 - y_1 = \text{change in } y = \text{rise}$$

and

$$x_2 - x_1 = \text{change in } x = \text{run}$$

The ratio of $(y_2 - y_1)$ to $(x_2 - x_1)$ represents the slope of the line that passes through the points (x_1, y_1) and (x_2, y_2).

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The Slope of a Line Passing Through Two Points

The **slope** m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 \neq x_2$.

When using the formula for slope, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). However, once you do this, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_2 - y_1}{x_1 - x_2}}_{\text{Incorrect}} \quad \times$$

For example, the slope of the line passing through the points (3, 4) and (5, 7) can be calculated as

$$m = \frac{7 - 4}{5 - 3} = \frac{3}{2}$$

or as

$$m = \frac{4 - 7}{3 - 5} = \frac{-3}{-2} = \frac{3}{2}.$$

EXAMPLE 2 **Finding the Slope of a Line Through Two Points**

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$ **b.** $(-1, 2)$ and $(2, 2)$

c. $(0, 4)$ and $(1, -1)$ **d.** $(3, 4)$ and $(3, 1)$

Solution

a. Letting $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (3, 1)$, you find that the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{3 - (-2)} = \frac{1}{5}.$$ See Figure 1.21.

b. The slope of the line passing through $(-1, 2)$ and $(2, 2)$ is

$$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0,$$ See Figure 1.22.

c. The slope of the line passing through $(0, 4)$ and $(1, -1)$ is

$$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5.$$ See Figure 1.23.

d. The slope of the line passing through $(3, 4)$ and $(3, 1)$ is

$$m = \frac{1 - 4}{3 - 3} = \frac{-3}{0}. \quad \times$$ See Figure 1.24.

Division by 0 is undefined, so the slope is undefined and the line is vertical.

REMARK In Figures 1.21 through 1.24, note the relationships between slope and the orientation of the line.

a. Positive slope: line rises from left to right

b. Zero slope: line is horizontal

c. Negative slope: line falls from left to right

d. Undefined slope: line is vertical

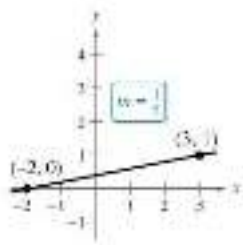


Figure 1.21

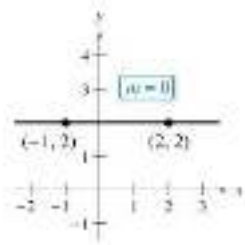


Figure 1.22

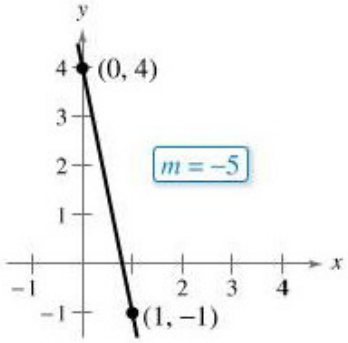


Figure 1.23

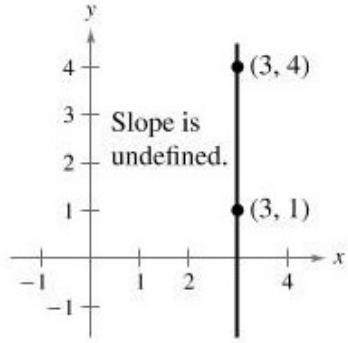


Figure 1.24

If (x_1, y_1) is a point on a line of slope m and (x, y) is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the **point-slope form** of the equation of a line

$$y - y_1 = m(x - x_1).$$

Point-Slope Form of the Equation of a Line

The equation of the line with slope m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

The point-slope form is useful for *finding* the equation of a line. You should remember this form.

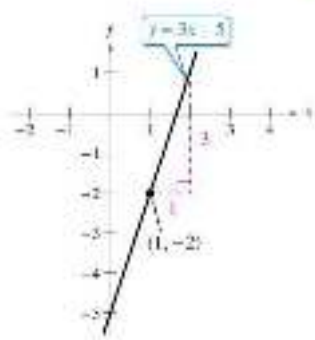


Figure 1.25

EXAMPLE 3 **Using the Point-Slope Form**

Find the slope-intercept form of the equation of the line that has a slope of 3 and passes through the point $(1, -2)$.

Solution Use the point-slope form with $m = 3$ and $(x_1, y_1) = (1, -2)$.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute for m, x_1, and y_1.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Write in slope-intercept form.

The slope-intercept form of the equation of the line is $y = 3x - 5$. Figure 1.25 shows the graph of this equation.

The point-slope form can be used to find an equation of the line passing through two points (x_1, y_1) and (x_2, y_2). To do this, first find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$

Then use the point-slope form to obtain the equation.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{Two-point form}$$

This is sometimes called the **two-point form** of the equation of a line.

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal. That is,

$$m_1 = m_2.$$

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,

$$m_1 = \frac{-1}{m_2}.$$

EXAMPLE 4 **Finding Parallel and Perpendicular Lines**

Find the slope-intercept form of the equations of the lines that pass through the point $(2, -1)$ and are (a) parallel to and (b) perpendicular to the line $2x - 3y = 5$.

Solution Write the equation of the given line in slope-intercept form.

$2x - 3y = 5$	Write original equation.
$-3y = -2x + 5$	Subtract $2x$ from each side.
$y = \frac{2}{3}x - \frac{5}{3}$	Write in slope-intercept form.

Notice that the line has a slope of $m = \frac{2}{3}$.

a. Any line parallel to the given line must also have a slope of $\frac{2}{3}$. Use the point-slope form with $m = \frac{2}{3}$ and $(x_1, y_1) = (2, -1)$.

$y - (-1) = \frac{2}{3}(x - 2)$	Write in point-slope form.
$3(y + 1) = 2(x - 2)$	Multiply each side by 3.
$3y + 3 = 2x - 4$	Distributive Property
$y = \frac{2}{3}x - \frac{7}{3}$	Write in slope-intercept form.

Notice the similarity between the slope-intercept form of this equation and the slope-intercept form of the given equation.

b. Any line perpendicular to the given line must have a slope of $-\frac{3}{2}$ [because $-\frac{3}{2}$ is the negative reciprocal of $\frac{2}{3}$]. Use the point-slope form with $m = -\frac{3}{2}$ and $(x_1, y_1) = (2, -1)$.

$y - (-1) = -\frac{3}{2}(x - 2)$	Write in point-slope form.
$2(y + 1) = -3(x - 2)$	Multiply each side by 2.
$2y + 2 = -3x + 6$	Distributive Property
$y = -\frac{3}{2}x + 2$	Write in slope-intercept form.

The graphs of all three equations are shown in Figure 1.26.

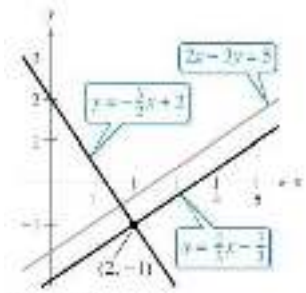


Figure 1.26

- In real life problem, the slope of a line can be interpreted as either a ration or a rate. When the x and y axis have the same unit of measure, the slope has no units and is a ratio. When the x and y axis have different units of measure, the slope is a rate or rate of change

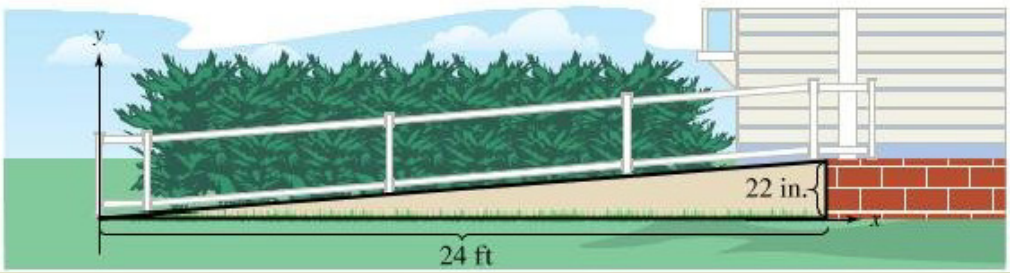
EXAMPLE 5 Using Slope as a Ratio

The maximum recommended slope of a wheelchair ramp is $\frac{1}{12}$. A business installs a wheelchair ramp that rises 22 inches over a horizontal length of 24 feet. Is the ramp steeper than recommended? *(Source: ADA Standards for Accessible Design)*

Solution The horizontal length of the ramp is 24 feet or $12(24) = 288$ inches (see figure). So, the slope of the ramp is

$$\text{Slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{22 \text{ in.}}{288 \text{ in.}} \approx 0.076.$$

Because $\frac{1}{12} \approx 0.083$, the slope of the ramp is not steeper than recommended.



Summary of Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$
6. Two-point form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Definition of Function

A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

Characteristics of a Function from Set *A* to Set *B*

1. Each element in A must be matched with an element in B.
2. Some elements in B may not be matched with any element in A.
3. Two or more elements in A may be matched with the same element in B.
4. An element in A (the domain) cannot be matched with two different elements in B.

Four Ways to Represent a Function

1. *Verbally* by a sentence that describes how the input variable is related to the output variable
2. *Numerically* by a table or a list of ordered pairs that matches input values with output values
3. *Graphically* by points in a coordinate plane in which the horizontal positions represent the input values and the vertical positions represent the output values
4. *Algebraically* by an equation in two variables

EXAMPLE 1 Testing for Functions

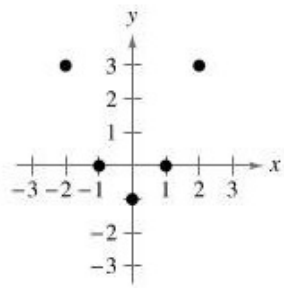
Determine whether the relation represents y as a function of x.

a. The input value x is the number of representatives from a state, and the output value y is the number of senators.

b.

Input, x	Output, y
2	11
2	10
3	8
4	5
5	1

c.



Solution

a. This verbal description *does* describe y as a function of x. Regardless of the value of x, the value of y is always 2. This is an example of a *constant function*.

b. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

c. The graph *does* describe y as a function of x. Each input value is matched with exactly one output value.

EXAMPLE 2 Testing for Functions Represented Algebraically

See LarsonPrecalculus.com for an interactive version of this type of example.

Determine whether each equation represents y as a function of x.

a. $x^2 + y = 1$

b. $-x + y^2 = 1$

Solution To determine whether y is a function of x, solve for y in terms of x.

a. Solving for y yields

$$x^2 + y = 1 \qquad \text{Write original equation.}$$
$$y = 1 - x^2. \qquad \text{Solve for } y.$$

To each value of x there corresponds exactly one value of y. So, y is a function of x.

b. Solving for y yields

$$-x + y^2 = 1 \qquad \text{Write original equation.}$$
$$y^2 = 1 + x \qquad \text{Add } x \text{ to each side.}$$
$$y = \pm\sqrt{1 + x}. \qquad \text{Solve for } y.$$

The $\pm$ indicates that to a given value of x there correspond two values of y. So, y is not a function of x.

When using an equation to represent a function, it is convenient to name the function for easy reference. For example, the equation $y = 1 - x^2$ describes y as a function of x. By renaming this function "f," you can write the input, output, and equation using **function notation.**

Input	**Output**	**Equation**
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as *the value of f at x* or simply *f of x*. The symbol $f(x)$ corresponds to the y-value for a given x. So, $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *value* of the function at x. For example, the function $f(x) = 3 - 2x$ has *function values* denoted by $f(-1)$, $f(0)$, $f(2)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

For $x = 2$, $\quad f(2) = 3 - 2(2) = 3 - 4 = -1.$

EXAMPLE 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Find each function value.

a. $g(2)$ **b.** $g(t)$ **c.** $g(x + 2)$

Solution

a. Replace x with 2 in $g(x) = -x^2 + 4x + 1$.

$$\begin{aligned} g(2) &= -(2)^2 + 4(2) + 1 \\ &= -4 + 8 + 1 \\ &= 5 \end{aligned}$$

b. Replace x with t.

$$\begin{aligned} g(t) &= -(t)^2 + 4(t) + 1 \\ &= -t^2 + 4t + 1 \end{aligned}$$

c. Replace x with $x + 2$.

$$\begin{aligned} g(x + 2) &= -(x + 2)^2 + 4(x + 2) + 1 \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 \\ &= -x^2 + 5 \end{aligned}$$

REMARK In Example 3(c), note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

A function defined by two or more equations over a specified domain is called a **piecewise-defined function.**

EXAMPLE 4 A Piecewise-Defined Function

Evaluate the function when $x = -1$, 0, and 1.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Solution Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain $f(-1) = (-1)^2 + 1 = 2$. For $x = 0$, use $f(x) = x - 1$ to obtain $f(0) = (0) - 1 = -1$. For $x = 1$, use $f(x) = x - 1$ to obtain $f(1) = (1) - 1 = 0$.

EXAMPLE 5 Finding Values for Which $f(x) = 0$

Find all real values of x for which $f(x) = 0$.

a. $f(x) = -2x + 10$ **b.** $f(x) = x^2 - 5x + 6$

Solution For each function, set $f(x) = 0$ and solve for x.

a. $-2x + 10 = 0$ Set $f(x)$ equal to 0.

$-2x = -10$ Subtract 10 from each side.

$x = 5$ Divide each side by -2.

So, $f(x) = 0$ when $x = 5$.

b. $x^2 - 5x + 6 = 0$ Set $f(x)$ equal to 0.

$(x - 2)(x - 3) = 0$ Factor.

$x - 2 = 0 \Rightarrow x = 2$ Set 1st factor equal to 0 and solve.

$x - 3 = 0 \Rightarrow x = 3$ Set 2nd factor equal to 0 and solve.

So, $f(x) = 0$ when $x = 2$ or $x = 3$.

EXAMPLE 6 Finding Values for Which $f(x) = g(x)$

Find the values of x for which $f(x) = g(x)$.

a. $f(x) = x^2 + 1$ and $g(x) = 3x - x^2$

b. $f(x) = x^2 - 1$ and $g(x) = -x^2 + x + 2$

Solution

a. $x^2 + 1 = 3x - x^2$ Set $f(x)$ equal to $g(x)$.

$2x^2 - 3x + 1 = 0$ Write in general form.

$(2x - 1)(x - 1) = 0$ Factor.

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ Set 1st factor equal to 0 and solve.

$x - 1 = 0 \Rightarrow x = 1$ Set 2nd factor equal to 0 and solve.

So, $f(x) = g(x)$ when $x = \dfrac{1}{2}$ or $x = 1$.

b. $x^2 - 1 = -x^2 + x + 2$ Set $f(x)$ equal to $g(x)$.

$2x^2 - x - 3 = 0$ Write in general form.

$(2x - 3)(x + 1) = 0$ Factor.

$2x - 3 = 0 \Rightarrow x = \frac{3}{2}$ Set 1st factor equal to 0 and solve.

$x + 1 = 0 \Rightarrow x = -1$ Set 2nd factor equal to 0 and solve.

So, $f(x) = g(x)$ when $x = \dfrac{3}{2}$ or $x = -1$.

The Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For example, the function

$$f(x) = \frac{1}{x^2 - 4}$$ Domain excludes x-values that result in division by zero.

has an implied domain consisting of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function

$$f(x) = \sqrt{x}$$ Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that cause division by zero *or* that result in the even root of a negative number.

EXAMPLE 7 Finding the Domains of Functions

Find the domain of each function.

a. f: $\{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = \dfrac{1}{x + 5}$

c. Volume of a sphere: $V = \frac{4}{3}\pi r^3$

d. $h(x) = \sqrt{4 - 3x}$

Solution

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}$$

b. Excluding x-values that yield zero in the denominator, the domain of g is the set of all real numbers x except $x = -5$.

c. This function represents the volume of a sphere, so the values of the radius r must be positive. The domain is the set of all real numbers r such that $r > 0$.

d. This function is defined only for x-values for which

$$4 - 3x \geq 0.$$

By solving this inequality, you can conclude that $x \leq \frac{4}{3}$. So, the domain is the interval $\left(-\infty, \frac{4}{3}\right]$.

EXAMPLE 8 The Dimensions of a Container

You work in the marketing department of a soft-drink company and are experimenting with a new can for iced tea that is slightly narrower and taller than a standard can. For your experimental can, the ratio of the height to the radius is 4.

a. Write the volume of the can as a function of the radius r.

b. Write the volume of the can as a function of the height h.

Solution

a. $V(r) = \pi r^2 h = \pi r^2(4r) = 4\pi r^3$ — Write V as a function of r.

b. $V(h) = \pi r^2 h = \pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{16}$ — Write V as a function of h.

EXAMPLE 9 The Path of a Baseball

A batter hits a baseball at a point 3 feet above ground at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). Will the baseball clear a 10-foot fence located 300 feet from home plate?

Algebraic Solution

Find the height of the baseball when $x = 300$.

$$f(x) = -0.0032x^2 + x + 3 \quad \text{Write original function.}$$

$$f(300) = -0.0032(300)^2 + 300 + 3 \quad \text{Substitute 300 for } x.$$

$$= 15 \quad \text{Simplify.}$$

When $x = 300$, the height of the baseball is 15 feet. So, the baseball will clear a 10-foot fence.

Difference Quotients

One of the basic definitions in calculus uses the ratio

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0.$$

This ratio is a **difference quotient,** as illustrated in Example 11.

REMARK You may find it easier to calculate the difference quotient in Example 11 by first finding $f(x+h)$, and then substituting the resulting expression into the difference quotient

$$\frac{f(x+h)-f(x)}{h}.$$

EXAMPLE 11 **Evaluating a Difference Quotient**

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h)-f(x)}{h}$.

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h}$$

$$= \frac{2xh + h^2 - 4h}{h} = \frac{h(2x + h - 4)}{h} = 2x + h - 4, \quad h \neq 0$$

Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

Function notation: $y = f(x)$

f is the *name* of the function.

y is the **dependent variable.**

x is the **independent variable.**

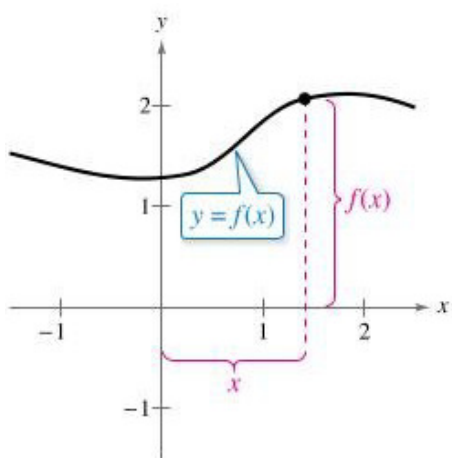
$f(x)$ is the *value of the function at* x.

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, then f is *defined* at x. If x is not in the domain of f, then f is *undefined* at x.

Range: The **range** of a function is the set of all values (outputs) taken on by the dependent variable (that is, the set of all function values).

Implied domain: If f is defined by an algebraic expression and the domain is not specified, then the **implied domain** consists of all real numbers for which the expression is defined.

- Graph of a function f = collection of ordered pairs (x. f(x)) such that x is in the domain of f
  - x is the directed distance from the y-axis
  - y = f(x) which is the directed distance from the x-axis



EXAMPLE 1 **Finding the Domain and Range of a Function**

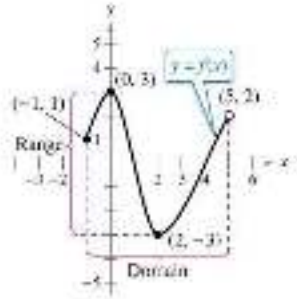


Figure 1.32

Use the graph of the function f, shown in Figure 1.32, to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

Solution

a. The closed dot at $(-1, 1)$ indicates that $x = -1$ is in the domain of f, whereas the open dot at $(5, 2)$ indicates that $x = 5$ is not in the domain. So, the domain of f is all x in the interval $[-1, 5)$.

b. One point on the graph of f is $(-1, 1)$, so $f(-1) = 1$. Another point on the graph of f is $(2, -3)$, so $f(2) = -3$.

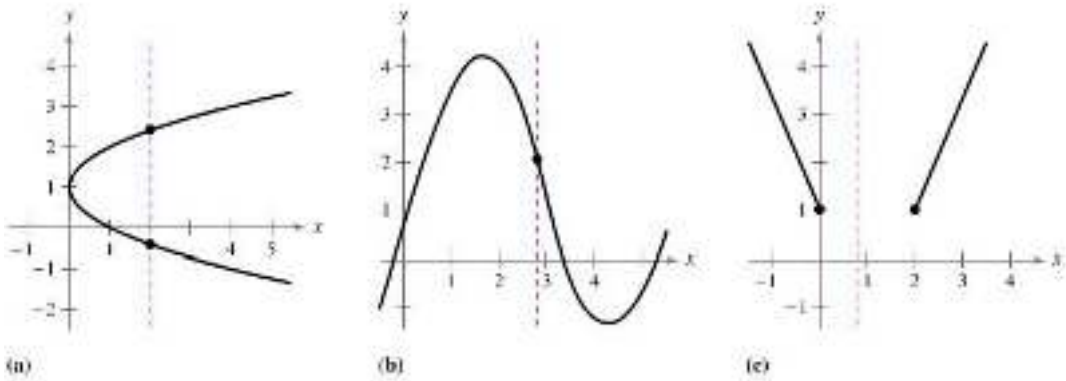
c. The graph does not extend below $f(2) = -3$ or above $f(0) = 3$, so the range of f is the interval $[-3, 3]$.

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.

EXAMPLE 2 **Vertical Line Test for Functions**

Use the Vertical Line Test to determine whether each graph represents y as a function of x.



Solution

a. This *is not* a graph of y as a function of x, because there are vertical lines that intersect the graph twice. That is, for a particular input x, there is more than one output y.

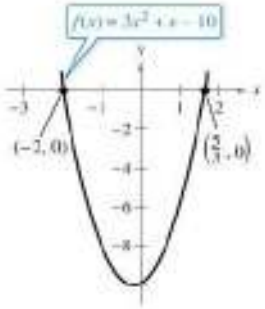
b. This *is* a graph of y as a function of x, because every vertical line intersects the graph at most once. That is, for a particular input x, there is at most one output y.

c. This *is* a graph of y as a function of x, because every vertical line intersects the graph at most once. That is, for a particular input x, there is at most one output y. [Note that when a vertical line does not intersect the graph, it simply means that the function is undefined for that particular value of x.]

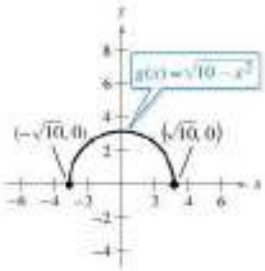
- Zeroes of a function = If the graph of a function of x has an x-intercept at (a, 0), then a is a zero of the function

Zeros of a Function

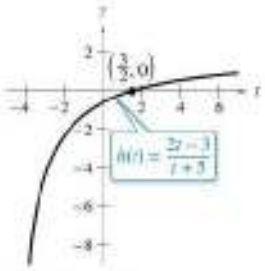
The **zeros of a function** $y = f(x)$ are the x-values for which $f(x) = 0$.



Zeros of f: $x = -2, x = \frac{5}{3}$
Figure 1.33



Zeros of g: $x = \pm\sqrt{10}$
Figure 1.34



Zero of h: $t = \frac{3}{2}$
Figure 1.35

EXAMPLE 3 **Finding the Zeros of Functions**

Find the zeros of each function algebraically.

a. $f(x) = 3x^2 + x - 10$

b. $g(x) = \sqrt{10 - x^2}$

c. $h(t) = \dfrac{2t - 3}{t + 5}$

Solution To find the zeros of a function, set the function equal to zero and solve for the independent variable.

a. $3x^2 + x - 10 = 0$ — Set $f(x)$ equal to 0.

$(3x - 5)(x + 2) = 0$ — Factor.

$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$ — Set 1st factor equal to 0 and solve.

$x + 2 = 0 \Rightarrow x = -2$ — Set 2nd factor equal to 0 and solve.

The zeros of f are $x = \frac{5}{3}$ and $x = -2$. In Figure 1.33, note that the graph of f has $\left(\frac{5}{3}, 0\right)$ and $(-2, 0)$ as its x-intercepts.

b. $\sqrt{10 - x^2} = 0$ — Set $g(x)$ equal to 0.

$10 - x^2 = 0$ — Square each side.

$10 = x^2$ — Add x^2 to each side.

$\pm\sqrt{10} = x$ — Extract square roots.

The zeros of g are $x = -\sqrt{10}$ and $x = \sqrt{10}$. In Figure 1.34, note that the graph of g has $(-\sqrt{10}, 0)$ and $(\sqrt{10}, 0)$ as its x-intercepts.

c. $\dfrac{2t - 3}{t + 5} = 0$ — Set $h(t)$ equal to 0.

$2t - 3 = 0$ — Multiply each side by $t + 5$.

$2t = 3$ — Add 3 to each side.

$t = \dfrac{3}{2}$ — Divide each side by 2.

The zero of h is $t = \frac{3}{2}$. In Figure 1.35, note that the graph of h has $\left(\frac{3}{2}, 0\right)$ as its t-intercept.

Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval when, for any x_1 and x_2 in the interval,

$$x_1 < x_2 \quad \text{implies} \quad f(x_1) < f(x_2).$$

A function f is **decreasing** on an interval when, for any x_1 and x_2 in the interval,

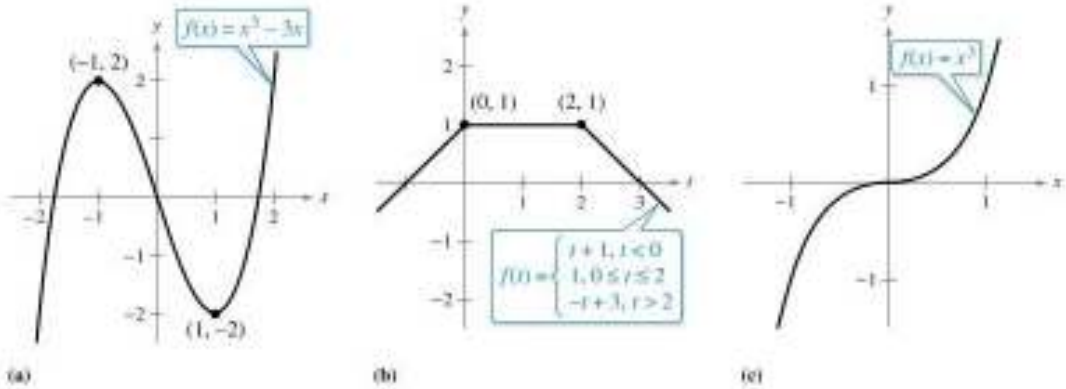
$$x_1 < x_2 \quad \text{implies} \quad f(x_1) > f(x_2).$$

A function f is **constant** on an interval when, for any x_1 and x_2 in the interval,

$$f(x_1) = f(x_2).$$

EXAMPLE 4 **Describing Function Behavior**

Determine the open intervals on which each function is increasing, decreasing, or constant.



Solution

a. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

b. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.

c. This function may appear to be constant on an interval near $x = 0$, but for all real values of x_1 and x_2, if $x_1 < x_2$, then $(x_1)^3 < (x_2)^3$. So, the function is increasing on the interval $(-\infty, \infty)$.

Definitions of Relative Minimum and Relative Maximum

A function value $f(a)$ is a **relative minimum** of f when there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is a **relative maximum** of f when there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

REMARK A relative minimum or relative maximum is also referred to as a local minimum or local maximum.

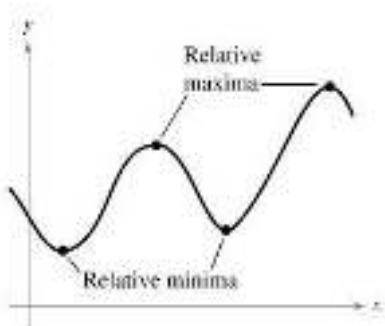


Figure 1.37

Average Rate of Change

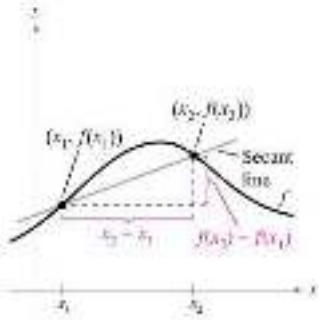


Figure 1.39

In Section 1.3, you learned that the slope of a line can be interpreted as a *rate of change*. For a nonlinear graph, the **average rate of change** between any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is the slope of the line through the two points (see Figure 1.39). The line through the two points is called a **secant line**, and the slope of this line is denoted as m_{sec}.

$$\text{Average rate of change of } f \text{ from } x_1 \text{ to } x_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} = m_{sec}$$

EXAMPLE 6 **Average Rate of Change of a Function**

Find the average rates of change of $f(x) = x^3 - 3x$ (a) from $x_1 = -2$ to $x_2 = -1$ and (b) from $x_1 = 0$ to $x_2 = 1$ (see Figure 1.40).

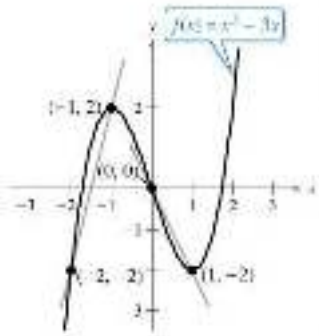


Figure 1.40

Solution

a. The average rate of change of f from $x_1 = -2$ to $x_2 = -1$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(-1) - f(-2)}{-1 - (-2)} = \frac{2 - (-2)}{1} = 4.$$

Secant line has positive slope.

b. The average rate of change of f from $x_1 = 0$ to $x_2 = 1$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(1) - f(0)}{1 - 0} = \frac{-2 - 0}{1} = -2$$

Secant line has negative slope.

EXAMPLE 7 **Finding Average Speed**

The distance s (in feet) a moving car is from a stoplight is given by the function

$$s(t) = 20t^{3/2}$$

where t is the time (in seconds). Find the average speed of the car (a) from $t_1 = 0$ to $t_2 = 4$ seconds and (b) from $t_1 = 4$ to $t_2 = 9$ seconds.

Solution

a. The average speed of the car from $t_1 = 0$ to $t_2 = 4$ seconds is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(4) - s(0)}{4 - 0} = \frac{160 - 0}{4} = 40 \text{ feet per second.}$$

b. The average speed of the car from $t_1 = 4$ to $t_2 = 9$ seconds is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(9) - s(4)}{9 - 4} = \frac{540 - 160}{5} = 76 \text{ feet per second.}$$

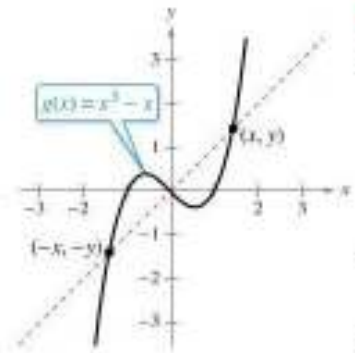
Even and Odd Functions

In Section 1.2, you studied different types of symmetry of a graph. In the terminology of functions, a function is said to be **even** when its graph is symmetric with respect to the y-axis and **odd** when its graph is symmetric with respect to the origin. The symmetry tests in Section 1.2 yield the tests for even and odd functions below.

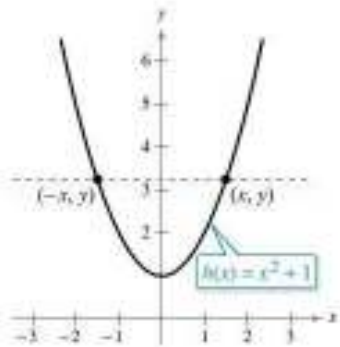
Tests for Even and Odd Functions

A function $y = f(x)$ is **even** when, for each x in the domain of f, $f(-x) = f(x)$.

A function $y = f(x)$ is **odd** when, for each x in the domain of f, $f(-x) = -f(x)$.



(a) Symmetric to origin: Odd Function



(b) Symmetric to y-axis: Even Function

Figure 1.41

EXAMPLE 8 **Even and Odd Functions**

a. The function $g(x) = x^3 - x$ is odd because $g(-x) = -g(x)$, as follows.

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) && \text{Substitute } -x \text{ for } x. \\ &= -x^3 + x && \text{Simplify.} \\ &= -(x^3 - x) && \text{Distributive Property} \\ &= -g(x) && \text{Test for odd function} \end{aligned}$$

b. The function $h(x) = x^2 + 1$ is even because $h(-x) = h(x)$, as follows.

$$h(-x) = (-x)^2 + 1 = x^2 + 1 = h(x) \qquad \text{Test for even function}$$

Figure 1.41 shows the graphs and symmetry of these two functions.

- Linear function:
  - The domain of the function is set of all real numbers
  - When $m \neq 0$, the range of the function is the set of all real numbers
  - The graph has an x-intercept at (-b/m, 0) and a y-intercept at (0,b)
  - The graph is increasing when m > 0, decreasing when m < 0, and constant when m = 0

EXAMPLE 1 Writing a Linear Function

Write the linear function f for which $f(1) = 3$ and $f(4) = 0$.

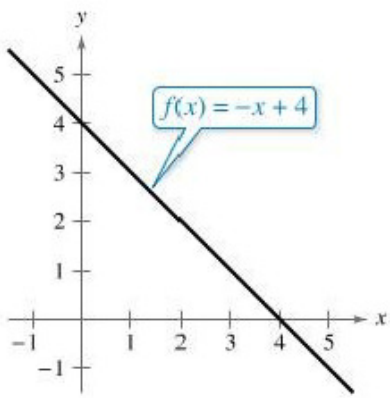
Solution To find the equation of the line that passes through $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (4, 0)$, first find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3}{4 - 1} = \frac{-3}{3} = -1$$

Next, use the point-slope form of the equation of a line.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - 3 = -1(x - 1)$	Substitute for x_1, y_1, and m.
$y = -x + 4$	Simplify.
$f(x) = -x + 4$	Function notation

The figure below shows the graph of this function.



There are two special types of linear functions, the **constant function** and the **identity function.** A constant function has the form

$$f(x) = c$$

and has a domain of all real numbers with a range consisting of a single real number c. The graph of a constant function is a horizontal line, as shown in Figure 1.42. The identity function has the form

$$f(x) = x.$$

Its domain and range are the set of all real numbers. The identity function has a slope of $m = 1$ and a y-intercept at $(0, 0)$. The graph of the identity function is a line for which each x-coordinate equals the corresponding y-coordinate. The graph is always increasing, as shown in Figure 1.43.

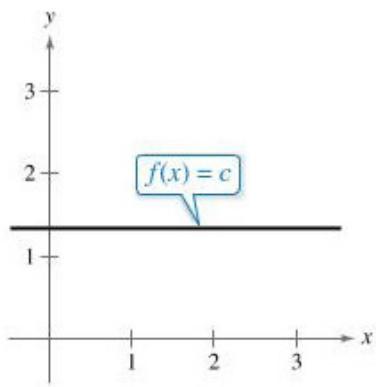


Figure 1.42

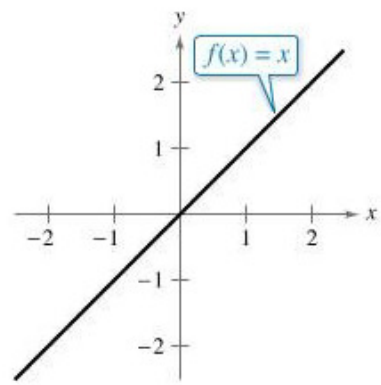


Figure 1.43

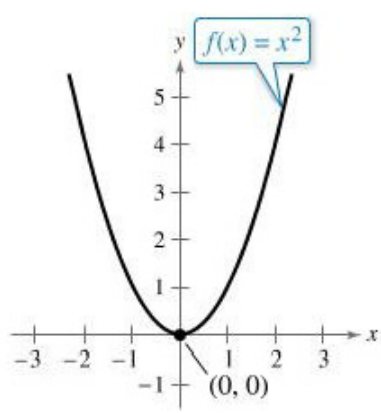
The graph of the **squaring function**

$$f(x) = x^2$$

is a U-shaped curve with the characteristics below.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all nonnegative real numbers.
- The function is even.
- The graph has an intercept at $(0, 0)$.
- The graph is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$.
- The graph is symmetric with respect to the y-axis.
- The graph has a relative minimum at $(0, 0)$.

The figure below shows the graph of the squaring function.

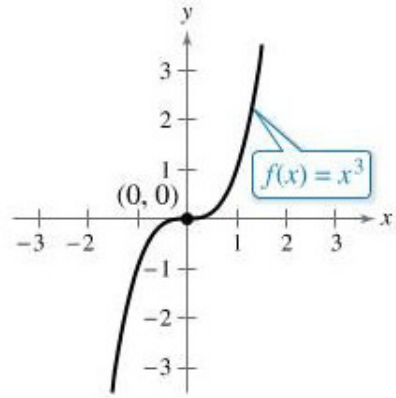


The graph of the *cubic* function

$$f(x) = x^3$$

has the characteristics below.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all real numbers.
- The function is odd.
- The graph has an intercept at (0, 0).
- The graph is increasing on the interval $(-\infty, \infty)$.
- The graph is symmetric with respect to the origin.



Cubic function

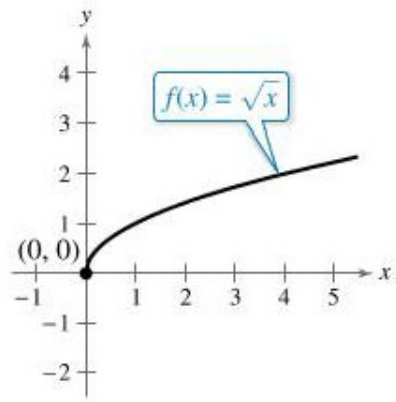
The figure shows the graph of the cubic function.

The graph of the *square root* function

$$f(x) = \sqrt{x}$$

has the characteristics below.

- The domain of the function is the set of all nonnegative real numbers.
- The range of the function is the set of all nonnegative real numbers.
- The graph has an intercept at (0, 0).
- The graph is increasing on the interval $(0, \infty)$.



Square root function

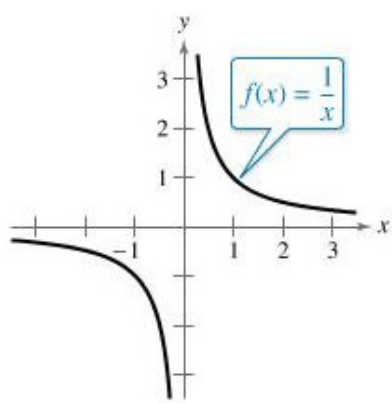
The figure shows the graph of the square root function.

The graph of the *reciprocal* function

$$f(x) = \frac{1}{x}$$

has the characteristics below.

- The domain of the function is $(-\infty, 0) \cup (0, \infty)$.
- The range of the function is $(-\infty, 0) \cup (0, \infty)$.
- The function is odd.
- The graph does not have any intercepts.
- The graph is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$.
- The graph is symmetric with respect to the origin.



Reciprocal function

The figure shows the graph of the reciprocal function.

Step and Piecewise-Defined Functions

Functions whose graphs resemble sets of stairsteps are known as **step functions.** One common type of step function is the **greatest integer function,** denoted by $[\![x]\!]$ and defined as

$$f(x) = [\![x]\!] = \textit{the greatest integer less than or equal to } x.$$

Here are several examples of evaluating the greatest integer function.

$$[\![-1]\!] = (\text{greatest integer} \le -1) = -1$$
$$[\![-\tfrac{1}{2}]\!] = (\text{greatest integer} \le -\tfrac{1}{2}) = -1$$
$$[\![\tfrac{1}{10}]\!] = (\text{greatest integer} \le \tfrac{1}{10}) = 0$$
$$[\![1.5]\!] = (\text{greatest integer} \le 1.5) = 1$$
$$[\![1.9]\!] = (\text{greatest integer} \le 1.9) = 1$$

The graph of the greatest integer function

$$f(x) = [\![x]\!]$$

has the characteristics below, as shown in Figure 1.44.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all integers.
- The graph has a y-intercept at $(0, 0)$ and x-intercepts in the interval $[0, 1)$.
- The graph is constant between each pair of consecutive integer values of x.
- The graph jumps vertically one unit at each integer value of x.

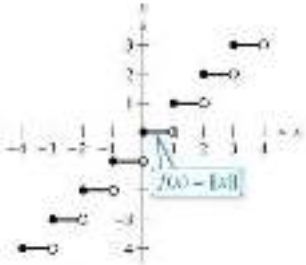


Figure 1.44

EXAMPLE 2 Evaluating a Step Function

Evaluate the function $f(x) = [\![x]\!] + 1$ when $x = -1, 2,$ and $\frac{5}{2}$.

Solution For $x = -1$, the greatest integer ≤ -1 is -1, so

$$f(-1) = [\![-1]\!] + 1 = -1 + 1 = 0.$$

For $x = 2$, the greatest integer ≤ 2 is 2, so

$$f(2) = [\![2]\!] + 1 = 2 + 1 = 3.$$

For $x = \frac{5}{2}$, the greatest integer $\le \frac{5}{2}$ is 2, so

$$f\left(\tfrac{5}{2}\right) = \left[\!\left[\tfrac{5}{2}\right]\!\right] + 1 = 2 + 1 = 3.$$

Verify your answers by examining the graph of $f(x) = [\![x]\!] + 1$ shown in Figure 1.45.

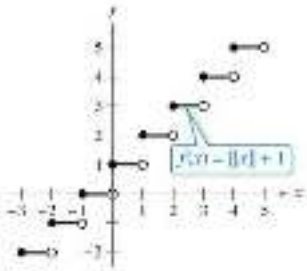


Figure 1.45

EXAMPLE 3 Graphing a Piecewise-Defined Function

See LarsonPrecalculus.com for an interactive version of this type of example.

Sketch the graph of $f(x) = \begin{cases} 2x - 3, & x \le 1 \\ -x + 4, & x > 1 \end{cases}$.

Solution This piecewise-defined function consists of two linear functions. At $x = 1$ and to the left of $x = 1$, the graph is the line $y = 2x - 3$, and to the right of $x = 1$, the graph is the line $y = -x + 4$, as shown in Figure 1.46. Notice that the point $(1, 5)$ is a solid dot and the point $(1, 3)$ is an open dot. This is because $f(1) = 2(1) + 3 = 5$.

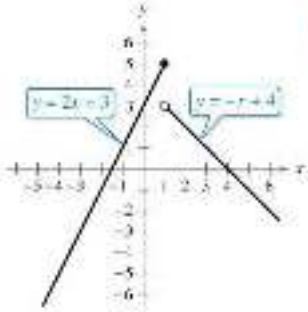
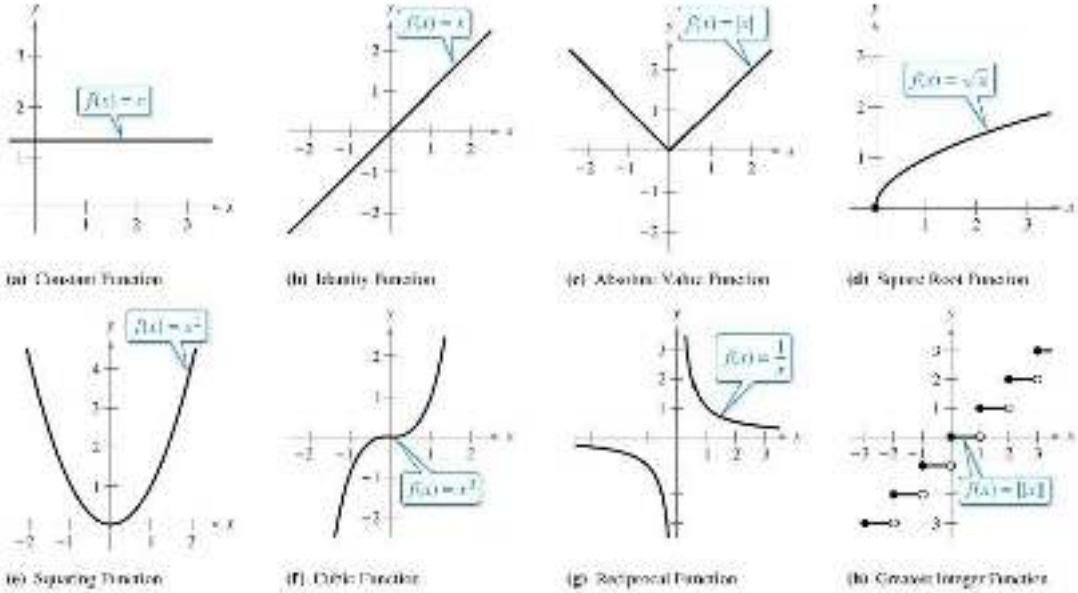


Figure 1.46



Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *up:* $h(x) = f(x) + c$
2. Vertical shift c units *down:* $h(x) = f(x) - c$
3. Horizontal shift c units to the *right:* $h(x) = f(x - c)$
4. Horizontal shift c units to the *left:* $h(x) = f(x + c)$

EXAMPLE 1 Shifting the Graph of a Function

Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $g(x) = x^3 - 1$

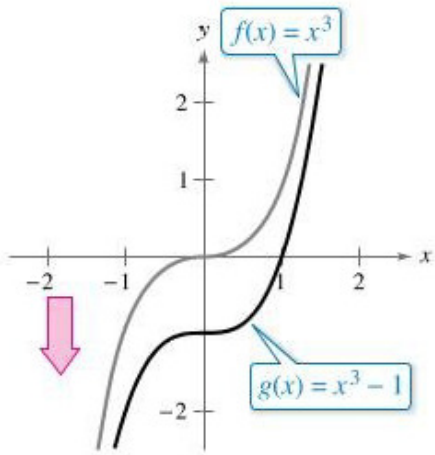
b. $h(x) = (x + 2)^3 + 1$

Solution

a. Relative to the graph of $f(x) = x^3$, the graph of

$$g(x) = x^3 - 1$$

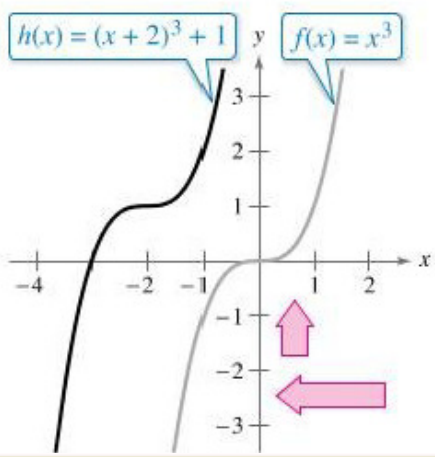
is a downward shift of one unit, as shown below.



b. Relative to the graph of $f(x) = x^3$, the graph of

$$h(x) = (x + 2)^3 + 1$$

is a left shift of two units and an upward shift of one unit, as shown below.



Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $h(x) = f(-x)$

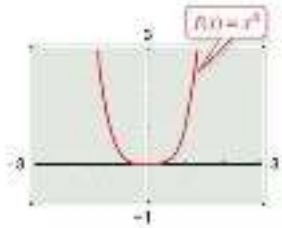


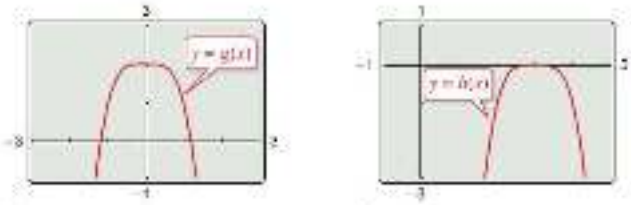
Figure 1.50

EXAMPLE 2 Writing Equations from Graphs

The graph of the function

$$f(x) = x^4$$

is shown in Figure 1.50. Each graph below is a transformation of the graph of f. Write an equation for the function represented by each graph.



(a) (b)

Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^4$. So, an equation for g is

$$g(x) = -x^4 + 2.$$

b. The graph of h is a right shift of three units *followed by* a reflection in the x-axis of the graph of $f(x) = x^4$. So, an equation for h is

$$h(x) = -(x - 3)^4.$$

EXAMPLE 3 Reflections and Shifts

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$ **c.** $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. The graph of g is a reflection of the graph of f in the x-axis because

$$g(x) = -\sqrt{x}$$
$$= -f(x).$$

b. The graph of h is a reflection of the graph of f in the y-axis because

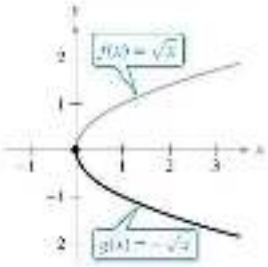
$$h(x) = \sqrt{-x}$$
$$= f(-x).$$

c. The graph of k is a left shift of two units followed by a reflection in the x-axis because

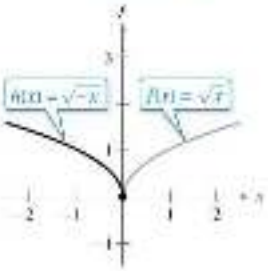
$$k(x) = -\sqrt{x+2}$$
$$= -f(x+2).$$

Graphical Solution

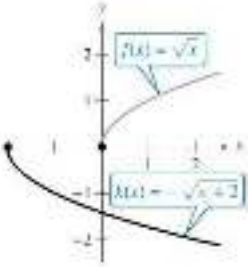
a. Graph f and g on the same set of coordinate axes. The graph of g is a reflection of the graph of f in the x-axis.



b. Graph f and h on the same set of coordinate axes. The graph of h is a reflection of the graph of f in the y axis.



c. Graph f and k on the same set of coordinate axes. The graph of k is a left shift of two units followed by a reflection in the x-axis of the graph of f.



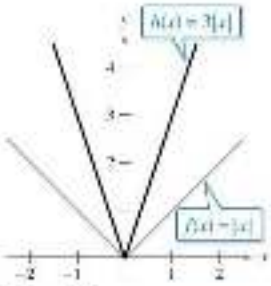


Figure 1.51

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the coordinate plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For example, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** when $c > 1$ and a **vertical shrink** when $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$, where the transformation is a **horizontal shrink** when $c > 1$ and a **horizontal stretch** when $0 < c < 1$.

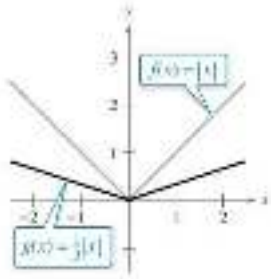


Figure 1.52

EXAMPLE 4 **Nonrigid Transformations**

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$ **b.** $g(x) = \frac{1}{3}|x|$

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of $h(x) = 3|x| = 3f(x)$ is a vertical stretch (each y-value is multiplied by 3). (See Figure 1.51.)

b. Similarly, the graph of $g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$ is a vertical shrink (each y-value is multiplied by $\frac{1}{3}$) of the graph of f. (See Figure 1.52.)

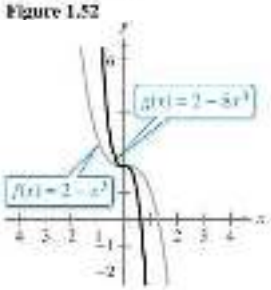


Figure 1.53

EXAMPLE 5 **Nonrigid Transformations**

Compare the graph of each function with the graph of $f(x) = 2 - x^3$.

a. $g(x) = f(2x)$ **b.** $h(x) = f(\frac{1}{2}x)$

Solution

a. Relative to the graph of $f(x) = 2 - x^3$, the graph of $g(x) = f(2x) = 2 - (2x)^3 = 2 - 8x^3$ is a horizontal shrink ($c > 1$). (See Figure 1.53.)

b. Similarly, the graph of $h(x) = f(\frac{1}{2}x) = 2 - (\frac{1}{2}x)^3 = 2 - \frac{1}{8}x^3$ is a horizontal stretch ($0 < c < 1$) of the graph of f. (See Figure 1.54.)

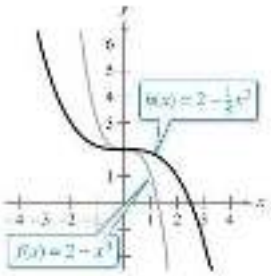


Figure 1.54

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. For example, the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined to form the sum, difference, product, and quotient of f and g.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4 \qquad \text{Sum}$$

$$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2 \qquad \text{Difference}$$

$$f(x)g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3 \qquad \text{Product}$$

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \qquad \text{Quotient}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the *sum, difference, product,* and *quotient* of f and g are defined as follows.

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

EXAMPLE 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 3$.

Solution The sum of f and g is

$$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x - 1) = x^2 + 4x.$$

When $x = 3$, the value of this sum is

$$(f + g)(3) = 3^2 + 4(3) = 21.$$

EXAMPLE 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Solution The difference of f and g is

$$(f - g)(x) = f(x) - g(x) = (2x + 1) - (x^2 + 2x - 1) = -x^2 + 2.$$

When $x = 2$, the value of this difference is

$$(f - g)(2) = -(2)^2 + 2 = -2.$$

EXAMPLE 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution The product of f and g is

$$(fg)(x) = f(x)g(x) = (x^2)(x - 3) = x^3 - 3x^2.$$

When $x = 4$, the value of this product is

$$(fg)(4) = 4^3 - 3(4)^2 = 16.$$

EXAMPLE 4 **Finding the Quotients of Two Functions**

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Then find the domains of f/g and g/f.

Solution The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains of f/g and g/f are as follows.

Domain of f/g: $[0, 2)$ Domain of g/f: $[0, 2]$

REMARK Note that the domain of f/g includes $x = 0$, but not $x = 2$, because $x = 2$ yields a zero in the denominator, whereas the domain of g/f includes $x = 2$, but not $x = 0$, because $x = 0$ yields a zero in the denominator.

Composition of Functions

Another way of combining two functions is to form the **composition** of one with the other. For example, if $f(x) = x^2$ and $g(x) = x + 1$, then the composition of f with g is

$$\begin{aligned} f(g(x)) &= f(x + 1) \\ &= (x + 1)^2. \end{aligned}$$

This composition is denoted as $f \circ g$ and reads as "f composed with g."

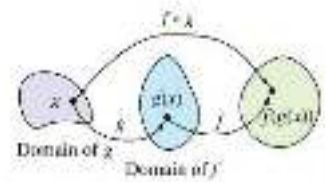


Figure 1.55

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.55.)

EXAMPLE 5 Compositions of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, find the following.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(g \circ f)(-2)$

Solution

a. The composition of f with g is as shown.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\
&= f(4 - x^2) && \text{Definition of } g(x) \\
&= (4 - x^2) + 2 && \text{Definition of } f(x) \\
&= -x^2 + 6 && \text{Simplify.}
\end{aligned}$$

b. The composition of g with f is as shown.

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\
&= g(x + 2) && \text{Definition of } f(x) \\
&= 4 - (x + 2)^2 && \text{Definition of } g(x) \\
&= 4 - (x^2 + 4x + 4) && \text{Expand.} \\
&= -x^2 - 4x && \text{Simplify.}
\end{aligned}$$

Note that, in this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

c. Evaluate the result of part (b) when $x = -2$.

$$\begin{aligned}
(g \circ f)(-2) &= -(-2)^2 - 4(-2) && \text{Substitute.} \\
&= -4 + 8 && \text{Simplify.} \\
&= 4 && \text{Simplify.}
\end{aligned}$$

REMARK The tables of values below help illustrate the composition $(f \circ g)(x)$ in Example 5(a).

x	0	1	2	3
$g(x)$	4	3	0	-5

$g(x)$	4	3	0	-5
$f(g(x))$	6	5	2	-3

x	0	1	2	3
$f(g(x))$	6	5	2	-3

Note that the first two tables are combined (or "composed") to produce the values in the third table.

EXAMPLE 6 **Finding the Domain of a Composite Function**

Find the domain of $f \circ g$ for the functions

$$f(x) = x^2 - 9 \quad \text{and} \quad g(x) = \sqrt{9 - x^2}.$$

Algebraic Solution

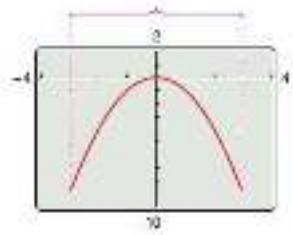
Find the composition of the functions.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{9 - x^2}) \\ &= (\sqrt{9 - x^2})^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2\end{aligned}$$

The domain of $f \circ g$ is restricted to the x-values in the domain of g for which $g(x)$ is in the domain of f. The domain of $f(x) = x^2 - 9$ is the set of all real numbers, which includes all real values of g. So, the domain of $f \circ g$ is the entire domain of $g(x) = \sqrt{9 - x^2}$, which is $[-3, 3]$.

Graphical Solution

Use a graphing utility to graph $f \circ g$.



From the graph, you can determine that the domain of $f \circ g$ is $[-3, 3]$.

In Examples 5 and 6, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. For example, the function $h(x) = (3x - 5)^3$ is the composition of $f(x) = x^3$ and $g(x) = 3x - 5$. That is,

$$h(x) = (3x - 5)^3 = [g(x)]^3 = f(g(x)).$$

Basically, to "decompose" a composite function, look for an "inner" function and an "outer" function. In the function h above, $g(x) = 3x - 5$ is the inner function and $f(x) = x^3$ is the outer function.

EXAMPLE 7 **Decomposing a Composite Function**

Write the function $h(x) = \dfrac{1}{(x - 2)^2}$ as a composition of two functions.

Solution Consider $g(x) = x - 2$ as the inner function and $f(x) = \dfrac{1}{x^2} = x^{-2}$ as the outer function. Then write

$$\begin{aligned}h(x) &= \frac{1}{(x - 2)^2} \\ &= (x - 2)^{-2} \\ &= f(x - 2) \\ &= f(g(x)).\end{aligned}$$

EXAMPLE 8 **Bacteria Count**

The number N of bacteria in a refrigerated food is given by

$$N(T) = 20T^2 - 80T + 500, \quad 2 \le T \le 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \quad 0 \le t \le 3$$

where t is the time in hours.

a. Find and interpret $(N \circ T)(t)$.

b. Find the time when the bacteria count reaches 2000.

Solution

a.
$$\begin{aligned}(N \circ T)(t) &= N(T(t)) \\ &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420\end{aligned}$$

The composite function $N \circ T$ represents the number of bacteria in the food as a function of the amount of time the food has been out of refrigeration.

b. The bacteria count reaches 2000 when $320t^2 + 420 = 2000$. By solving this equation algebraically, you find that the count reaches 2000 when $t \approx 2.2$ hours. Note that the negative solution $t \approx -2.2$ hours is rejected because it is not in the domain of the composite function.

Summary

In this week, we learned the Cartesian plane, Pythagorean theorem, distance formula, graph intervals/intercepts, graph symmetry, what a function is, average rate of change, various parent functions found throughout algebra and so much more.

Week 9 Graph Sketching & Kinematics Reading Note 3

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-21 5:54 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Graph Sketching & Kinematics Reading	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: May 21, 2020

Essential Question:

What is a function and what are its applications to kinematics (simple motion)?

Questions/Cues:

- How do we calculate average speed?
- What are distance-time graphs?
- What is instantaneous speed?
- What is average velocity?
- What are displacement-time graphs?
- What is acceleration?
- What are velocity-time graphs?
- What is the area underneath the graph and how is it related to an object's displacement?
- What are the suvat equations?
- What is stopping distance?
- What is acceleration due to gravity or the acceleration of free fall?

Notes

Calculating average speed

The **average speed** v of an object can be calculated from the distance x travelled and the time t taken using the equation

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

You can write this in algebraic form as

$$v = \frac{\Delta x}{\Delta t}$$

The Greek capital letter Δ (delta) means 'change in'. From the SI base units for distance and time, the unit of speed is m s^{-1}.

Worked example: Fine or not?

A car travels 2.5 km in 1 minute 22 seconds. The average speed limit for the road is 50 mph ($22\,m\,s^{-1}$). Has the driver exceeded this average speed limit?

Step 1: Identify the equation and list the known values.

$$v = \frac{\Delta x}{\Delta t}$$

$\Delta x = 2500\,m$, $\Delta t = 1$ minute 22 seconds $= 60 + 22 = 82\,s$

Step 2: Substitute the values into the equation and calculate the answer.

$$v = \frac{2500}{82} = 30.49\,m\,s^{-1}$$

$$v = 30\,m\,s^{-1}\ (2\ s.f.)$$

Yes, the driver has exceeded the average speed limit.

Distance–time graphs

Graphs of distance against time are used to represent the motion of objects.

- Distance is plotted on the y-axis (vertical axis).
- Time is plotted on the x-axis (horizontal axis).

In a distance–time graph, a stationary object is represented by a horizontal straight line. An object moving at a **constant speed** is represented by a straight, sloping line. The **gradient** of that line is equal to the distance travelled divided by the time taken, $\Delta x/\Delta t$, in other words, to the speed of the object.

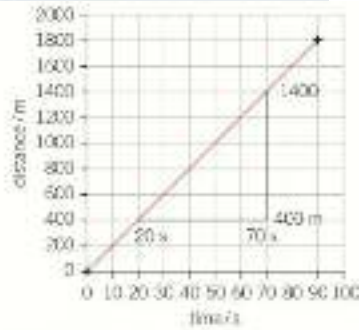
- speed = gradient of a distance–time graph

In Figure 2

change in distance $\Delta x = 1400 - 400 = 1000\,m$

change in time $\Delta t = 70 - 20 = 50\,s$

speed $v = \frac{\Delta x}{\Delta t} = \frac{1000}{50} = 20\,m\,s^{-1}$



▲ **Figure 2** *A distance–time graph for an object moving at constant speed – the gradient of the graph represents the speed*

The vector nature of velocity

Displacement s is a vector quantity, unlike distance, which is scalar. Displacement has both magnitude and direction. Speed is a scalar quantity calculated from distance, but velocity is a vector quantity calculated from displacement. The **average velocity** v of an object can be calculated from the change in displacement and the time taken.

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{time taken}}$$

You can write this in algebraic form as

$$v = \frac{\Delta s}{\Delta t}$$

where Δs is the change in displacement and Δt is the time taken. The SI unit for velocity is m s^{-1}.

The worked example below shows that average speed and average velocity are very different quantities.

Worked example: Speed and velocity

a Leeds is about 21 km south of Harrogate, but the distance by road is about 24 km (Figure 2). It takes 37 minutes to travel from Harrogate to Leeds by road. Calculate the average speed and the average velocity.

Step 1: Identify the equations needed.

$$\text{average speed } v = \frac{\Delta x}{\Delta t}, \text{ average velocity } v = \frac{\Delta s}{\Delta t}$$

Step 2: Substitute the values in SI units into the equations and calculate the answers.

$$\text{time taken } \Delta t = 60 \times 37 = 2220\,\text{s}$$

$$\text{average speed } v = \frac{\Delta x}{\Delta t} = \frac{24\,000}{2220} = 10.8\,\text{m s}^{-1} = 11\,\text{m s}^{-1} \text{ (2 s.f.)}$$

$$\text{average velocity } v = \frac{\Delta s}{\Delta t} = \frac{21\,000}{2220} = 9.5\,\text{m s}^{-1} \text{ (2 s.f.)}$$

The magnitude of the average velocity is $9.5\,\text{m s}^{-1}$ and its direction is due south from Harrogate.

b What would happen to the magnitude of the average velocity if the journey was from Harrogate to Leeds and then back to Harrogate?

The overall change in displacement would be zero and therefore the average velocity would be zero.

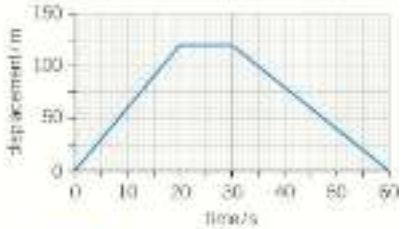
Displacement–time graphs

Graphs of displacement against time are used to represent the motion of objects.

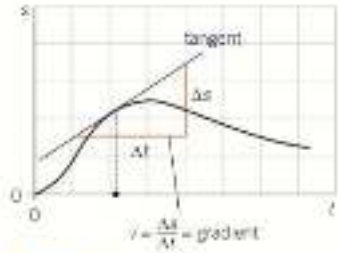
- Displacement is plotted on the y-axis (vertical axis).
- Time is plotted on the x-axis (horizontal axis).

Figure 3 shows the displacement–time graph for a car travelling along a straight road. The car is travelling at a constant velocity between $t = 0$ and $t = 20$ s, as can be seen from the first straight-line section of the graph. The horizontal section of the graph between $t = 20$ s and $t = 30$ s shows that the displacement of the car remains constant. Therefore, the car must be stationary. After $t = 30$ s, the graph is still a straight line but has a negative slope. The displacement of the car is getting smaller with time. The car must therefore be returning at a constant velocity.

You can determine the velocity of an object from the gradient of its displacement–time (s–t) graph. If the graph is not a straight line, draw a tangent to the graph, then calculate the gradient of this tangent for the instantaneous velocity, as illustrated in Figure 4.



▲ **Figure 3** *A displacement–time graph for a car journey*



▲ **Figure 4** *Velocity can be determined from the gradient of the displacement–time graph*

Worked example: Forwards and backwards

Use Figure 3 to determine the velocity of the car at $t = 10$ s and $t = 40$ s.

Step 1: Identify the equation needed and how to obtain the values from the graph.

$$\text{velocity } v = \frac{\Delta s}{\Delta t}$$

The right-hand terms are equivalent to the change on the y-axis (s) divided by the change on the x-axis (t).

Step 2: Substitute the values into the equation and calculate the answer.

The velocity at $t = 10$ s can be determined from the gradient of the straight-line graph between $t = 0$ and $t = 20$ s.

$$\text{velocity } v = \frac{\Delta s}{\Delta t} = \frac{120-0}{20-0} = 6.0\,\text{m s}^{-1}$$

The velocity at $t = 40$ s can be determined from the gradient of the straight-line graph between $t = 30$ s and $t = 60$ s.

$$\text{velocity } v = \frac{\Delta s}{\Delta t} = \frac{0-120}{60-30} = -4.0\,\text{m s}^{-1}$$

The negative sign for the velocity shows that the car is travelling in the opposite direction to its motion between 0 and 20 s.

Determining acceleration

The **acceleration** of an object is defined as the rate of change of velocity. In mathematical form, the acceleration a is

$$a = \frac{\Delta v}{\Delta t}$$

where Δv is the change in velocity and Δt is the time taken for the change. The unit of acceleration is $m\,s^{-2}$. Since acceleration is determined from velocity, it too is a vector quantity – it has magnitude and direction. A negative acceleration is often called deceleration.

Acceleration can be determined by calculation, or from a velocity–time (v–t) graph.

Calculating acceleration

You can calculate acceleration if you know the change in velocity of an object and the time taken for this change.

Worked example: 0 to 62, then slam the brakes on

a A Bugatti Veyron can accelerate from 0 to 100 km/h ($27.8\,m\,s^{-1}$) in 2.46 s. Calculate its average acceleration.

Step 1: Identify the equation needed.

$$a = \frac{\Delta v}{\Delta t}$$

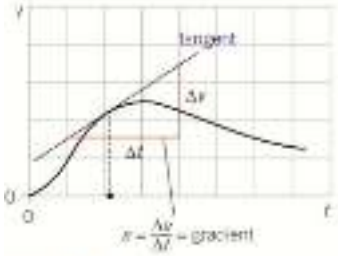
Step 2: Substitute the values into the equation and calculate the answer.

$$a = \frac{\Delta v}{\Delta t} = \frac{27.8-0}{2.46} = 13.3\,m\,s^{-2}$$

b The car takes 2.34 s to stop from 100 km/h under braking. Calculate its acceleration, assuming that this is constant during braking.

$$a = \frac{\Delta v}{\Delta t} = \frac{0-27.8}{2.34} = -11.9\,m\,s^{-2}$$

The negative sign means that the velocity of the car is decreasing over time – it is decelerating.



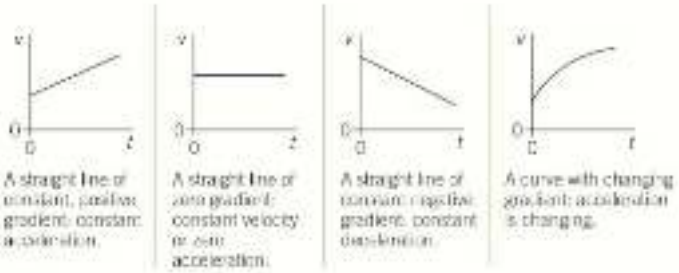
▲ **Figure 3** *Acceleration can be determined from the gradient of the velocity–time graph*

Velocity–time graphs

Since $a = \frac{\Delta v}{\Delta t}$, it follows that the acceleration of an object can be determined from the gradient of a velocity–time graph. You have seen how to determine gradients for straight-line graphs and for non-linear graphs in the previous topics in this chapter; the only difference in this case is that the y-axis of the graph represents velocity v rather than displacement s.

- acceleration = gradient of velocity–time graph

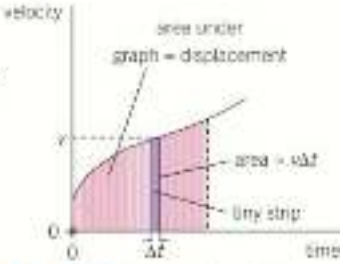
Figure 4 shows how the motion of an object can be deduced from the velocity–time graph.



▲ **Figure 4** *Interpreting velocity–time graphs*

Area under the graph

In the previous topic you saw that acceleration can be determined from the gradient of a velocity–time graph. In addition, we can read the displacement of the object from the area under the graph. Figure 2 shows why this is so.



▲ **Figure 2** *The area under the velocity–time graph is equal to displacement*

You will recall that the average velocity v is given by the equation

$$v = \frac{\Delta s}{\Delta t}$$

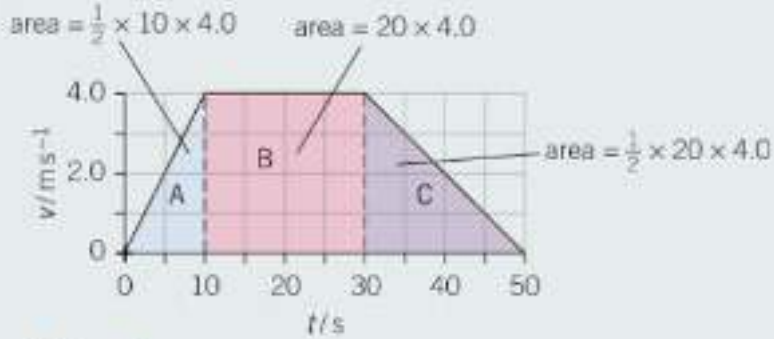
where Δs is the displacement in the time interval Δt. For the instantaneous velocity, assume that Δt is very small indeed – the velocity of the object is not going to change much. The change in the displacement $\Delta s = v\Delta t$. If you look at the graph in Figure 2, this is the area of the very thin rectangular strip marked under the graph. Therefore, the change in displacement is equal to the area of this strip. If you add similar strips for a longer interval of time, then clearly the area under the velocity–time graph is the total displacement of the object.

Calculating displacement for constant accelerations

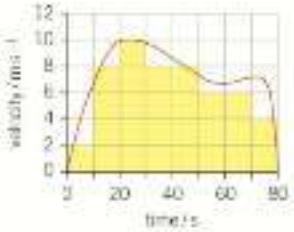
Displacement is easy to calculate when the acceleration is constant, because the areas can be broken down into rectangles and right-angled triangles. This is illustrated in the worked example below for the short journey of a cyclist.

Worked example: Displacement of a cyclist

The velocity–time graph for a cyclist travelling along a straight road is shown in Figure 3.



▲ **Figure 3**



▲ **Figure 4** *Calculating the area under a non-linear velocity–time graph*

Calculate the total distance travelled by the cyclist (the cyclist's displacement) in the period of 50 s.

Step 1: Identify the method needed.

distance travelled = area between the graph and the time axis

Step 2: Calculate the answer.

distance travelled = area of triangle **A** + area of rectangle **B** + area of triangle **C**

$$= \left(\tfrac{1}{2} \times 10 \times 4.0\right) + \left(20 \times 4.0\right) + \left(\tfrac{1}{2} \times 20 \times 4.0\right)$$

$$= 20 + 80 + 40$$

$$= 140\,\text{m}$$

Step 3: There is an alternative method to determine the area under the graph. Calculate the area of the trapezium using the formula

area = $\frac{1}{2}$ × (sum of the parallel sides) × vertical height

$$\text{distance} = \tfrac{1}{2} \times (50 + 20) \times 4.0 = 140\,\text{m}$$

Calculating displacement for changing accelerations

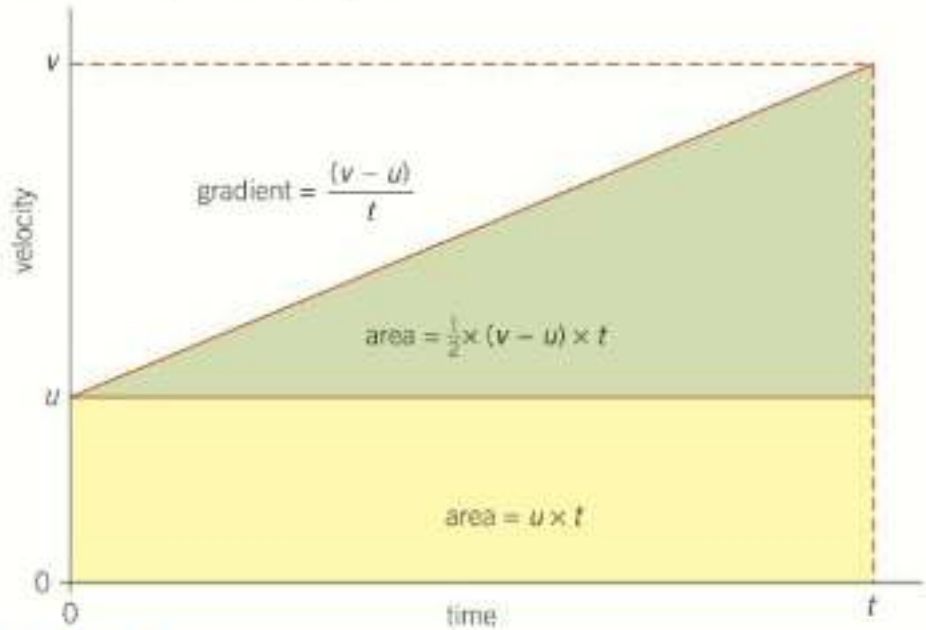
For non-linear velocity–time graphs, you can determine the area under the graph by counting squares. Taking Figure 4 as an example, you would start by counting the squares that are complete or nearly complete (yellow). Then count the remaining squares that lie mostly beneath the graph. Omit squares that are mostly above the graph.

Equations of motion: the *suvat* equations

You need four equations to calculate quantities involving motion in a *straight line* at a constant acceleration. These equations of motion are often informally referred to as the '*suvat* equations' after the symbols for the quantities involved.

Deriving the equations of motion

Figure 2 shows the velocity–time graph for an accelerating object. The initial velocity of the object is u. After a time t the final velocity of the object is v. The object has a constant acceleration a, as you can see from the straight-line graph.



▲ **Figure 2** *Velocity–time graph showing how the* suvat *equations are derived*

Equation without s

From the graph in Figure 2

$$a = \frac{\Delta v}{\Delta t} = \frac{v - u}{t}$$

This can be rearranged to give

$$\boxed{v = u + at} \qquad (1)$$

Equation without v

You will recall that the area under the velocity–time graph is equal to the displacement s.

- the rectangular area $= ut$
- the triangular area $= \frac{1}{2} \times (v - u) \times t$

▼ **Table 1** *The* suvat *quantities*

Symbol	Quantity
s	displacement (or distance travelled)
u	initial velocity
v	final velocity
a	acceleration
t	time taken for the change in velocity

From Equation **1** above, $(v - u) = at$. If you substitute this into the expression for the area of the triangle, you get $\frac{1}{2} \times at \times t$. With ut for the area of the rectangle, this gives the total area s.

$$\boxed{s = ut + \frac{1}{2}at^2} \quad (2)$$

Study tip

The four *suvat* equations are

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

Equation without *a*

If you treat the area under the graph as the area of a trapezium (with u and v as the parallel sides of the trapezium, and t as the perpendicular separation between them), this becomes

$$\boxed{s = \frac{1}{2}(u + v)t} \quad (3)$$

In other words, the displacement s is the average velocity, $\left(\frac{u+v}{2}\right)$, multiplied by the time t.

Equation without *t*

You need Equations **1** and **3** to derive the last useful equation of motion.

According to Equation **1** the time t is given by the equation

$$t = \frac{(v - u)}{a}$$

This equation for t can be substituted into Equation **3** to give

$$s = \frac{1}{2}(u + v) \times \frac{(v - u)}{a}$$

Rearranging this gives

$$(u + v)(v - u) = 2as$$

$$v^2 - u^2 = 2as$$

$$\boxed{v^2 = u^2 + 2as} \quad (4)$$

Equation without *u*

There is one more equation, $s = vt - \frac{1}{2}at^2$.

See if you can derive it from the velocity–time graph.

Worked example: Car accelerating from a standing start

A car on a straight road accelerates from rest to a velocity of $12\,\mathrm{m\,s^{-1}}$ in a time of 9.0 s. Calculate the acceleration of the car and the distance travelled in this time.

Step 1: Write down the quantities given in the order *suvat* and identify the equations needed.

$$s = ?,\ u = 0,\ v = 12\,\mathrm{m\,s^{-1}},\ a = ?,\ t = 9.0\,\mathrm{s}$$

Use Equation **1** to calculate the acceleration a and Equation **3** to calculate s.

Step 2: Substitute the values into the equations and calculate the answers.

Acceleration can be calculated from Equation **1**.

$$v = u + at$$

$$12 = 0 + a \times 9.0$$

$$a = \frac{12 - 0}{9.0} = 1.33\,\text{m}\,\text{s}^{-2}$$

The distance travelled is the displacement s along the straight road. Equation **3** can be used to calculate s.

$$s = \frac{1}{2}(v + u)t$$

$$s = \frac{1}{2} \times (12 + 0) \times 9.0$$

$$s = 54\,\text{m}$$

Worked example: Particle accelerating

A particle travels a distance of 16 m as it accelerates from $4.0\,\text{m}\,\text{s}^{-1}$ to $12\,\text{m}\,\text{s}^{-1}$. Calculate its acceleration.

Step 1: Again, start with the *suvat* values and identify the equation needed.

$s = 16\,\text{m}$, $u = 4.0\,\text{m}\,\text{s}^{-1}$, $v = 12\,\text{m}\,\text{s}^{-1}$, $a = ?$, $t = ?$

The equation for v, u, and s is Equation **4.** We can use this to calculate the acceleration a.

Step 2: Substitute the values into the equation and calculate the answer.

$$v^2 = u^2 + 2as$$

$$12^2 = 4.0^2 + 2 \times a \times 16$$

$$a = \frac{12^2 - 4.0^2}{2 \times 16} = 4.0\,\text{m}\,\text{s}^{-2}$$

Worked example: Falling to Earth

An apple falls from rest in a tree towards soft ground from a height of 1.50 m. Objects falling to Earth have an acceleration (free fall) of $9.81\,\mathrm{m\,s^{-2}}$. Calculate the time taken for the apple to reach the ground. Assume air resistance has negligible effect on the motion.

Step 1: List the *suvat* values and identify the equation needed.

$$s = 1.50\,\mathrm{m},\ u = 0\,\mathrm{m\,s^{-1}},\ a = 9.81\,\mathrm{m\,s^{-2}},\ t = ?$$

Use Equation **2** to calculate the time t.

Step 2: Substitute the values into the equation and calculate the answer.

$$s = ut + \frac{1}{2}at^2$$

$$1.50 = (0 \times t) + \frac{1}{2} \times 9.81 \times t^2$$

$$t^2 = \frac{2 \times 1.50}{9.81} = 0.306\ (3\ \text{s.f.})$$

$$t = 0.553\,\mathrm{s}$$

In this calculation, the equation is effectively $s = \frac{1}{2}at^2$, because $u = 0$.

Worked example: What goes up

A paper clip is flicked vertically up in the air at $6.0\,\mathrm{m\,s^{-1}}$ (Figure 4). Calculate its maximum height. Assume air resistance has negligible effect on the motion.

Step 1: List the known *suvat* values and identify the equation needed.

The paper clip will decelerate as it moves vertically, therefore, a must be *negative*. At maximum height it will stop momentarily, therefore, $v = 0$.

$$s = ?,\ u = 6.0\,\mathrm{m\,s^{-1}},\ v = 0\,\mathrm{m\,s^{-1}},\ a = -9.81\,\mathrm{m\,s^{-2}}$$

The equation for v, u, and s is Equation **4.** We can use this to calculate the height s.

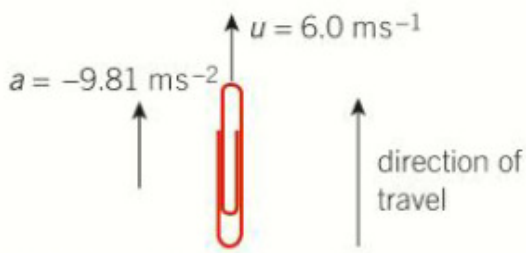
Step 2: Substitute the values into the equation and calculate the answer.

$$v^2 = u^2 + 2as$$

$$0 = u^2 + 2as$$

$$s = -\frac{u^2}{2a} = -\frac{6.0^2}{2 \times -9.81}$$

$$s = 1.83\,\mathrm{m} = 1.8\,\mathrm{m}\ (2\ \mathrm{s.f.})$$



▲ **Figure 4**

Components of stopping distances

The **stopping distance** is the total distance travelled from when the driver first sees a reason to stop, to when the vehicle stops. It has two components:

- **thinking distance**, the distance travelled between the moment when you first see a reason to stop, to the moment when you use the brake
- **braking distance**, the distance travelled from the time the brake is applied until the vehicle stops.

Many factors influence these distances, including the speed of the vehicle, the condition of the brakes, tyres, and road, the weather conditions, and the alertness of the driver.

Thinking distance

It takes time for a driver to react to a need to stop. For a vehicle moving at constant speed

$$\text{thinking distance} = \text{speed} \times \text{reaction time}$$

Worked example: Reaction time

In the UK Highway Code, the thinking distance at 30 mph ($13.4\,\text{m}\,\text{s}^{-1}$) is shown as 9.0 m. Calculate the corresponding reaction time.

Step 1: Identify the equation needed.

$$\text{reaction time} = \frac{\text{thinking distance}}{\text{speed}}$$

Step 2: Substitute the values into the equation and calculate the answer.

$$\text{reaction time} = \frac{9.0}{13.4} = 0.67\,\text{s (2 s.f.)}$$

The greater the speed or the reaction time, the further a vehicle will travel before its driver applies the brakes. Assuming a constant reaction time of 0.67 s, the thinking distance will be about 21 m at the UK national speed limit of 70 mph ($31.1\,\text{m}\,\text{s}^{-1}$). This is equivalent to the total length of five average cars lined up.

Braking distance

In the UK Highway Code, the braking distance at 30 mph ($13.4\,m\,s^{-1}$) is shown as 14.0 m. If you assume constant deceleration from $13.4\,m\,s^{-1}$ to $0\,m\,s^{-1}$, you can use one of the equations of motion to determine the magnitude of the deceleration.

▼ **Table 1** *Thinking, braking, and overall stopping distances according to the Highway Code*

Speed / mph	20	30	40	50	60	70
Speed / $m\,s^{-1}$	8.9	13.4	17.8	22.2	26.7	31.1
Thinking distance / m	6	9	12	15	18	21
Braking distance / m	6	14	24	38	55	75
Stopping distance / m	12	23	36	53	73	96

Worked example: Braking distance

Step 1: Once again, start with the *suvat* quantities and identify the equation you need.

$s = 14.0\,\text{m}$

$u = 13.4\,\text{m s}^{-1}$

$v = 0$

$a = ?$

Use the equation $v^2 = u^2 + 2as$.

Step 2: Substitute the values into the equation and calculate the answer.

$$a = \frac{v^2 - u^2}{2s}$$

$v = 0$

Therefore

$$a = -\frac{u^2}{2s}$$

$$= -\frac{13.4^2}{2 \times 14.0}$$

$$= -6.4\,\text{m s}^{-2}\ (2\ \text{s.f.})$$

The magnitude of the deceleration is about $6.4\,\text{m s}^{-2}$.

Acceleration due to gravity

Objects with mass exert a gravitational force on each other. The Earth is so massive that its gravitational pull is enough to keep us on its surface. An object released on the Earth will accelerate vertically downwards towards the centre of the Earth. When an object is accelerating under gravity, with no other force acting on it, it is said to be in **free fall**. The **acceleration of free fall** is denoted by the label g (not g, which means grams). Since g is an acceleration, it has the unit $m\,s^{-2}$.

Value for g close to Earth's surface

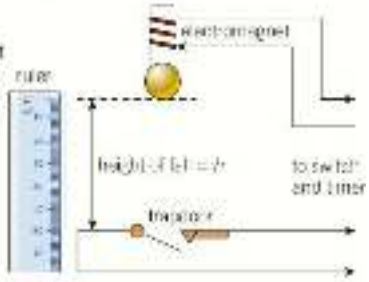
The value for g varies depending upon factors including altitude, latitude, and the geology of an area. For example, g is $9.825\,m\,s^{-2}$ in Helsinki, $9.816\,m\,s^{-2}$ in London, but only $9.776\,m\,s^{-2}$ in Singapore. A value of $9.81\,m\,s^{-2}$ is generally used.

Determining g

The basic idea behind determining g in the laboratory is to drop a heavy ball over a known distance and time its descent. The problem is that it all happens very quickly, about 0.45 s for a 1.0 m fall. Methods for measuring g are described here.

Electromagnet and trapdoor

An electromagnet holds a small steel ball above a trapdoor (Figure 2). When the current is switched off, a timer is triggered, the electromagnet demagnetises, and the ball falls. When it hits the trapdoor, the electrical contact is broken and the timer stops. The value for g is calculated from the height of the fall and the time taken.



▲ **Figure 2** *Determining g using an electromagnet and timer*

Worked example: An experimental value for g

A ball drops 85.6 cm from an electromagnet to a trapdoor in 0.421 s. Use this information to determine a value for g.

Step 1: List the *suvat* values and identify the equation needed.

$s = 0.856\,\text{m}$, $u = 0\,\text{m s}^{-1}$, $a = g = ?$, $t = 0.421\,\text{s}$ (the distance must be converted into the SI unit m.)

Using $s = ut + \frac{1}{2}at^2$, we have

$$s = \frac{1}{2}at^2 \text{ because } u = 0$$

Step 2: Substitute the values into the equation and calculate the answer.

$$a = \frac{2s}{t^2} = \frac{2 \times 0.856}{0.421^2}$$

$$a = 9.66\,\text{m s}^{-2} \text{ (3 s.f.)}$$

The inaccuracy in this experiment is caused by the presence of air resistance and the slight delay in the release of the steel ball because of the finite time taken for the magnet to demagnetise. The accuracy may be improved by using a heavier ball and a much longer drop.

Summary

In this week, we learned about average/instantaneous speed, average velocity, distance/displacement time graphs, velocity-time graphs, the suvat equations, the acceleration due to gravity and so much more.

Week 10 Graph Sketching & Kinematics Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-05-22 5:57 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Graph Sketching & Kinematics Continued	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: May 22, 2020

Essential Question:

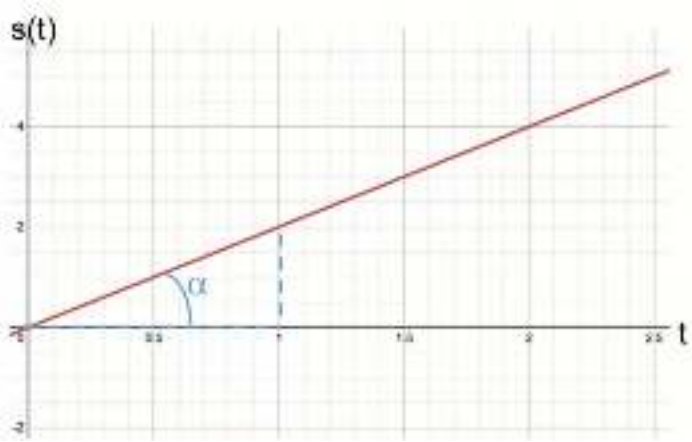
What is a function and what are its applications to kinematics (simple motion)?

Questions/Cues:

- What is the uniform motion on a straight line in the case of constant velocity?
- What is uniformly accelerated motion in the case of a projectile?

Notes

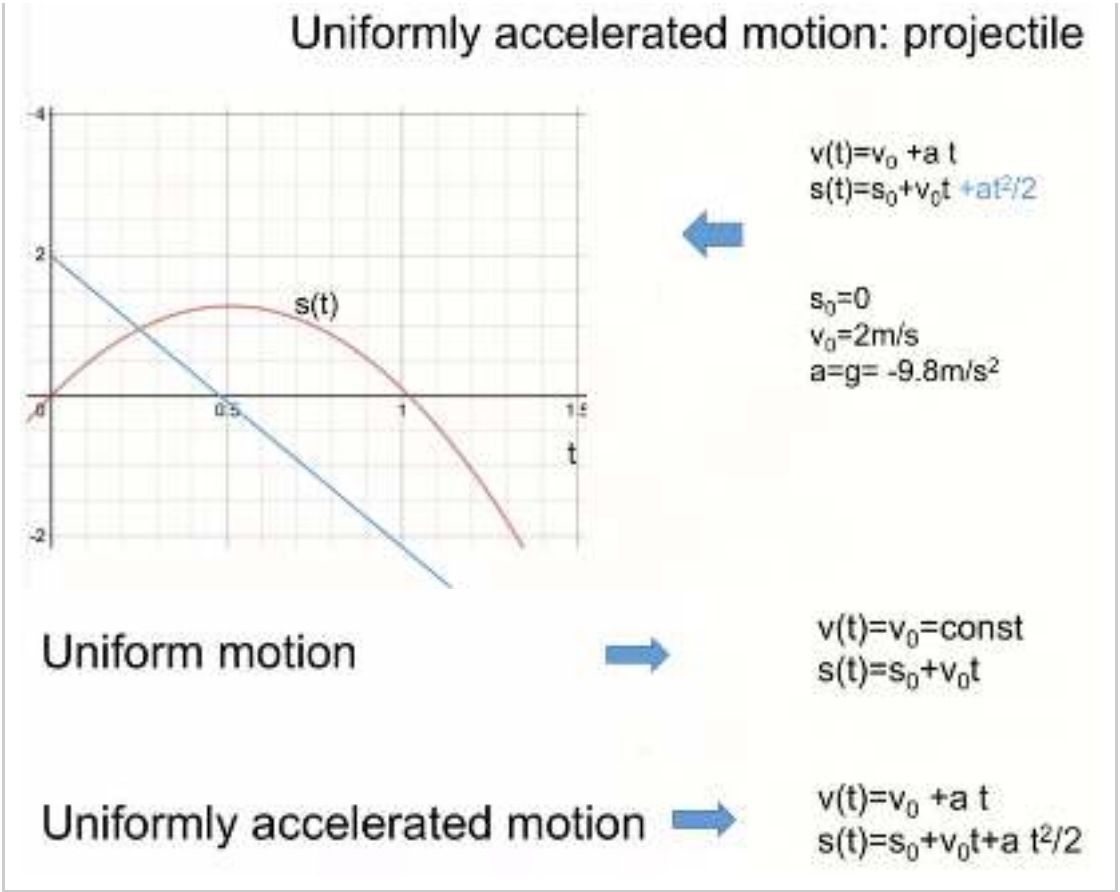
Uniform motion on a straight line: constant velocity



$v(t)=v_0=const$
$s(t)=s_0+v_0t$

$s_0=0$
$v_0=2m/s$

$\tan(\alpha)=2/1=v_0$



Summary

In this week, we learned about uniform motion and uniformly accelerated motion.

Week 11 Trigonometric Functions Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-03 3:15 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Trigonometric Functions	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 03, 2020

Essential Question:

What is the unit circle and the various trig functions associated with it?

Questions/Cues:

- What are the trig functions for a right angle triangle?
- How do we consider the trig functions for angles greater than 90 degrees?
- How do the sine and cosine change signs change across the four quadrants established by the unit circle?
- What are some special angles when mentioning the unit circle?
- What are the other trig functions and the inverse trig functions?
- What are the properties of trig functions?

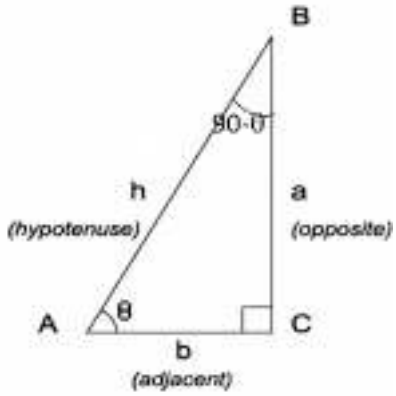
Notes

From right triangles we defined.

1)$\text{Sin}(\theta)=a/h=\frac{opposite}{hypotenuse}$

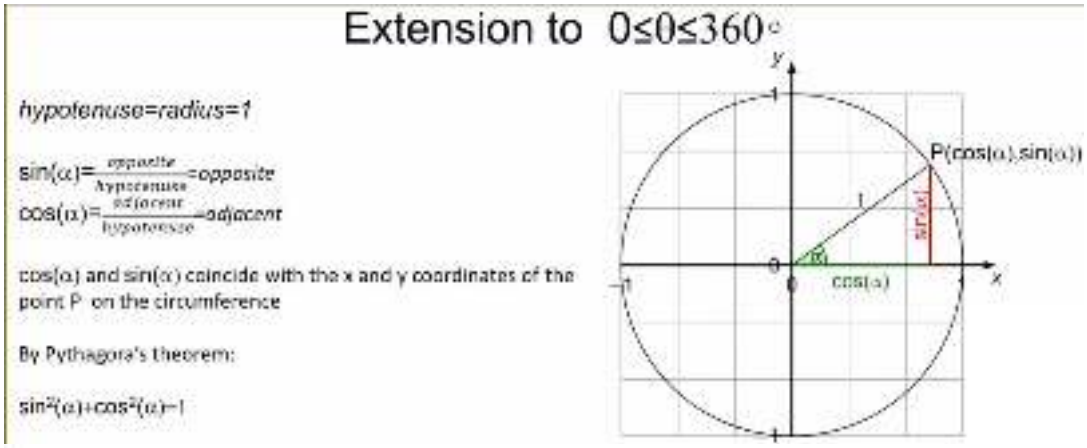
2)$\text{Cos}(\theta)=b/h=\frac{adjacent}{hypotenuse}$

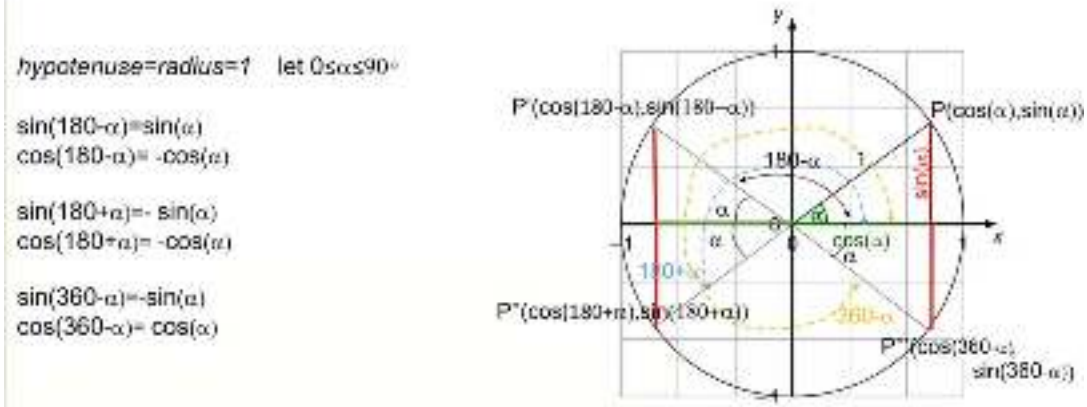
3)$\text{Tan}(\theta)=a/b=\frac{opposite}{adjacent}$

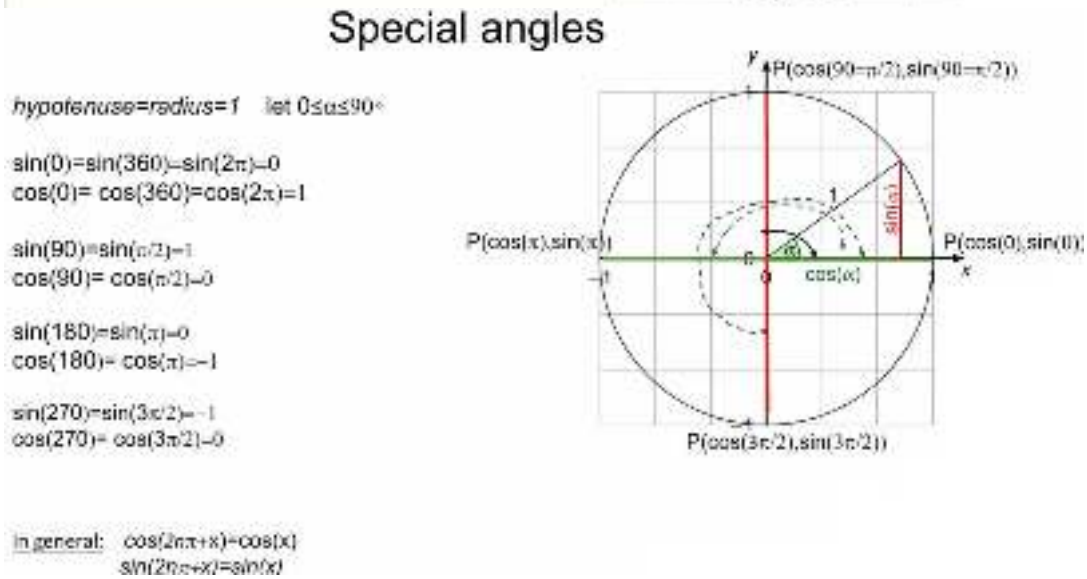


Limited to angles $0 \le \theta \le 90^\circ$

- Where the other two angles expect the 90 degree angle have to add up to 90 degrees because the sum of the all the internal angles must be equal to 180 degrees







- sine and cosine both are periodic after 360 degrees or 2 pi radians, which means they repeat.

Definitions

Secant : $\sec(\theta) = 1/\cos(\theta)$

Cosecant: $\csc(\theta) = 1/\sin(\theta)$

Cotangent: $\cot(\theta) = 1/\tan(\theta)$

Inverse functions: if $y=f(x) \rightarrow x=f^{-1}(y)$ <u>f must be bijective</u>

ArcSin: if $\mathrm{Sin}(\theta)=y \rightarrow \theta=\mathrm{Sin}^{-1}(y)=\mathrm{ArcSin}(y)$

ArcCos: if $\mathrm{Cos}(\theta)=y \rightarrow \theta=\mathrm{Cos}^{-1}(y)=\mathrm{ArcCos}(y)$

ArcTan: if $\mathrm{Tan}(\theta)=y \rightarrow \theta=\mathrm{Tan}^{-1}(y)=\mathrm{ArcTan}(y)$

Properties

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \rightarrow \quad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \rightarrow \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\sin(\alpha/2) = \sqrt{\frac{1-\cos(\alpha)}{2}}$

$\cos(\alpha/2) = \sqrt{\frac{1+\cos(\alpha)}{2}}$

Summary

In this week, we learned about the unit circle, the trig functions for angles greater than 90 degrees, the fundamental identity of trigonometry, special trig angles, inverse trig functions and the properties of trig functions.

Week 12 Trigonometric Functions Continued Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-08 6:34 PM
Author:	SUKHJIT MANN		

Cornell Notes	**Topic:** Trigonometric Functions Continued	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 08, 2020

Essential Question:

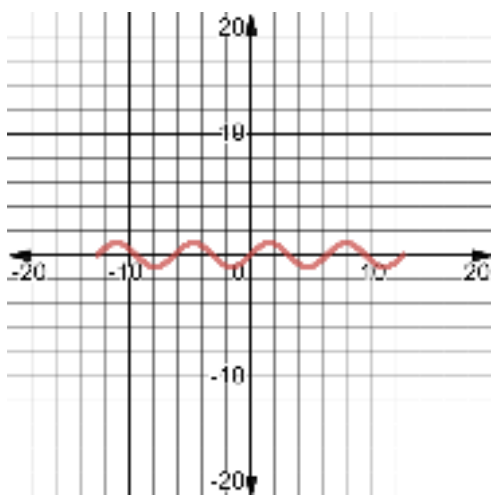
What are properties for the various trig functions?

Questions/Cues:

- What are the properties for sin(x)?
- What are properties of cos(x)?
- What are properties of tan(x)?
- What are the properties of 2 sin(x)?
- What are the properties of cos(2x)?
- How is frequency defined?
- How do you consider the $\sin(\omega x)$?

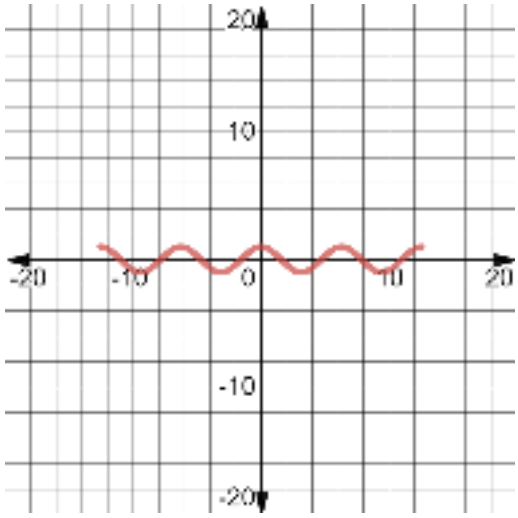
Notes

- Sin(x) = The period for this function is 2 pi, which means the curve repeats every 2 pi



  - If we place dashed lines or as they are called asymptotes at the zeros of sin(x), we see that the distance from one line to the next is 2 pi
  - The y-values for the sine curve are between -1 and 1 (including the endpoints). These are the peaks and troughs of the curve

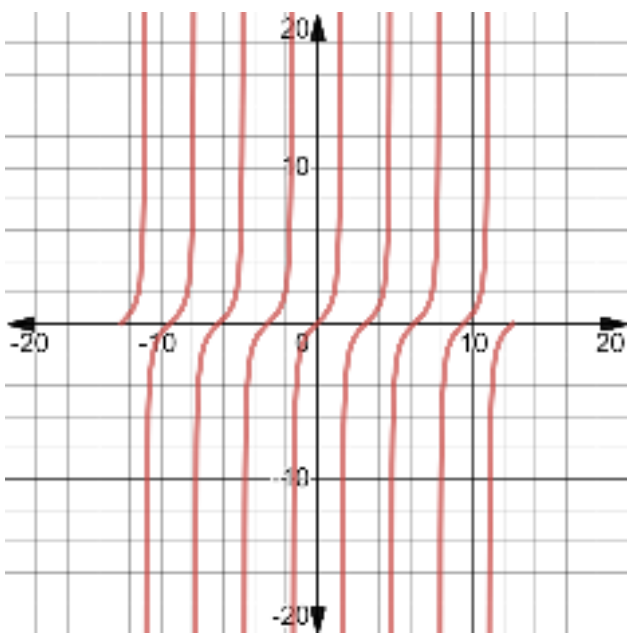
- Cos(x) = The period for this function is 2 pi, which means the curve repeats every 2 pi



  - We can see from the graph that the curve for cos(x) is just the sine curve shifted over pi/2
  - The y-values for the cosine curve are between -1 and 1 (including the endpoints). These are the peaks and troughs of the curve

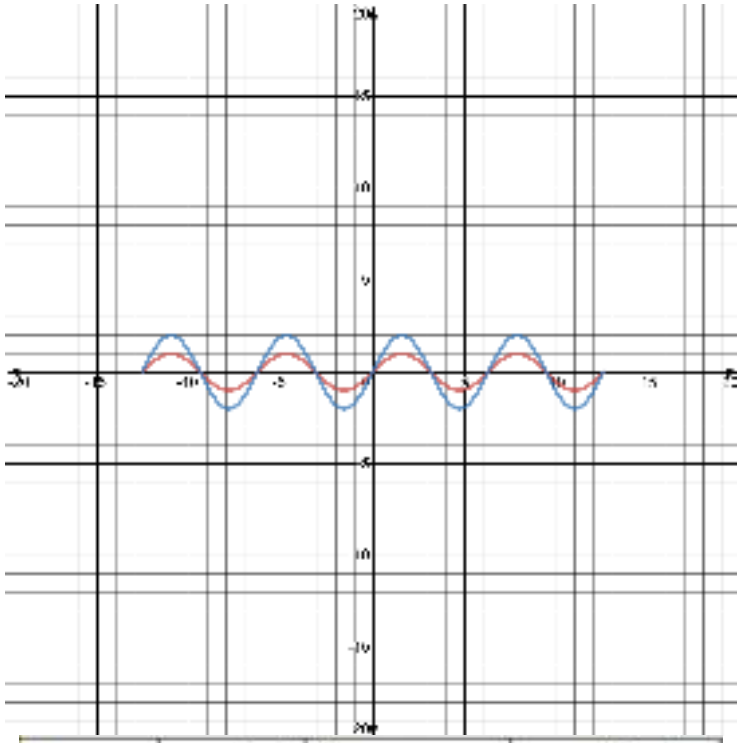
	period	frequency	amplitude
$\sin x$	2π	$\frac{1}{2\pi}$	1
$\cos x$	2π	$\frac{1}{2\pi}$	1

    - Period is between two maximums and half the period is between one maximum and one minimum
- Tan(x) = sin(x)/cos(x)



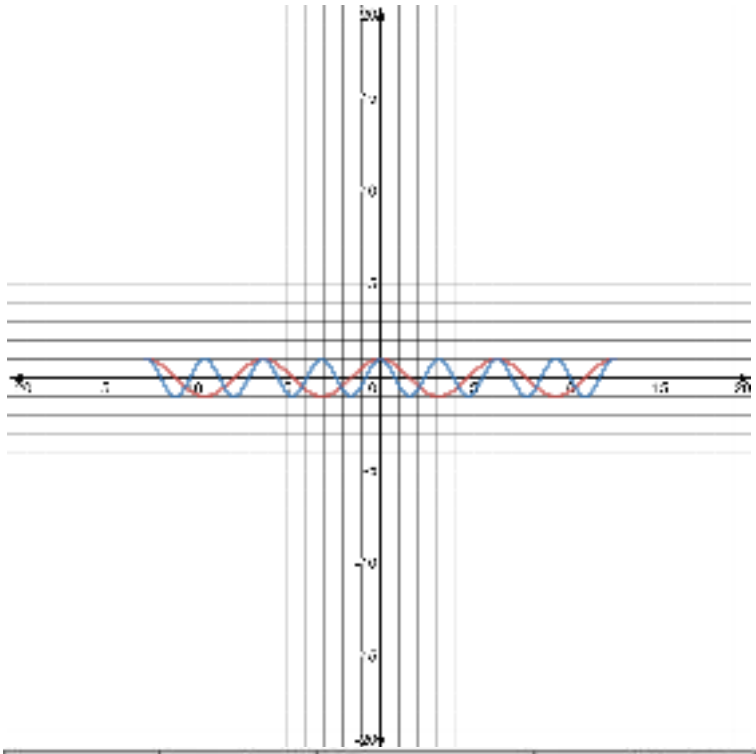
  - Is periodic like the other functions, with a period of pi
  - Is divergent when cos(x) = 0, undefined
    - Has vertical asymptotes there
    - For any odd multiple of pi/2

        - Will have positive divergence when sine is positive or negative when sine is negative
    - $\tan(\pi - x) = \tan(x)$
    - $\tan(\pi + x) = -\tan(x)$
- 2 sin(x) = Has the same period as sin(x), but the amplitude is doubled
    - Having a coefficient in front, affects the amplitude for the function



	period	frequency	amplitude
$\sin x$	2π	$\frac{1}{2\pi}$	1
$2\sin x$	2π	$\frac{1}{2\pi}$	2

- Cos(2x) = The amplitude is the same as the cosine curve, but since the distance between two maximums is half of the original, the period is pi instead of 2pi. So the curve is more condensed horizontally and more periods are covered in one cycle



	period	frequency	amplitude
$\cos x$	2π	$\frac{1}{2\pi}$	1
$\cos 2x$	π	$\frac{1}{\pi}$	1

- Frequency (f)= $\frac{1}{period} = \frac{1}{T}$

$$\sin(\omega x)$$
$$y = \omega x$$
$$\sin(y) \rightarrow \begin{Bmatrix} T_y = 2\pi \\ f_y = \frac{1}{T_y} = \frac{1}{2\pi} \end{Bmatrix}$$
$$\text{Since } x = \frac{y}{\omega} \Rightarrow T_x = \frac{T_y}{\omega} = \frac{2\pi}{\omega}$$
$$f_x = \frac{1}{T_x} = \frac{\omega}{2\pi}$$

Summary

In this week, we learned about the properties of the sine, cosine and tangent functions, characteristics like period, frequency and amplitude. Lastly we touched on various examples with differing characteristics.

Week 13 Exponential and Logarithmic functions lecture note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-21 3:16 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Exponential and Logarithmic functions**	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 21, 2020

Essential Question:

What are the exponential and logarithmic functions?

Questions/Cues:

- What are laws of indices/exponents used to simplify exponential expressions?
    - From this, what is the form of the exponential function?
- What are various graphs of the exponential function like?
- What are the properties of the exponential function?

Notes

Definitions

Integer $a^n = a \times a \times \ldots \times a$ ($\rightarrow \sqrt[n]{}$ inverse)

$\rightarrow$ $a^{n+m} = a^n \times a^m$ $a \gtrless 0$

prove also: 2) $(a^n)^m = a^{nm}$

3) $(a\,b)^n = a^n b^n$

Rational power $a^{n/m} = (\sqrt[m]{a})^n$ or $\sqrt[m]{a^n}$

First is defined <u>only for m odd</u> if $a<0$,

second for any m and a if n even

$\rightarrow$ Always defined if $a>0$

$(4)^{3/2} = (\sqrt{4})^3 = \pm 8$ by convention 8

Definitions

$a^{n/m+q/k}$?

$a^{n/m+q/k} = a^{(nk+mq)/(mk)} = \sqrt[mk]{a^{nk} a^{mq}} = \sqrt[mk]{a^{nk}}\ \sqrt[mk]{a^{mq}} = a^{n/m}\ a^{q/k}$

Extension to all real numbers:

for any real number x you can truncate it

Ex. $\pi = 3.1415\ldots \approx 31415/10000$

$x \rightarrow y(x) = a^x$ <u>with $a>0$</u>

<u>Properties</u>:

$a^{x+y} = a^x \times a^y$

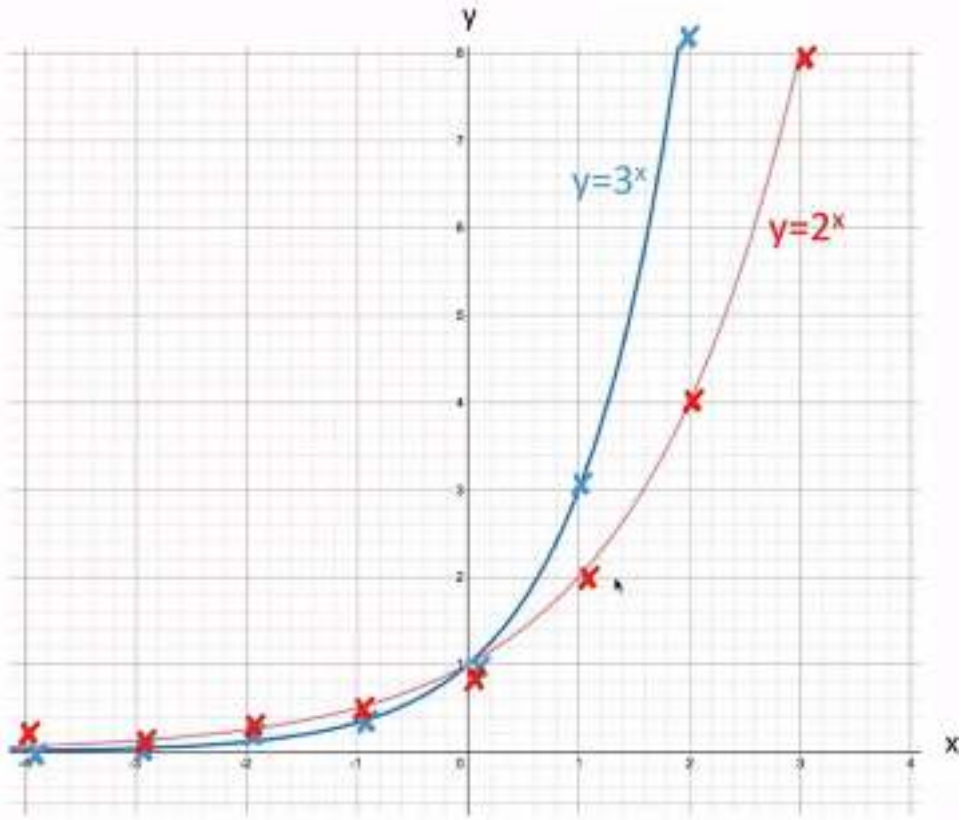
$(a^x)^y = a^{xy}$ $(ab)^x = a^x b^x$

Graphs

Using a table of values plot graphs of $f(x) = a^x$ for $a = 2, 3$ for $-4 \le x \le 4$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	$\frac{1}{16} \cong 0.06$	$\frac{1}{8} \cong 0.13$	$\frac{1}{4} = 0.25$	$\frac{1}{2} = 0.5$	1	2	4	8	16

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 3^x$	$\frac{1}{81} \cong 0.01$	$\frac{1}{27} \cong 0.04$	$\frac{1}{9} \cong 0.1$	$\frac{1}{3} \cong 0.33$	1	3	9	27	81

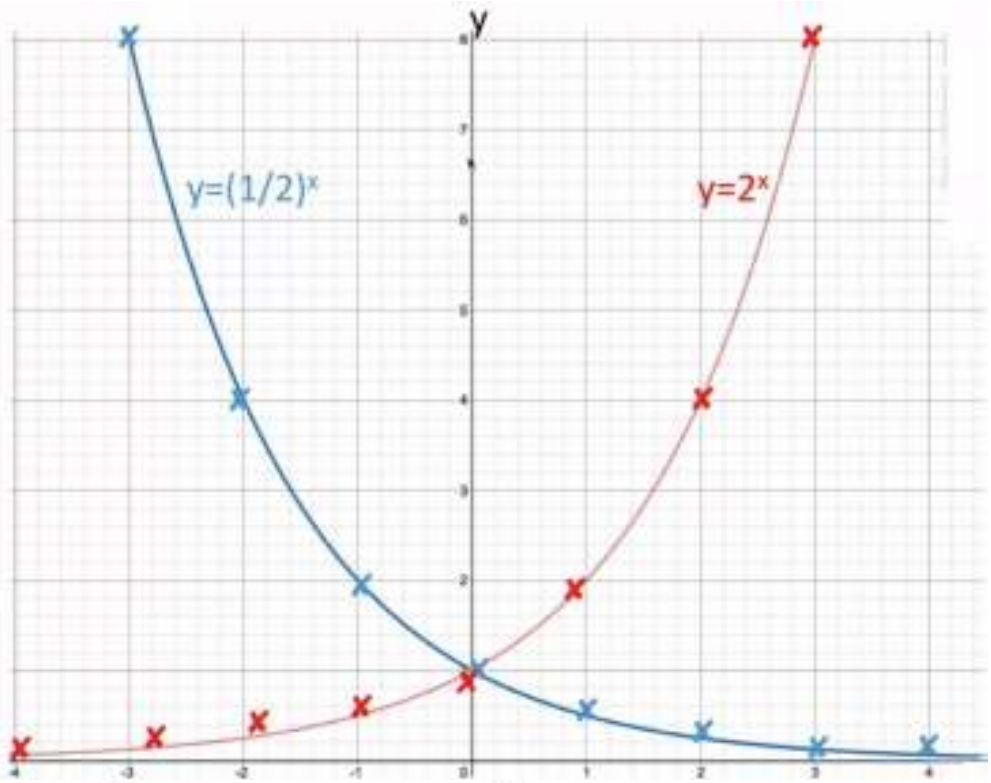


Graphs

Using a table of values plot graphs of $f(x) = a^x$ for $a = 2, ½$ for $-4 \le x \le 4$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	$\frac{1}{16} \cong 0.06$	$\frac{1}{8} \cong 0.13$	$\frac{1}{4} = 0.25$	$\frac{1}{2} = 0.5$	1	2	4	8	16

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = \left(\frac{1}{2}\right)^x$	16	8	4	2	1	$\frac{1}{2} = 0.5$	$\frac{1}{4} = 0.25$	$\frac{1}{8} \cong 0.13$	$\frac{1}{16} \cong 0.06$



Summary properties

a) $f(x) = a^x$ only defined $a > 0$ (so $f(x) = a^x$ is not defined for $a < 0$ (what happens for $a = 0$?))
b) For all a, $f(x) = a^x$ has a y − intercept of 1, that is the graph passes through (0,1)
c) For all a, the graph of $f(x) = a^x$ pass through $(1, a)$
d) For $a > 1$ $f(x) = a^x$ is increasing
e) For $a < 1$ $f(x) = a^x$ is decreasing (and for $a = 1$?)
f) $f(x) = a^x > 0$ for all x
g) the x−*axis* is an asymptote
h) For $a > 1$ the bigger a is the more rapidly $f(x) = a^x$ increases
i) For $a < 1$ the smaller a is the more rapidly $f(x) = a^x$ decreases

Summary

In this week, we learned about the laws of exponents, what the exponential function is, the different graphs of the exponential function and the properties of the exponential function.

Week 13 Exponential and Logarithmic functions Reading Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-20 6:40 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Exponential and Logarithmic functions**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 20, 2020

Essential Question:

What are the exponential and logarithmic functions?

Questions/Cues:

- What is an exponential expression?
- What are laws of indices/exponents used to simplify exponential expressions?
- What is the exponential function and its properties?
- What is the function e^{-x} and its properties?
- What are logarithms?
- How do you calculate logarithms to bases other than 10 or e?
- What are the laws of logarithms?
- How do you solve equations involving logarithms?
- What is the logarithmic function and its properties?

Notes

- Exponential expression = of the form a^x, where the number a is the base and is the power or index
    - One of the most commonly used values for the base is the non-terminating number 2.71828..., denoted by the letter e
        - e^x is found throughout modelling problems in nature, i.e. population growth, spread of bacteria and radioactive decay

$$e^a e^b = e^{a+b}$$

$$\frac{e^a}{e^b} = e^{a-b}$$

$$e^0 = 1$$

$$(e^a)^b = e^{ab}$$

WORKED EXAMPLES

19.2 Simplify the following exponential expressions:

(a) e^2e^4 (b) $\dfrac{e^4}{e^3}$ (c) $(e^2)^{2.5}$

Solution

(a) $e^2e^4 = e^{2+4} = e^6$

(b) $\dfrac{e^4}{e^3} = e^{4-3} = e^1 = e$

(c) $(e^2)^{2.5} = e^{2\times 2.5} = e^5$

19.3 Simplify the following exponential expressions:

(a) e^xe^{3x} (b) $\dfrac{e^{4t}}{e^{3t}}$ (c) $(e^t)^4$

Solution

(a) $e^xe^{3x} = e^{x+3x} = e^{4x}$

(b) $\dfrac{e^{4t}}{e^{3t}} = e^{4t-3t} = e^t$

(c) $(e^t)^4 = e^{4t}$

19.4 Simplify

(a) $\sqrt{e^{6t}}$ (b) $e^{3y}(1+e^y) - e^{4y}$ (c) $(2e^t)^2(3e^{-t})$

Solution

(a) $\sqrt{e^{6t}} = (e^{6t})^{\frac{1}{2}} = e^{6t/2} = e^{3t}$

$\sqrt{a} = a^{\frac{1}{2}}$; that is, a square root sign is equivalent to the power $\frac{1}{2}$.

(b) $e^{3y}(1+e^y) - e^{4y} = e^{3y} + e^{3y}e^y - e^{4y}$

$= e^{3y} + e^{4y} - e^{4y}$

$= e^{3y}$

(c) $(2e^t)^2(3e^{-t}) = (2e^t)(2e^t)(3e^{-t})$

$= 2\cdot 2\cdot 3\cdot e^te^te^{-t}$

$= 12e^{t+t-t}$

$= 12e^t$

19.5 (a) Verify that

$$(e^t+1)^2 = e^{2t} + 2e^t + 1$$

(b) Hence simplify $\sqrt{e^{2t} + 2e^t + 1}$.

Solution

(a) $(e^t+1)^2 = (e^t+1)(e^t+1)$

$= e^te^t + e^t + e^t + 1$

$= e^{2t} + 2e^t + 1$

(b) $\sqrt{e^{2t} + 2e^t + 1} = \sqrt{(e^t+1)^2} = e^t + 1$

- Exponential Function = has the form $y = e^x$ and following properties:

(a) The exponential function is never negative.

(b) When $x = 0$, the function value is 1.

(c) As x increases, then e^x increases. This is known as **exponential growth.**

(d) By choosing x large enough, the value of e^x can be made larger than any given number. We say e^x increases without bound as x increases.

(e) As x becomes large and negative, e^x gets nearer and nearer to 0.

Table 19.1

x	-3	-2	-1	0	1	2	3
e^x	0.0498	0.1353	0.3679	1	2.7183	7.3891	20.086

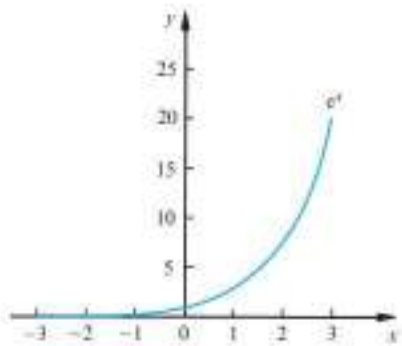


Figure 19.1
A graph of $y = e^x$

WORKED EXAMPLE

19.6 Sketch a graph of $y = 2e^x$ and $y = e^{2x}$ for $-3 \leqslant x \leqslant 3$.

Solution Table 19.2 gives values of $2e^x$ and e^{2x}, and the graphs are shown in Figure 19.2.

Table 19.2

x	-3	-2	-1	0	1	2	3
$2e^x$	0.0996	0.2707	0.7358	2	5.4366	14.7781	40.1711
e^{2x}	0.0025	0.0183	0.1353	1	7.3891	54.5982	403.429

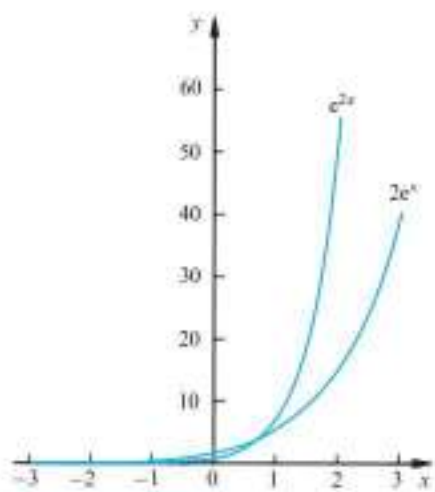


Figure 19.2
Graphs of $y = e^{2x}$ and $y = 2e^x$

Note that when $x > 0$, the graph of $y = e^{2x}$ rises more rapidly than that of $y = 2e^x$.

We now turn our attention to an associated function: $y = e^{-x}$. Values are listed in Table 19.3 and the function is illustrated in Figure 19.3. From the table and the graph we note the following properties of $y = e^{-x}$:

(a) The function is never negative.

(b) When $x = 0$, the function has a value of 1.

(c) As x increases, then e^{-x} decreases, getting nearer to 0. This is known as **exponential decay**.

Table 19.3

x	-3	-2	-1	0	1	2	3
$-x$	3	2	1	0	-1	-2	-3
e^{-x}	20.086	7.3891	2.7183	1	0.3679	0.1353	0.0498

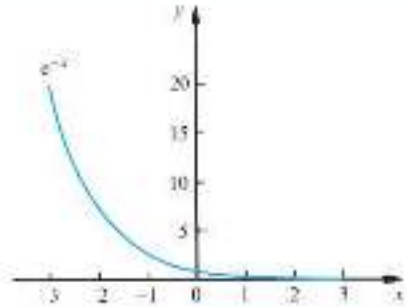


Figure 19.3
Graph of $y = e^{-x}$

WORKED EXAMPLE

19.7 (a) Plot $y = e^{x/2}$ and $y = 2e^{-x}$ for $-1 \leqslant x \leqslant 1$.

(b) Hence solve the equation

$$e^{x/2} - 2e^{-x} = 0$$

Solution (a) A table of values is drawn up for $e^{x/2}$ and $2e^{-x}$ so that the graphs can be drawn. Table 19.4 lists the values and Figure 19.4 shows the graphs of the functions.

Table 19.4

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
$e^{x/2}$	0.607	0.687	0.779	0.882	1	1.133	1.284	1.455	1.649
$2e^{-x}$	5.437	4.234	3.297	2.568	2	1.558	1.213	0.945	0.736

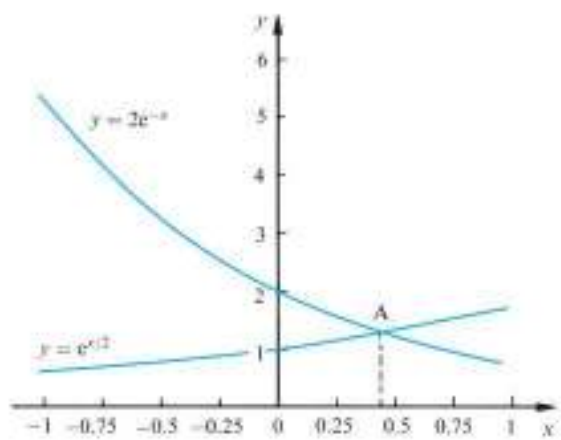


Figure 19.4
The graphs of $y = e^{x/2}$ and $y = 2e^{-x}$ intersect at A

(b) We note that the graphs intersect; the point of intersection is A. The equation

$$e^{x/2} - 2e^{-x} = 0$$

is equivalent to

$$e^{x/2} = 2e^{-x}$$

The graphs of $y = e^{x/2}$ and $y = 2e^{-x}$ intersect at A, where $x = 0.44$. Hence $x = 0.44$ is an approximate solution of $e^{x/2} - 2e^{-x} = 0$. The exact answer can be shown to be 0.46 (2 d.p.).

Given an equation such as $125 = 5^3$, we call 5 the base and 3 the power or index. We can use **logarithms** to write the equation in another form. The logarithm form is

$$\log_5 125 = 3$$

This is read as 'logarithm to the base 5 of 125 is 3'. In general if

$$y = a^x$$

then

$$\log_a y = x$$

Key point $y = a^x$ and $\log_a y = x$ are equivalent.

'The logarithm to the base a of y is x' is equivalent to saying that 'y is a to the power x'. The word logarithm is usually shortened to just 'log'.

WORKED EXAMPLE

20.1 Write down the logarithmic form of the following:

(a) $16 = 4^2$ (b) $8 = 2^3$ (c) $25 = 5^2$

Solution (a) $16 = 4^2$ may be written as

$$2 = \log_4 16$$

that is, 2 is the logarithm to the base 4 of 16.

(b) $8 = 2^3$ may be expressed as

$$3 = \log_2 8$$

which is read as '3 is the logarithm to the base 2 of 8'.

(c) $25 = 5^2$ may be expressed as

$$2 = \log_5 25$$

that is, 2 is the logarithm to the base 5 of 25.

Given an equation such as $16 = 4^2$ then 'taking logs' to base 4 will result in the logarithmic form $\log_4 16 = 2$.

WORKED EXAMPLE

20.2 Write the exponential form of the following:

(a) $\log_2 16 = 4$ (b) $\log_3 27 = 3$ (c) $\log_5 125 = 3$ (d) $\log_{10} 100 = 2$

Solution (a) Here the base is 2 and so we may write $16 = 2^4$.

(b) The base is 3 and so $27 = 3^3$.

(c) The base is 5 and so $125 = 5^3$.

(d) The base is 10 and so $100 = 10^2$.

Recall that e is the constant 2.71828..., which occurs frequently in natural phenomena.

Although the base of a logarithm can be any positive number other than 1, the commonly used bases are 10 and e. Logarithms to base 10 are often denoted by 'log' or '$\log_{10}$'; logarithms to base e are denoted by 'ln' or '$\log_e$' and referred to as **natural logarithms**. Most scientific calculators possess 'log' and 'ln' buttons, which are used to evaluate logarithms to base 10 and base e.

20.4 Given $10^{0.6990} = 5$ evaluate the following:

(a) $10^{1.6990}$ (b) $\log 5$ (c) $\log 500$

Solution

(a) $10^{1.6990} = 10^1 . 10^{0.6990} = 10(5) = 50$

(b) We are given $10^{0.6990} = 5$ and by taking logs to the base 10 we obtain $0.6990 = \log 5$.

(c) From $10^{0.6990} = 5$ we can see that

$$100(5) = 100(10^{0.6990})$$

and so

$$500 = 10^2 . 10^{0.6990} = 10^{2.6990}$$

Taking logs to the base 10 gives

$$\log 500 = 2.6990$$

Key point

$$\log_a X = \frac{\log_{10} X}{\log_{10} a} \qquad \log_a X = \frac{\ln X}{\ln a}$$

WORKED EXAMPLE

20.5 Evaluate (a) $\log_6 19$, (b) $\log_7 29$.

Solution

(a) We use the formula

$$\log_a X = \frac{\log_{10} X}{\log_{10} a}$$

Comparing $\log_6 19$ with $\log_a X$ we see $X = 19$ and $a = 6$. So

$$\log_6 19 = \frac{\log_{10} 19}{\log_{10} 6} = \frac{1.2788}{0.7782} = 1.6433$$

We could have equally well used the formula

$$\log_a X = \frac{\ln X}{\ln a}$$

With this formula we obtain

$$\log_6 19 = \frac{\ln 19}{\ln 6} = \frac{2.9444}{1.7918} = 1.6433$$

(b) Comparing $\log_a X$ with $\log_7 29$ we see $X = 29$ and $a = 7$. Hence

$$\log_7 29 = \frac{\log_{10} 29}{\log_{10} 7} = \frac{1.4624}{0.8451} = 1.7304$$

Alternatively we use

$$\log_7 29 = \frac{\ln 29}{\ln 7} = \frac{3.3673}{1.9459} = 1.7304$$

By considering the formula

$$\log_a X = \frac{\log_{10} X}{\log_{10} a}$$

with $X = a$ we obtain

$$\log_a a = \frac{\log_{10} a}{\log_{10} a} = 1$$

Key point $\log_a a = 1$

This same result could be derived by writing the logarithm form of $a = a^1$.

Key point $\log A + \log B = \log AB$

This law holds true for any base. However, in any one calculation all bases must be the same.

WORKED EXAMPLES

20.7 Simplify to a single log term

(a) $\log 9 + \log x$
(b) $\log t + \log 4t$
(c) $\log 3x^2 + \log 2x$

Solution

(a) $\log 9 + \log x = \log 9x$

(b) $\log t + \log 4t = \log(t.4t) = \log 4t^2$

(c) $\log 3x^2 + \log 2x = \log(3x^2.2x) = \log 6x^3$

20.8 Simplify

(a) $\log 7 + \log 3 + \log 2$
(b) $\log 3x + \log x + \log 4x$

Solution

(a) We know $\log 7 + \log 3 = \log(7 \times 3) = \log 21$, and so

$$\begin{aligned}\log 7 + \log 3 + \log 2 &= \log 21 + \log 2\\ &= \log(21 \times 2) = \log 42\end{aligned}$$

(b) We have

$$\log 3x + \log x = \log(3x.x) = \log 3x^2$$

and so

$$\begin{aligned}\log 3x + \log x + \log 4x &= \log 3x^2 + \log 4x\\ &= \log(3x^2.4x) = \log 12x^3\end{aligned}$$

Key point

$$\log A - \log B = \log\left(\frac{A}{B}\right)$$

20.10 Use the second law of logarithms to simplify the following to a single log term:

(a) $\log 20 - \log 10$ (b) $\log 500 - \log 75$ (c) $\log 4x^3 - \log 2x$
(d) $\log 5y^3 - \log y$

Solution (a) Using the second law of logarithms we have

$$\log 20 - \log 10 = \log\left(\frac{20}{10}\right) = \log 2$$

(b) $$\log 500 - \log 75 = \log\left(\frac{500}{75}\right) = \log\left(\frac{20}{3}\right)$$

(c) $$\log 4x^3 - \log 2x = \log\left(\frac{4x^3}{2x}\right) = \log 2x^2$$

(d) $$\log 5y^3 - \log y = \log\left(\frac{5y^3}{y}\right) = \log 5y^2$$

20.11 Simplify

(a) $\log 20 + \log 3 - \log 6$
(b) $\log 18 - \log 24 + \log 2$

Solution (a) Using the first law of logarithms we see that

$$\log 20 + \log 3 = \log 60$$

and so

$$\log 20 + \log 3 - \log 6 = \log 60 - \log 6$$

Using the second law of logarithms we see that

$$\log 60 - \log 6 = \log\left(\frac{60}{6}\right) = \log 10$$

Hence

$$\log 20 + \log 3 - \log 6 = \log 10$$

(b) $$\begin{aligned}\log 18 - \log 24 + \log 2 &= \log\left(\frac{18}{24}\right) + \log 2\\ &= \log\left(\frac{3}{4}\right) + \log 2\\ &= \log\left(\frac{3}{4} \times 2\right)\\ &= \log 1.5\end{aligned}$$

20.12 Simplify

(a) $\log 2 + \log 3x - \log 2x$
(b) $\log 5y^2 + \log 4y - \log 10y^2$

Solution (a)
$$\log 2 + \log 3x - \log 2x = \log(2 \times 3x) - \log 2x$$
$$= \log 6x - \log 2x$$
$$= \log\left(\frac{6x}{2x}\right) = \log 3$$

(b)
$$\log 5y^2 + \log 4y - \log 10y^2 = \log(5y^2.4y) - \log 10y^2$$
$$= \log 20y^3 - \log 10y^2$$
$$= \log\left(\frac{20y^3}{10y^2}\right) = \log 2y$$

We consider a special case of the second law. Consider $\log A - \log A$. This is clearly 0. However, using the second law we may write

$$\log A - \log A = \log\left(\frac{A}{A}\right) = \log 1$$

Thus

Key point $\log 1 = 0$

In any base, the logarithm of 1 equals 0.

Key point $n \log A = \log A^n$

This law applies if n is integer, fractional, positive or negative.

WORKED EXAMPLES

20.14 Write the following as a single logarithmic expression:

(a) $3\log 2$ (b) $2\log 3$ (c) $4\log 3$

Solution (a) $3\log 2 = \log 2^3 = \log 8$

(b) $2\log 3 = \log 3^2 = \log 9$

(c) $4\log 3 = \log 3^4 = \log 81$

20.15 Write as a single log term

(a) $\frac{1}{2}\log 16$ (b) $-\log 4$ (c) $-2\log 2$ (d) $-\frac{1}{2}\log 0.5$

Solution (a) $\frac{1}{2}\log 16 = \log 16^{\frac{1}{2}} = \log\sqrt{16} = \log 4$

(b) $-\log 4 = -1.\log 4 = \log 4^{-1} = \log\left(\frac{1}{4}\right) = \log 0.25$

(c) $-2\log 2 = \log 2^{-2} = \log\left(\frac{1}{2^2}\right) = \log\left(\frac{1}{4}\right) = \log 0.25$

(d) $-\frac{1}{2}\log 0.5 = -\frac{1}{2}\log\left(\frac{1}{2}\right) = \log\left(\frac{1}{2}\right)^{-\frac{1}{2}} = \log 2^{\frac{1}{2}} = \log\sqrt{2}$

20.16 Simplify

(a) $3\log x - \log x^2$
(b) $3\log t^3 - 4\log t^2$
(c) $\log Y - 3\log 2Y + 2\log 4Y$

Solution (a) $3\log x - \log x^2 = \log x^3 - \log x^2$

$$= \log\left(\frac{x^3}{x^2}\right)$$

$$= \log x$$

(b) $3\log t^3 - 4\log t^2 = \log(t^3)^3 - \log(t^2)^4$

$$= \log t^9 - \log t^8$$

$$= \log\left(\frac{t^9}{t^8}\right)$$

$$= \log t$$

(c) $\log Y - 3\log 2Y + 2\log 4Y = \log Y - \log(2Y)^3 + \log(4Y)^2$

$$= \log Y - \log 8Y^3 + \log 16Y^2$$

$$= \log\left(\frac{Y.16Y^2}{8Y^3}\right)$$

$$= \log 2$$

20.17 Simplify

(a) $2\log 3x - \frac{1}{2}\log 16x^2$

(b) $\frac{3}{2}\log 4x^2 - \log\left(\frac{1}{x}\right)$

(c) $2\log\left(\frac{2}{x^2}\right) - 3\log\left(\frac{2}{x}\right)$

Solution (a)

$$\begin{aligned} 2\log 3x - \frac{1}{2}\log 16x^2 &= \log(3x)^2 - \log(16x^2)^{\frac{1}{2}} \\ &= \log 9x^2 - \log 4x \\ &= \log\left(\frac{9x^2}{4x}\right) \\ &= \log\left(\frac{9x}{4}\right) \end{aligned}$$

(b)

$$\begin{aligned} \frac{3}{2}\log 4x^2 - \log\left(\frac{1}{x}\right) &= \log(4x^2)^{\frac{3}{2}} - \log(x^{-1}) \\ &= \log 8x^3 + \log x \\ &= \log 8x^4 \end{aligned}$$

(c)

$$\begin{aligned} 2\log\left(\frac{2}{x^2}\right) - 3\log\left(\frac{2}{x}\right) &= \log\left(\frac{2}{x^2}\right)^2 - \log\left(\frac{2}{x}\right)^3 \\ &= \log\left(\frac{4}{x^4}\right) - \log\left(\frac{8}{x^3}\right) \\ &= \log\left(\frac{4/x^4}{8/x^3}\right) \\ &= \log\left(\frac{1}{2x}\right) \end{aligned}$$

20.18 Solve the following equations:

(a) $10^x = 59$ (b) $10^x = 0.37$ (c) $e^x = 100$ (d) $e^x = 0.5$

Solution (a) $10^x = 59$
Taking logs to base 10 gives

$$x = \log 59 = 1.7709$$

(b) $10^x = 0.37$
Taking logs to base 10 gives

$$x = \log 0.37 = -0.4318$$

(c) $e^x = 100$
Taking logs to base e gives

$$x = \ln 100 = 4.6052$$

(d) $e^x = 0.5$
Taking logs to base e we have

$$x = \ln 0.5 = -0.6931$$

20.19 Solve the following equations:

(a) $\log x = 1.76$ (b) $\ln x = -0.5$ (c) $\log(3x) = 0.76$

(d) $\ln\left(\frac{x}{2}\right) = 2.6$ (e) $\log(2x - 4) = 1.1$ (f) $\ln(7 - 3x) = 1.75$

Solution We note that if $\log_a X = n$, then $X = a^n$.

(a) $\log x = 1.76$ and so $x = 10^{1.76}$. Using the 'x^y' button of a scientific calculator we find

$$x = 10^{1.76} = 57.5440$$

(b) $\ln x = -0.5$

$$x = e^{-0.5} = 0.6065$$

(c) $\log 3x = 0.76$

$$3x = 10^{0.76}$$

$$x = \frac{10^{0.76}}{3} = 1.9181$$

(d) $\ln\left(\frac{x}{2}\right) = 2.6$

$$\frac{x}{2} = e^{2.6}$$

$$x = 2e^{2.6} = 26.9275$$

(e) $\log(2x - 4) = 1.1$

$$2x - 4 = 10^{1.1}$$

$$2x = 10^{1.1} + 4$$

$$x = \frac{10^{1.1} + 4}{2} = 8.2946$$

(f) $\ln(7 - 3x) = 1.75$

$$7 - 3x = e^{1.75}$$

$$3x = 7 - e^{1.75}$$

$$x = \frac{7 - e^{1.75}}{3} = 0.4151$$

20.20 Solve the following equations:

(a) $e^{3+x}.e^{x} = 1000$ (b) $\ln\left(\frac{x}{3} + 1\right) + \ln\left(\frac{1}{x}\right) = -1$

(c) $3e^{x-1} = 75$ (d) $\log(x + 2) + \log(x - 2) = 1.3$

Solution (a) Using the laws of logarithms we have

$$e^{3+x}.e^{x} = e^{3+2x}$$

Hence

$$e^{3+2x} = 1000$$

$$3 + 2x = \ln 1000$$

$$2x = \ln(1000) - 3$$

$$x = \frac{\ln(1000) - 3}{2} = 1.9539$$

(b) Using the laws of logarithms we may write

$$\ln\left(\frac{x}{3}+1\right)+\ln\left(\frac{1}{x}\right)=\ln\left(\left(\frac{x}{3}+1\right)\frac{1}{x}\right)=\ln\left(\frac{1}{3}+\frac{1}{x}\right)$$

So

$$\ln\left(\frac{1}{3}+\frac{1}{x}\right)=-1$$

$$\frac{1}{3}+\frac{1}{x}=e^{-1}$$

$$\frac{1}{x}=e^{-1}-\frac{1}{3}$$

$$\frac{1}{x}=0.0345$$

$$x=28.947$$

(c) $3e^{x-1}=75$

$$e^{x-1}=25$$

$$x-1=\ln 25$$

$$x=\ln 25+1=4.2189$$

(d) Using the first law we may write

$$\log(x+2)+\log(x-2)=\log(x+2)(x-2)=\log(x^2-4)$$

Hence

$$\log(x^2-4)=1.3$$

$$x^2-4=10^{1.3}$$

$$x^2=10^{1.3}+4$$

$$x=\sqrt{10^{1.3}+4}=4.8941$$

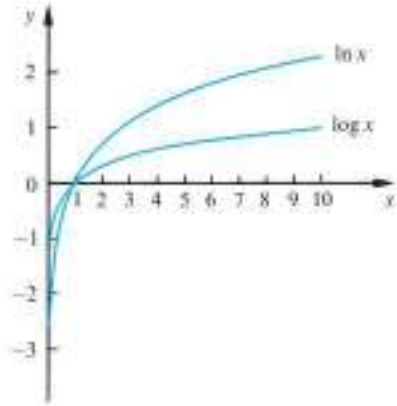
- Logarithmic Function= has the form $y=\log x$ or $y=\ln x$ and has the following properties:

(a) As x increases, the values of $\log x$ and $\ln x$ increase.
(b) $\log 1=\ln 1=0$.
(c) As x approaches 0 the values of $\log x$ and $\ln x$ increase negatively.
(d) When $x<1$, the values of $\log x$ and $\ln x$ are negative.
(e) $\log x$ and $\ln x$ are not defined when $x\leqslant 0$.

Table 20.1

x	0.01	0.1	0.5	1	2	5	10	100
$\log x$	-2	-1	-0.30	0	0.30	0.70	1	2
$\ln x$	-4.61	-2.30	-0.69	0	0.69	1.61	2.30	4.61

Figure 20.1
Graphs of $y = \log x$ and $y = \ln x$



Summary

In this week, we learned about what exponential expressions and logarithms are, how to calculate a base other than 10 or e, the laws of logarithms, how to solve equations involving logarithms and exponential expressions and finally what the exponential and logarithmic functions and their respective properties.

Week 14 Exponential and Logarithmic functions continued lecture note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-21 4:24 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** Exponential and Logarithmic functions continued	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 21, 2020

Essential Question:

What are the exponential and logarithmic functions?

Questions/Cues:

- What are some definitions/properties pertaining to logarithms?
- What are the different graphs of the logarithmic function?
- What are the properties of the logarithmic function?

Notes

Definitions

if $x = f(y) = a^y \xrightarrow{\text{inverse}} y = f^{-1}(x) = \log_a x$

$\rightarrow f(f^{-1}(x)) = f^{-1}(f(x)) = x \rightarrow a^{\log_a x} = x \rightarrow \log_a(a^x) = x$

Always defined if a>0

Properties

$\log_a(x \times y) = \log_a(x) + \log_a(y)$

proof: replace identities $a^{\log_a x} = x \quad a^{\log_a y} = y$

$\log_a(x \times y) = \log_a(a^{\log_a x} \times a^{\log_a y}) = \log_a(a^{(\log_a x + \log_a y)}) =$

Properties

$\log_a(x^b) = b \times \log_a(x)$

proof: replace $a^{\log_a x} = x$

$\rightarrow \log_a((a^{\log_a x})^b) = \log_a(a^{b \log_a x}) = b \log_a(x)$

$\log_a x = \frac{\log_c(x)}{\log_c(a)}$

proof: replace identity $a^{\log_a x} = x$

$\rightarrow \log_c(a^{\log_a x}) = \log_a(x) \log_c(a)$

$\rightarrow \frac{\log_c(x)}{\log_c(a)} = \frac{\log_a(x) \log_c(a)}{\log_c(a)} = \log_a(x)$

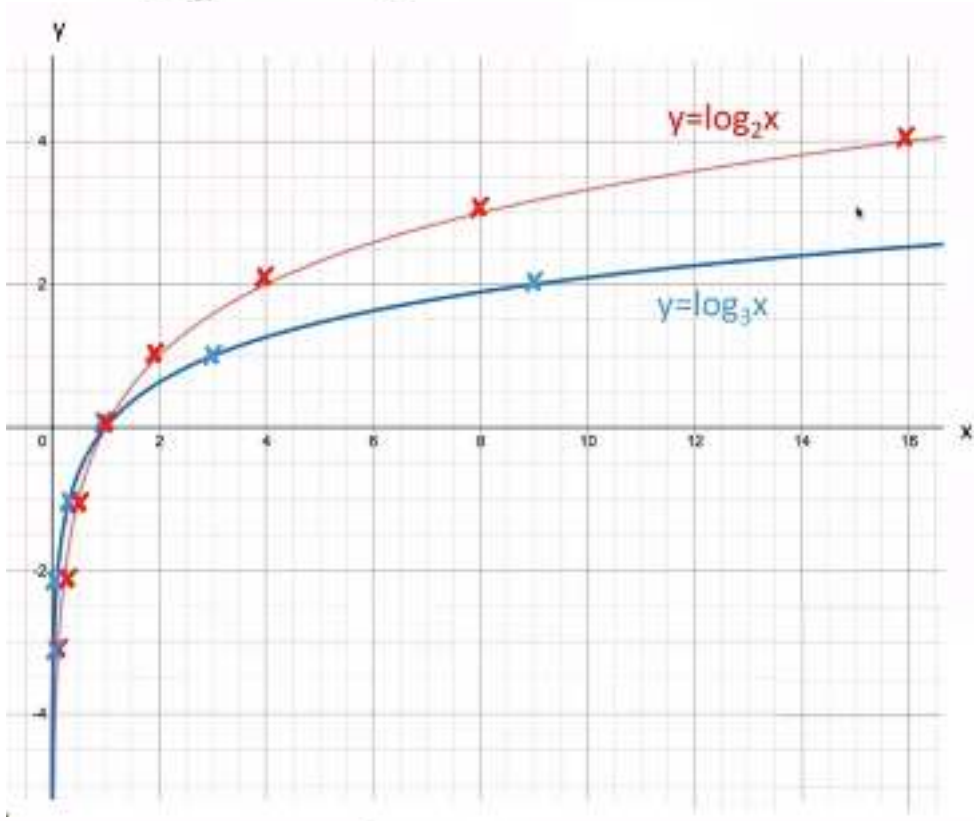
Graphs

Using a table of values plot graphs of $f(x) = \log_a x$ for $a = 2, 3,$

x	-2	0	1	2	4	8	16	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$f(x) = \log_2 x$	undefined	undefined	0	1	2	3	4	-1	-2	-3

x	-3	0	1	3	9	27	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$
$f(x) = \log_3 x$	undefined	undefined	0	1	2	3	-1	-2	-3

$\log_a x$ only defined for x>0

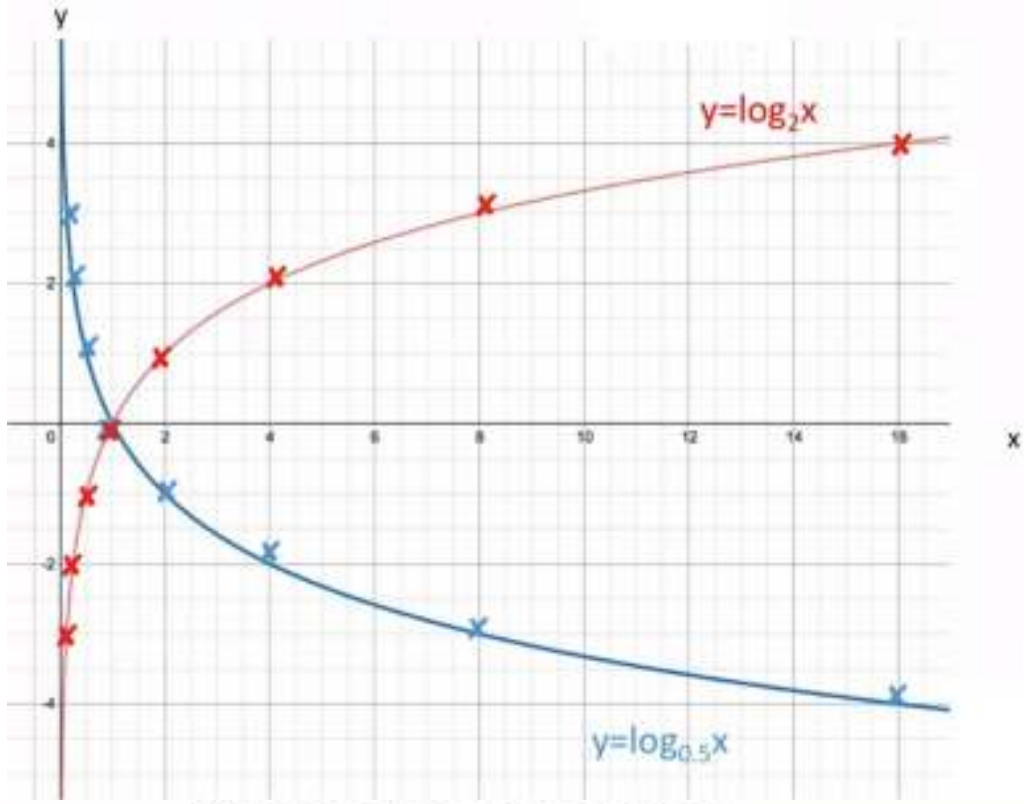


Graphs

Using a table of values plot graphs of $f(x) = \log_a x$ for $a = 2, ½$

x	-2	0	1	2	4	8	16	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$f(x) = \log_2 x$	undefined	undefined	0	1	2	3	4	-1	-2	-3

x	-2	0	1	2	4	8	16	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$f(x) = \log_{\frac{1}{2}} x$	undefined	undefined	0	-1	-2	-3	-4	1	2	3



Summary properties

a) $f(x)=\log_a x$ is only defined for $a>0$
b) $f(x)=\log_a x$ is only defined for $x>0$
c) For all a, $f(x)=\log_a x$ has an $x-$intercept of 1, that is the graph passes through (1,0)
d) For all a, the graph of $f(x)=\log_a x$ passes through $(a,1)$
e) For a>1 $f(x)=\log_a x$ is increasing
f) For a<1 $f(x)=\log_a x$ is decreasing (and for a=1?)
g) the $y-axis$ is an asymptote
h) For $a>1$ the bigger a is the more slowly $f(x)=\log_a x$ increases
i) For $a<1$ the smaller a is the more slowly $f(x)=\log_a x$ decreases
j) lg(x) or Log(x) indicates $\log_{10}(x)$
and ln(x) or log(x) denotes $\log_e(x)$ e=2.71828…

$a^x \quad a>0$

•1) $a^{x+y} = a^x \cdot a^y$

•2) $(a^x)^y = a^{x\cdot y}$

3) $a^0 = 1$

•4) $a^{-n} = \frac{1}{a^n}$

•) $\frac{e^{3x}}{e^x} = e^{3x}\left(\frac{1}{e^x}\right) = e^{3x}\cdot e^{-x} = e^{3x-x} = e^{2x}$ $\quad \left(\frac{1}{e^x} = e^{-x}\right)$

•) $\sqrt{e^{6t}} = (e^{6t})^{\frac{1}{2}} = e^{6\cdot t\cdot\frac{1}{2}} = e^{3t}$

•) $\left(\frac{1}{4}\right)^{-x} = \left(\frac{1}{4}\right)^{-1\cdot x} = \left(\left(\frac{1}{4}\right)^{-1}\right)^x = 4^x$

$$3^{x+1} = 9 \rightarrow x\ ?$$

$$3^{x+1} = 3^{2} \Rightarrow x+1 = 2 \Rightarrow x = 1$$

$$3^{x+1} = 10$$

$$\log_3 x \rightleftarrows 3^x \Rightarrow \log_3(3^2) = 2$$

$$\log_3(3^{x+1}) = \log_3 10$$

$$x+1 = \log_3 10 \Rightarrow x = \log_3 10 - 1$$

$$\log_a x \quad \begin{cases} x > 0 \\ a > 0 \end{cases}$$

1) $\log_a(b \cdot c) = \log_a b + \log_a c$

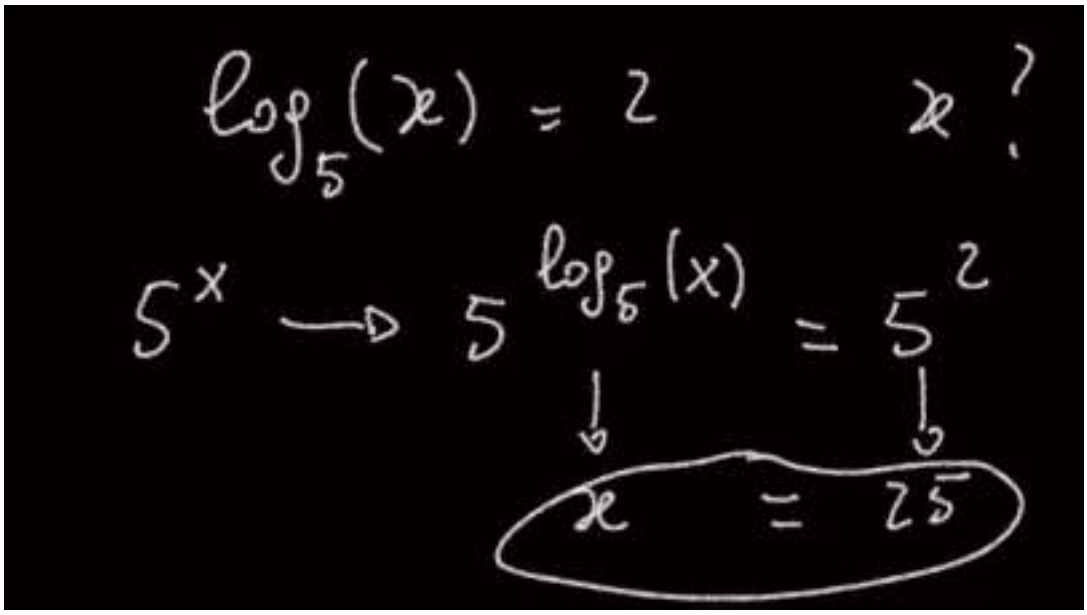
2) $\log_a(b^c) = c \cdot \log_a b$

3) $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$

4) $\log_a b = \dfrac{\log_c b}{\log_c a}$

•) $\log_{10}(10 \cdot x) = \log_{10} 10 + \log_{10} x$
$= 1 + \log_{10} x$

•) $\log_3 10 = \dfrac{\log_{10} 10}{\log_{10} 3} = \dfrac{1}{\log_{10} 3}$



Summary

In this week, we learned about some definitions/properties pertaining to logarithms, the form of a logarithmic function, the different graphs of the logarithmic function and the properties of the logarithmic function.

Week 15 Limits and differentiation Lecture note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-23 4:58 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Limits and differentiation**	Course: BSc Computer Science
		Class: Computational Mathematics[Lecture]
		Date: July 23, 2020

Essential Question:

What are limits and derivatives and how do they relate to the notion of continuity of a function?

Questions/Cues:

- What is the definition of a limit for a sequence?
- What is the definition of a limit and continuity for a function?
- What is the slope of a straight line?
- What is the definition of the derivative?

Notes

Definition of Limit for a sequence

	n=0	1	2	3	4	5	6	7	8	.....100.....
Examples $a_n=n/(n+1)$	0,	1/2,	2/3,	3/4,	4/5,	5/6,	6/7,	7/8,	8/9	.....100/101....

In general: $\lim_{n\to\infty} a_n=L$ if $\forall\ \varepsilon>0\ \exists\ N$: for $n>N$ $|a_n-L|<\varepsilon$

$\lim_{n\to\infty} a_n=L=1$ if $\varepsilon=1/5$ let $N=4$ for $n>N$ $\rightarrow 4/5<a_n<1 \rightarrow |a_n-1|<\varepsilon=\frac{1}{5}$

if $\varepsilon=1/9$ let $N=8$ for $n>N$ $\rightarrow 8/9<a_n<1 \rightarrow |a_n-1|<\varepsilon=\frac{1}{9}$

$\rightarrow$ if $\varepsilon=1/k$ let $N=k-1$ for $n>N$ $\rightarrow \frac{k-1}{k}<a_n<1 \rightarrow |a_n-1|<\varepsilon=\frac{1}{k}$

If limit exists finite, the sequence is <u>convergent</u>

$a_n=n/(n+1)$ converges to 1

Definition of Limit for a sequence

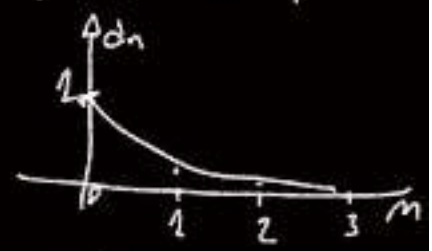
If limit doesn't exist the sequence is said to be <u>divergent</u>.
Examples:

		n=0	1	2	3	4	5	
1)	$a_n=1\times 3^n$	1,	3,	9,	27,	81,	243....	diverges to ∞

		n=0	1	2	3	4	5	6	7....	
2)	$a_n=\sin(\pi\, n/2)=$	0,	1,	0,	-1,	0,	1,	0,	-1...	does not converge

1) $\lim\limits_{n\to\infty} a_n = \dfrac{1}{(n+1)^3} = 0$

n	$(n+1)^3$	a_n
0	1	1
1	8	1/8
2	27	1/27
3	64	1/64
⋮	⋮	⋮



2) $\lim\limits_{n\to\infty} a_n = -2 + (-1)^n$

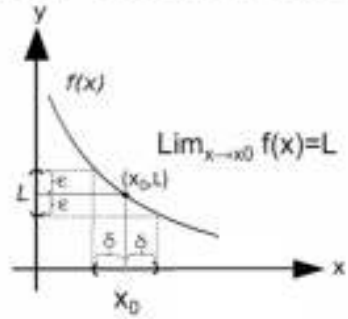
n	a_n
0	-1
1	-3
2	-1
3	-3
4	-1
5	-3
⋮	⋮

3) $\lim\limits_{n\to\infty} \dfrac{n^2+2n+5}{3n^2+2} = \dfrac{\cancel{n^2}\left(1+\overset{\to 0}{\frac{2}{n}}+\overset{\to 0}{\frac{5}{n^2}}\right)}{\cancel{n^2}\left(3+\underset{\to 0}{\frac{2}{n^2}}\right)} = \dfrac{1}{3}$

Definitions of limit and continuity for a function

$\text{Lim}_{x\to x0} f(x)=L$

if $\forall\ \varepsilon>0\ \ \exists\ \delta>0$: for $|x-x_0|<\delta\ \rightarrow |f(x)-L|<\varepsilon$



If limit exists finite and coincides with the value of the function in x_0 , i.e. if

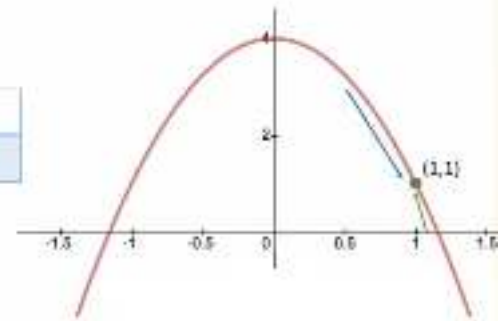
$f(x_0)=\lim_{x\to x0} f(x)=L$

→ the function is said to be <u>continuous</u> in x_0

Example

$f(x)=4-3x^2$ calculate $\lim_{x\to 1} f(x)$

x=0.5	0.9	0.99	0.999	1	1.001	1.01	1.1
f(x)=3.25	1.57	1.06	1.006	?	0.993	0.94	0.37



Left and right limits:

left limit : $\lim_{x\to x_0^-} f(x)$ (blu arrow/numbers)

right limit: $\lim_{x\to x_0^+} f(x)$ (green arrow/numbers)

Limit exists if and only if $\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)=L$

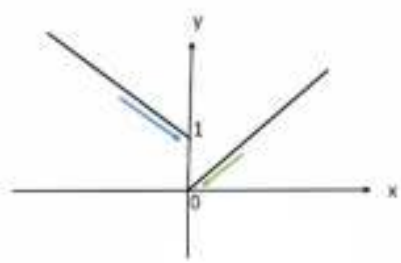
in our case $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=1=\lim_{x\to 1} f(x)$

f(1)=4-3=1 → function is <u>continuous in x=1</u>

Discontinuous functions

Example $y=f(x)=\begin{cases} x & x\geq 0 \\ 1-x & x<0 \end{cases}$ $\lim_{x\to 0} f(x)$?

x=-1	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
f(x)=2	1.5	1.1	1.01	?	0.01	0.1	0.5



$\lim_{x\to 0^-} f(x)=1$ $\lim_{x\to 0^+} f(x)=0$

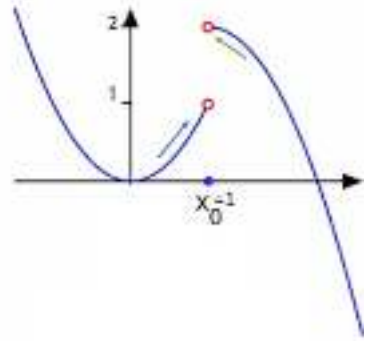
Left and right limits are different

→ limit does not exist, → <u>f not continuous in x=0</u>

Discontinuous functions

$$y=f(x)=\begin{cases} x^2 & x<1 \\ 1+2x-x^2 & x \geq 1 \end{cases} \qquad \lim_{x\to 1} f(x)?$$

x=0	0.5	0.9	0.99	1	1.01	1.1	1.5
f(x)=0	0.25	0.81	0.98	?	1.99	1.9	1.75



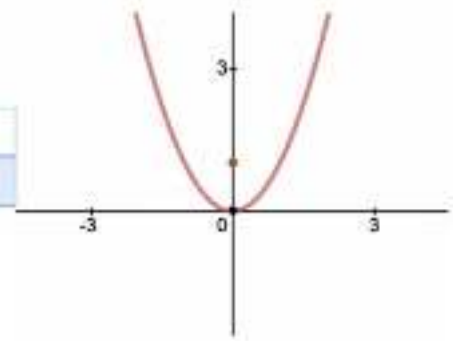
$\lim_{x\to 1^-} f(x)=1 \qquad \lim_{x\to 1^+} f(x)=2$

Left and right limits are different
→ limit does not exist, → f not continuous in $x_0=1$

Discontinuous functions

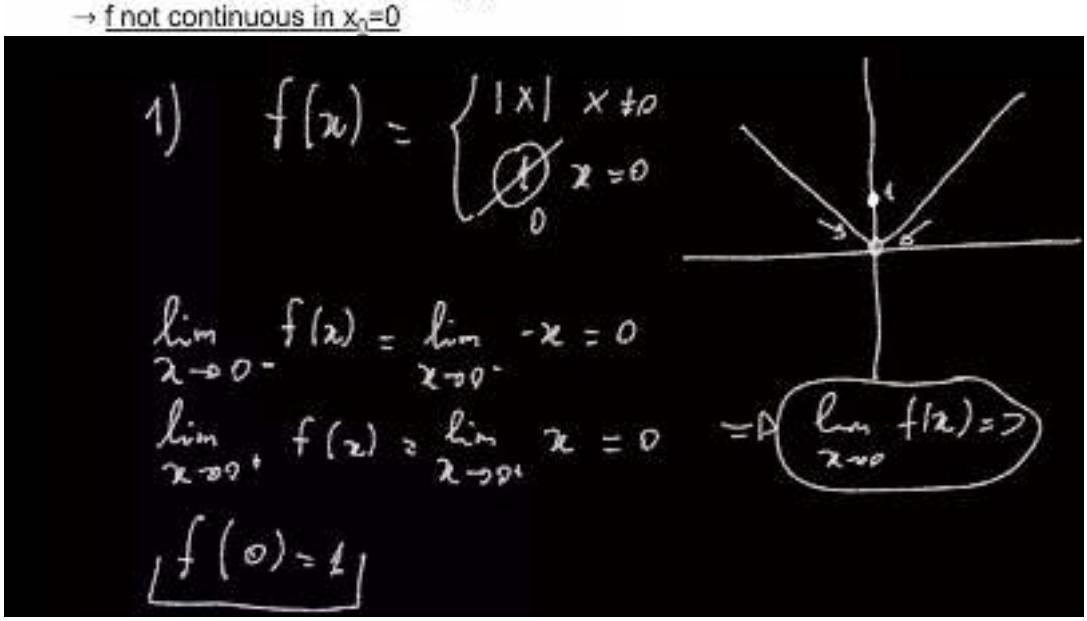
$$y=f(x)=\begin{cases} x^2 & x<0 \\ 1 & x=0 \\ x^2 & x>0 \end{cases} \qquad \lim_{x\to 0} f(x)?$$

x=-1	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
f(x)=1	0.25	0.01	0.0001	?	0.0001	0.01	0.25



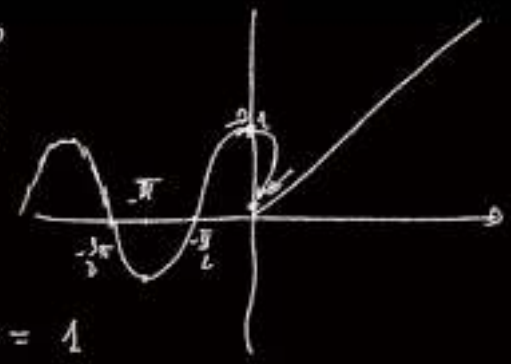
$\lim_{x\to 0^-} f(x)=0=\lim_{x\to 0^+} f(x) \neq f(0)=1$

Left and right limits are equal
→ limit exists but different from f(0)
→ f not continuous in $x_0=0$



- This is called a removable discontinuity

2) $f(x) = \begin{cases} \cos x & x < 0 \\ x & x \geq 0 \end{cases}$

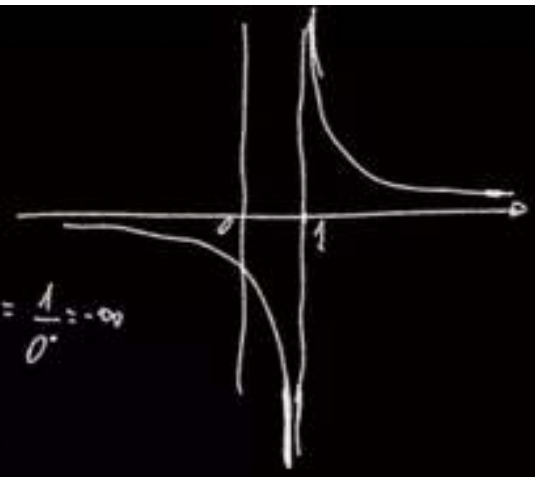


$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos x = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

- This is a jump discontinuity

3) $f(x) = \dfrac{1}{x-1}$

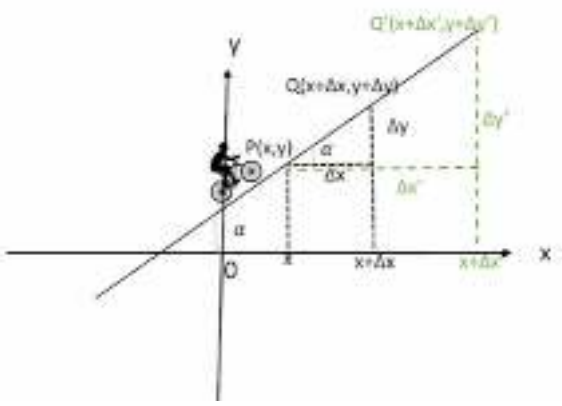


$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x-1} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to 1^+} f(x) = \frac{1}{0^+} = +\infty$$

- This is called an essential singularity or infinite discontinuity

Defining the slope (or gradient)

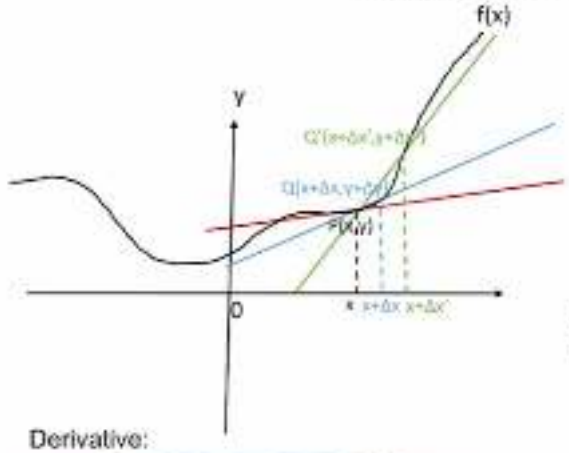


Straight line y=f(x)=mx+k m=tan α

$\Delta y = \Delta x \tan \alpha \rightarrow \tan \alpha = \Delta y / \Delta x$

$\Delta y' = \Delta x' \tan \alpha \rightarrow \tan \alpha = \Delta y' / \Delta x'$

Definition of derivative



P(x,y) Q'(x+ Δx', y+ Δy')

y=f(x) y+ Δy'=f(x+ Δx') →Δy'=f(x+ Δx')-f(x)

Δy'=Δx' tan α' → tan α'=Δy'/Δx'

for Q(x+ Δx, y+ Δy) →Δy=f(x+ Δx)-f(x)

Δy=Δx tan α → tan α=Δy/Δx

Slope tangent in P (red line) →

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Derivative:

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$y = f(x) = x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{\cancel{x} + h - \cancel{x}}{h} = 1$$

$$f'(x) = \frac{d(x)}{dx} = 1$$

$$f(x) = \frac{1}{x} \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \frac{1}{x+h} \qquad \frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$\frac{\frac{x - (x+h)}{(x+h)x}}{h} = \frac{-\cancel{h}}{\cancel{h}x(x+h)} = -\frac{1}{x(x+h)}$$

$$\lim_{h \to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2} \Rightarrow f'(x) = -\frac{1}{x^2}$$

$$y = f(x) = x^n$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h}$$

$$(x+h)^n = \left(x^n + n x^{n-1} \cdot h + C_n x^{n-2} \cdot h^2 + C x^{n-3} h^3 + \dots + h^n\right) =$$

$$= \frac{\cancel{x^n} + n x^{n-1} \cdot h + C h^2 - \cancel{x^n}}{h}$$

$$= \frac{n x^{n-1} \cancel{h}}{\cancel{h}} = n x^{n-1} \Rightarrow \frac{d x^\alpha}{dx} = \alpha x^{\alpha - 1}$$

$$f(x) = e^x$$

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} =$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h} = e^x \underbrace{\lim_{h \to 0} \frac{e^h - 1}{h}}_{=1}$$

$$f'(x) = e^x$$

$$e^{\alpha x} \to f'(x) = \alpha e^{\alpha x}$$

$$f(x) = \log x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\log(x+h) - \log(x)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{hx}{x}}$$

$$q = \frac{h}{x} \qquad \lim_{q \to 0} \frac{1}{x} \frac{\log(1+q)}{q} = \frac{1}{x} \underbrace{\lim_{q \to 0} \frac{\log(1+q)}{q}}_{=1}$$

$$\Rightarrow f'(x) = \frac{1}{x}$$

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

$$\lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$\lim_{h \to 0} \sin x \frac{(\cos h - 1)}{h} + \cos x \frac{\sin h}{h} =$$

$$= \sin x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{=0} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{=1}$$

$$f'(x) = \cos x \qquad \to f(x) = \cos x \quad f'(x) = -\sin x$$

Summary

In this week, we learned about limit of a sequence, the limit/continuity of a function, discontinuous functions, the slope of a straight, the derivative and the derivative from first principles of some common functions.

Week 15 Limits and differentiation Reading note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-22 5:24 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** Limits and differentiation	Course: BSc Computer Science
		Class: Computational Mathematics[Reading]
		Date: July 22, 2020

Essential Question:

What are limits and derivatives and how do they relate to the notion of continuity of a function?

Questions/Cues:

- What is the definition of a limit?
- Under which conditions do limits not exist?
- What are some basic limits?
- What are the properties of limits?
- What are the limits of Polynomial and Rational Functions?
- What is the dividing out technique?
- What is the rationalization technique?
- What are one-sided limits?
- How do we confirm the existence of a limit?
- What is the squeeze theorem?
- What is calculus?
- What is the tangent line to a graph?
- What is the definition of the slope of a graph?
- What is the definition of a derivative?
- What is the definition of limits at infinity?
- How are limits at infinity defined for rational functions?
- What is the limit of a sequence?
- What are the summation formulas and properties?
- How do we calculate the exact area of a plane region?

Notes

- Limit =
  If $f(x)$ becomes arbitrarily close to a unique number L as x approaches

c from either side, then the limit of $f(x)$ as x approaches c is L. This is written as $\lim_{x \to c} f(x) = L$.

- Or alternatively as, $f(x) \to L$ as $x \to c$, which is read as "$f(x)$ approaches L as x approaches c."

EXAMPLE 2 **Estimating a Limit Numerically**

Use a table to estimate the limit numerically: $\lim_{x \to 2} (3x - 2)$.

Solution Let $f(x) = 3x - 2$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 2 from the left and one that approaches 2 from the right.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300

From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, we estimate the limit to be 4. Figure 12.1 verifies this conclusion.

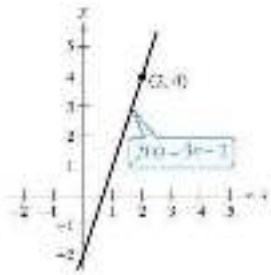


Figure 12.1

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Use a table to estimate the limit numerically: $\lim_{x \to 5} (3 - 2x)$.

In Figure 12.1, note that the graph of $f(x) = 3x - 2$ is continuous. For graphs that are not continuous, finding a limit can be more challenging.

EXAMPLE 3 **Estimating a Limit Numerically**

Use a table to estimate the limit numerically.

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$$

Solution Let $f(x) = x/(\sqrt{x+1} - 1)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 0 from the left and one that approaches 0 from the right.

x	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

From the table, it appears that the limit is 2. Figure 12.2 verifies this conclusion.

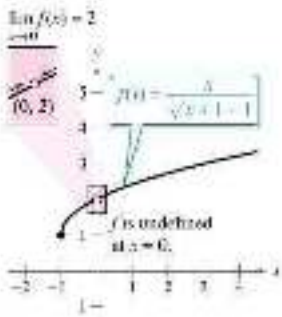


Figure 12.2

REMARK In Example 3, note that $f(0)$ is undefined, so it is not possible to *reach* the limit. In Example 2, note that $f(2) = 4$, so it is possible to reach the limit.

In Example 3, note that $f(x)$ has a limit when $x \to 0$ even though the function is not defined when $x = 0$. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

EXAMPLE 4 **Estimating a Limit**

Estimate the limit: $\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1}$.

Numerical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one that approaches 1 from the left and one that approaches 1 from the right.

x	0.9	0.99	0.999	1.0
$f(x)$	1.8100	1.9801	1.9980	?

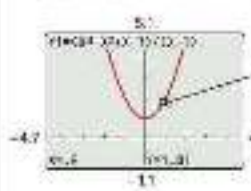
x	1.001	1.01	1.1
$f(x)$	2.0020	2.0301	2.2100

From the table, it appears that the limit is 2.

Graphical Solution

Use a graphing utility to graph

$$f(x) = (x^3 - x^2 + x - 1)/(x - 1).$$



Use the *trace* feature to determine that as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2 from the left and from the right.

From the graph, estimate the limit to be 2. As you use the *trace* feature, notice that there is no value given for y when $x = 1$, and that there is a hole or break in the graph at $x = 1$.

EXAMPLE 5 **Using a Graph to Find a Limit**

Find the limit of $f(x)$ as x approaches 3.

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 10, & x = 3 \end{cases}$$

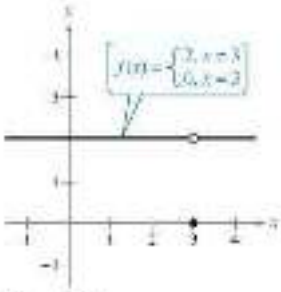


Figure 12.3

Solution Because $f(x) = 2$ for all x other than $x = 3$ and the value of $f(3)$ is immaterial, it follows that the limit is 2 (see Figure 12.3). So, write

$$\lim_{x \to 3} f(x) = 2.$$

✓ ***Checkpoint*** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find the limit of $f(x)$ as x approaches 2.

$$f(x) = \begin{cases} -3, & x \neq 2 \\ 10, & x = 2 \end{cases}$$

In Example 5, the fact that $f(3) = 10$ has no bearing on the existence or value of the limit as x approaches 3. For example, if the function were defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

then the limit as x approaches 3 would still equal 2.

Limits That Fail to Exist

Next, you will examine some limits that fail to exist.

EXAMPLE 6 **Comparing Left and Right Behavior**

Show that the limit does not exist.

$$\lim_{x \to 0} \frac{|x|}{x}$$

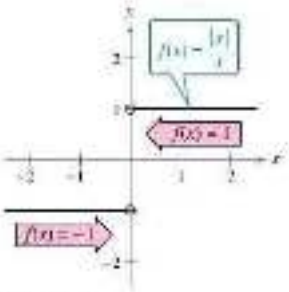


Figure 12.4

Solution Consider the graph of $f(x) = |x|/x$, shown in Figure 12.4. Notice that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \quad x < 0.$$

This means that no matter how close x gets to 0, there are both positive and negative x-values that yield $f(x) = 1$ and $f(x) = -1$, respectively. This implies that the limit does not exist.

EXAMPLE 7 **Unbounded Behavior**

Discuss the existence of the limit.

$$\lim_{x \to 0} \frac{1}{x^2}$$

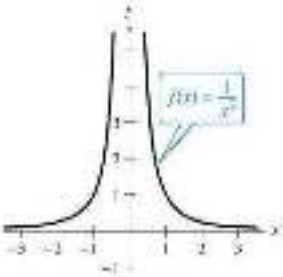


Figure 12.5

Solution Let $f(x) = 1/x^2$. In Figure 12.5, note that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that choosing x close enough to 0 enables you to force $f(x)$ to be as large as you want. For example, $f(x)$ is larger than 100 when you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \implies f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000 by choosing x that is within $\frac{1}{1000}$ of 0, as shown below.

$$0 < |x| < \frac{1}{1000} \implies f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a unique real number L as x approaches 0, the limit does not exist.

EXAMPLE 8 **Oscillating Behavior**

See LarsonPrecalculus.com for an interactive version of this type of example.

Discuss the existence of the limit.

$$\lim_{x \to 0} \sin \frac{1}{x}$$

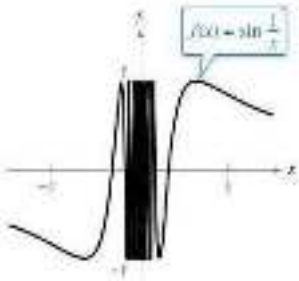


Figure 12.6

Solution Let $f(x) = \sin(1/x)$. Notice in Figure 12.6 that as x approaches 0, $f(x)$ oscillates between -1 and 1. So, the limit does not exist because no matter how close you are to 0, it is possible to choose values of x_1 and x_2 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as shown in the table.

x	$-\frac{2}{\pi}$	$-\frac{2}{3\pi}$	$-\frac{2}{5\pi}$	0	$\frac{2}{5\pi}$	$\frac{2}{3\pi}$	$\frac{2}{\pi}$
$\sin \frac{1}{x}$	-1	1	-1	?	1	-1	1

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist when any of the conditions listed below are true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c. — Example 6
2. $f(x)$ increases or decreases without bound as x approaches c. — Example 7
3. $f(x)$ oscillates between two fixed values as x approaches c. — Example 8

Properties of Limits and Direct Substitution

Sometimes, as in Example 2, the limit of $f(x)$ as $x \to c$ is $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

There are many "well-behaved" functions, such as polynomial functions and rational functions with nonzero denominators, that have this property. The list below includes some basic limits.

Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$ — Limit of a constant function
2. $\lim_{x \to c} x = c$ — Limit of the identity function
3. $\lim_{x \to c} x^n = c^n$ — Limit of a power function
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, valid for all c when n is odd and valid for $c > 0$ when n is even — Limit of a radical function

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K.$$

1. Scalar multiple: $\lim_{x \to c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

EXAMPLE 9 Direct Substitution and Properties of Limits

Find each limit.

a. $\lim_{x \to 4} x^2$ **b.** $\lim_{x \to 4} 5x$ **c.** $\lim_{x \to \pi} \dfrac{\tan x}{x}$

d. $\lim_{x \to 9} \sqrt{x}$ **e.** $\lim_{x \to \pi} (x \cos x)$ **f.** $\lim_{x \to 3} (x + 4)^2$

Solution Use the properties of limits and direct substitution to evaluate each limit.

a. $\lim_{x \to 4} x^2 = (4)^2 = 16$ — Use direct substitution.

b. $\lim_{x \to 4} 5x = 5 \lim_{x \to 4} x = 5(4) = 20$ — Use the Scalar Multiple Property and direct substitution.

c. $\lim_{x \to \pi} \dfrac{\tan x}{x} = \dfrac{\lim_{x \to \pi} \tan x}{\lim_{x \to \pi} x} = \dfrac{\tan \pi}{\pi} = \dfrac{0}{\pi} = 0$ — Use the Quotient Property and direct substitution.

d. $\lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3$ — Use direct substitution.

e. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi(\cos \pi) = -\pi$ — Use the Product Property and direct substitution.

f. $\lim_{x \to 3} (x + 4)^2 = \left[\left(\lim_{x \to 3} x\right) + \left(\lim_{x \to 3} 4\right)\right]^2 = (3 + 4)^2 = 49$ — Use the Power and Sum Properties and direct substitution.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

2. If r is a rational function $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

EXAMPLE 10 **Evaluating Limits by Direct Substitution**

Find each limit.

a. $\lim_{x \to -1} (x^2 + x - 6)$ **b.** $\lim_{x \to -1} \frac{x^3 - 5x}{x}$ **c.** $\lim_{x \to -1} \frac{x^2 + x - 6}{x + 3}$

Solution The first function is a polynomial function. The second and third functions are rational functions (with nonzero denominators at $x = -1$). So, you can evaluate the limits by direct substitution.

a. $\lim_{x \to -1} (x^2 + x - 6) = (-1)^2 + (-1) - 6 = -6$

b. $\lim_{x \to -1} \frac{x^3 - 5x}{x} = \frac{(-1)^3 - 5(-1)}{-1} = -\frac{4}{1} = -4$

c. $\lim_{x \to -1} \frac{x^2 + x - 6}{x + 3} = \frac{(-1)^2 + (-1) - 6}{-1 + 3} = -\frac{6}{2} = -3$

Dividing Out Technique

In Section 12.1, you studied several types of functions whose limits can be evaluated by direct substitution. In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails. For example, consider the limit

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}.$$

Direct substitution produces 0 in both the numerator and denominator.

$$(-3)^2 + (-3) - 6 = 0 \qquad \text{Numerator is 0 when } x = -3.$$

$$-3 + 3 = 0 \qquad \text{Denominator is 0 when } x = -3.$$

The resulting fraction, $\frac{0}{0}$, has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. By using a table, however, it appears that the limit of the function as x approaches -3 is -5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\frac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

When you attempt to evaluate a limit of a rational function by direct substitution and encounter the indeterminate form $\frac{0}{0}$, the numerator and denominator must have a common factor. After factoring and dividing out, use direct substitution again. Examples 1 and 2 show this **dividing out technique.**

EXAMPLE 1 Dividing Out Technique

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the limit: $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$.

Solution From the discussion above, you know that direct substitution fails. So, begin by factoring the numerator and dividing out any common factors.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \qquad \text{Factor numerator.}$$

$$= \lim_{x \to -3} (x - 2) \qquad \text{Divide out common factor and simplify.}$$

$$= -5 \qquad \text{Direct substitution}$$

EXAMPLE 2 Dividing Out Technique

Find the limit.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

REMARK To obtain the factorization of the denominator, divide by $(x - 1)$ or factor by grouping.

$$x^3 - x^2 + x - 1$$
$$= x^2(x - 1) + (x - 1)$$
$$= (x - 1)(x^2 + 1)$$

Solution Begin by substituting $x = 1$ into the numerator and denominator.

$$1 - 1 = 0 \qquad \text{Numerator is 0 when } x = 1.$$

$$1^3 - 1^2 + 1 - 1 = 0 \qquad \text{Denominator is 0 when } x = 1.$$

Both the numerator and denominator are zero when $x = 1$, so direct substitution will not yield the limit. To find the limit, factor the numerator and denominator, divide out any common factors, and then use direct substitution again.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)} \qquad \text{Factor denominator.}$$

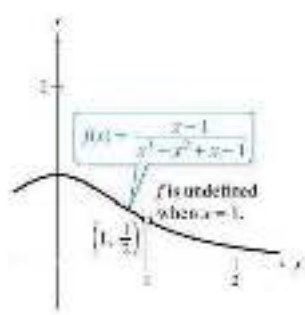
$$= \lim_{x \to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(x^2 + 1)} \qquad \text{Divide out common factor.}$$

$$= \lim_{x \to 1} \frac{1}{x^2 + 1} \qquad \text{Simplify.}$$

$$= \frac{1}{1^2 + 1} \qquad \text{Direct substitution}$$

$$= \frac{1}{2} \qquad \text{Simplify.}$$

The graph below verifies this result.



Rationalizing Technique

▷ **ALGEBRA HELP** To review techniques for rationalizing numerators and denominators, see Appendix A.2.

A way to find the limits of some functions is to first rationalize the numerator of the function. This is the **rationalizing technique.** Recall that to rationalize a numerator of the form $a \pm b\sqrt{m}$ or $b\sqrt{m} \pm a$, multiply the numerator and denominator by the conjugate of the numerator. For example, the conjugate of $\sqrt{x} + 4$ is $\sqrt{x} - 4$.

EXAMPLE 3 **Rationalizing Technique**

Find the limit.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

Solution By direct substitution, you obtain the indeterminate form $\frac{0}{0}$.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{0+1} - 1}{0} = \frac{0}{0} \qquad \text{Indeterminate form}$$

In this case, rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}
\frac{\sqrt{x+1} - 1}{x} &= \left(\frac{\sqrt{x+1} - 1}{x}\right)\left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}\right) \\
&= \frac{(x+1) - 1}{x(\sqrt{x+1} + 1)} && \text{Multiply.} \\
&= \frac{x}{x(\sqrt{x+1} + 1)} && \text{Simplify.} \\
&= \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1} + 1)} && \text{Divide out common factor.} \\
&= \frac{1}{\sqrt{x+1} + 1}, \quad x \neq 0 && \text{Simplify.}
\end{aligned}$$

Now, evaluate the limit by direct substitution.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{\sqrt{0+1} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

To verify your conclusion that the limit is $\frac{1}{2}$, construct a table, such as the one shown below, or sketch a graph, as shown in Figure 12.7.

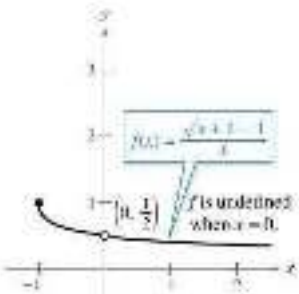


Figure 12.7

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

One-Sided Limits

In Section 12.1, you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left side of c than it approaches from the right side of c. This type of behavior can be described more concisely with the concept of a **one-sided limit.**

$$\lim_{x \to c^-} f(x) = L_1 \text{ or } f(x) \to L_1 \text{ as } x \to c^- \qquad \text{Limit from the left}$$

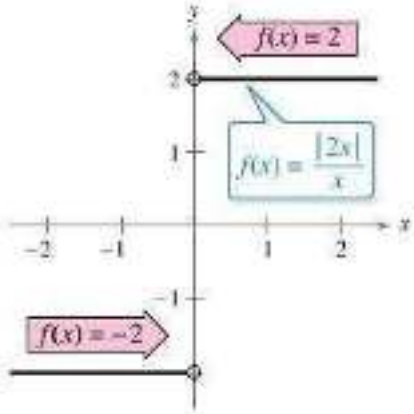
$$\lim_{x \to c^+} f(x) = L_2 \text{ or } f(x) \to L_2 \text{ as } x \to c^+ \qquad \text{Limit from the right}$$

EXAMPLE 6 Evaluating One-Sided Limits

Find the limit of $f(x)$ as x approaches 0 from the left and from the right.

$$f(x) = \frac{|2x|}{x}$$

Solution From the graph of f, shown below, notice that $f(x) = -2$ for all $x < 0$.



So, the limit from the left is

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2.$$ Limit from the left: $f(x) \to -2$ as $x \to 0^-$

Also from the graph, notice that $f(x) = 2$ for all $x > 0$, so the limit from the right is

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2.$$ Limit from the right: $f(x) \to 2$ as $x \to 0^+$

In Example 6, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \to c$ does not exist. For the limit of a function to exist as $x \to c$, it must be true that both one-sided limits exist and are equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

if and only if both the left and right limits *exist* and are *equal* to L.

EXAMPLE 7 Evaluating One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

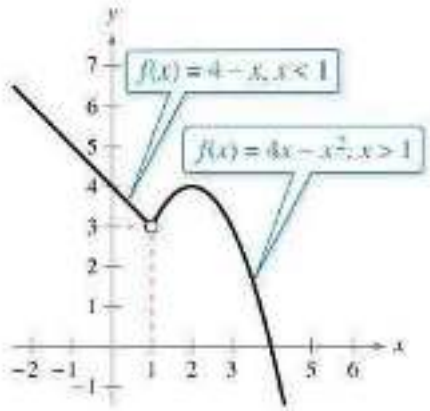
Solution Remember that you are concerned about the value of f *near* $x = 1$ rather than *at* $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$. Use direct substitution to obtain

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (4 - x) \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

For $x > 1$, $f(x)$ is given by $4x - x^2$. Use direct substitution to obtain

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (4x - x^2) \\ &= 4(1) - 1^2 \\ &= 3. \end{aligned}$$

Both one-sided limits exist and are equal to 3, so it follows that $\lim_{x \to 1} f(x) = 3$. The graph at the right confirms this conclusion.



EXAMPLE 8 Comparing Limits from the Left and Right

For 2-day shipping, a delivery service charges \$24 for the first pound and \$4 for each additional pound or portion of a pound. Let x represent the weight (in pounds) of a package and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as $x \to 2$ does not exist.

$$f(x) = \begin{cases} \$24, & 0 < x \le 1 \\ \$28, & 1 < x \le 2 \\ \$32, & 2 < x \le 3 \end{cases}$$

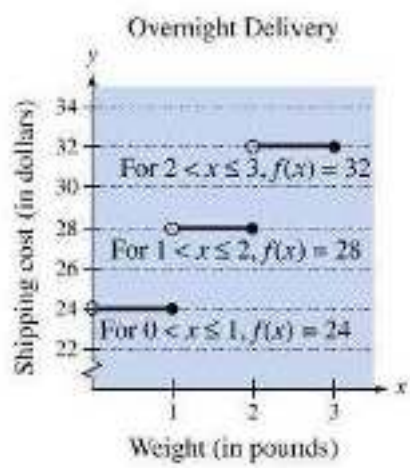
Solution The graph of f is at the right. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x \to 2^-} f(x) = 28$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x \to 2^+} f(x) = 32.$$

These one-sided limits are not equal, so the limit of $f(x)$ as $x \to 2$ does not exist.



Limits from Calculus

▷ **ALGEBRA HELP** To review evaluating difference quotients, see Section 1.4.

In the next example, you will study an important type of limit from calculus—the limit of a *difference quotient*.

EXAMPLE 9 **Evaluating a Limit from Calculus**

For the function

$$f(x) = x^2 - 1$$

find

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}.$$

Solution Begin by substituting for $f(3 + h)$ and $f(3)$ and simplifying.

$$\begin{aligned}\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} &= \lim_{h \to 0} \frac{[(3 + h)^2 - 1] - (3^2 - 1)}{h} \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h} \\ &= \lim_{h \to 0} \frac{6h + h^2}{h}\end{aligned}$$

By factoring and dividing out, you obtain

$$\begin{aligned}\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} &= \lim_{h \to 0} \frac{h(6 + h)}{h} \\ &= \lim_{h \to 0} (6 + h) \\ &= 6 + 0 \\ &= 6.\end{aligned}$$

So, the limit is 6.

▷ **REMARK** Note that for any x-value, the limit of a difference quotient is an expression of the form

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Direct substitution into the difference quotient always produces the indeterminate form $\frac{0}{0}$.

$$\begin{aligned}\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} &= \frac{f(x + 0) - f(x)}{0} \\ &= \frac{f(x) - f(x)}{0} \\ &= \frac{0}{0}\end{aligned}$$

The theorem below concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 12.11.

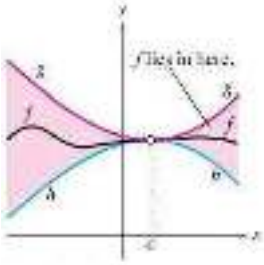
The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.

$h(x) \le f(x) \le g(x)$



The Squeeze Theorem
Figure 12.11

EXAMPLE 10 **A Special Trigonometric Limit**

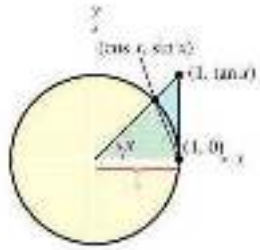
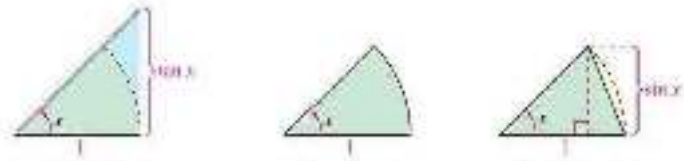


Figure 12.12

Find the limit: $\lim_{x \to 0} \frac{\sin x}{x}$.

Solution One way to find this limit is to consider a sector of a circle of radius 1 with central angle x, "squeezed" between two triangles (see Figure 12.12), where x is an acute positive angle measured in radians.



Area of triangle	$\leq$	Area of sector	$\leq$	Area of triangle
$\frac{\cos x}{2}$	$\leq$	$\frac{x}{2}$	$\leq$	$\frac{\sin x}{2}$

Multiplying each expression by $2/\sin x$ produces

$$\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1$$

and taking reciprocals and reversing the inequality symbols yields

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

Because $\cos x = \cos(-x)$ and $(\sin x)/x = [\sin(-x)]/(-x)$, this inequality is valid for *all* nonzero x in the open interval $(-\pi/2, \pi/2)$. Finally, $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} 1 = 1$, so apply the Squeeze Theorem to conclude that $\lim_{x \to 0} (\sin x)/x = 1$.

REMARK Recall from Section 4.1 that the area of a sector of a circle of radius r is given by

$$A = \frac{1}{2}r^2\theta$$

where θ is the measure of the central angle in radians.

- Calculus = branch of mathematics that studies rates of change of functions.
  - The **tangent line** to the graph of a function f at a point $P(x_1, y_1)$ is the line whose slope best approximates the slope of the graph at the point.
    - The problem of finding the slope of a graph at a point is the same as finding the slope of the tangent line at the point

EXAMPLE 1 **Visually Approximating the Slope of a Graph**

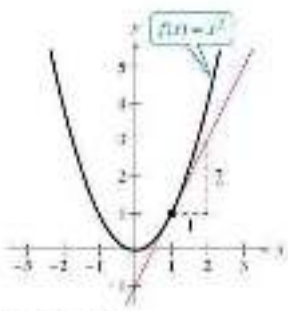


Figure 12.13

Use Figure 12.13 to approximate the slope of the graph of $f(x) = x^2$ at the point (1, 1).

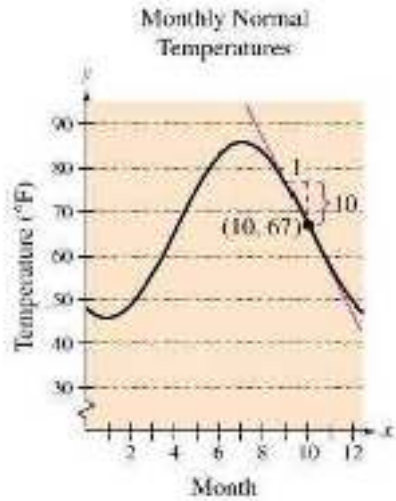
Solution From the graph of $f(x) = x^2$, notice that the tangent line at (1, 1) rises approximately two units for each unit change in x. So, you can estimate the slope of the tangent line at (1, 1) to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{2}{1} = 2.$$

The tangent line at the point (1, 1) has a slope of about 2, so the graph of f has a slope of about 2 at the point (1, 1).

EXAMPLE 2 Visually Approximating the Slope of a Graph

The figure at the right graphically depicts the monthly normal temperatures (in degrees Fahrenheit) for Dallas, Texas. Approximate the slope of this graph at the point shown and give a physical interpretation of the result. *(Source: National Climatic Data Center)*



Solution From the graph, the tangent line at the given point falls approximately 10 units for each one-unit change in x. So, you can estimate the slope at the given point to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$\approx \frac{-10}{1}$$

$$= -10 \text{ degrees per month.}$$

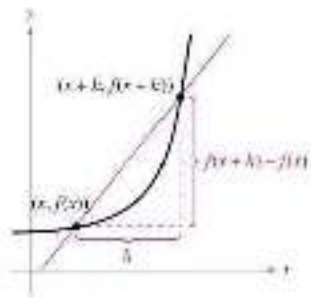
This means that the monthly normal temperature in November is about 10 degrees lower than the monthly normal temperature in October.

Slope and the Limit Process

In Examples 1 and 2, you approximated the slope of a graph at a point by creating a graph and then "eyeballing" the tangent line at the point of tangency. A more precise method of approximating tangent lines uses a **secant line** through the point of tangency and a second point on the graph, as shown at the right. If

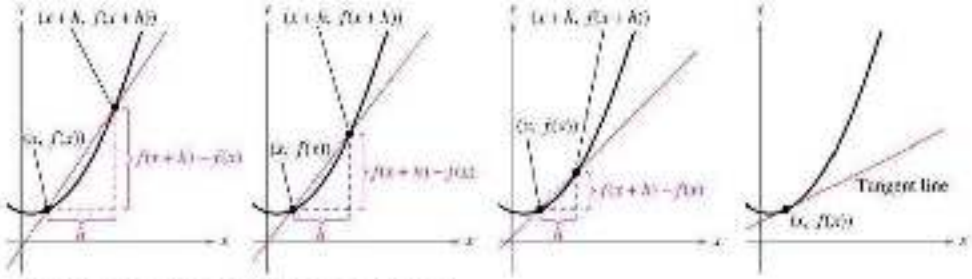
$$(x, f(x)) \quad \text{and} \quad (x + h, f(x + h))$$

are two points on the graph of f and $(x, f(x))$ is the point of tangency, then the slope of the secant line through the two points is



$$m_{sec} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x + h) - f(x)}{h}. \qquad \text{Slope of secant line}$$

Notice that the right side of this equation is a difference quotient. The denominator h is the *change in* x, and the numerator is the *change in* y. Using this method, you obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in the figures below.



As h approaches 0, the secant line approaches the tangent line.

Using the limit process, you can find the exact slope of the tangent line at $(x, f(x))$.

Definition of the Slope of a Graph

The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$m = \lim_{h \to 0} m_{sec}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

EXAMPLE 3 **Finding the Slope of a Graph**

Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$.

Solution Find an expression that represents the slope of a secant line at $(-2, 4)$.

$$m_{sec} = \frac{f(-2+h) - f(-2)}{h} \quad \text{Set up difference quotient.}$$

$$= \frac{(-2+h)^2 - (-2)^2}{h} \quad \text{Substitute into } f(x) = x^2.$$

$$= \frac{4 - 4h + h^2 - 4}{h} \quad \text{Expand terms.}$$

$$= \frac{-4h + h^2}{h} \quad \text{Simplify.}$$

$$= \frac{h(-4+h)}{h} \quad \text{Factor and divide out.}$$

$$= -4 + h, \quad h \neq 0 \quad \text{Simplify.}$$

Next, find the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{sec}$$

$$= \lim_{h \to 0} (-4 + h)$$

$$= -4$$

The graph has a slope of -4 at the point $(-2, 4)$, as shown in Figure 12.14.

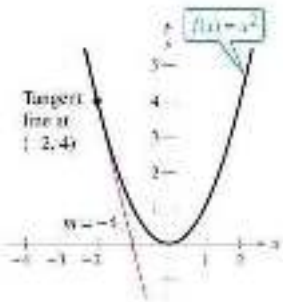


Figure 12.14

EXAMPLE 4 **Finding the Slope of a Graph**

Find the slope of $f(x) = -2x + 4$.

Solution

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{Set up difference quotient.}$$

$$= \lim_{h \to 0} \frac{[-2(x+h) + 4] - (-2x + 4)}{h} \quad \text{Substitute into } f(x) = -2x + 4.$$

$$= \lim_{h \to 0} \frac{-2x - 2h + 4 + 2x - 4}{h} \quad \text{Expand terms.}$$

$$= \lim_{h \to 0} \frac{-2h}{h} \quad \text{Divide out.}$$

$$= -2 \quad \text{Simplify.}$$

You know from your study of linear functions that the line $f(x) = -2x + 4$ has a slope of -2, as shown in Figure 12.15. This conclusion is consistent with that obtained by the limit definition of slope, as shown above.

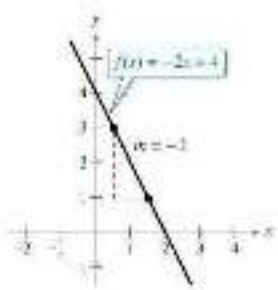


Figure 12.15

It is important that you see the difference between the ways the difference quotients were set up in Examples 3 and 4. In Example 3, you found the slope of a graph at a specific point $(c, f(c))$. To find the slope in such a case, use the form of the difference quotient below.

$$m = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$ Slope at specific point

In Example 4, however, you found a *formula* for the slope at *any* point on the graph. In such cases, you should use x, rather than c, in the difference quotient.

$$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$ Formula for slope

This form will always produce a function of x, which can then be evaluated to find the slope at any desired point on the graph.

EXAMPLE 5 Finding a Formula for the Slope of a Graph

Find a formula for the slope of the graph of $f(x) = x^2 + 1$. What are the slopes at the points $(-1, 2)$ and $(2, 5)$?

Solution

$$m_{sec} = \frac{f(x + h) - f(x)}{h}$$ Set up difference quotient.

$$= \frac{[(x + h)^2 + 1] - (x^2 + 1)}{h}$$ Substitute into $f(x) = x^2 + 1$.

$$= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h}$$ Expand terms.

$$= \frac{2xh + h^2}{h}$$ Simplify.

$$= \frac{\cancel{h}(2x + h)}{\cancel{h}}$$ Factor and divide out.

$$= 2x + h, \quad h \neq 0$$ Simplify.

Next, find the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{sec}$$ Formula for slope

$$= \lim_{h \to 0} (2x + h)$$

$$= 2x$$

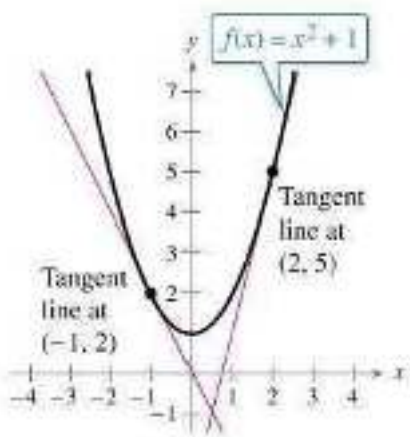
Using the formula $m = 2x$ for the slope at $(x, f(x))$, find the slope at the specified points. At $(-1, 2)$, the slope is

$$m = 2(-1) = -2$$

and at $(2, 5)$, the slope is

$$m = 2(2) = 4.$$

The graph of f is at the right.



Definition of a Derivative

The **derivative** of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

EXAMPLE 6 **Finding a Derivative**

Find the derivative of

$$f(x) = 3x^2 - 2x.$$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 2(x+h)] - (3x^2 - 2x)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}} \\
&= \lim_{h \to 0} (6x + 3h - 2) \\
&= 6x + 3(0) - 2 \\
&= 6x - 2
\end{aligned}$$

So, the derivative of $f(x) = 3x^2 - 2x$ is

$$f'(x) = 6x - 2.$$

EXAMPLE 7 **Using the Derivative**

See LarsonPrecalculus.com for an interactive version of this type of example.

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Direct substitution yields the indeterminate form $\frac{0}{0}$, so use the rationalizing technique discussed in Section 12.2 to find the limit.

$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

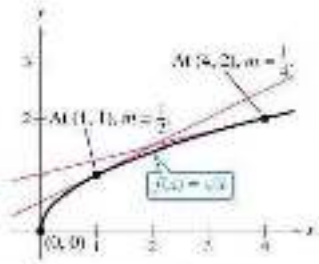
At the point $(1, 1)$, the slope is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

At the point $(4, 2)$, the slope is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Figure 12.16 shows the graph of f.



The slope of f at $(x, f(x))$, $x > 0$, is $m = 1/[2\sqrt{x}]$.
Figure 12.16

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, then the statements

$$\lim_{x \to -\infty} f(x) = L_1 \qquad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x \to \infty} f(x) = L_2 \qquad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity.** The first statement is read "*the limit of $f(x)$ as x approaches $-\infty$ is L_1,*" and the second is read "*the limit of $f(x)$ as x approaches ∞ is L_2.*"

Limits at Infinity

If r is a positive real number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the right}$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the left}$$

Find the limit.

$$\lim_{x\to\infty}\left(4-\frac{3}{x^2}\right)$$

Algebraic Solution

Use the properties of limits listed in Section 12.1.

$$\begin{aligned}\lim_{x\to\infty}\left(4-\frac{3}{x^2}\right) &= \lim_{x\to\infty} 4 - \lim_{x\to\infty}\frac{3}{x^2}\\ &= \lim_{x\to\infty} 4 - 3\left(\lim_{x\to\infty}\frac{1}{x^2}\right)\\ &= 4 - 3(0)\\ &= 4\end{aligned}$$

So, the limit of $f(x) = 4 - \dfrac{3}{x^2}$ as x approaches ∞ is 4.

EXAMPLE 2 **Evaluating Limits at Infinity**

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the limit as x approaches ∞ (if it exists) for each function.

a. $f(x) = \dfrac{-2x+3}{3x^2+1}$

b. $f(x) = \dfrac{-2x^2+3}{3x^2+1}$

c. $f(x) = \dfrac{-2x^3+3}{3x^2+1}$

Solution In each case, begin by dividing both the numerator and denominator by x^2, the highest power of x in the denominator.

a. $$\lim_{x\to\infty}\frac{-2x+3}{3x^2+1} = \lim_{x\to\infty}\frac{-\frac{2}{x}+\frac{3}{x^2}}{3+\frac{1}{x^2}} = \frac{-0+0}{3+0} = 0$$

b. $$\lim_{x\to\infty}\frac{-2x^2+3}{3x^2+1} = \lim_{x\to\infty}\frac{-2+\frac{3}{x^2}}{3+\frac{1}{x^2}} = \frac{-2+0}{3+0} = -\frac{2}{3}$$

c. $$\lim_{x\to\infty}\frac{-2x^3+3}{3x^2+1} = \lim_{x\to\infty}\frac{-2x+\frac{3}{x^2}}{3+\frac{1}{x^2}}$$

In this case, the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

Limits at Infinity for Rational Functions

Consider the rational function

$$f(x) = \frac{N(x)}{D(x)}$$

where

$$N(x) = a_n x^n + \cdots + a_0 \quad \text{and} \quad D(x) = b_m x^m + \cdots + b_0.$$

The limit of $f(x)$ as x approaches positive or negative infinity is as follows.

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, then the limit does not exist.

EXAMPLE 3 Finding the Average Cost

You are manufacturing mobile phone protective cases that cost \$4.50 per case to produce. Your initial investment is \$20,000, which implies that the total cost C of producing x cases is given by $C = 4.50x + 20{,}000$. The average cost $\overline{C}$ per case is given by

$$\overline{C} = \frac{C}{x} = \frac{4.50x + 20{,}000}{x}.$$

Find the average cost per case when (a) $x = 1000$, (b) $x = 10{,}000$, and (c) $x = 100{,}000$. (d) What is the limit of $\overline{C}$ as x approaches infinity?

Solution

a. When $x = 1000$,

$$\overline{C} = \frac{4.50(1000) + 20{,}000}{1000}$$

$$= \$24.50.$$

b. When $x = 10{,}000$,

$$\overline{C} = \frac{4.50(10{,}000) + 20{,}000}{10{,}000}$$

$$= \$6.50.$$

c. When $x = 100{,}000$,

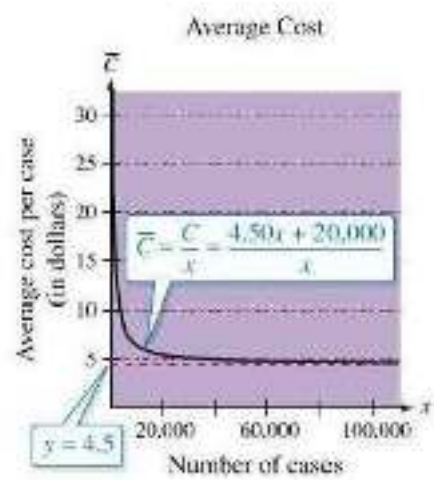
$$\overline{C} = \frac{4.50(100{,}000) + 20{,}000}{100{,}000}$$

$$= \$4.70.$$

d. As x approaches infinity, the limit of $\overline{C}$ is

$$\lim_{x \to \infty} \frac{4.50x + 20{,}000}{x} = \$4.50.$$

The graph of $\overline{C}$ is at the right.



As $x \to \infty$, the average cost per case approaches \$4.50.

Limit of a Sequence

Let f be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n \to \infty} a_n = L.$$

EXAMPLE 4 Finding the Limit of a Sequence

Write the first five terms of each sequence and find the limit of the sequence. (Assume that n begins with 1.)

a. $a_n = \dfrac{2n+1}{n+4}$ **b.** $a_n = \dfrac{2n+1}{n^2+4}$ **c.** $a_n = \dfrac{2n^2+1}{4n^2}$

Solution

a. The first five terms are $a_1 = \frac{3}{5}$, $a_2 = \frac{5}{6}$, $a_3 = 1$, $a_4 = \frac{9}{8}$, and $a_5 = \frac{11}{9}$. The limit of the sequence is

$$\lim_{n \to \infty} \frac{2n+1}{n+4} = 2.$$

b. The first five terms are $a_1 = \frac{3}{5}$, $a_2 = \frac{5}{8}$, $a_3 = \frac{7}{13}$, $a_4 = \frac{9}{20}$, and $a_5 = \frac{11}{29}$. The limit of the sequence is

$$\lim_{n \to \infty} \frac{2n+1}{n^2+4} = 0.$$

c. The first five terms are $a_1 = \frac{3}{4}$, $a_2 = \frac{9}{16}$, $a_3 = \frac{19}{36}$, $a_4 = \frac{33}{64}$, and $a_5 = \frac{51}{100}$. The limit of the sequence is

$$\lim_{n \to \infty} \frac{2n^2+1}{4n^2} = \frac{1}{2}.$$

EXAMPLE 5 Finding the Limit of a Sequence

Find the limit of the sequence whose nth term is

$$a_n = \frac{8}{n^2}\left[\frac{n(n+1)(2n+1)}{6}\right].$$

Algebraic Solution

Begin by writing the nth term in standard rational function form—as the ratio of two polynomials.

$$a_n = \frac{8}{n^2}\left[\frac{n(n+1)(2n+1)}{6}\right] \qquad \text{Write original } n\text{th term.}$$

$$= \frac{8n(n+1)(2n+1)}{6n^2} \qquad \text{Multiply factors.}$$

$$= \frac{8n^3+12n^2+4n}{3n^2} \qquad \text{Write in standard rational form.}$$

This form shows that the degree of the numerator is equal to the degree of the denominator. So, the limit of the sequence is the ratio of the leading coefficients.

$$\lim_{n \to \infty} \frac{8n^3+12n^2+4n}{3n^2} = \frac{8}{3}$$

Numerical Solution

Construct a table that shows the value of a_n as n becomes larger and larger.

n	a_n
1	8
10	3.08
100	2.7068
1000	2.6707
10,000	2.6671
100,000	2.6667
1,000,000	2.6667

Notice from the table that as n approaches ∞, a_n gets closer and closer to

$$2.6667 \approx \frac{8}{3}.$$

Limits of Summations

In Section 9.3, you used the concept of a limit to obtain a formula for the sum S of an infinite geometric series

$$S = a_1 + a_1 r + a_1 r^2 + \cdots = \sum_{i=1}^{\infty} a_1 r^{i-1} = \frac{a_1}{1-r}, \quad |r| < 1.$$

Using limit notation, this sum can be written as

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=1}^{n} a_1 r^{i-1} \\ &= \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} && \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right) \\ &= \frac{a_1}{1-r}, && \lim_{n\to\infty} r^n = 0 \text{ for } |r| < 1 \end{aligned}$$

The summation formulas and properties listed below are useful for evaluating finite and infinite summations.

Summation Formulas and Properties

1. $\sum_{i=1}^{n} c = cn$, c is a constant.
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
5. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
6. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$, k is a constant.

EXAMPLE 1 **Evaluating a Summation**

Evaluate the summation.

$$\sum_{i=1}^{200} i = 1 + 2 + 3 + 4 + \cdots + 200$$

Solution Using the second summation formula with $n = 200$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{200} i = \frac{200(200+1)}{2} = \frac{40{,}200}{2} = 20{,}100.$$

EXAMPLE 2 Evaluating a Summation

Evaluate the summation

$$S = \sum_{i=1}^{n} \frac{i+2}{n^2} = \frac{3}{n^2} + \frac{4}{n^2} + \frac{5}{n^2} + \cdots + \frac{n+2}{n^2}$$

for $n = 10, 100, 1000$, and $10{,}000$.

Solution Begin by applying summation formulas and properties to simplify S. In the second line of the solution, note that

$$\frac{1}{n^2}$$

factors out of the sum because n is considered to be constant. You cannot factor i out of the first summation in the third line of the solution because i is the index of summation.

$$S = \sum_{i=1}^{n} \frac{i+2}{n^2} \qquad \text{Write original form of summation.}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(i+2) \qquad \text{Factor constant } 1/n^2 \text{ out of sum.}$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2\right) \qquad \text{Write as two sums.}$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + 2n\right] \qquad \text{Apply Formulas 1 and 2.}$$

$$= \frac{1}{n^2}\left(\frac{n^2+5n}{2}\right) \qquad \text{Add fractions.}$$

$$= \frac{n+5}{2n} \qquad \text{Simplify.}$$

Now, evaluate the sum by substituting the appropriate values of n. The table below shows the results.

n	10	100	1000	10,000
$\sum_{i=1}^{n} \frac{i+2}{n^2} = \frac{n+5}{2n}$	0.75	0.525	0.5025	0.50025

In Example 2, note that the sum appears to approach a limit as n increases. To find the limit of

$$\frac{n+5}{2n}$$

as n approaches infinity, use the techniques from Section 12.4 to write

$$\lim_{n\to\infty} \frac{n+5}{2n} = \frac{1}{2}.$$

EXAMPLE 3 Finding the Limit of a Summation

Find the limit of $S(n)$ as $n \to \infty$.

$$S(n) = \sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)^2\left(\frac{1}{n}\right)$$

Solution Begin by rewriting the summation in rational form.

$$S(n) = \sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)^2\left(\frac{1}{n}\right)$$ Write original form of summation.

$$= \sum_{i=1}^{n}\left(\frac{n^2 + 2ni + i^2}{n^2}\right)\left(\frac{1}{n}\right)$$ Square $[1 + (i/n)]$ and write as a single fraction.

$$= \frac{1}{n^3}\sum_{i=1}^{n}(n^2 + 2ni + i^2)$$ Factor constant $1/n^3$ out of the sum.

$$= \frac{1}{n^3}\left(\sum_{i=1}^{n}n^2 + \sum_{i=1}^{n}2ni + \sum_{i=1}^{n}i^2\right)$$ Write as three sums.

$$= \frac{1}{n^3}\left\{n^3 + 2n\left[\frac{n(n+1)}{2}\right] + \frac{n(n+1)(2n+1)}{6}\right\}$$ Use summation formulas.

$$= \frac{14n^3 + 9n^2 + n}{6n^3}$$ Simplify.

Now, use this rational form to find the limit as $n \to \infty$.

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\frac{14n^3 + 9n^2 + n}{6n^3}$$

$$= \frac{14}{6}$$

$$= \frac{7}{3}$$

EXAMPLE 4 **Approximating the Area of a Region**

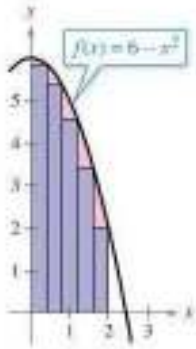


Figure 12.18

See LarsonPrecalculus.com for an interactive version of this type of example.

Use the five rectangles in Figure 12.18 to approximate the area of the region bounded by the graph of $f(x) = 6 - x^2$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution The length of the interval along the x-axis is 2 and there are five rectangles, so the width of each rectangle is $\frac{2}{5}$. To obtain the height of each rectangle, evaluate $f(x)$ at the right endpoint of each interval. The five intervals are

$$\left[0, \tfrac{2}{5}\right], \quad \left[\tfrac{2}{5}, \tfrac{4}{5}\right], \quad \left[\tfrac{4}{5}, \tfrac{6}{5}\right], \quad \left[\tfrac{6}{5}, \tfrac{8}{5}\right], \quad \text{and} \quad \left[\tfrac{8}{5}, \tfrac{10}{5}\right].$$

Notice that the right endpoint of each interval is $\frac{2}{5}i$ for $i = 1, 2, 3, 4, 5$. The sum of the areas of the five rectangles is

$$\begin{aligned}
\sum_{i=1}^{5} \overbrace{f\left(\frac{2i}{5}\right)\left(\frac{2}{5}\right)}^{\text{Height Width}} &= \sum_{i=1}^{5}\left[6 - \left(\frac{2i}{5}\right)^2\right]\left(\frac{2}{5}\right) \\
&= \frac{2}{5}\left(\sum_{i=1}^{5} 6 - \frac{4}{25}\sum_{i=1}^{5} i^2\right) \\
&= \frac{2}{5}\left[6(5) - \frac{4}{25} \cdot \frac{5(5+1)(10+1)}{6}\right] \\
&= \frac{2}{5}\left(30 - \frac{44}{5}\right) \\
&= \frac{212}{25}.
\end{aligned}$$

So, the area of the region is approximately $212/25 = 8.48$ square units.

By increasing the number of rectangles used in Example 4, you obtain a more accurate approximation of the area of the region. For instance, using 25 rectangles each of width $\frac{2}{25}$, you can approximate the area to be $A \approx 9.17$ square units. The table below includes even better approximations.

Number of Rectangles	5	25	100	1000	5000
Approximate Area	8.48	9.17	9.29	9.33	9.33

The Exact Area of a Plane Region

Based on the procedure illustrated in Example 4, the *exact* **area of a plane region** R can be found by using the limit process to increase the number n of approximating rectangles without bound.

Area of a Plane Region

Let f be continuous and nonnegative on the interval $[a, b]$. The area A of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{f\left(a + \frac{(b-a)i}{n}\right)}_{\text{Height}} \underbrace{\left(\frac{b-a}{n}\right)}_{\text{Width}}.$$

EXAMPLE 5 **Finding the Area of a Region**

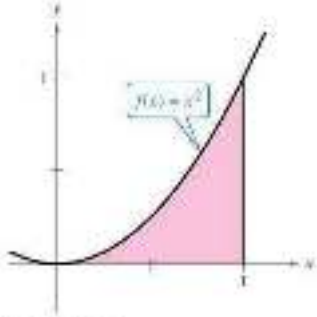


Figure 12.20

Find the area of the region bounded by the graph of $f(x) = x^2$ and the x axis between $x = 0$ and $x = 1$, as shown in Figure 12.20.

Solution Begin by finding the dimensions of the rectangles.

$$\text{Width: } \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$\text{Height: } f\left(a + \frac{(b-a)i}{n}\right) = f\left(0 + \frac{(1-0)i}{n}\right) = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$\begin{aligned}
A &\approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) && \text{Summation form} \\
&= \sum_{i=1}^{n} \frac{i^2}{n^3} \\
&= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{2n^3 + 3n^2 + n}{6n^3} && \text{Rational form}
\end{aligned}$$

Finally, find the exact area by taking the limit as n approaches ∞.

$$A = \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3}$$

So, the area of the region is $\frac{1}{3}$ square unit.

EXAMPLE 6 **Finding the Area of a Region**

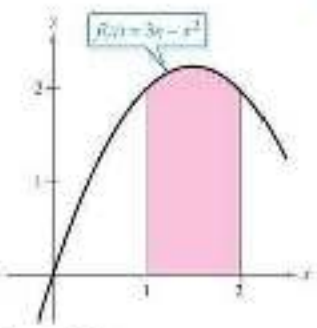


Figure 12.21

Find the area of the region bounded by the graph of $f(x) = 3x - x^2$, the x-axis, and the lines $x = 1$ and $x = 2$, as shown in Figure 12.21.

Solution Begin by finding the dimensions of the rectangles.

$$\text{Width: } \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$\begin{aligned}
\text{Height: } f\left(a + \frac{(b-a)i}{n}\right) &= f\left(1 + \frac{i}{n}\right) \\
&= 3\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2 \\
&= 2 + \frac{i}{n} - \frac{i^2}{n^2}
\end{aligned}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$\begin{aligned}
A &\approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right) \\
&= \sum_{i=1}^{n} \left(2 + \frac{i}{n} - \frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} 2 + \frac{1}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{1}{n}(2n) + \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{12n^3}{6n^3} + \frac{3n^2 + 3n^2}{6n^3} - \frac{2n^3 + 3n^2 + n}{6n^3} \\
&= \frac{13n^2 - n}{6n^3}
\end{aligned}$$

Finally, find the exact area by taking the limit as n approaches ∞.

$$A = \lim_{n \to \infty} \frac{13n^2 - n}{6n^3} = \frac{13}{6}$$

Summary

In this week, we learned about limits, conditions for the existence of limits, properties of limits, one-sided limits, the tangent line to a graph, what a derivative is, limits at infinity, the

limit of a sequence, summation formulas/properties, and finally how to calculate the exact area of a plane region.

Week 15 Limits and differentiation Reading note 2

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-22 7:34 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** Limits and differentiation	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 22, 2020

Essential Question:

What are limits and derivatives and how do they relate to the notion of continuity of a function?

Questions/Cues:

- What is the gradient of a curve at a point?
- What is the formula for the gradient function?
- What is the formula for the gradient function of $y = x^n$?
- What are the gradient functions of some of the most common functions?
- What are some rules for finding gradient functions?
- What is the definition of the second derivative?
- What are Stationary points?
- What is the second derivative test for maximum and minimum points?
- What is the product rule?
- What is the quotient rule?
- What is the chain rule?

Notes

- Gradient of a curve at a point = is the gradient of the tangent at that point

Given a function $y = f(x)$ we denote its gradient function by $\frac{dy}{dx}$ or simply by y'.

  - Also called the first derivative or simply the derivative. Process of finding the gradient function is known as differentiation.
    - Because the gradient function measures how rapidly a graph is changing, it's also referred to as the rate of change of y.

For any function of the form $y = x^n$ the gradient function is found from the following formula:

Key point If $y = x^n$ then $y' = nx^{n-1}$.

WORKED EXAMPLES

34.1 Find the gradient function of (a) $y = x^3$, (b) $y = x^4$.

Solution (a) Comparing $y = x^3$ with $y = x^n$ we see that $n = 3$. Then $y' = 3x^{3-1} = 3x^2$.

(b) Applying the formula with $n = 4$ we find that if $y = x^4$ then $y' = 4x^{4-1} = 4x^3$.

34.2 Find the gradient function of (a) $y = x^2$, (b) $y = x$.

Solution (a) Applying the formula with $n = 2$ we find that if $y = x^2$ then $y' = 2x^{2-1} = 2x^1$. Because x^1 is simply x we find that the gradient function is $y' = 2x$.

(b) Applying the formula with $n = 1$ we find that if $y = x^1$ then $y' = 1x^{1-1} = 1x^0$. Because x^0 is simply 1 we find that the gradient function is $y' = 1$.

WORKED EXAMPLES

34.3 Find the gradient of $y = x^2$ at the points where

(a) $x = -1$ (b) $x = 0$ (c) $x = 2$ (d) $x = 3$

Solution From Worked Example 34.2, or from the formula, we know that the gradient function of $y = x^2$ is given by $y' = 2x$.

(a) When $x = -1$ the gradient of the graph is then $y'(-1) = 2(-1) = -2$. The fact that the gradient is negative means that the curve is falling at the point.

(b) When $x = 0$ the gradient is $y'(0) = 2(0) = 0$. The gradient of the curve is zero at this point. This means that the curve is neither falling nor rising.

(c) When $x = 2$ the gradient is $y'(2) = 2(2) = 4$. The fact that the gradient is positive means that the curve is rising.

(d) When $x = 3$ the gradient is $y'(3) = 2(3) = 6$ and so the curve is rising here. Comparing this answer with that of part (c) we conclude that the curve is rising more rapidly at $x = 3$ than at $x = 2$, where the gradient was found to be 4.

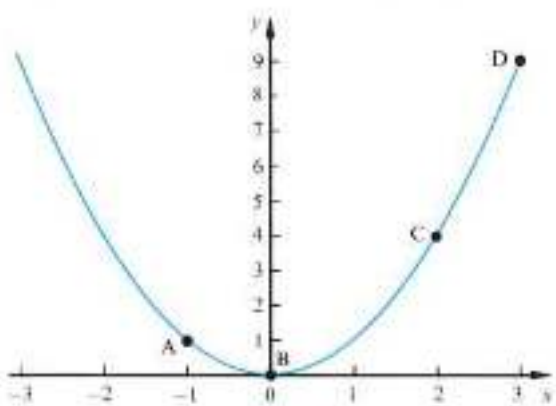


Figure 34.2
Graph of $y = x^2$

The graph of $y = x^2$ is shown in Figure 34.2. If we compare our results with the graph we see that the curve is indeed falling when $x = -1$ (point A) and is rising when $x = 2$ (point C) and $x = 3$ (point D). At the point where $x = 0$ (point B) the curve is neither rising nor falling.

34.4 Find the gradient function of $y = x^{-3}$. Hence find the gradient of $y = x^{-3}$ when $x = 4$.

Solution Using the formula with $n = -3$ we find that if $y = x^{-3}$ then $y' = -3x^{-3-1} = -3x^{-4}$. Because x^{-4} can also be written as $1/x^4$ we could write $y' = -3(1/x^4)$ or $-3/x^4$. When $x = 4$ we find $y'(4) = -3/4^4 = -\frac{3}{256}$. This number is very small and negative, which means that when $x = 4$ the curve is falling, but only slowly.

34.5 Find the gradient function of $y = 1$.

Solution Before we use the formula to calculate the gradient function let us think about the graph of $y = 1$. Whatever the value of x, this function takes the value 1. Its graph must then be a horizontal line – it neither rises nor falls. We conclude that the gradient function must be zero, that is $y' = 0$. To obtain the same result using the formula we must rewrite 1 as x^0. Then, using the formula with $n = 0$, we find that if $y = x^0$ then $y' = 0x^{0-1} = 0$.

$y = f(x)$	$y' = f'(x)$	Notes
constant	0	
x	1	
x^2	$2x$	
x^n	nx^{n-1}	
e^x	e^x	
e^{kx}	ke^{kx}	k is a constant
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\sin kx$	$k \cos kx$	k is a constant
$\cos kx$	$-k \sin kx$	k is a constant
$\ln kx$	$1/x$	k is a constant

- When finding the gradient functions of trigonometric functions, the angle x must always be measured in radians

WORKED EXAMPLES

34.6 Use Table 34.1 to find the gradient function y' when y is

(a) $\sin x$ (b) $\sin 2x$ (c) $\cos 3x$ (d) e^x

Solution (a) Directly from the table we see that if $y = \sin x$ then its gradient function is given by $y' = \cos x$. This result occurs frequently and is worth remembering.

(b) Using the table and taking $k = 2$ we find that if $y = \sin 2x$ then $y' = 2\cos 2x$.

(c) Using the table and taking $k = 3$ we find that if $y = \cos 3x$ then $y' = -3\sin 3x$.

(d) Using the table we see directly that if $y = e^x$ then $y' = e^x$. Note that the exponential function e^x is very special because its gradient function y' is the same as y.

34.7 Find the gradient function of $y = e^{-x}$. Hence find the gradient of the graph of y at the point where $x = 1$.

Solution Noting that $e^{-x} = e^{-1x}$ and using Table 34.1 with $k = -1$ we find that if $y = e^{-x}$ then $y' = -1e^{-x} = -e^{-x}$. Using a calculator to evaluate this when $x = 1$ we find $y'(1) = -e^{-1} = -0.368$.

34.8 Find the gradient function of $y = \sin 4x$ where $x = 0.3$.

Solution Using Table 34.1 with $k = 4$ gives $y' = 4\cos 4x$. Remembering to measure x in radians we evaluate this when $x = 0.3$:

$$y'(0.3) = 4\cos 4(0.3) = 4\cos 1.2 = 1.4494$$

Key point *Rule 1:* If $y = f(x) + g(x)$ then $y' = f'(x) + g'(x)$.

In words this says that to find the gradient function of a sum of two functions we simply find the two gradient functions separately and add these together.

WORKED EXAMPLE

34.9 Find the gradient function of $y = x^2 + x^4$.

Solution The gradient function of x^2 is $2x$. The gradient function of x^4 is $4x^3$. Therefore

$$\text{if } y = x^2 + x^4 \quad \text{then} \quad y' = 2x + 4x^3$$

Key point *Rule 2:*
If $y = f(x) - g(x)$ then $y' = f'(x) - g'(x)$.

WORKED EXAMPLE

34.10 Find the gradient function of $y = x^5 - x^7$.

Solution We find the gradient function of each term separately and subtract them. That is, $y' = 5x^4 - 7x^6$.

Key point *Rule 3:* If $y = kf(x)$, where k is a number, then $y' = kf'(x)$.

WORKED EXAMPLE

34.11 Find the gradient function of $y = 3x^2$.

Solution This function is 3 times x^2. The gradient function of x^2 is $2x$. Therefore, using Rule 3, we find that

$$\text{if } y = 3x^2 \quad \text{then } y' = 3(2x) = 6x$$

WORKED EXAMPLES

34.12 Find the derivative of $y = 4x^2 + 3x^{-3}$.

Solution The derivative of $4x^2$ is $4(2x) = 8x$. The derivative of $3x^{-3}$ is $3(-3x^{-4}) = -9x^{-4}$. Therefore, if $y = 4x^2 + 3x^{-3}$ then $y' = 8x - 9x^{-4}$.

34.13 Find the derivative of

(a) $y = 4\sin t - 3\cos 2t$ (b) $y = \dfrac{e^{2t}}{3} + 6 + \dfrac{\ln(2t)}{5}$

Solution (a) We differentiate each quantity in turn using Table 34.1:

$$y' = 4\cos t - 3(-2\sin 2t) = 4\cos t + 6\sin 2t$$

(b) Writing y as $\frac{1}{3}e^{2t} + 6 + \frac{1}{5}\ln(2t)$ we find

$$y' = \frac{2}{3}e^{2t} + 0 + \frac{1}{5}\left(\frac{1}{t}\right) = \frac{2e^{2t}}{3} + \frac{1}{5t}$$

Key point

y'' or $\dfrac{d^2y}{dx^2}$ is found by differentiating y'.

WORKED EXAMPLES

34.14 Find the first and second derivatives of $y = x^4$.

Solution From Table 34.1, if $y = x^4$ then $y' = 4x^3$. The second derivative is found by differentiating the first derivative. Therefore

$$\text{if } y' = 4x^3 \qquad \text{then } y'' = 4(3x^2) = 12x^2$$

34.15 Find the first and second derivatives of $y = 3x^2 - 7x + 2$.

Solution Using Table 34.1 we find $y' = 6x - 7$. The derivative of the constant 2 equals 0. Differentiating again we find $y'' = 6$, because the derivative of the constant -7 equals 0.

34.16 If $y = \sin x$, find

(a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$

Solution (a) Recall that $\frac{dy}{dx}$ is the first derivative of y. From Table 34.1 this is given by

$$\frac{dy}{dx} = \cos x$$

(b) $\frac{d^2y}{dx^2}$ is the second derivative of y. This is found by differentiating the first derivative. From Table 34.1 we find that the derivative of $\cos x$ is $-\sin x$, and so

$$\frac{d^2y}{dx^2} = -\sin x$$

Key point Stationary points are located by setting the gradient function equal to zero, that is $y' = 0$.

- If to the left of the point the curve is rising and to the right it's falling, then the point is a maximum turning point or simply a maximum
- If to the left of the point the curve is falling and to the right it's rising, then the point is a minimum turning point or simply a minimum
- If at the point, the slope of the curve is momentarily zero and then it continues rising or falling, then the point is a point of inflexion or inflection

WORKED EXAMPLE

34.17 Find the stationary points of $y = 3x^2 - 6x + 8$.

Solution We first determine the gradient function y' by differentiating y. This is found to be $y' = 6x - 6$. Stationary points occur when the gradient is zero, that is when $y' = 6x - 6 = 0$. From this

$$6x - 6 = 0$$
$$6x = 6$$
$$x = 1$$

When $x = 1$ the gradient is zero. When $x = 1$, $y = 5$. Therefore $(1, 5)$ is a stationary point. At this stage we cannot tell whether to expect a maximum, minimum or point of inflexion; all we know is that one of these occurs when $x = 1$. However, a sketch of the graph of $y = 3x^2 - 6x + 8$ is shown in Figure 34.4, which reveals that the point is a minimum.

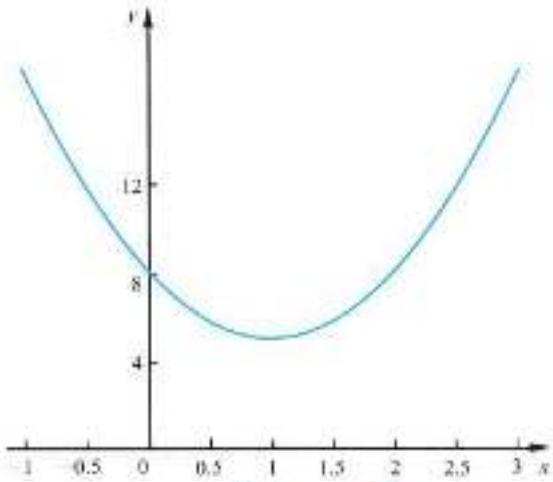


Figure 34.4
There is a minimum turning point at $x = 1$

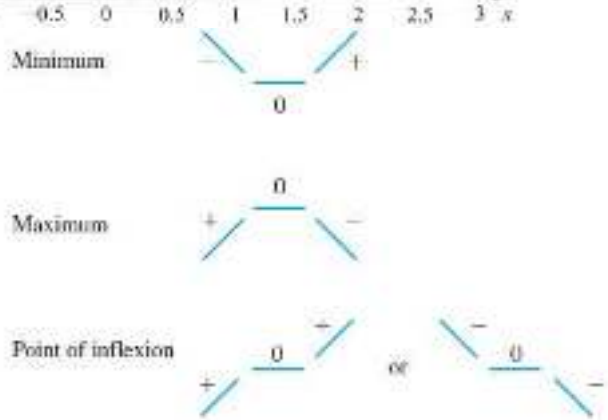


Figure 34.5
The sign of the gradient function close to a stationary point

34.18 Find the location and nature of the stationary points of $y = 2x^3 - 6x^2 - 18x$.

Solution In order to find the location of the stationary points we must calculate the gradient function. This is $y' = 6x^2 - 12x - 18$. At a stationary point $y' = 0$ and so the equation that must be solved is $6x^2 - 12x - 18 = 0$. This quadratic equation is solved as follows:

$$6x^2 - 12x - 18 = 0$$

Factorising:

$$6(x^2 - 2x - 3) = 0$$

$$6(x + 1)(x - 3) = 0$$

so that $x = -1$ and 3

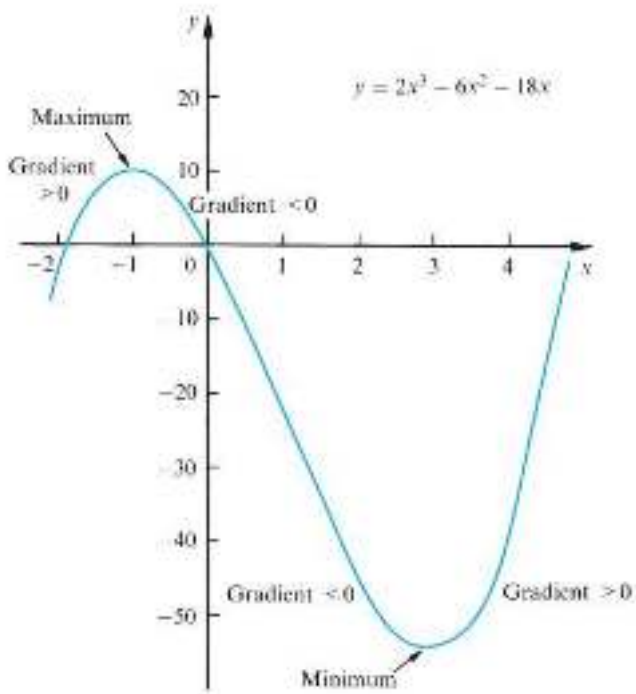
The stationary points are located at $x = -1$ and $x = 3$. At these values, $y = 10$ and $y = -54$ respectively. We now find their nature:

When $x = -1$ A little to the left of $x = -1$, say at $x = -2$, we calculate the gradient of the graph using the gradient function. That is, $y'(-2) = 6(-2)^2 - 12(-2) - 18 = 24 + 24 - 18 = 30$. Therefore the graph is rising when $x = -2$. A little to the right of $x = -1$, say at $x = 0$, we calculate the gradient of the graph using the gradient function. That is, $y'(0) = 6(0)^2 - 12(0) - 18 = -18$. Therefore the graph is falling when $x = 0$. From this information we conclude that the turning point at $x = -1$ must be a maximum.

When $x = 3$ A little to the left of $x = 3$, say at $x = 2$, we calculate the gradient of the graph using the gradient function. That is, $y'(2) = 6(2)^2 - 12(2) - 18 = 24 - 24 - 18 = -18$. Therefore the graph is falling when $x = 2$. A little to the right of $x = 3$, say at $x = 4$, we calculate the gradient of the graph using the gradient function. That is, $y'(4) = 6(4)^2 - 12(4) - 18 = 96 - 48 - 18 = 30$. Therefore the graph is rising when $x = 4$. From this information we conclude that the turning point at $x = 3$ must be a minimum.

To show this behaviour a graph of $y = 2x^3 - 6x^2 - 18x$ is shown in Figure 34.6 where these turning points can be clearly seen.

Figure 34.6
Graph of $y = 2x^3 - 6x^2 - 18x$



34.19 Find the location and nature of the stationary points of $y = x^3 - 3x^2 + 3x - 1$.

Solution First we find the gradient function: $y' = 3x^2 - 6x + 3$. At a stationary point $y' = 0$ and so

$$3x^2 - 6x + 3 = 0$$

$$3(x^2 - 2x + 1) = 0$$

$$3(x-1)(x-1) = 0$$

$$\text{and so } x = 1$$

When $x = 1$, $y = 0$.

We conclude that there is only one stationary point, and this is at $(1, 0)$. To determine its nature we look at the gradient function on either side of

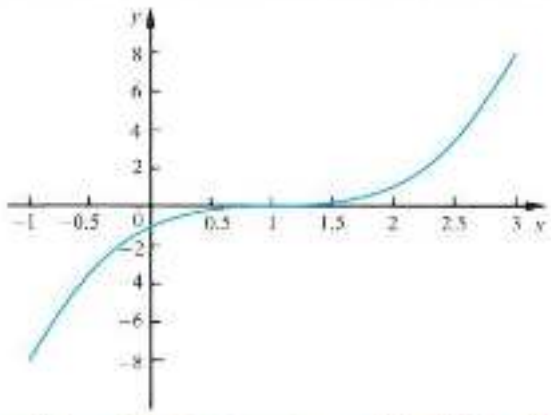


Figure 34.7 Graph of $y = x^3 - 3x^2 + 3x - 1$ showing the point of inflexion

this point. At $x = 0$, say, $y'(0) = 3$. At $x = 2$, say, $y'(2) = 3(2^2) - 6(2) + 3 = 3$. The gradient function changes from positive to zero to positive as we move through the stationary point. We conclude that the stationary point is a point of inflexion. For completeness a graph is sketched in Figure 34.7.

Key point

If y'' is positive at the stationary point, the point is a minimum.
If y'' is negative at the stationary point, the point is a maximum.
If y'' is equal to zero, this test does not tell us anything and the previous method should be used.

WORKED EXAMPLE

34.20 Locate the stationary points of

$$y = \frac{x^3}{3} + \frac{x^2}{2} - 12x + 5$$

and determine their nature using the second-derivative test.

Solution We first find the gradient function: $y' = x^2 + x - 12$. Setting this equal to zero to locate the stationary points we find

$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x = -4 \text{ and } 3$$

When $x = -4$, $y = \frac{109}{6}$. When $x = 3$, $y = -\frac{25}{2}$.

There are two stationary points: at $(-4, \frac{109}{6})$ and $(3, -\frac{25}{2})$. To apply the second-derivative test we need to find y''. This is found by differentiating y' to give $y'' = 2x + 1$. We now evaluate this at each stationary point in turn:

$x = -4$ When $x = -4$, $y''(-4) = 2(-4) + 1 = -7$. This is negative and we conclude from the second-derivative test that the point is a maximum point.

$x = 3$ When $x = 3$, $y''(3) = 2(3) + 1 = 7$. This is positive and we conclude from the second-derivative test that the point is a minimum point. Remember, if in doubt, that these results could be confirmed by sketching a graph of the function.

Key point

The product rule: if $y = uv$ then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

35.1 Use the product rule to differentiate $y = x^3 \sin 2x$.

Solution We let $u(x) = x^3$ and $v(x) = \sin 2x$.

Then clearly

$$y = uv$$

and using Table 34.1 we have

$$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = 2\cos 2x$$

Using the product rule we have

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$
$$= 3x^2 \sin 2x + x^3(2\cos 2x)$$
$$= x^2(3\sin 2x + 2x\cos 2x)$$

35.2 Use the product rule to differentiate $y = e^{2x} \cos x$.

Solution We let

$$u(x) = e^{2x} \qquad v(x) = \cos x$$

so that

$$y = e^{2x} \cos x = uv$$

Using Table 34.1 we have

$$\frac{du}{dx} = 2e^{2x} \qquad \frac{dv}{dx} = -\sin x$$

Using the product rule we have

$$\frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}$$

$$= 2e^{2x} \cos x + e^{2x}(-\sin x)$$

$$= e^{2x}(2\cos x - \sin x)$$

35.3 Use the product rule to differentiate $y = x \ln x$.

Solution Let $u = x$, $v = \ln x$. Then

$$y = x \ln x = uv$$

and

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{x}$$

So using the product rule

$$\frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}$$

$$= 1 \ln x + x\left(\frac{1}{x}\right)$$

$$= \ln x + 1$$

35.4 Find the gradient of $y = 2x^2e^{-x}$ when $x = 1$.

Solution Let $u = 2x^2$, $v = e^{-x}$ so that

$$y = 2x^2e^{-x} = uv$$

and

$$\frac{du}{dx} = 4x \qquad \frac{dv}{dx} = -e^{-x}$$

Using the product rule we see

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx}\\ &= 4xe^{-x} + 2x^2(-e^{-x})\\ &= 2xe^{-x}(2 - x)\end{aligned}$$

Recall that the derivative, $\frac{dy}{dx}$, is the gradient of y. When $x = 1$

$$\frac{dy}{dx} = 2e^{-1}(2 - 1) = 0.7358$$

so that the gradient of $y = 2x^2e^{-x}$ when $x = 1$ is 0.7358.

Key point

The quotient rule: if

$$y = \frac{u}{v}$$

then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

35.5 Use the quotient rule to differentiate

$$y = \frac{\sin x}{x^2}$$

Solution We let $u(x) = \sin x$ and $v(x) = x^2$. Then

$$y = \frac{u}{v} \quad \text{and} \quad \frac{du}{dx} = \cos x \quad \frac{dv}{dx} = 2x$$

Using the quotient rule we have

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{x^2 \cos x - \sin x(2x)}{(x^2)^2}$$

$$= \frac{x(x \cos x - 2 \sin x)}{x^4}$$

$$= \frac{x \cos x - 2 \sin x}{x^3}$$

35.6 Use the quotient rule to differentiate $y = \dfrac{x+1}{x^2+1}$.

Solution We let

$$u(x) = x + 1 \qquad v(x) = x^2 + 1$$

and so clearly

$$y = \frac{u}{v}$$

Now

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$$

Applying the quotient rule we have

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(x^2+1)1 - (x+1)2x}{(x^2+1)^2}$$

$$= \frac{x^2 + 1 - 2x^2 - 2x}{(x^2+1)^2}$$

$$= \frac{1 - 2x - x^2}{(x^2+1)^2}$$

35.7 Use the quotient rule to differentiate $y = \dfrac{e^x + x}{\sin x}$.

Solution We let

$$u(x) = e^x + x \qquad v(x) = \sin x$$

and so

$$\frac{du}{dx} = e^x + 1 \qquad \frac{dv}{dx} = \cos x$$

Clearly

$$y = \frac{u}{v}$$

and so

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{\sin x(e^x + 1) - (e^x + x)\cos x}{(\sin x)^2}$$

The chain rule

Suppose we are given a function $y(x)$ where the variable x is itself a function of another variable, t, say. We say that y is a **function of a function**. For example, suppose

$$y(x) = \cos x \qquad \text{and} \qquad x(t) = t^2$$

Then we can write

$$y = \cos(t^2)$$

There will be occasions when it is necessary to calculate $\frac{dy}{dt}$. This can be done by first finding $\frac{dy}{dx}$ and $\frac{dx}{dt}$ and then using the chain rule.

Key point The **chain rule** states that if $y = y(x)$ and $x = x(t)$, then

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

WORKED EXAMPLE

35.8 Use the chain rule to find $\frac{dy}{dt}$ when $y = \cos x$ and $x = t^2$.

Solution When $y = \cos x$ our previous knowledge of differentiation (or use of Table 34.1) tells us that $\frac{dy}{dx} = -\sin x$. Also, when $x = t^2$, $\frac{dx}{dt} = 2t$. Using the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= (-\sin x) \times 2t$$

Using the fact that $x = t^2$ enables us to write this as

$$\frac{dy}{dt} = -2t \sin t^2$$

WORKED EXAMPLES

35.9 If $y = (7t + 3)^4$ find $\frac{dy}{dt}$.

Solution Note that if we introduce a new variable x and make the substitution $7t + 3 = x$ then y takes the much simpler form $y = x^4$. Then, from $y = x^4$,

$$\frac{dy}{dx} = 4x^3$$

and from $x = 7t + 3$,

$$\frac{dx}{dt} = 7$$

Using the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= 4x^3 \times 7$$

$$= 28x^3$$

$$= 28(7t + 3)^3$$

35.10 If $y = (3t^2 + 4)^{1/2}$, find the derivative $\frac{dy}{dt}$.

Solution If we let $3t^2 + 4 = x$, then y looks much simpler: $y = x^{1/2}$. Then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \qquad \text{and} \qquad \frac{dx}{dt} = 6t$$

Using the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \frac{1}{2}x^{-1/2} \times 6t$$

$$= 3t(3t^2 + 4)^{-1/2}$$

35.11 Find the gradient of the function $y = e^{(t^2)}$ when $t = 0.5$.

Solution Note that by writing $t^2 = x$ then $y = e^x$. Using the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= e^x \times 2t$$

$$= 2te^{(t^2)}$$

This is the gradient function of $y = e^{(t^2)}$. So, when $t = 0.5$ the value of the gradient is $(2)(0.5)e^{(0.5^2)} = e^{0.25} = 1.284$.

Summary

In this week, we learned about gradient of a curve at a point, the formula for the gradient function, the gradient functions of some of the most common functions, the second derivative, stationary points, the second derivative test, product rule, quotient rule and finally the chain rule.

Week 16 Limits and differentiation continued Lecture note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-23 8:34 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Limits and differentiation continued**	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 23, 2020

Essential Question:

What are limits and derivatives and how do they relate to the notion of continuity of a function?

Questions/Cues:

- What are some properties of the derivative?
- What is L'Hopital Rule?
- How does the derivative of a function relate to its minima and maxima?
- What are inflection points?

Notes

Properties:

1)derivative of sum of two functions

Let f(x)=g(x)+h(x)

$f'(x)=g'(x)+h'(x)$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{g(x+\Delta x)+h(x+\Delta x)-g(x)-h(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}+\frac{h(x+\Delta x)-h(x)}{\Delta x}=g'(x)+h'(x)$$

2)n^{th} derivative

$$f''(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}\rightarrow f^{n}(x)=\lim_{\Delta x\to 0}\frac{f^{n-1}(x+\Delta x)-f^{n-1}(x)}{\Delta x}$$

Derivative of product of two functions:

$f(x)=g(x)h(x)$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{g(x+\Delta x)h(x+\Delta x)-g(x)h(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{g(x+\Delta x)h(x+\Delta x)\overbrace{-g(x)h(x+\Delta x)+g(x)h(x+\Delta x)}^{=0}-g(x)h(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}h(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x)h(x+\Delta x)-g(x)h(x)}{\Delta x}$$

$$=h(x)g'(x)+\lim_{\Delta x\to 0}g(x)\frac{h(x+\Delta x)-h(x)}{\Delta x}$$

$$=h(x)g'(x)+g(x)h'(x)$$

Derivative from first principles

Example: $y=f(x)=x^2$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Let $\Delta x=h$

$$\lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}=\lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h}$$

$$=\lim_{h\to 0}\frac{(2xh+h^2)}{h}=\lim_{h\to 0}(2x+h)=2x$$

$f'(x)=2x$

Derivative of ratio of two functions:

$f(x)=g(x)/h(x)$

$$f'(x)=\frac{g'(x)h(x)-g(x)h'(x)}{h^2(x)}$$

Example: $f(x)= x^2/(2x+1) =g(x)/h(x)$

$g(x)=x^2 \quad \rightarrow \quad g'(x)=2x$
$h(x)=(2x+1) \quad \rightarrow \quad h'(x)=2$

$$f'(x)=\frac{2x(2x+1)-x^2\,2}{(2x+1)2}=\frac{4x^2+2x-2x^2}{(2x+1)2}=\frac{2x(x+1)}{(2x+1)2}$$

Derivative of composed function:

Composed function: two functions g and h applied in succession
Example:
$h(x) \quad x \rightarrow log(x)$
$g(x): \; x \rightarrow sin(x)$
Composed functions $\rightarrow g(h(x)) = sin(log(x)) \qquad h(g(x)) = log(sin(x))$

$f(x)=g(h(x)) \quad \rightarrow f'(x) = g'(h(x))h'(x)$

Example: f(x)= e^{x^2}

$h(x)=x^2 \quad \rightarrow \quad h'(x)= 2x$
$g(x)=e^x \quad \rightarrow \quad g'(x)= e^x$

In g'(h(x)) need to replace h(x) in place of x in g'(x)

$\rightarrow g'(h(x))=e^{h(x)}=e^{x^2}$

$\rightarrow f'(x)=e^{x^2}h'(x)=2xe^{x^2}$

1) $f(x) = g(x) + h(x) \rightarrow f'(x) = g'(x) + h'(x)$

$g(x) = x^2 \quad h(x) = e^x \qquad g'(x) = 2x$
$f(x) = x^2 + e^x \qquad h'(x) = e^x$

$f'(x) = 2x + e^x$

2) $f(x) = h(x)\,g(x)$

$$f'(x) = h'(x)\,g(x) + h(x)\,g'(x)$$

$$f(x) = \underbrace{x}_{h(x)}\,\overbrace{(\sin x + 1)}^{g(x)}$$

$$\begin{cases} h'(x) = 1 \\ g'(x) = \cos x \end{cases}$$

$$f'(x) = 1 \cdot (\sin x + 1) + x \cdot \cos x =$$

$$= \sin x + 1 + x\cos x$$

$$f(x) = \frac{g(x)}{h(x)} \rightarrow \underbrace{f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}}$$

$$f(x) = \frac{\cos x}{x^2+3}$$

$g(x) = \cos x \rightarrow g'(x) = -\sin x$

$h(x) = \underline{x^2+3} \rightarrow h'(x) = 2x$

$$f'(x) = \frac{-\sin x\,(x^2+3) - \cos x\,(2x)}{(x^2+3)^2} =$$

$$= -\frac{\sin x}{x^2+3} - \frac{2x\cos x}{(x^2+3)^2}$$

$$f(x) = g(h(x)) \rightarrow f'(x) = g'(h(x))\,h'(x)$$

$$f(x) = \cos(\ln x)$$

$g(x) = \cos x$

$h(x) = \ln x$

$g'(x) = -\sin(x)$

$h'(x) = \frac{1}{x}$

$$f'(x) = -\frac{\sin(\ln x)}{x}$$

$f(x)$	$f'(x)$
x^{α}	$\alpha x^{\alpha-1}$
$e^{\alpha x}$	$\alpha e^{\alpha x}$
$\log(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

$$\tan(x) \quad \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\overbrace{\cos x \cdot \cos x - (-\sin x)\cdot \sin x}^{\cos^2 x + \sin^2 x = 1}}{\cos^2 x} = \frac{1}{\cos^2 x}$$

L'Hopital's rule

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \quad \text{if} \quad \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \begin{cases} 0 \\ \pm\infty \end{cases}$$

$$g'(x) \neq 0 \quad x \in I / \{x_0\}$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

1) $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$

2) $\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$

3) $\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$

4) $\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{-\sin x}{2x} = -\frac{1}{2} \left(\lim_{x \to 0} \frac{\sin x}{x} \right)$ $= 1$

$= -\frac{1}{2}$

$$\lim_{x \to 0} \frac{e^x - 1}{x^2 + \sin x} = \lim_{x \to 0} \frac{e^x}{2x + \cos x} = \frac{1}{1} = 1$$

$$\lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \log(x^x)}$$

$$\boxed{x^x = e^{\log(x^x)}}$$

$$= \lim_{x \to 0} e^{x \log x}$$

$$\lim_{x \to 0} e^{f(x)} = e^{\lim_{x \to 0} f(x)}$$

$$\lim_{x \to 0} x \log x = \lim_{x \to 0} \frac{\log x}{\frac{1}{x}} =$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} -x^2 \cdot \frac{1}{x} = -x \to 0$$

$$\lim_{x \to 0} x^x = 1$$

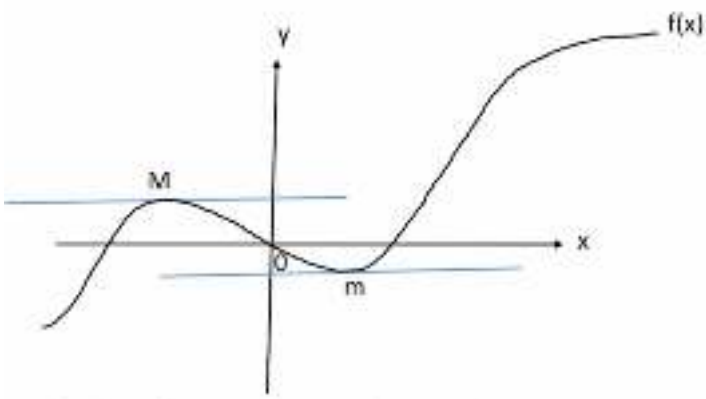
$$\lim_{n \to \infty} \frac{n^3}{e^n} \to \frac{x^3}{e^x}$$

$$\lim_{x \to \infty} \frac{x^3}{e^x} = \lim_{x \to \infty} \frac{3x^2}{e^x} =$$

$$= \lim_{x \to \infty} \frac{6x}{e^x} = \lim_{x \to \infty} \frac{6}{e^x} = 0$$

$$\boxed{\lim_{n \to \infty} \frac{n^3}{e^n} = 0}$$

Maxima and minima of function f(x)

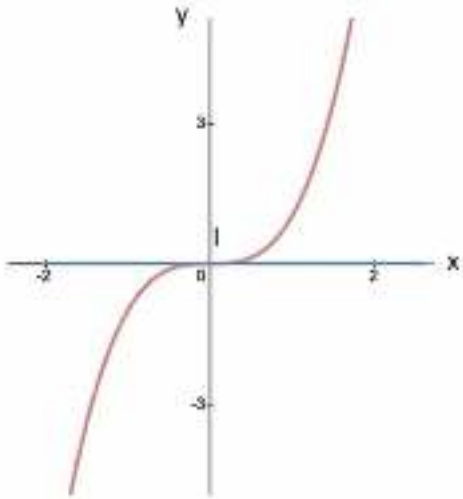


The slope of tangent line in M and m is zero (line is horizontal -> slope=tan(0)=0)

→ f'(x)=0 in points of local Max and min

- Because the derivative alone equaling zero not sufficient enough to determine the minima and maxima of the function

Inflection points



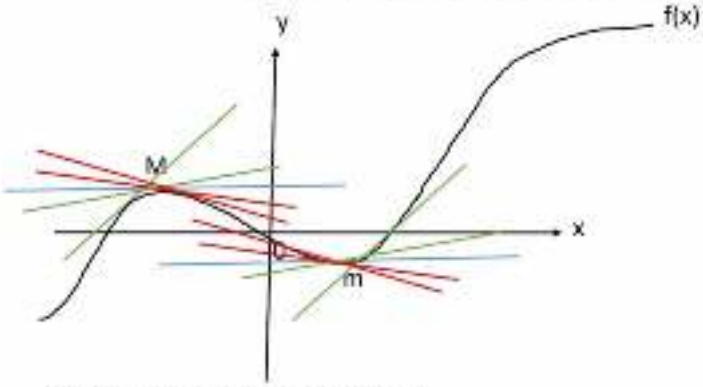
$y=f(x)=x^3$ $(d(x^n)/dx \rightarrow nx^{n-1})$
$f'(x)=3x^2$
$f'(x)=0$ $3x^2=0 \rightarrow x=0$
$f''(x)=6x \rightarrow f''(x=0)=0$

Inflection point
neither Max nor min
marks change of concavity

$f'(x)=0$ necessary but not sufficient condition
for x to be either local Max or min of the function

Points where $f''(x)=0$ are called inflection points
In which the concavity of the function changes

Maxima and minima of function f(x)



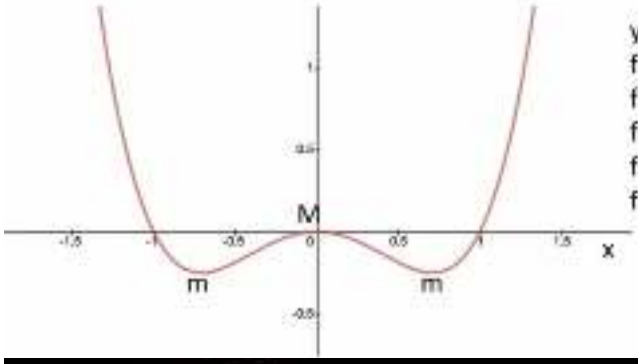
Close to M slope of tangent is positive
first (green) then negative (red)
→ the slope decreases

→ $f''(x)<0$ in a Maximum

In m slope of tangent is negative first
(red) then positive(green)
→ the slope increases

→ $f''(x)>0$ in a Minimum

Maxima and minima of function f(x)



$y=f(x)=x^4-x^2$

$f'(x)=4x^3-2x$

$f'(x)=2x(2x^2-1) \rightarrow x=0\ ;\ x=\pm 1/\sqrt{2}$

$f''(x)=12x^2-2$

$f''(x=0)=-2<0 \rightarrow$ Maximum

$f''(x=1/\sqrt{2})=f''(x=-1/\sqrt{2})=12/2-2=4>0$

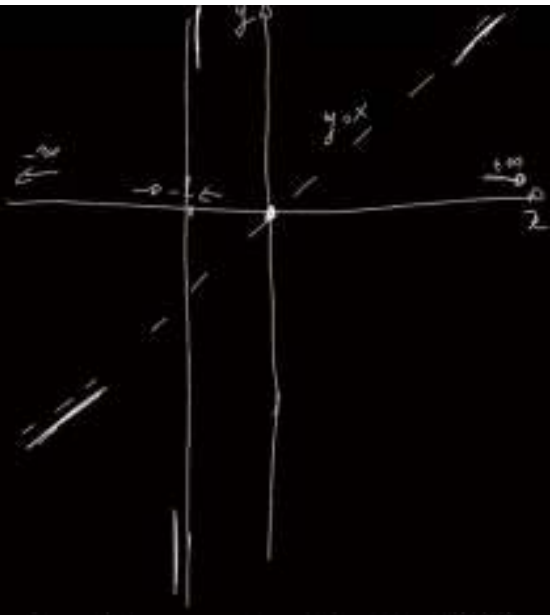
minima

$$f(x)=\frac{x^2}{x+1}$$

$$\begin{cases} \lim\limits_{x\to+\infty} \dfrac{x^2}{x+1} \approx \dfrac{x^2}{x}=x \\ \lim\limits_{x\to-\infty} \dfrac{x^2}{x+1} \approx \dfrac{x^2}{x}=x \end{cases}$$

$x=-1$ V. asym

$(x=0,\ y=0)$



$$f'(x)=\frac{d}{dx}\left(\frac{x^2}{x+1}\right)$$

$$f(x)=\frac{h(x)}{g(x)} \rightarrow f'(x)=\frac{h'(x)g(x)-h(x)g'(x)}{(g(x))^2}$$

$h(x)=x^2 \rightarrow h'(x)=2x$

$g(x)=x+1 \rightarrow g'(x)=1$

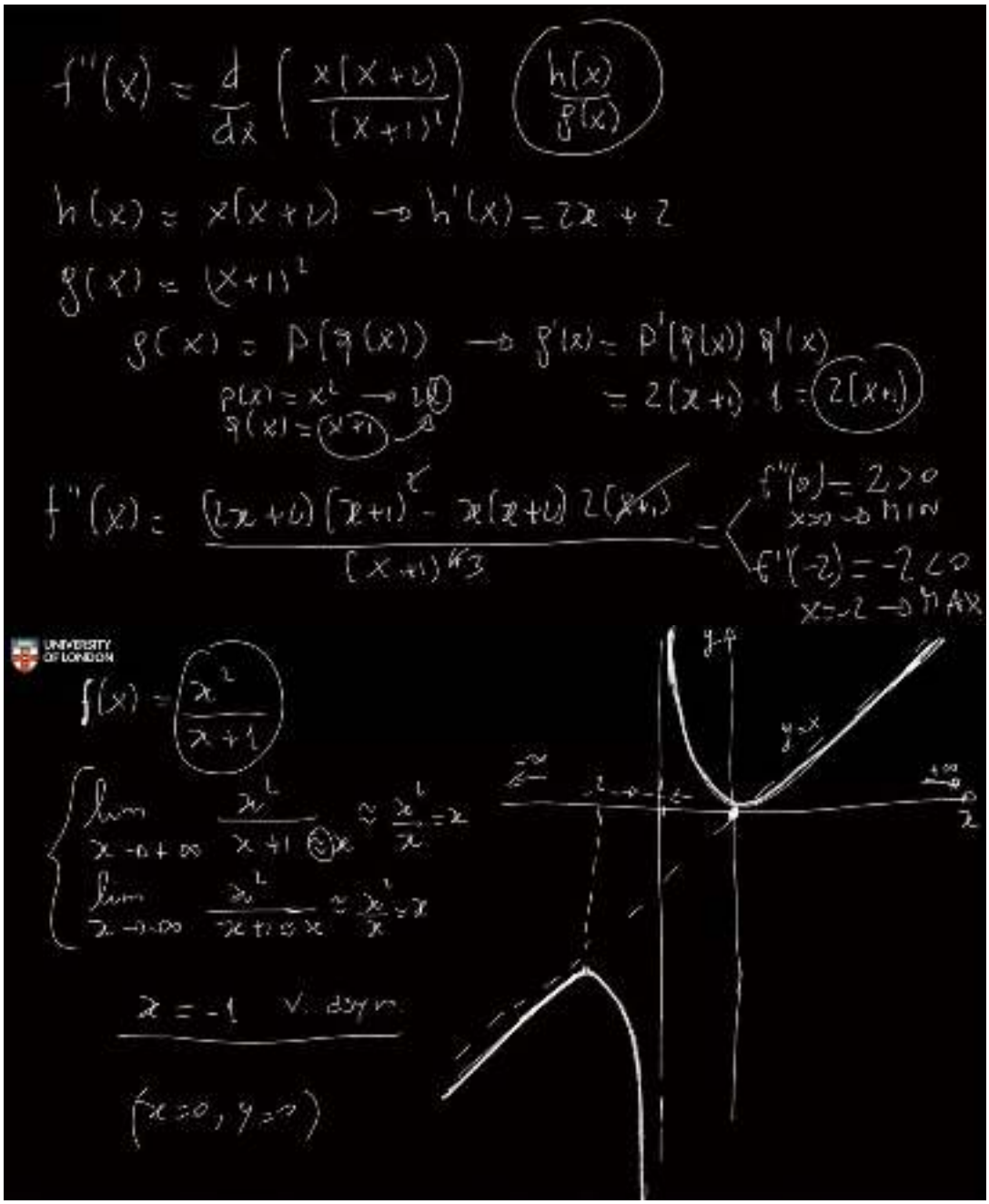
$$f'(x)=\frac{2x\cdot(x+1)-x^2\cdot 1}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2}=0\ ?$$

$x(x+2)=0$ — $x=0$, $x=-2$

MAX ?

MIN ?

INFLECTION ?



Summary

In this week, we learned about the properties of the derivative, L'Hopital's Rule, the relation of the minima/maxima of a function to its first and second derivative, and finally points of inflection.

Week 17 Algebra, Vectors, and Matrices Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-25 6:40 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices**	Course: BSc Computer Science
		Class: Computational Mathematics[Lecture]
		Date: July 25, 2020

Essential Question:

What are vectors and matrices?

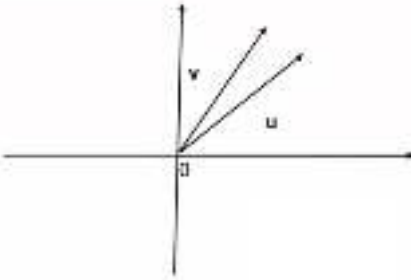
Questions/Cues:

- What is a vector space and field?
- What is scalar multiplication?
- What is the sum/difference of two vectors?
- How do we use the unit vectors to represent a general vector?
- What is the scalar product of two vectors?
- What is the cross product of two vectors?
- How do we perform the rotation of a vector?
- What is meant by a linear operation?
- What is a matrix?
- What are some operations which be performed on matrices?
- What is a translation?
- What are homogeneous coordinates?

Notes

Vector Space and Vectors

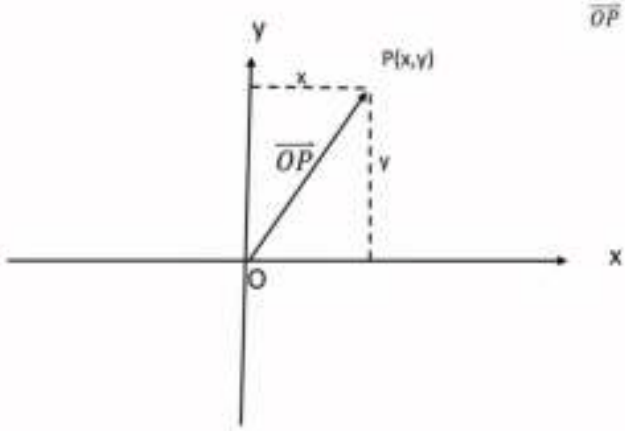
Given a 'Field' usually **R** the set of real numbers (but can be also **C**, complex numbers) a Vector Space on **R** is an abstract Space of objects v called vectors for which you can define a sum and a product with a 'scalar' (a number in **R**) , that fulfill the following properties:



Associativity	$u + (v + w) = (u + v) + w$
Commutativity	$u + v = v + u$
Identity or null vector	$v + 0 = v$
Inverse	$v + (-v) = 0$
Compatibility	$a(bv) = (ab)v$
Distributivity	$a(u + v) = au + av$
Distributivity	$(a + b)v = av + bv$

Vectors: examples

'Euclidean vectors'
Position or displacement vectors
$\overrightarrow{OP}$

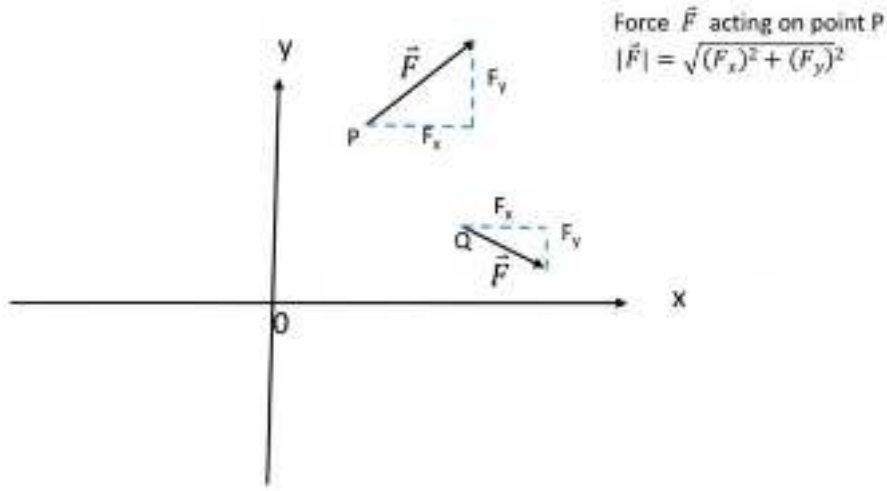


$$|\overrightarrow{OP}| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$x_0 = y_0 = 0$

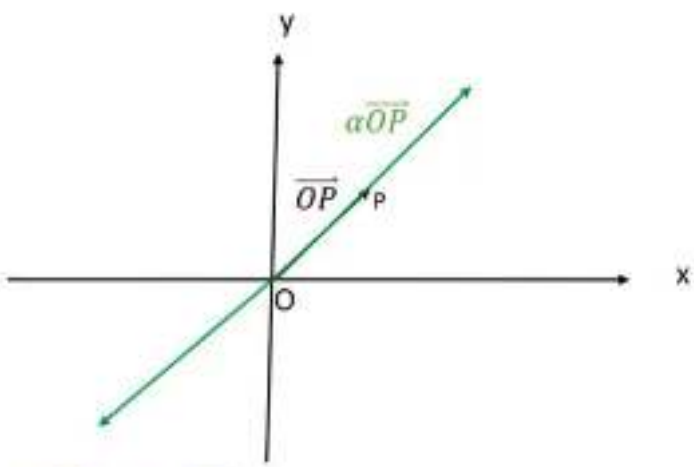
$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = (x,y) = [x,y]$

Vector Spaces: examples



Other examples:
Rgb vectors (1,227,1) for determining colour of pixel

Multiplication by a scalar

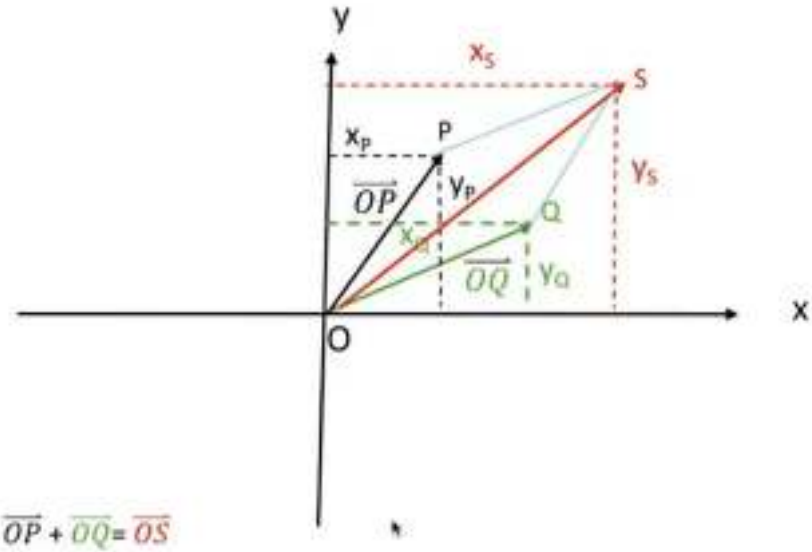


$|\alpha\,\overrightarrow{OP}| = |\alpha|\,|\overrightarrow{OP}|$

$$\begin{pmatrix} x_P \\ y_P \end{pmatrix} \rightarrow \begin{pmatrix} \alpha x_P \\ \alpha y_P \end{pmatrix}$$

- Multiplying by a negative constant changes the direction of the vector
- Scalar multiplication increases the magnitude of the vector by a factor by the absolute value of the constant

Sum of vectors



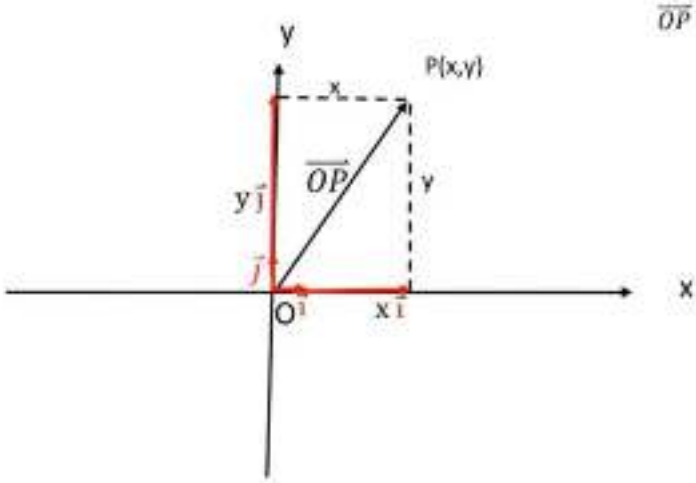
$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OS}$

$$\begin{pmatrix} x_P \\ y_P \end{pmatrix} + \begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = \begin{pmatrix} x_S = x_P + x_Q \\ y_S = y_P + y_Q \end{pmatrix}$$

Parallelogram rule

- The difference of two vector can be found by summing together **OP** and **OQ**, but with the x and y coordinates of **OQ** having a negative in front

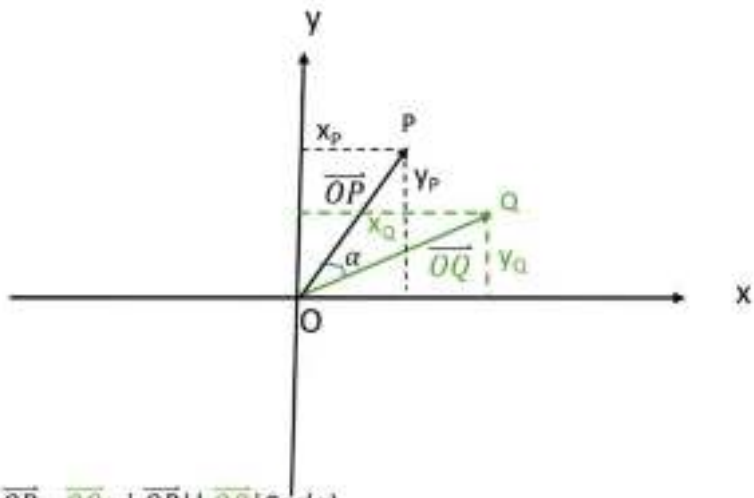
'Euclidean vectors'
Position or displacement vectors
$\overrightarrow{OP}$



$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = (x,y) = [x,y] = x\vec{i} + y\vec{j}$

- i is the unit vector (vector of magnitude 1) in the x-direction a vector of length unit you are using on the x-axis. j is the unit vector in y-direction

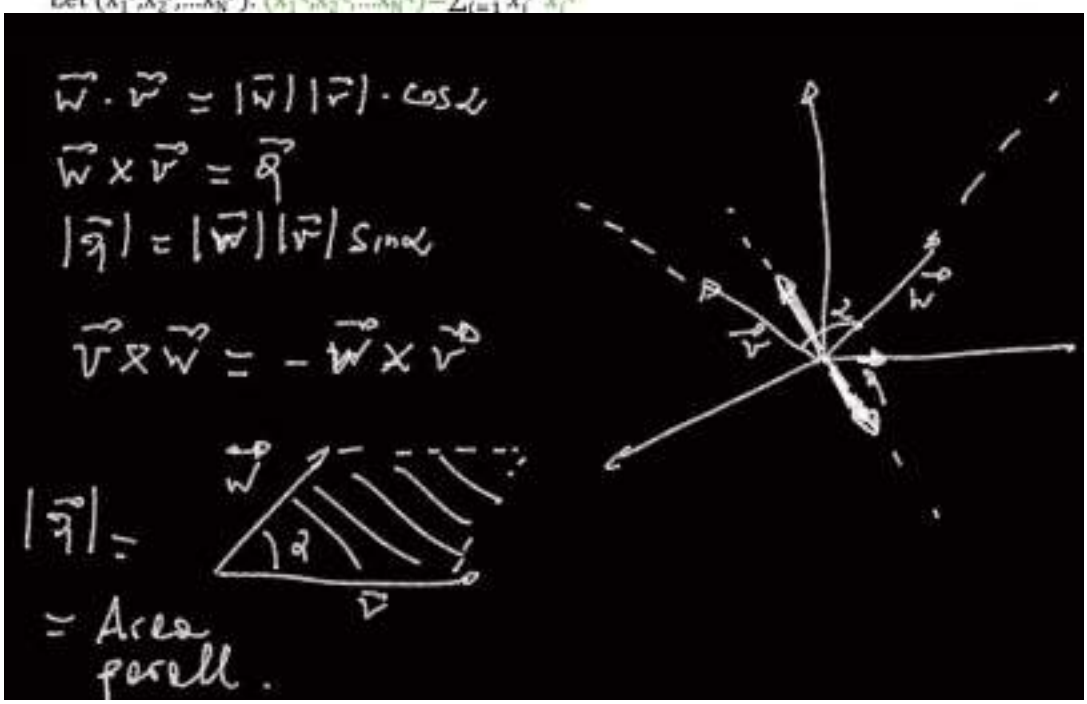
Scalar product of vectors



$\overrightarrow{OP} \cdot \overrightarrow{OQ} = |\overrightarrow{OP}||\overrightarrow{OQ}|\cos(\alpha)$

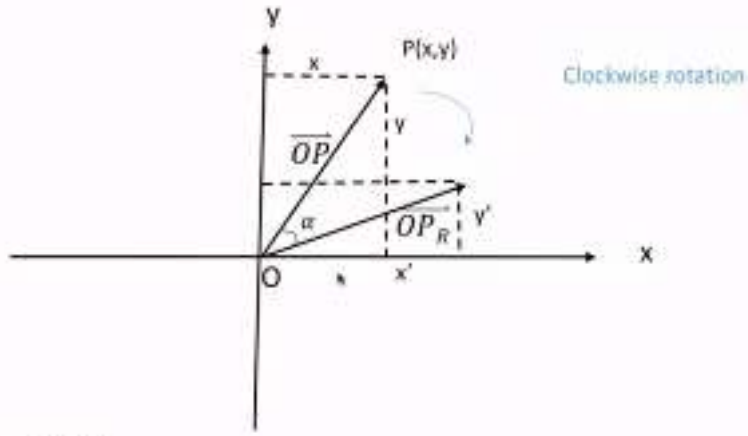
$\begin{pmatrix} x_P \\ y_P \end{pmatrix} \cdot \begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = x_P x_Q + y_P y_Q$

Let $(x_1^P, x_2^P, \ldots x_N^P)$. $(x_1^Q, x_2^Q, \ldots x_N^Q) = \sum_{i=1}^{N} x_i^P x_i^Q$



- Area of a parallelogram = b X h

Linear Transformations: Rotations of a vector



$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\overrightarrow{OP_R} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\,\mathrm{Cos}(\alpha) + y\,\mathrm{Sin}(\alpha) \\ -x\,\mathrm{Sin}(\alpha) + y\,\mathrm{Cos}(\alpha) \end{pmatrix}$

Prove:

$\sqrt{(x)^2 + (y)^2} = \sqrt{(x')^2 + (y')^2}$

Linear Transformations: Rotations of a vector

$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\overrightarrow{OP_R} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\mathrm{Cos}(\alpha) + y\mathrm{Sin}(\alpha) \\ -x\mathrm{Sin}(\alpha) + y\mathrm{Cos}(\alpha) \end{pmatrix}$

Linear Operator $\boldsymbol{M}$ s.t. $\overrightarrow{OP_R} = \boldsymbol{M}\,\overrightarrow{OP}$

Linear means if $\overrightarrow{OP} = a\,\overrightarrow{OQ} + b\,\overrightarrow{OR}$

$\rightarrow \boldsymbol{M}\,\overrightarrow{OP} = \boldsymbol{M}(a\,\overrightarrow{OQ} + b\,\overrightarrow{OR}) = a\,\boldsymbol{M}\,\overrightarrow{OQ} + b\,\boldsymbol{M}\,\overrightarrow{OR}$

$Q(x) \rightarrow (x^2)$ <u>not linear:</u>

if $x = (ax_1 + bx_2)$

$\rightarrow Q(x) \rightarrow (a\,x_1 + b\,x_2)^2 \neq a\,x^2_1 + b\,x^2_2 = aQ(x_1) + bQ(x_2)$

- A linear operation applied to a vector means that the components of the vector are recombined, which means they are either added or subtracted together or multiplied by a scalar (constant)

Linear Transformations: Matrices

As a vector can be represented as $\begin{pmatrix} x \\ y \end{pmatrix}$

a Linear Operator can be represented as a matrix

What is a matrix?

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \dots & n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \end{array}$$

Each element has two indeces a_{ij}

2x2 Matrix: $\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

2 rows and 2 columns

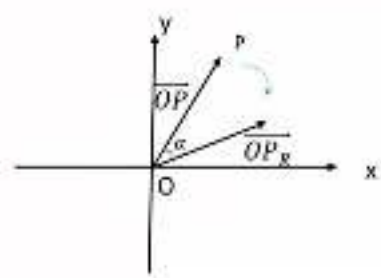
Applied to a column vector (row times column rule)

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m_{11}x + m_{12}y \\ m_{21}x + m_{22}y \end{pmatrix}$$

Linear Transformation Example: Rotation of a vector

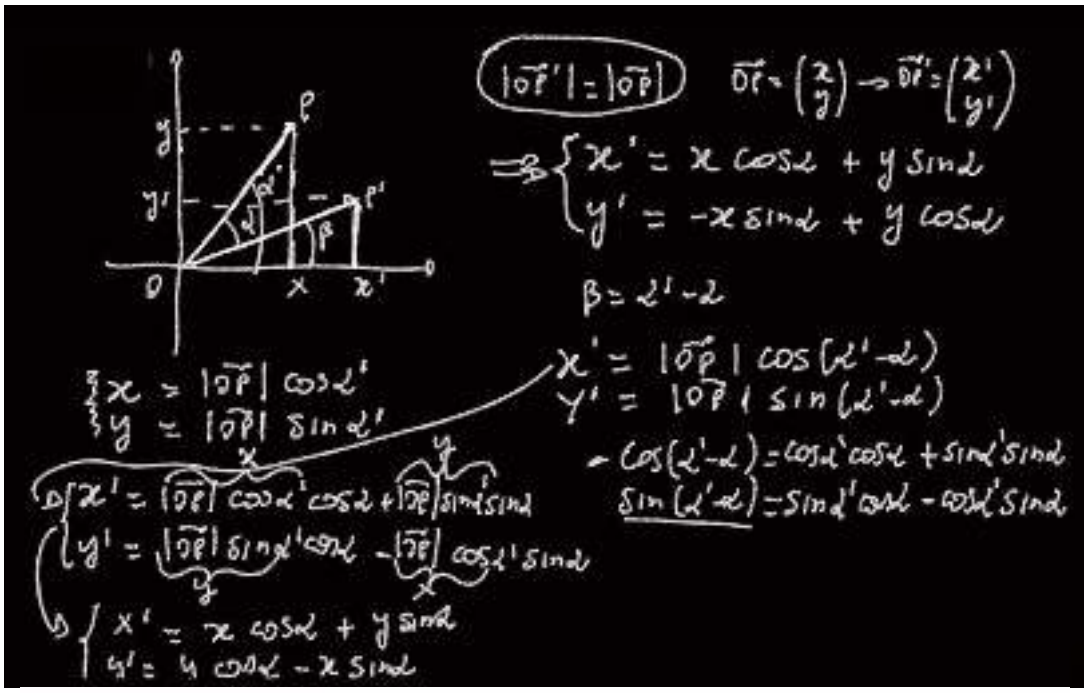
$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\overrightarrow{OP_R} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\mathrm{Cos}(\alpha) + y\mathrm{Sin}(\alpha) \\ -x\mathrm{Sin}(\alpha) + y\mathrm{Cos}(\alpha) \end{pmatrix}$



Matrix $M = \begin{pmatrix} \mathrm{Cos}(\alpha) & \mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\alpha) & \mathrm{Cos}(\alpha) \end{pmatrix}$ applied to a vector:

$$M\overrightarrow{OP} = \begin{pmatrix} \mathrm{Cos}(\alpha) & \mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\alpha) & \mathrm{Cos}(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\mathrm{Cos}(\alpha) + y\mathrm{Sin}(\alpha) \\ -x\mathrm{Sin}(\alpha) + y\mathrm{Cos}(\alpha) \end{pmatrix} = \overrightarrow{OP_R}$$



Operations with matrices

Moltiplication by a scalar

$$a\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} a\,m_{11} & a\,m_{12} \\ a\,m_{21} & a\,m_{22} \end{pmatrix}$$

Sum of two matrices

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} m_{11}+a_{11} & m_{12}+a_{12} \\ m_{21}+a_{21} & m_{22}+a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 8 & 0 \end{pmatrix}$$

Operations with matrices

Multiplication of two matrices

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1\times0+2\times3 & 1\times2+2\times0 \\ 5\times0+0\times3 & 5\times2+0\times0 \end{pmatrix}$$

$$=\begin{pmatrix} 6 & 2 \\ 0 & 10 \end{pmatrix}$$

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} m_{11}a_{11}+m_{12}a_{21} & m_{21}a_{11}+m_{22}a_{21} \\ m_{11}a_{12}+m_{12}a_{22} & m_{21}a_{12}+m_{22}a_{22} \end{pmatrix}$$

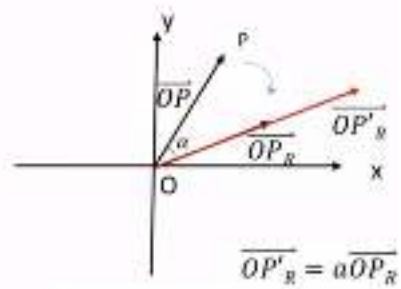
Examples: Multiplication by a scalar

We saw that $\overrightarrow{OP_R} = \boldsymbol{M}\,\overrightarrow{OP}$

$$\boldsymbol{M}=\begin{pmatrix} \mathrm{Cos}(\alpha) & \mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\alpha) & \mathrm{Cos}(\alpha) \end{pmatrix}$$

$$\boldsymbol{M'} = a\,\boldsymbol{M} = \begin{pmatrix} a\,\mathrm{Cos}(\alpha) & a\,\mathrm{Sin}(\alpha) \\ -a\,\mathrm{Sin}(\alpha) & a\,\mathrm{Cos}(\alpha) \end{pmatrix}$$

$$\overrightarrow{OP'_R} = \boldsymbol{M'}\,\overrightarrow{OP} = a\boldsymbol{M}\,\overrightarrow{OP} = a\overrightarrow{OP_R}$$



Examples: Product of matrices

We saw that $\overrightarrow{OP_R} = M\,\overrightarrow{OP}$

$$M = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

$$\overrightarrow{OP_{R'}} = M'\,\overrightarrow{OP_R} = M_T\overrightarrow{OP}$$

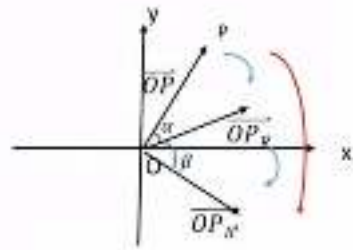
$$M' = \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{pmatrix}$$

$$\overrightarrow{OP_{R'}} = M'\,\overrightarrow{OP_R} = M'\,(M\,\overrightarrow{OP}) = (M'\cdot M)\,\overrightarrow{OP} = M_T\overrightarrow{OP}$$

$$M_T = M'\cdot M = \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{pmatrix}\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\beta)\cos(\alpha)-\sin(\beta)\sin(\alpha) & \cos(\beta)\sin(\alpha)+\sin(\beta)\sin(\alpha) \\ -\sin(\beta)\cos(\alpha)-\cos(\beta)\sin(\alpha) & \cos(\beta)\cos(\alpha)-\sin(\beta)\sin(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\beta+\alpha) & \sin(\beta+\alpha) \\ -\sin(\beta+\alpha) & \cos(\beta+\alpha) \end{pmatrix} = M'\cdot M$$



implements a rotation of angle $\alpha+\beta$

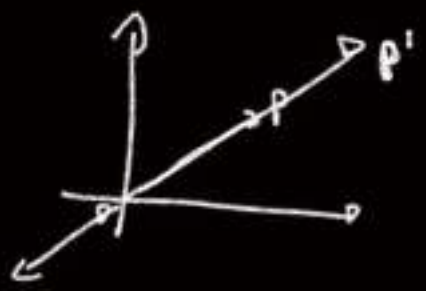
- The composition of two linear transformations is equal to the product of the matrices of the single two transformations. If you do the product of the matrices associated with two single transformations, the product of those two matrices implements the composition of those two transformations.

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{L} \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = L(\vec{v})$$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$L\begin{pmatrix} x \\ y \end{pmatrix}$$

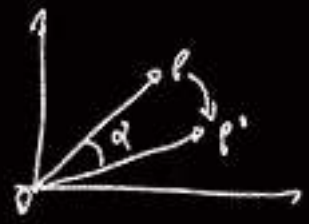
1) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$



$|\vec{OP'}| = |\alpha|\,|\vec{OP}|$

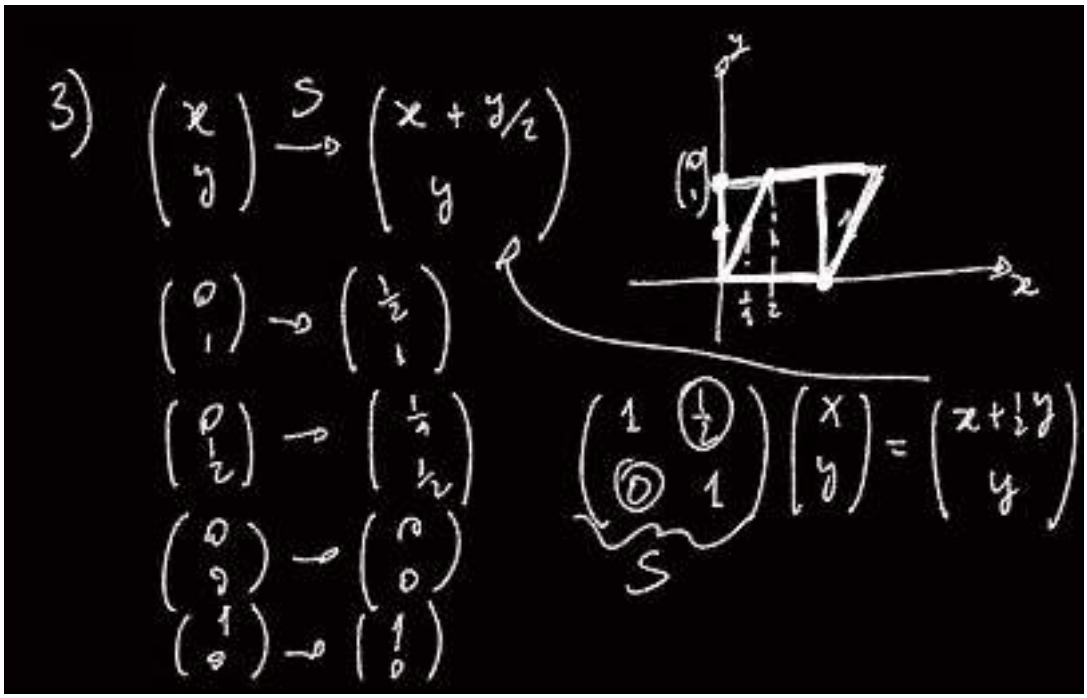
$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

2) $\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{R} \begin{pmatrix} x\cos\alpha + y\sin\alpha \\ -x\sin\alpha + y\cos\alpha \end{pmatrix}$

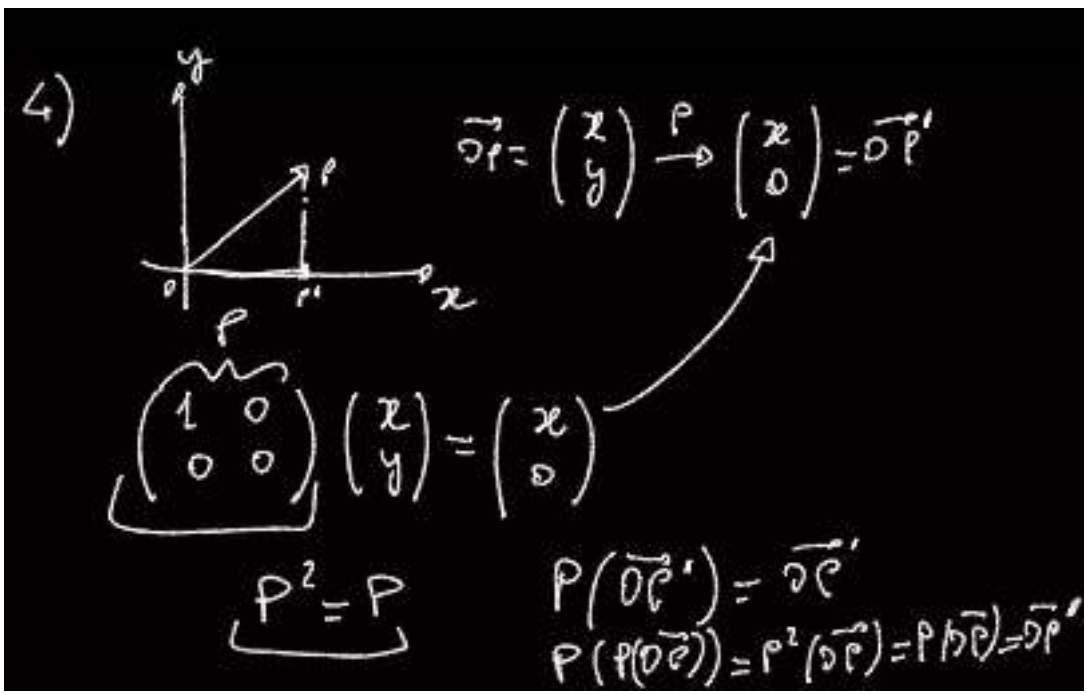


$|\vec{OP'}| = |\vec{OP}|$

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\alpha + y\sin\alpha \\ -x\sin\alpha + y\cos\alpha \end{pmatrix}$$



- A shear transformation fixes a direction, which in two-dimensions means just a straight line with direction fixed. It displaces all points of the vector along the direction by an amount proportional to the distance of that point from that direction, so from that straight line.
- x component is shifted by an amount proportional to y, which is the distance from the x-axis

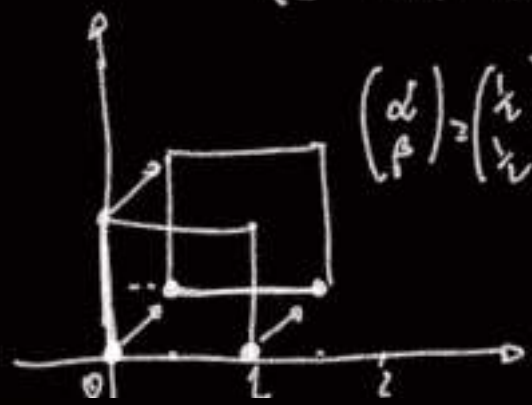


- For a projection you have choose a direction of reference, so in this case a straight line. It takes a vector and transforms it into another vector, which has a component, the same component of the initial vector along the direction of reference and has 0 as a component along the direction perpendicular to the direction reference

- Translation= a transformation that takes a vector of component xy into a vector in which the components of the original vector where a constant is added

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{T} \begin{pmatrix} x+\alpha \\ y+\beta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

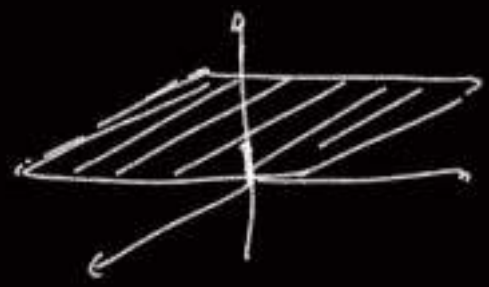


$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

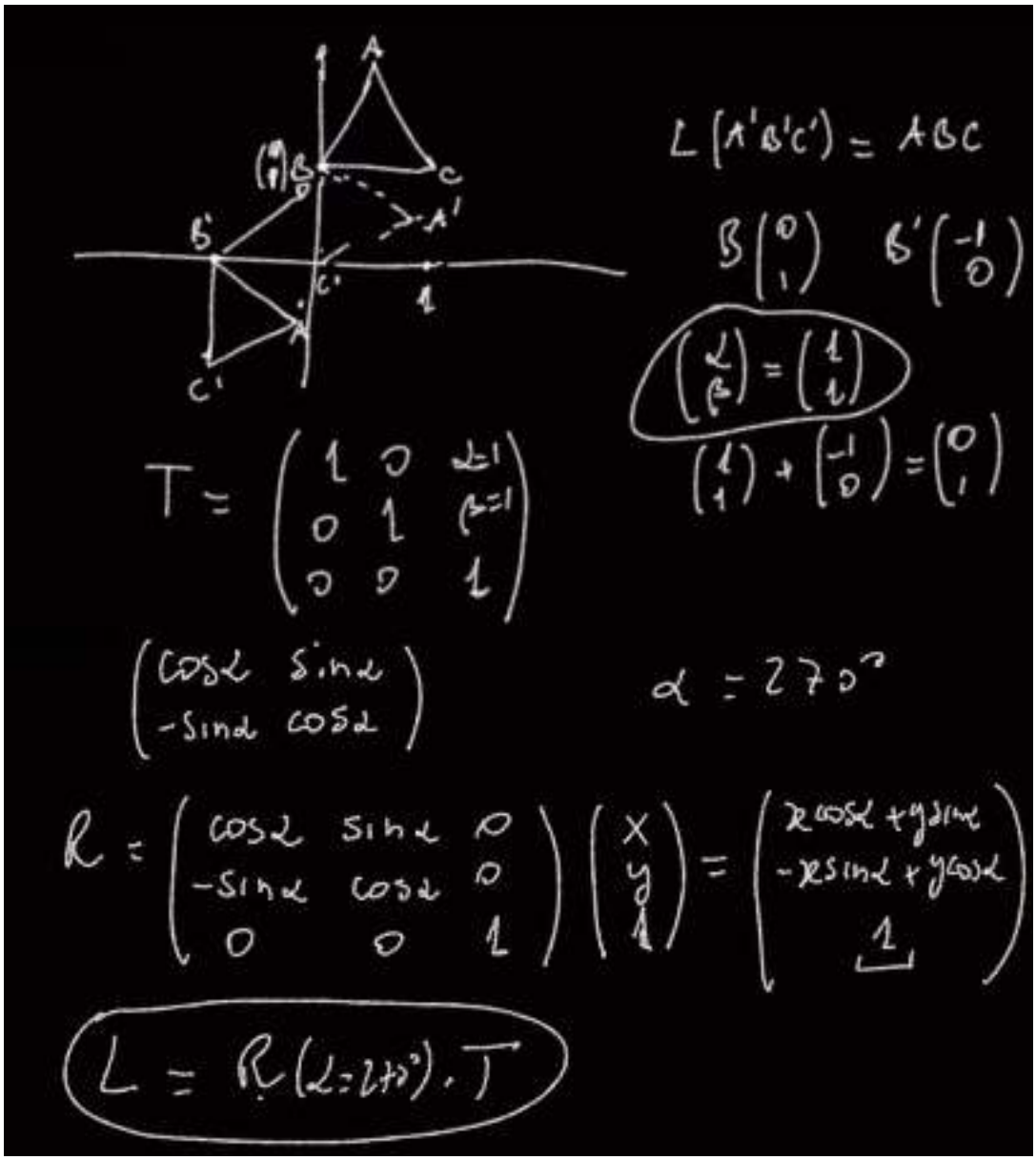
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \\ \boxed{1} \end{pmatrix}$$



$$T = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+\alpha \\ y+\beta \\ \underline{1} \end{pmatrix}$$



Summary

In this week, we learned about what a vector space and field are, what scalar multiplication is, how to calculate the sum/difference of two vectors, how to use unit vector notation to represent a general vector, what the scalar/cross of two vectors is, linear transformations (rotation, shears), what a matrix is, matrix operations, what a translation operation is and finally what homogeneous coordinates are.

Week 17 Algebra, Vectors, and Matrices Reading Note 1

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-24 6:48 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices**	Course: BSc Computer Science
		Class: Computational Mathematics[Reading]
		Date: July 24, 2020

Essential Question:

What are vectors and matrices?

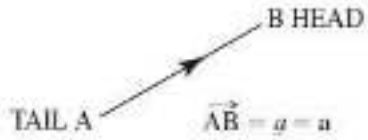
Questions/Cues:

- What is magnitude?
- What is a scalar?
- What is a vector?
- What is a unit vector?
- How do we multiply a vector by a scalar and what does it mean?
- How do we add two vectors together?
- How do we subtract two vectors?
- How do we use coordinates to describe vectors?
- What is the scalar product?

Notes

- Magnitude = size of a quantity, for example, 3 km or 10 amps
- Scalar = Quantities only described by their magnitude
    - Temperature, length, volume, and density are examples of scalars
- Vector = a quantity has both magnitude and direction, for example a force of 10 newtons acting vertically down
    - Often represented graphically by a straight line with an arrowhead to show the direction. Where the length of the line represents the magnitude and the direction of the line shows the direction of the vector

Figure 26.1
The line AB represents the vector a



- Vector from A to B can be written $\overrightarrow{AB}$, where B is called the head and A is called the tail of the vector
- The magnitude is written as $\left|\overrightarrow{AB}\right|$ or $|a|$
- Two vectors are equal if they have the same magnitude and same direction

- Unit vector = has a magnitude of 1, denoted by $\widehat{a}$

It is possible to multiply a vector, say **a**, by a positive number (scalar), say k. The result is a new vector whose direction is the same as **a** and whose magnitude is k times that of the original vector **a**. So, for example, if **a** has magnitude 4, then the vector 3**a** has the same direction as **a** and magnitude of 12. The vector $\frac{1}{2}$**a** has magnitude 2. Figure 26.2 illustrates this.

Figure 26.2
The vector 3**a** has the same direction as **a** and three times the magnitude. The vector $\frac{1}{2}$**a** has the same direction and half the magnitude

A vector may also be multiplied by a negative number. Again, if **a** has magnitude 4 then −3**a** has magnitude 12. The direction of −3**a** is opposite to that of **a**. Notice that the negative sign has the effect of reversing the direction. The vector −**a** has the same magnitude as that of **a** but the opposite direction. Figure 26.3 illustrates this.

Figure 26.3
The vector −3**a** has the opposite direction to **a** and three times the magnitude. The vector −**a** has the same magnitude as **a** and the opposite direction

Recall that the magnitude of **a** is denoted by |**a**|. Note that this is a scalar quantity. Then a unit vector in the direction of **a** is given by

$$\text{unit vector in direction of } \mathbf{a} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

For example, given **a** has magnitude 4, then a unit vector in the direction of **a** is $\frac{1}{4}$**a**. If **b** has magnitude $\frac{1}{2}$ then a unit vector in the direction of **b** is 2**b**. Figure 26.4 illustrates some unit vectors.

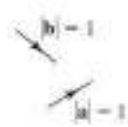


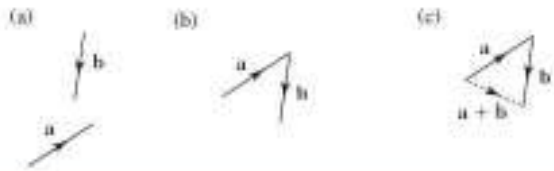
Figure 26.4
Unit vectors have a magnitude of 1

Adding vectors

If **a** and **b** are two vectors then we can form the vector **a** + **b**. We do this by moving **b**, still maintaining its length and direction, so that its tail coincides with the head of **a**. Figures 26.5(a) and (b) illustrate this. Then the vector **a** + **b** goes from the tail of **a** to the head of **b**. Figure 26.5 (c) illustrates this.

The method of adding vectors illustrated in Figure 26.5 is known as the **triangle law of addition**. The sum, **a** + **b**, is the same as **b** + **a**. The sum **a** + **b** is also known as the **resultant** of **a** and **b**.

Figure 26.5
The tail of **b** is moved to join the head of **a**

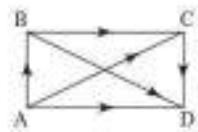


Key point The resultant of **a** and **b** is the sum **a** + **b**.

WORKED EXAMPLE

26.1 Figure 26.6 illustrates a set of vectors. Use the triangle law of addition to find the resultant of (a) $\overrightarrow{AB}$ and $\overrightarrow{BC}$, (b) $\overrightarrow{BC}$ and $\overrightarrow{CD}$, (c) $\overrightarrow{AB}$ and $\overrightarrow{BD}$, (d) $\overrightarrow{AC}$ and $\overrightarrow{CD}$.

Figure 26.6
Figure for Worked Example 26.1



Solution

(a) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

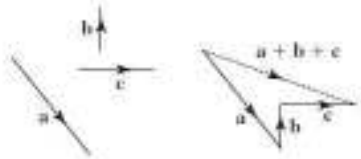
(b) $\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

(c) $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

(d) $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

The same principle used for adding two vectors can be applied to finding the sum of three or more vectors. Consider the vectors as shown in Figure 26.7.

Figure 26.7
Addition of three vectors



By positioning the tail of **b** at the head of **a** and the tail of **c** at the head of **b**, the sum **a** + **b** + **c** can be found. It is a vector from the tail of **a** to the head of **c** as illustrated.

Subtraction of vectors

We wish to find the difference of two vectors, that is **a** − **b**. We do this by calculating **a** + (− **b**); that is, we add the vectors **a** and −**b**. Recall that −**b** is a vector in the opposite direction to **b** and with the same magnitude.

WORKED EXAMPLE

26.2 The vectors **a** and **b** are shown in Figure 26.8. On a diagram show (a) **a** − **b**, (b) **b** − **a**.

Figure 26.8
The vectors **a** and **b** for Worked Example 26.2

Solution (a) We form the vector −**b**. This has the same magnitude (length) as **b** and the opposite direction. It is illustrated in Figure 26.9. This vector, −**b**, is now added to **a** by positioning the tail of −**b** at the head of **a** (see Figure 26.9). The sum **a** + (−**b**) is then determined using the triangle law of addition and is shown as a dotted line.

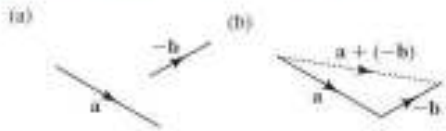


Figure 26.9
The vector **a** − **b** is found by adding **a** and (−**b**)

(b) To find **b** − **a** we calculate **b** + (−**a**). The vector −**a** has the same magnitude as **a** and the opposite direction. The tail of −**a** is positioned at the head of **b** and the resultant **b** + (−**a**) found using the triangle law of addition. Figure 26.10 illustrates this.

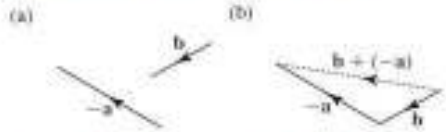


Figure 26.10
The vector **b** − **a** is found by adding **b** and (−**a**)

From Figures 26.8 and 26.9 we note that **b** − **a** = −(**a** − **b**).

We begin by introducing a **unit vector** in the x direction and a unit vector in the y direction. These are denoted by **i** and **j** respectively.

Key point The unit vectors in the x and y directions are **i** and **j** respectively.

Consider Figure 26.12 which shows the x–y plane, the point P with coordinates (2, 4) and the vector $\overrightarrow{OP}$.

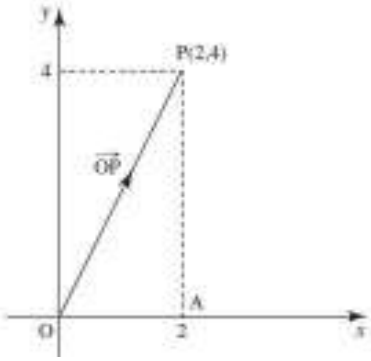


Figure 26.12
The vector $\overrightarrow{OP}$ is 2**i** + 4**j**

From the triangle law of addition,

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

Now the magnitude of $\overrightarrow{OA}$ is 2 and the direction of $\overrightarrow{OA}$ is the positive x direction, so

$$\overrightarrow{OA} = 2\mathbf{i}$$

Similarly $\overrightarrow{AP}$ is in the y direction and has magnitude 4. Hence

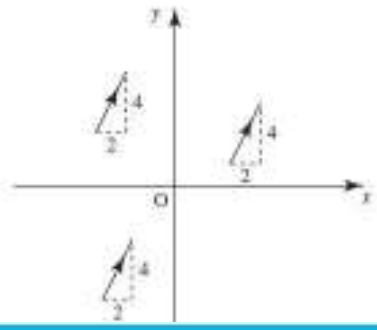
$$\overrightarrow{AP} = 4\mathbf{j}$$

Then finally

$$\overrightarrow{OP} = 2\mathbf{i} + 4\mathbf{j}$$

The quantities 2 and 4 are the **Cartesian components** of the vector $\overrightarrow{OP}$. Note that $2\mathbf{i} + 4\mathbf{j}$ is any vector comprising 2 units in the x direction and 4 units in the y direction. Figure 26.13 illustrates three such vectors.

Figure 26.13
Each vector is $2\mathbf{i} + 4\mathbf{j}$ regardless of position

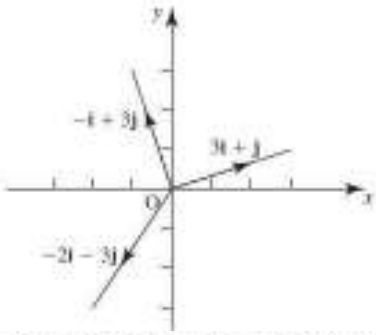


WORKED EXAMPLES

26.3 Sketch the vectors (a) $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, (b) $\mathbf{b} = -2\mathbf{i} - 3\mathbf{j}$, (c) $\mathbf{c} = -\mathbf{i} + 3\mathbf{j}$, positioning the tail of each vector at the origin.

Solution Figure 26.14 illustrates the required vectors.

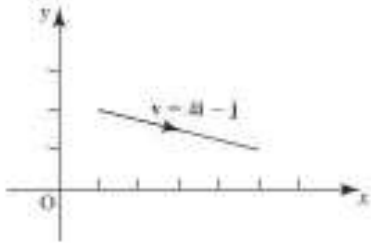
Figure 26.14
Vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -2\mathbf{i} - 3\mathbf{j}$, $\mathbf{c} = -\mathbf{i} + 3\mathbf{j}$



26.4 Sketch the vector $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$ with the point (1, 2) as the tail of the vector.

Solution Figure 26.15 illustrates the vector $\mathbf{v}$ with its tail at the point (1, 2).

Figure 26.15
The vector $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$ starts at the point (1, 2)



WORKED EXAMPLE

26.5 Given $\mathbf{a} = 7\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{c} = -3\mathbf{i} + \mathbf{j}$, find (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{b} + \mathbf{c} + \mathbf{a}$, (c) $\mathbf{a} - \mathbf{b}$, (d) $\mathbf{c} - \mathbf{a}$.

Solution We separately add or subtract the x and y components:

$$\begin{aligned}
\text{(a)}\quad \mathbf{a} + \mathbf{b} &= 7\mathbf{i} + 3\mathbf{j} + \mathbf{i} - 2\mathbf{j} \\
&= 8\mathbf{i} + 3\mathbf{j} - 2\mathbf{j} \quad \text{combining the i parts} \\
&= 8\mathbf{i} + \mathbf{j} \quad \text{combining the j parts} \\
\text{(b)}\quad \mathbf{b} + \mathbf{c} + \mathbf{a} &= \mathbf{i} - 2\mathbf{j} + (-3\mathbf{i} + \mathbf{j}) + 7\mathbf{i} + 3\mathbf{j} \\
&= 5\mathbf{i} + 2\mathbf{j} \\
\text{(c)}\quad \mathbf{a} - \mathbf{b} &= 7\mathbf{i} + 3\mathbf{j} - (\mathbf{i} - 2\mathbf{j}) \\
&= 7\mathbf{i} + 3\mathbf{j} - \mathbf{i} + 2\mathbf{j} \\
&= 6\mathbf{i} + 5\mathbf{j} \\
\text{(d)}\quad \mathbf{c} - \mathbf{a} &= -3\mathbf{i} + \mathbf{j} - (7\mathbf{i} + 3\mathbf{j}) \\
&= -3\mathbf{i} + \mathbf{j} - 7\mathbf{i} - 3\mathbf{j} \\
&= -10\mathbf{i} - 2\mathbf{j}
\end{aligned}$$

WORKED EXAMPLE

26.6 Given $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + \mathbf{j}$, find (a) $2\mathbf{v}$, (b) $-3\mathbf{w}$, (c) $4\mathbf{v} + 5\mathbf{w}$.

Solution

$$\begin{aligned}
\text{(a)}\quad 2\mathbf{v} &= 2(4\mathbf{i} + 3\mathbf{j}) \\
&= 8\mathbf{i} + 6\mathbf{j} \\
\text{(b)}\quad -3\mathbf{w} &= -3(-2\mathbf{i} + \mathbf{j}) \\
&= 6\mathbf{i} - 3\mathbf{j} \\
\text{(c)}\quad 4\mathbf{v} + 5\mathbf{w} &= 4(4\mathbf{i} + 3\mathbf{j}) + 5(-2\mathbf{i} - \mathbf{j}) \\
&= 16\mathbf{i} + 12\mathbf{j} - 10\mathbf{i} + 5\mathbf{j} \\
&= 6\mathbf{i} + 17\mathbf{j}
\end{aligned}$$

The magnitude of a vector was described as the 'length' of the vector. This idea can be made exact when using Cartesian coordinates. Figure 26.16 illustrates the vector $\mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

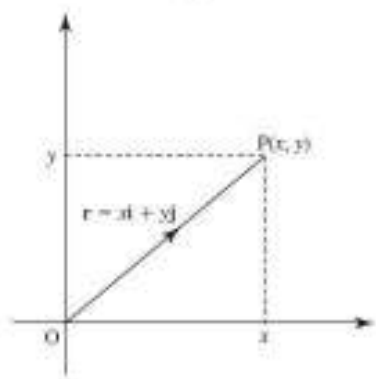


Figure 26.16
The vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

The length (magnitude) of the vector is OP, that is $|\mathbf{r}| = \text{OP}$. Using Pythagoras' theorem we have

$$\text{OP}^2 = x^2 + y^2$$

so

$$|\mathbf{r}| = \text{OP} = \sqrt{x^2 + y^2}$$

Key point If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, then $|\mathbf{r}| = \sqrt{x^2 + y^2}$

WORKED EXAMPLE

26.7 Given the point A has coordinates (2, 7) and B has coordinates (4, 11) find

(a) the vector $\overrightarrow{AB}$

(b) the vector $\overrightarrow{BA}$

(c) the magnitude of $|\overrightarrow{AB}|$

Solution Figure 26.17 illustrates the situation.

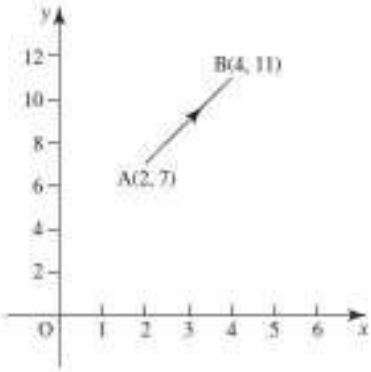


Figure 26.17
The vector $\overrightarrow{AB}$ for Worked Example 26.7

(a) Let $\overrightarrow{OA} = \mathbf{a} = 2\mathbf{i} + 7\mathbf{j}$ and $\overrightarrow{OB} = 4\mathbf{i} + 11\mathbf{j}$. From the triangle law of addition

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\mathbf{a} + \overrightarrow{AB} = \mathbf{b}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$= 4\mathbf{i} + 11\mathbf{j} - (2\mathbf{i} + 7\mathbf{j})$$

$$= 2\mathbf{i} + 4\mathbf{j}$$

(b) $\overrightarrow{BA} = -\overrightarrow{AB}$

$$= -(2\mathbf{i} + 4\mathbf{j})$$

$$= -2\mathbf{i} - 4\mathbf{j}$$

(c) $\overrightarrow{AB} = 2\mathbf{i} + 4\mathbf{j}$

So

$$|\overrightarrow{AB}| = \sqrt{2^2 + 4^2} = \sqrt{20}$$

Key point

Given two vectors, **a** and **b**, their scalar product, denoted by $\mathbf{a} \cdot \mathbf{b}$, is given by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between **a** and **b**.

This product has applications in engineering. The scalar product is also known as the **dot product**.

WORKED EXAMPLE

26.8 Show that (a) $\mathbf{i} \cdot \mathbf{i} = 1$, (b) $\mathbf{i} \cdot \mathbf{j} = 0$.

Solution Recall that $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions respectively.

(a) So

$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} &= |\mathbf{i}||\mathbf{i}| \cos 0^\circ \quad \text{since the angle between } \mathbf{i} \text{ and itself is } 0^\circ \\ &= (1)(1)(1) \\ &= 1 \end{aligned}$$

(b) $$\begin{aligned} \mathbf{i} \cdot \mathbf{j} &= |\mathbf{i}||\mathbf{j}| \cos 90^\circ \quad \text{since the angle between } \mathbf{i} \text{ and } \mathbf{j} \text{ is } 90^\circ \\ &= (1)(1)(0) \\ &= 0 \end{aligned}$$

Similarly it is easy to show that $\mathbf{j} \cdot \mathbf{j} = 1$, $\mathbf{j} \cdot \mathbf{i} = 0$.

Key point $\mathbf{i} \cdot \mathbf{i} = 1 \qquad \mathbf{j} \cdot \mathbf{j} = 1 \qquad \mathbf{i} \cdot \mathbf{j} = 0 \qquad \mathbf{j} \cdot \mathbf{i} = 0$

We use the above Key point to find the scalar product of two vectors from their Cartesian form. This does not require the value of the angle between the two vectors.

WORKED EXAMPLE

26.9 Given $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - \mathbf{j}$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (3\mathbf{i} + 2\mathbf{j}) \cdot (4\mathbf{i} - \mathbf{j}) \\ &= 3\mathbf{i} \cdot (4\mathbf{i} - \mathbf{j}) + 2\mathbf{j} \cdot (4\mathbf{i} - \mathbf{j}) \\ &= 12\mathbf{i} \cdot \mathbf{i} - 3\mathbf{i} \cdot \mathbf{j} + 8\mathbf{j} \cdot \mathbf{i} - 2\mathbf{j} \cdot \mathbf{j} \end{aligned}$$

Recalling $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$, $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$, we see that

$$\mathbf{a} \cdot \mathbf{b} = 12(1) - 3(0) + 8(0) - 2(1) = 10$$

We can use the technique of the above example to deduce the following general result:

Key point If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

That is, the $\mathbf{i}$ components are multiplied together, the $\mathbf{j}$ components are multiplied together and the sum of these products is the scalar product.

WORKED EXAMPLE

26.10 Given $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$

(a) Find the scalar product $\mathbf{a} \cdot \mathbf{b}$.

(b) Hence find the angle between $\mathbf{a}$ and $\mathbf{b}$.

Solution (a) We find the scalar product:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (4\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} + 3\mathbf{j}) \\ &= 4(2) + 1(3) \\ &= 11\end{aligned}$$

(b) The scalar product is also given by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Now $|\mathbf{a}| = \sqrt{4^2 + 1^2} = \sqrt{17}$, $|\mathbf{b}| = \sqrt{2^2 + 3^2} = \sqrt{13}$. We have found the value of $\mathbf{a} \cdot \mathbf{b}$ in part (a). Hence using the values for $\mathbf{a} \cdot \mathbf{b}$, $|\mathbf{a}|$ and $|\mathbf{b}|$ we have

$$11 = \sqrt{17}\sqrt{13}\cos\theta$$

$$\cos\theta = \frac{11}{\sqrt{17}\sqrt{13}} = 0.7399$$

Using the inverse cosine function of a calculator we find that $\theta = 42.3°$.

Summary

In this week, we learned about what magnitude is, what a scalar and a vector quantity are, what a unit vector is, what scalar multiplication is, vector addition/subtraction, how to use coordinates to describe vector and finally what the scalar product of two vectors is.

Week 17 Algebra, Vectors, and Matrices Reading Note 2

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-27 7:26 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices**	Course: BSc Computer Science
		Class: Computational Mathematics[Reading]
		Date: July 27, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- Are transformations commutative?
- What are the different notations in which transformations can be defined?
- What are the rules for matrix multiplication?
- What is the determinant of a matrix?
- What are homogeneous coordinates?
- What is the notation for a 2D translation and 2D scaling?
- What is a 2D reflection?
- What is a 2D shearing?
- What is 2D rotation?
- How is 2D scaling performed from a matrix perspective?
- How is 2D reflection performed from a matrix perspective?
- How is 2D rotation performed for an arbitrary point?

Notes

7.2 Matrices

Matrix notation was investigated by the British mathematician Arthur Cayley around 1858. Caley formalized matrix algebra, along with the American mathematicians Benjamin and Charles Pierce. Also, by the start of the 19th century Carl Gauss (1777–1855) had proved that transformations were not commutative, i.e. $T_1 \times T_2 \neq T_2 \times T_1$, and Caley's matrix notation would clarify such observations. For example, consider the transformation T_1:

$$\mathbf{T}_1 \quad \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{7.4}$$

and another transformation T_2 that transforms T_1:

$$\mathbf{T}_2 \times \mathbf{T}_1 \quad \begin{aligned} x'' &= Ax' + By' \\ y'' &= Cx' + Dy' \end{aligned} \tag{7.5}$$

If we substitute the full definition of T_1 we get

$$\mathbf{T}_2 \times \mathbf{T}_1 \quad \begin{aligned} x'' &= A(ax + by) + B(cx + dy) \\ y'' &= C(ax + by) + D(cx + dy) \end{aligned} \tag{7.6}$$

which simplifies to

$$\mathbf{T}_2 \times \mathbf{T}_1 \quad \begin{aligned} x'' &= (Aa + Bc)x + (Ab + Bd)y \\ y'' &= (Ca + Dc)x + (Cb + Dd)y \end{aligned} \tag{7.7}$$

Caley proposed separating the constants from the variables, as follows:

$$\mathbf{T}_1 \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.8}$$

where the square matrix of constants in the middle determines the transformation. The algebraic form is recreated by taking the top variable x', introducing the $=$ sign, and multiplying the top row of constants $[a\ b]$ individually by the last column vector containing x and y. We then examine the second variable y', introduce the $=$ sign, and multiply the bottom row of constants $[c\ d]$ individually by the last *column* vector containing x and y, to create

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{7.9}$$

Using Caley's notation, the product $\mathrm{T}_2 \times \mathrm{T}_1$ is

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{7.10}$$

But the notation also intimated that

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.11}$$

and when we multiply the two *inner matrices* together they must produce

$$\begin{aligned} x'' &= (Aa + Bc)x + (Ab + Bd)y \\ y'' &= (Ca + Dc)x + (Cb + Dd)y \end{aligned} \tag{7.12}$$

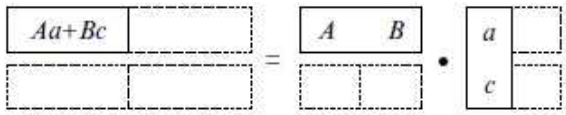
or in matrix form

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.13}$$

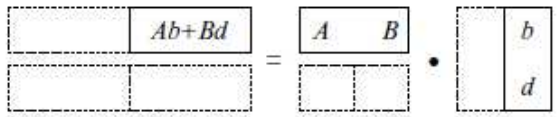
otherwise the two systems of notation will be inconsistent. This implies that

$$\begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{7.14}$$

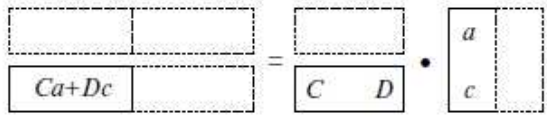
which demonstrates how matrices must be multiplied. Here are the rules for matrix multiplication:



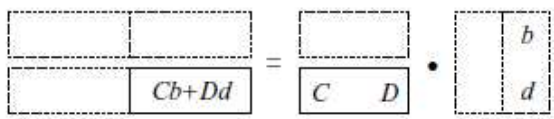
1 The top left-hand corner element $Aa + Bc$ is the product of the top row of the first matrix by the left column of the second matrix.



2 The top right-hand element $Ab + Bd$ is the product of the top row of the first matrix by the right column of the second matrix.



3 The bottom left-hand element $Ca + Dc$ is the product of the bottom row of the first matrix by the left column of the second matrix.



4 The bottom right-hand element $Cb + Dd$ is the product of the bottom row of the first matrix by the right column of the second matrix.

It is now a trivial exercise to confirm Gauss's observation that $\mathrm{T}_1 \times \mathrm{T}_2 \neq \mathrm{T}_2 \times \mathrm{T}_1$, because if we reverse the transforms $\mathrm{T}_2 \times \mathrm{T}_1$ to $\mathrm{T}_1 \times \mathrm{T}_2$ we get

$$\begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{7.15}$$

which shows conclusively that the product of two transforms is not commutative.

7.2.2 The Determinant of a Matrix

The *determinant* of a 2×2 matrix is a scalar quantity computed. Given a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

its determinant is $ad - cb$ and is represented by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{7.18}$$

For example, the determinant of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ is $3 \times 2 - 1 \times 2 = 4$

Basically, homogeneous coordinates define a point in a plane using three coordinates instead of two. Initially, Plücker located a homogeneous point relative to the sides of a triangle, but later revised his notation to the one employed in contemporary mathematics and computer graphics. This states that for a point P with coordinates (x, y) there exists a homogeneous point (x, y, t) such that $X = x/t$ and $Y = y/t$. For example, the point (3, 4) has homogeneous coordinates (6, 8, 2), because $3 = 6/2$ and $4 = 8/2$. But the homogeneous point (6, 8, 2) is not unique to (3, 4); (12, 16, 4), (15, 20, 5) and (300, 400, 100) are all possible homogeneous coordinates for (3, 4).

The reason why this coordinate system is called 'homogeneous' is because it is possible to transform functions such as $f(x, y)$ into the form $f(x/t, y/t)$ without disturbing the degree of the curve. To the non-mathematician this may not seem anything to get excited about, but in the field of projective geometry it is a very powerful concept.

For our purposes, we can imagine that a collection of homogeneous points of the form (x, y, t) exist on an xy-plane where t is the z-coordinate, as illustrated in Figure 7.4. The figure shows a triangle on the $t = 1$ plane, and a similar triangle, much larger, on a more distant plane. Thus instead of working in two dimensions, we can work on an arbitrary xy-plane in three dimensions. The t- or z-coordinate of the plane is immaterial because the x- and y-coordinates are eventually scaled by t. However, to keep things simple it seems a good idea to choose $t = 1$. This means that the point (x, y) has homogeneous coordinates $(x, y, 1)$, making scaling unnecessary.

If we substitute 3D homogeneous coordinates for traditional 2D Cartesian coordinates, we must attach a 1 to every (x, y) pair. When a point $(x, y, 1)$ is transformed, it will emerge as $(x', y', 1)$, and we discard the 1. This may seem a futile exercise, but it resolves the problem of creating a translation transformation.

Consider the following transformation on the homogeneous point $(x, y, 1)$:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.19}$$

This expands to

$$\begin{aligned} x' &= ax + by + c \\ y' &= dx + ey + f \\ 1 &= 1 \end{aligned} \tag{7.20}$$

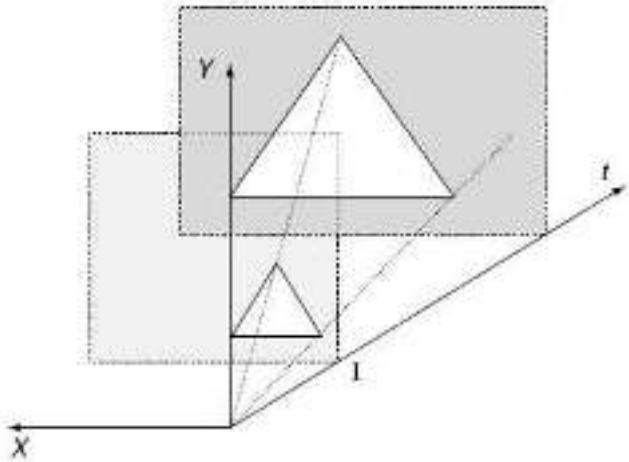


Fig. 7.4. 2D homogeneous coordinates can be visualized as a plane in 3D space, generally where $t = 1$, for convenience.

which solves the above problem of adding a constant.

7.3.1 2D Translation

The algebraic and matrix notation for 2D translation is

$$\begin{aligned} x' &= x + t_x \\ y' &= y + t_y \end{aligned} \tag{7.21}$$

or, using matrices,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.22}$$

7.3.2 2D Scaling

The algebraic and matrix notation for 2D scaling is

$$\begin{aligned} x' &= s_x x \\ y' &= s_y y \end{aligned} \tag{7.23}$$

or, using matrices,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.24}$$

The scaling action is relative to the origin, i.e. the point (0,0) remains (0,0) All other points move away from the origin. To scale relative to another point (p_x, p_y) we first subtract (p_x, p_y) from (x, y) respectively. This effectively translates the reference point (p_x, p_y) back to the origin. Second, we perform the scaling operation, and third, add (p_x, p_y) back to (x, y) respectively, to compensate for the original subtraction. Algebraically this is

$$\begin{aligned} x' &= s_x(x - p_x) + p_x \\ y' &= s_y(y - p_y) + p_y \end{aligned} \tag{7.25}$$

which simplifies to

$$\begin{aligned} x' &= s_x x + p_x(1 - s_x) \\ y' &= s_y y + p_y(1 - s_y) \end{aligned} \tag{7.26}$$

or in a homogeneous matrix form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & p_x(1 - s_x) \\ 0 & s_y & p_y(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.27}$$

For example, to scale a shape by 2 relative to the point (1, 1) the matrix is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

7.3.3 2D Reflections

The matrix notation for reflecting about the y-axis is:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.28}$$

or about the x-axis

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.29}$$

However, to make a reflection about an arbitrary vertical or horizontal axis we need to introduce some more algebraic deception. For example, to make a reflection about the vertical axis $x = 1$, we first subtract 1 from the x-coordinate. This effectively makes the $x = 1$ axis coincident with the major y-axis. Next we perform the reflection by reversing the sign of the modified

x-coordinate. And finally, we add 1 to the reflected coordinate to compensate for the original subtraction. Algebraically, the three steps are

$$\begin{aligned} x_1 &= x - 1 \\ x_2 &= -(x-1) \\ x' &= -(x-1) + 1 \end{aligned}$$

which simplifies to

$$\begin{aligned} x' &= -x + 2 \\ y' &= y \end{aligned} \tag{7.30}$$

or in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.31}$$

Figure 7.5 illustrates this process.

In general, to reflect a shape about an arbitrary y-axis, $y = a_x$, the following transform is required:

$$\begin{aligned} x' &= -(x - a_x) + a_x \quad = -x + 2a_x \\ y' &= y \end{aligned} \tag{7.32}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.33}$$

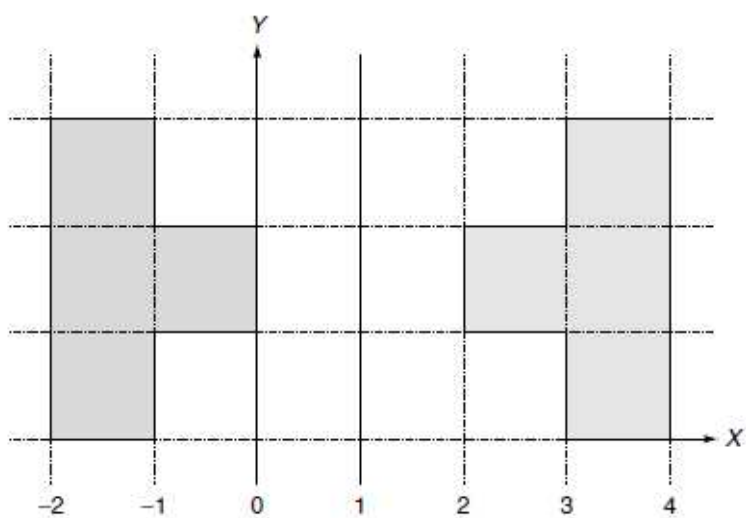


Fig. 7.5. The shape on the right is reflected about the $x = 1$ axis.

Similarly, this transform is used for reflections about an arbitrary x-axis, $y = a_y$:

$$\begin{aligned} x' &= x \\ y' &= -(y - a_y) + a_y = -y + 2a_y \end{aligned} \tag{7.34}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2a_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.35}$$

7.3.4 2D Shearing

A shape is sheared by leaning it over at an angle β. Figure 7.6 illustrates the geometry, and we see that the y-coordinate remains unchanged but the x-coordinate is a function of y and $\tan(\beta)$.

$$\begin{aligned} x' &= x + y\tan(\beta) \\ y' &= y \end{aligned} \tag{7.36}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan(\beta) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.37}$$

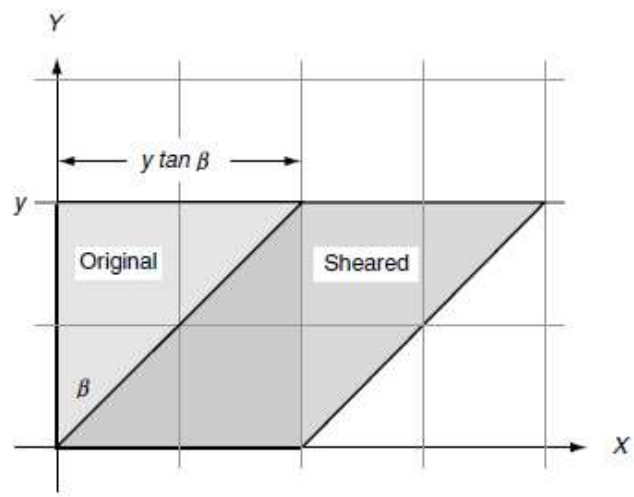


Fig. 7.6. The original square shape is sheared to the right by an angle β, and the horizontal shift is proportional to $ytan(\beta)$.

7.3.5 2D Rotation

Figure 7.7 shows a point $P(x, y)$ which is to be rotated by an angle β about the origin to $P'(x', y')$. It can be seen that

$$\begin{aligned} x' &= R\cos(\theta + \beta) \\ y' &= R\sin(\theta + \beta) \end{aligned} \tag{7.38}$$

therefore

$$\begin{aligned} x' &= R(\cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta)) \\ y' &= R(\sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta)) \\ x' &= R\left(\frac{x}{R}\cos(\beta) - \frac{y}{R}\sin(\beta)\right) \\ y' &= R\left(\frac{y}{R}\cos(\beta) + \frac{x}{R}\sin(\beta)\right) \\ x' &= x\cos(\beta) - y\sin(\beta) \\ y' &= x\sin(\beta) + y\cos(\beta) \end{aligned} \tag{7.39}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.40}$$

For example, to rotate a point by 90° the matrix becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

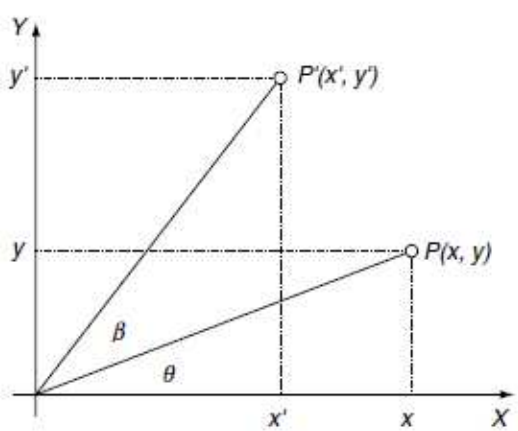


Fig. 7.7. The point $P(x, y)$ is rotated through an angle β to $P'(x', y')$.

Thus the point (1, 0) becomes (0, 1). If we rotate by 360° the matrix becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Such a matrix has a null effect and is called an *identity matrix*.

To rotate a point (x, y) about an arbitrary point (p_x, p_y) we first subtract (p_x, p_y) from the coordinates (x, y) respectively. This enables us to perform the rotation about the origin. Second, we perform the rotation, and third, we add (p_x, p_y) to compensate for the original subtraction. Here are the steps:

1 Subtract (p_x, p_y):

$$\begin{aligned} x_1 &= (x - p_x) \\ y_1 &= (y - p_y) \end{aligned}$$

2 Rotate β about the origin:

$$\begin{aligned} x_2 &= (x - p_x)\cos(\beta) - (y - p_y)\sin(\beta) \\ y_2 &= (x - p_x)\sin(\beta) + (y - p_y)\cos(\beta) \end{aligned}$$

3 Add (p_x, p_y):

$$\begin{aligned} x' &= (x - p_x)\cos(\beta) - (y - p_y)\sin(\beta) + p_x \\ y' &= (x - p_x)\sin(\beta) + (y - p_y)\cos(\beta) + p_y \end{aligned}$$

Simplifying,

$$\begin{aligned} x' &= x\cos(\beta) - y\sin(\beta) + p_x(1 - \cos(\beta)) + p_y\sin(\beta) \\ y' &= x\sin(\beta) + y\cos(\beta) + p_y(1 - \cos(\beta)) - p_x\sin(\beta) \end{aligned} \tag{7.41}$$

and, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & p_x(1 - \cos(\beta)) + p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & p_y(1 - \cos(\beta)) - p_x\sin(\beta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.42}$$

If we now consider rotating a point 90° about the point (1, 1) the matrix operation becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

A simple test is to substitute the point (2, 1) for (x, y): it is transformed correctly to (1, 2).

7.3.6 2D Scaling

The strategy we used to scale a point (x, y) relative to some arbitrary point (p_x, p_y) was to first, translate $(-p_x, -p_y)$; second, perform the scaling; and third, translate (p_x, p_y). These three transforms can be represented in matrix form as follows:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [\text{translate}(p_x, p_y)] \cdot [\text{scale}(s_x, s_y)] \cdot [\text{translate}(-p_x, -p_y)] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

which expands to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad (7.43)$$

Note the sequence of the transforms, as this often causes confusion. The first transform acting on the point $(x, y, 1)$ is translate $(-p_x, -p_y)$, followed by scale (s_x, s_y), followed by translate (p_x, p_y). If they are placed in any other sequence, you will discover, like Gauss, that transforms are not commutative!

We can now concatenate these matrices into a single matrix by multiplying them together. This can be done in any sequence, so long as we preserve the original order. Let's start with scale (s_x, s_y) and translate $(-p_x, -p_y)$. This produces

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & -s_x p_x \\ 0 & s_y & -s_y p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and finally

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & p_x(1 - s_x) \\ 0 & s_y & p_y(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad (7.44)$$

which is the same as the previous transform (7.27).

7.3.7 2D Reflections

A reflection about the y-axis is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.45}$$

Therefore, using matrices, we can reason that a reflection transform about an arbitrary axis $x = a_x$, parallel with the y-axis, is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [\text{translate}(a_x, 0)] \cdot [\text{reflection}] \cdot [\text{translate}(-a_x, 0)] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

which expands to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can now concatenate these matrices into a single matrix by multiplying them together. Let's begin by multiplying the reflection and the translate $(-a_x, 0)$ matrices together. This produces

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and finally

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.46}$$

which is the same as the previous transform (7.33).

7.3.8 2D Rotation about an Arbitrary Point

A rotation about the origin is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.47}$$

Therefore, using matrices, we can develop a rotation about an arbitrary point (p_x, p_y) as follows:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [\text{translate}(p_x,\ p_y)] \cdot [\text{rotate } \beta] \cdot [\text{translate}(-p_x, -p_y)] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

which expands to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can now concatenate these matrices into a single matrix by multiplying them together. Let's begin by multiplying the rotate β and the translate $(-p_x, -p_y)$ matrices together. This produces

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) & -p_x\cos(\beta) + p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & -p_x\sin(\beta) - p_y\cos(\beta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and finally

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & p_x(1-\cos(\beta)) + p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & p_y(1-\cos(\beta)) - p_x\sin(\beta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad (7.48)$$

which is the same as the previous transform (7.42).

I hope it is now is obvious to the reader that one can derive all sorts of transforms either algebraically, or by using matrices – it is just a question of convenience.

Summary

In this week, we learned about the rules of matrix multiplication, what linear transformations (rotation, shears, scaling, reflection) are in 2 dimensions, what a matrix is, what a translation operation is in 2 dimensions,and finally what homogeneous coordinates are.

Week 18 Algebra, Vectors, and Matrices continued Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-29 6:39 PM
Author:	SUKHJIT MANN		
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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices continued**	Course: BSc Computer Science
		Class: Computational Mathematics[Lecture]
		Date: July 29, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- What is the determinant of a 2 x 2 matrix?
- What is the determinant of a 3 x 3 matrix?
- What is the method to calculate the determinant of a matrix greater than 2 x 2 in general?
- What is the identity transformation and the identity matrix?
- What is the inverse transformation and the inverse matrix?
- How do we use matrices to solve a system of linear equations?
- What is Gauss-Jordan Elimination?

Notes

$$M=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

det M= $m_{11}m_{22}-m_{12}m_{21}$

Example:

$$M=\begin{pmatrix} 1 & 4 \\ 8 & 0 \end{pmatrix}$$

det M=0-32=-32

Three by three matrix

$$M=\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{matrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{matrix}$$

$$\det M = m_{11}m_{22}m_{33}+m_{12}m_{23}m_{31}+m_{13}m_{21}m_{32}$$
$$-m_{13}m_{22}m_{31}-m_{11}m_{23}m_{32}-m_{12}m_{21}m_{33}$$

Example

$$M=\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \begin{matrix} 1 & 2 \\ 1 & 2 \\ 0 & 1 \end{matrix} \qquad \det M=2+0+0-0-3-2=-3$$

Three by three (and beyond) matrix, alternative method

$$M=\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1^+ & 2^- & 0^+ \\ 1^- & 2^+ & 3^- \\ 0^+ & 1^- & 0^+ \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1^+ & 2^- & 0^+ \\ 1^- & 2^+ & 3^- \\ 0^+ & 1^- & 0^+ \end{pmatrix} = +\,0 \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 0$$

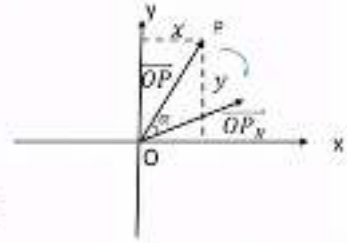
$$\Rightarrow \begin{pmatrix} 1^+ & 2^- & 0^+ \\ 1^- & 2^+ & 3^- \\ 0^+ & 1^- & 0^+ \end{pmatrix} = -\,3 \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = -3$$

$$\Rightarrow \begin{pmatrix} 1^+ & 2^- & 0^+ \\ 1^- & 2^+ & 3^- \\ 0^+ & 1^- & 0^+ \end{pmatrix} = +\,0 \det \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 0$$

$$\Rightarrow \qquad \det M = 0 - 3 + 0 = -3$$

Identity Transformation and Identity Matrix

$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad M = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \Rightarrow \overrightarrow{OP_R} = M\,\overrightarrow{OP}$



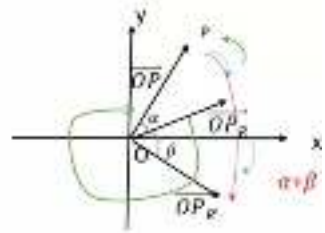
$M\,(\alpha=0) = \begin{pmatrix} \cos(0) & \sin(0) \\ -\sin(0) & \cos(0) \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$\overrightarrow{OP_R} = I\,\overrightarrow{OP} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \overrightarrow{OP}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{n\times n} \begin{pmatrix} 1 & 0 \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & 0 \cdots & 1 \end{pmatrix}$$

Inverse Transformation and Inverse Matrix

We saw that $\overrightarrow{OP_R} = M\,\overrightarrow{OP}$ $\quad M = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$



$\overrightarrow{OP_{R'}} = M'\,\overrightarrow{OP_R}$ $\quad$ with $M' = \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{pmatrix}$

$\overrightarrow{OP_{R'}} = (M' \cdot M)\,\overrightarrow{OP} = M_T\,\overrightarrow{OP}$ $\quad M_T = (M' \cdot M) = \begin{pmatrix} \cos(\beta+\alpha) & \sin(\beta+\alpha) \\ -\sin(\beta+\alpha) & \cos(\beta+\alpha) \end{pmatrix}$

Let $\beta = -\alpha$ or (equivalently $\beta = 360^\circ - \alpha$) $M_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\Rightarrow \quad \overrightarrow{OP_{R'}} = M_T\,\overrightarrow{OP} = (M' \cdot M)\,\overrightarrow{OP} = I\,\overrightarrow{OP} = \overrightarrow{OP}$

If M' exists such that M' ·M=M ·M'=I
M is invertible and M' is the <u>inverse</u> of M
(M' → M⁻¹)

M is <u>invertible if and only if detM≠0</u> $\quad$ det $M^{-1} = 1/\det M$

System of Linear Equations

$$\begin{matrix} x+2y=1 \\ x-3y=-2 \end{matrix} \Longleftrightarrow \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x+2y \\ x-3y \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\boldsymbol{M}=\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \Rightarrow \boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- <u>$\boldsymbol{M}$ is invertible if and only if $\det\boldsymbol{M}\neq 0$</u>
  if $\det \boldsymbol{M}=0 \rightarrow$ $\boldsymbol{M}$ is not invertible
- <u>A linear system of equations has solutions if and only if $\det\boldsymbol{M}\neq 0$</u>

Let $\boldsymbol{M}^{-1}$ be the inverse of M

$$\boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \boldsymbol{M}^{-1}\boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}=\boldsymbol{M}^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

but $\boldsymbol{M}^{-1}\boldsymbol{M}=\boldsymbol{I}$ $\Rightarrow$ $\boldsymbol{I}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\boldsymbol{M}^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Solving the system: Inverse Matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ x-3y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{M}^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

How to find the inverse

If $\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \Rightarrow \boldsymbol{M}^{-1} = \frac{1}{detM}\begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$

if $\boldsymbol{M} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$ $\mathrm{detM} = -5 \Rightarrow \boldsymbol{M}^{-1} = -\frac{1}{5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{M}^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} - \frac{4}{5} \\ \frac{1}{5} + \frac{2}{5} \end{pmatrix} = \begin{pmatrix} -1/5 \\ 3/5 \end{pmatrix}$$

x=-1/5 y=3/5

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det A = -1 \neq 0$$

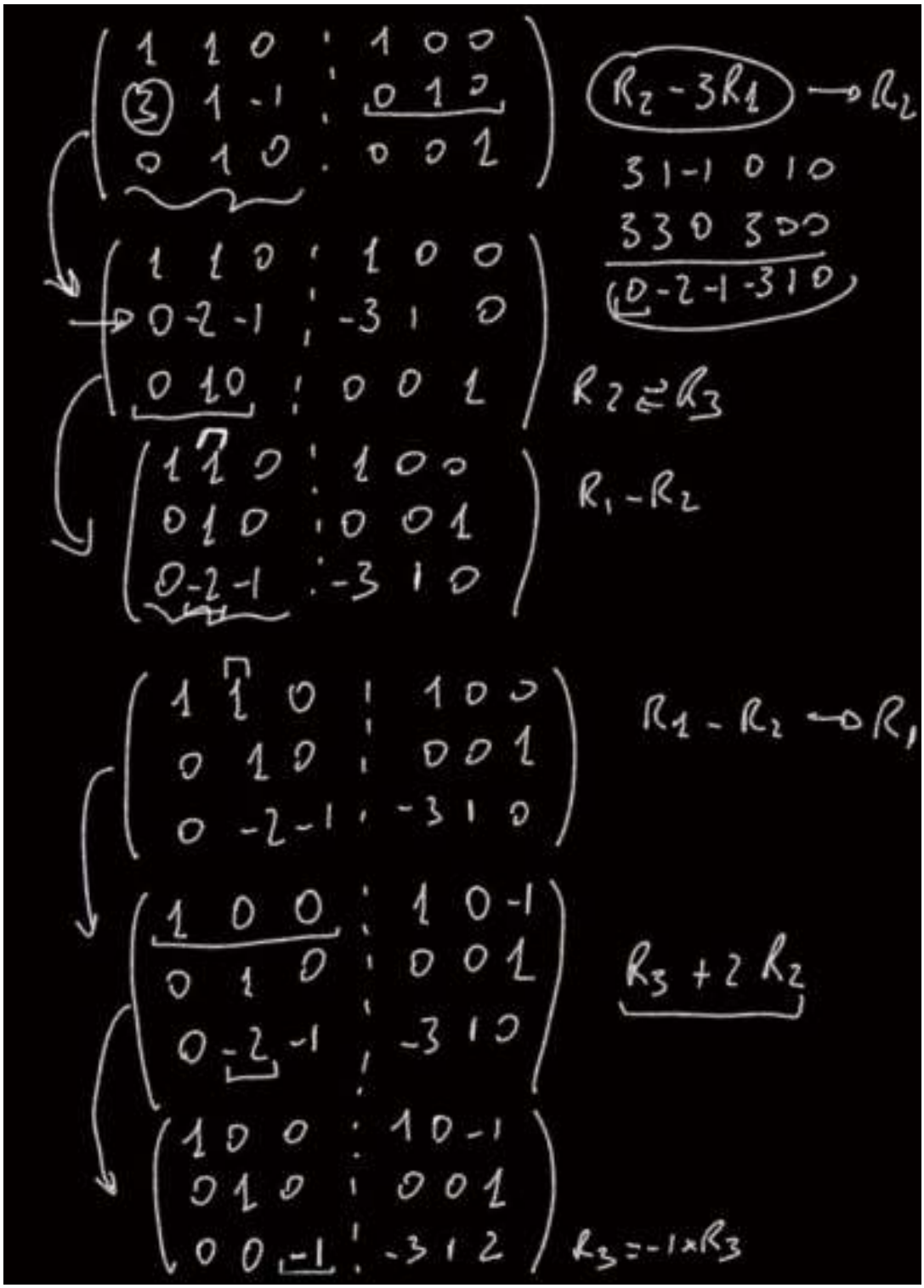
$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

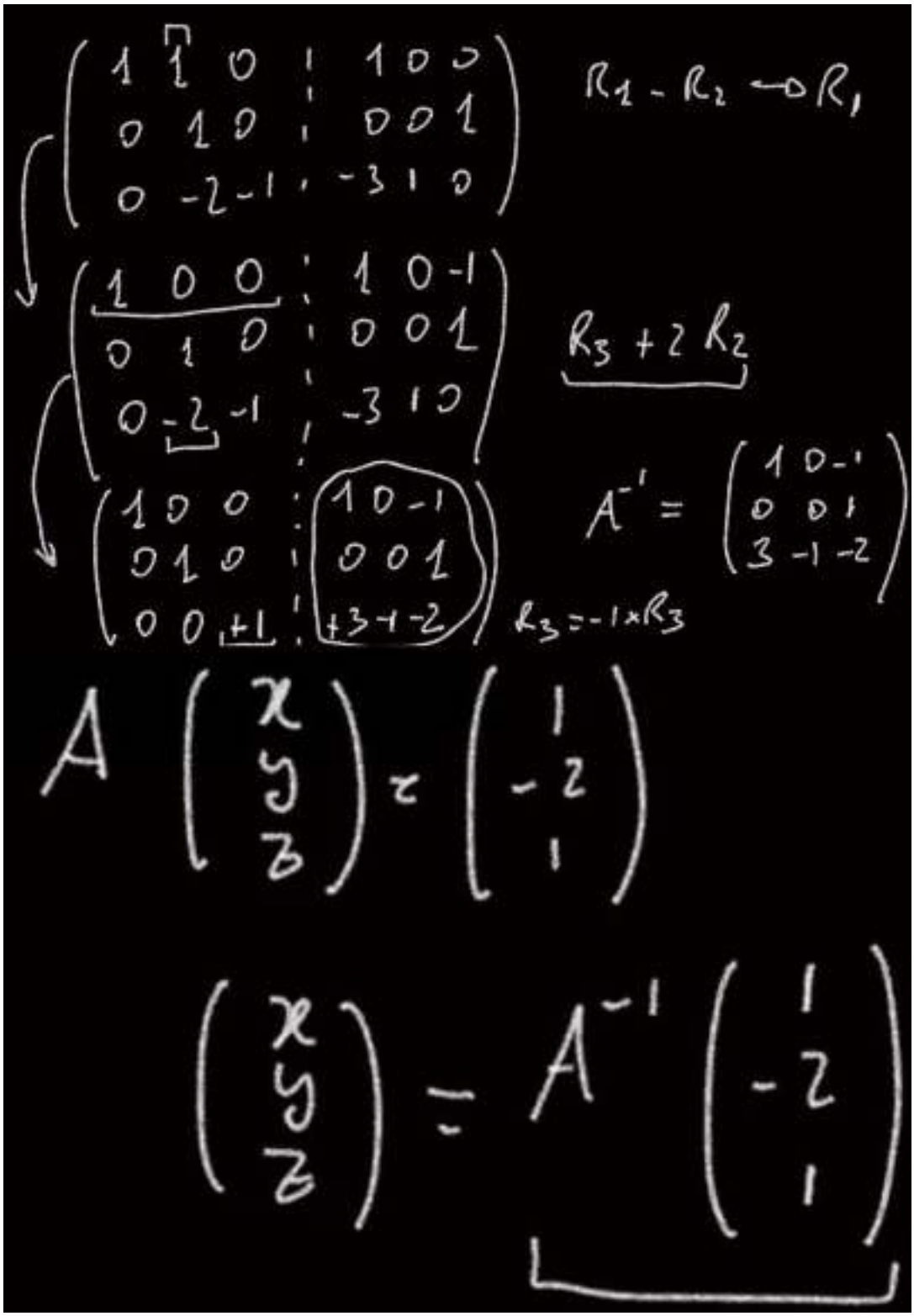
$R_1 \to R_1 + \alpha R_2 + \beta R_3$

$R_1 \rightleftarrows R_2$

$R_1 \to \alpha R_1$

- The aim of Gauss-Jordan Elimination is to obtain the identity matrix on the left hand side of the augmented matrix (of the set of colon). This can be achieved using the fact that rows can added/subtracted together and replaced, swapped and replaced or multiplied by a constant and replaced





Summary

In this week, we learned about the determinant of a 2 x 2 and 3 x 3 matrix, the general method to calculate the determinant of a matrix greater than 2 x 2, the identity transformation and identity matrix, the inverse transformation and inverse matrix, and finally how to solve systems of linear equations by implementing Gauss-Jordan Elimination.

Week 18 Algebra, Vectors, and Matrices continued Reading Note 1

Notebook:	Computational Mathematics		
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Author:	SUKHJIT MANN		
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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices continued**	Course: BSc Computer Science
		Class: Computational Mathematics[Reading]
		Date: July 28, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- What is a matrix?
- What are the augmented and coefficient matrices?
- What are elementary row operations?
- What is Row-Echelon and Reduced Row-Echelon Form?
- What is Gaussian Elimination with Back-Substitution?
- What is a system of linear equations with no solution?
- What is Gauss-Jordan Elimination?
- How do we represent matrices?
- What is Matrix addition?
- What is scalar multiplication?
- What are the properties of Matrix Addition and Scalar Multiplication?
- What is the zero matrix and the additive identity?
- What is matrix multiplication?
- What are properties of matrix multiplication?
- What is the definition of the identity matrix?
- What is the inverse of a matrix?
- What are the steps to finding an inverse matrix?
- What is formula for the inverse of a 2 x 2 matrix?
- What is a system of equation with a unique solution?
- What is the determinant of a 2 x 2 matrix?
- What are Cofactors and Minors?
- What is the determinant of a matrix greater than 2 x 2?

Notes

Definition of Matrix

If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

in which each **entry** a_{ij} of the matrix is a number. An $m \times n$ matrix has m rows and n columns.

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the **augmented matrix** of the system. Moreover, the matrix derived from the coefficients of the system (but not including the constant terms) is the **coefficient matrix** of the system.

$$\textit{System:} \begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x \qquad - 4z = 6 \end{cases}$$

$$\textit{Augmented matrix:} \begin{bmatrix} 1 & -4 & 3 & \vdots & 5 \\ -1 & 3 & -1 & \vdots & -3 \\ 2 & 0 & -4 & \vdots & 6 \end{bmatrix} \qquad \textit{Coefficient matrix:} \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

Note the use of 0 for the coefficient of the missing y-variable in the third equation, and also note the fourth column of constant terms in the augmented matrix.

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by vertically aligning the variables in the equations and using zeros for the coefficients of the missing variables.

The entry in the ith row and jth column of a matrix is denoted by the *double subscript* notation a_{ij}. For example, a_{23} refers to the entry in the second row, third column. A matrix having m rows and n columns is said to be of **dimension** $m \times n$. If $m = n$, then the matrix is **square** of dimension $m \times m$ (or $n \times n$). For a square matrix, the entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **main diagonal** entries. A matrix with only one row is called a **row matrix,** and a matrix with only one column is called a **column matrix.**

EXAMPLE 1 Dimensions of Matrices

Determine the dimension of each matrix.

a. $[2]$ **b.** $\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$

Solution

a. This matrix has *one* row and *one* column. The dimension of the matrix is 1×1.

b. This matrix has *one* row and *four* columns. The dimension of the matrix is 1×4.

c. This matrix has *two* rows and *two* columns. The dimension of the matrix is 2×2.

d. This matrix has *three* rows and *two* columns. The dimension of the matrix is 3×2.

EXAMPLE 2 Writing an Augmented Matrix

Write the augmented matrix for the system of linear equations.

$$\begin{cases} x + 3y - w = 9 \\ -y + 4z + 2w = -2 \\ x - 5z - 6w = 0 \\ 2x + 4y - 3z = 4 \end{cases}$$

What is the dimension of the augmented matrix?

Solution

Begin by rewriting the linear system and aligning the variables.

$$\begin{cases} x + 3y \qquad - w = 9 \\ \quad -y + 4z + 2w = -2 \\ x \qquad - 5z - 6w = 0 \\ 2x + 4y - 3z \qquad = 4 \end{cases}$$

Next, use the coefficients and constant terms as the matrix entries. Include zeros for the coefficients of the missing variables.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 1 & 3 & 0 & -1 & \vdots & 9 \\ 0 & -1 & 4 & 2 & \vdots & -2 \\ 1 & 0 & -5 & -6 & \vdots & 0 \\ 2 & 4 & -3 & 0 & \vdots & 4 \end{bmatrix}$$

The augmented matrix has four rows and five columns, so it is a 4×5 matrix. The notation R_n is used to designate each row in the matrix. For example, Row 1 is represented by R_1.

Elementary Row Operations

In Section 7.3, you studied three operations that can be used on a system of linear equations to produce an equivalent system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix of a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a sequence of elementary row operations.

Elementary Row Operations

Operation	Notation
1. Interchange two rows.	$R_a \leftrightarrow R_b$
2. Multiply a row by a nonzero constant.	$cR_a \quad (c \neq 0)$
3. Add a multiple of a row to another row.	$cR_a + R_b$

EXAMPLE 3 Elementary Row Operations

a. Interchange the first and second rows of the original matrix.

Original Matrix **New Row-Equivalent Matrix**

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \qquad \begin{matrix} R_2 \\ R_1 \\ \end{matrix}\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

b. Multiply the first row of the original matrix by $\frac{1}{2}$.

Original Matrix **New Row-Equivalent Matrix**

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \qquad \begin{matrix} \frac{1}{2}R_1 \rightarrow \\ \\ \end{matrix}\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

c. Add -2 times the first row of the original matrix to the third row.

Original Matrix **New Row-Equivalent Matrix**

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \qquad \begin{matrix} \\ \\ -2R_1 + R_3 \rightarrow \end{matrix}\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

Note that the elementary row operation is written beside the row that is *changed*.

EXAMPLE 4 Comparing Linear Systems and Matrix Operations

Linear System

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Associated Augmented Matrix

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

Add the first equation to the second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Add the first row to the second row: $R_1 + R_2$.

$$R_1 + R_2 \rightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

Add -2 times the first equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Add -2 times the first row to the third row: $-2R_1 + R_3$.

$$-2R_1 + R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & -1 & -1 & \vdots & -1 \end{bmatrix}$$

Add the second equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

Add the second row to the third row: $R_2 + R_3$.

$$R_2 + R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix}$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiply the third row by $\frac{1}{2}$: $\frac{1}{2}R_3$.

$$\frac{1}{2}R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

At this point, use back-substitution to find x and y.

$y + 3(2) = 5$	Substitute 2 for z.
$y = -1$	Solve for y.
$x - 2(-1) + 3(2) = 9$	Substitute -1 for y and 2 for z.
$x = 1$	Solve for x.

The solution is $(1, -1, 2)$.

The last matrix in Example 4 is in *row-echelon form*. The term *echelon* refers to the stair-step pattern formed by the nonzero entries of the matrix. The row-echelon form and *reduced row-echelon form* of matrices are described below.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

It is worth noting that the row-echelon form of a matrix is not unique. That is, two different sequences of elementary row operations may yield different row-echelon forms. The *reduced* row-echelon form of a matrix, however, is unique.

EXAMPLE 5 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether it is in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ **f.** $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because a row of all zeros occurs above a row that is not all zeros. The matrix in (e) is not in row-echelon form because the first nonzero entry in Row 2 is not a leading 1.

Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 5, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$.

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which the elementary row operations are performed is important. You should operate from left to right by columns, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form and use back-substitution to find the solution.

EXAMPLE 6 **Gaussian Elimination with Back-Substitution**

Solve the system

$$\begin{cases} y + z - 2w = -3 \\ x + 2y - z = 2 \\ 2x + 4y + z - 3w = -2 \\ x - 4y - 7z - w = -19 \end{cases}$$

Solution

$$\begin{bmatrix} 0 & 1 & 1 & -2 & \vdots & -3 \\ 1 & 2 & -1 & 0 & \vdots & 2 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix}$$

Write augmented matrix.

$$\begin{array}{r} R_2 \\ R_1 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix}$$

Interchange R_1 and R_2 so first column has leading 1 in upper left corner.

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \\ -R_1 + R_4 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & -6 & -6 & -1 & \vdots & -21 \end{bmatrix}$$

Perform operations on R_3 and R_4 so first column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \\ 6R_2 + R_4 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{bmatrix}$$

Perform operations on R_4 so second column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \frac{1}{3}R_3 \rightarrow \\ -\frac{1}{13}R_4 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$

Perform operations on R_3 and R_4 so third and fourth columns have leading 1's.

The matrix is now in row-echelon form, and the corresponding system is

$$\begin{cases} x + 2y - z = 2 \\ y + z - 2w = -3 \\ z - w = -2 \\ w = 3 \end{cases}$$

Using back-substitution, the solution is $(-1, 2, 1, 3)$.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row of all zeros except for the last entry, then the system has no solution, or is *inconsistent*.

EXAMPLE 7 A System with No Solution

Solve the system $\begin{cases} x - y + 2z = 4 \\ x + z = 6 \\ 2x - 3y + 5z = 4 \\ 3x + 2y - z = 1 \end{cases}$.

Solution

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 6 \\ 2 & -3 & 5 & \vdots & 4 \\ 3 & 2 & -1 & \vdots & 1 \end{bmatrix} \quad \text{Write augmented matrix.}$$

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \\ -3R_1 + R_4 \rightarrow \end{array} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -4 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix} \quad \text{Perform row operations.}$$

$$\begin{array}{r} \\ \\ R_2 + R_3 \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & -2 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix} \quad \text{Perform row operations.}$$

Note that the third row of this matrix consists entirely of zeros except for the last entry. This means that the original system of linear equations is inconsistent. You can see why this is true by converting back to a system of linear equations.

$$\begin{cases} x - y + 2z = 4 \\ y - z = 2 \\ 0 = -2 \\ 5y - 7z = -11 \end{cases}$$

The third equation is not possible, so the system has no solution.

Gauss-Jordan Elimination

With Gaussian elimination, elementary row operations are applied to a matrix to obtain a (row-equivalent) row-echelon form of the matrix. A second method of elimination, called **Gauss-Jordan elimination,** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until the *reduced* row-echelon form is obtained. This procedure is demonstrated in Example 8.

EXAMPLE 8 Gauss-Jordan Elimination

See LarsonPrecalculus.com for an interactive version of this type of example.

Use Gauss-Jordan elimination to solve the system $\begin{cases} x - 2y + 3z = 9 \\ -x + 3y \qquad = -4. \\ 2x - 5y + 5z = 17 \end{cases}$

Solution In Example 4, Gaussian elimination was used to obtain the row-echelon form of the linear system above.

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Now, rather than using back-substitution, apply elementary row operations until you obtain zeros above each of the leading 1's.

$$2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 9 & \vdots & 19 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Perform operations on R_1 so second column has a zero above its leading 1.

$$\begin{matrix} -9R_3 + R_1 \rightarrow \\ -3R_3 + R_2 \rightarrow \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Perform operations on R_1 and R_2 so third column has zeros above its leading 1.

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{cases} x = 1 \\ y = -1, \\ z = 2 \end{cases}$$

So, the solution is $(1, -1, 2)$.

The elimination procedures described in this section sometimes result in fractional coefficients. For example, consider the system

$$\begin{cases} 2x - 5y + 5z = 17 \\ 3x - 2y + 3z = 11, \\ -3x + 3y \qquad = -6 \end{cases}$$

Multiplying the first row by $\frac{1}{2}$ to produce a leading 1 results in fractional coefficients. You can sometimes avoid fractions by judiciously choosing the order in which you apply elementary row operations.

EXAMPLE 9 A System with an Infinite Number of Solutions

Solve the system $\begin{cases} 2x + 4y - 2z = 0 \\ 3x + 5y \qquad = 1 \end{cases}$.

Solution

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$\tfrac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$-3R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 3 & \vdots & 1 \end{bmatrix}$$

$$-R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

$$-2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x + 5z = 2 \\ y - 3z = -1 \end{cases}.$$

Solving for x and y in terms of z, you have

$$x = -5z + 2 \quad \text{and} \quad y = 3z - 1.$$

To write a solution of the system that does not use any of the three variables of the system, let a represent any real number and let $z = a$. Substitute a for z in the equations for x and y.

$$x = -5z + 2 = -5a + 2 \quad \text{and} \quad y = 3z - 1 = 3a - 1$$

So, the solution set can be written as an ordered triple of the form

$$(-5a + 2, 3a - 1, a)$$

where a is any real number. Remember that a solution set of this form represents an infinite number of solutions. Substitute values for a to obtain a few solutions. Then check each solution in the original system of equations.

Representation of Matrices

1. A matrix can be denoted by an uppercase letter such as A, B, or C.
2. A matrix can be denoted by a representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.
3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** when they have the same dimension $(m \times n)$ and $a_{ij} = b_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$. In other words, two matrices are equal when their corresponding entries are equal.

EXAMPLE 1 **Equality of Matrices**

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the matrix equation $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$.

Solution Two matrices are equal when their corresponding entries are equal, so $a_{11} = 2$, $a_{12} = -1$, $a_{21} = -3$, and $a_{22} = 0$.

Be sure you see that for two matrices to be equal, they must have the same dimension *and* their corresponding entries must be equal. For example,

$$\begin{bmatrix} 2 & -1 \\ \sqrt{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0.5 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of dimension $m \times n$, then their sum is the $m \times n$ matrix

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different dimensions is undefined.

EXAMPLE 2 **Addition of Matrices**

a. $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

d. The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined because A is of dimension 3×3 and B is of dimension 3×2.

In operations with matrices, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. To multiply a matrix A by a scalar c, multiply each entry in A by c.

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** of A by c is the $m \times n$ matrix

$$cA = [ca_{ij}].$$

The symbol $-A$ represents the **negation** of A, which is the scalar product $(-1)A$. Moreover, if A and B are of the same dimension, then $A - B$ represents the sum of A and $(-1)B$. That is,

$$A - B = A + (-1)B.$$

Subtraction of matrices

EXAMPLE 3 Operations with Matrices

For the matrices below, find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Solution

a. $$3A = 3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$ Scalar multiplication

$$= \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix}$$ Multiply each entry by 3.

$$= \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$ Simplify.

b. $$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$ Definition of negation

$$= \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$ Multiply each entry by -1.

c. $$3A - B = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$ $3A - B = 3A + (-1)B$

$$= \begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$ Add corresponding entries.

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. The example below shows factoring the scalar $\frac{1}{2}$ out of a matrix.

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1) & \frac{1}{2}(-3) \\ \frac{1}{2}(5) & \frac{1}{2}(1) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}$$

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers.

Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

1. $A + B = B + A$	Commutative Property of Matrix Addition
2. $A + (B + C) = (A + B) + C$	Associative Property of Matrix Addition
3. $(cd)A = c(dA)$	Associative Property of Scalar Multiplication
4. $1A = A$	Scalar Identity Property
5. $c(A + B) = cA + cB$	Distributive Property
6. $(c + d)A = cA + dA$	Distributive Property

Note that the Associative Property of Matrix Addition allows you to write expressions such as $A + B + C$ without ambiguity because the same sum occurs no matter how the matrices are grouped. This same reasoning applies to sums of four or more matrices.

EXAMPLE 4 **Addition of More than Two Matrices**

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Add corresponding entries.

EXAMPLE 5 **Evaluating an Expression**

$$\begin{aligned} 3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right) &= 3\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 0 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 12 & -6 \\ 9 & 21 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix} \end{aligned}$$

In Example 5, you could add the two matrices first and then multiply the resulting matrix by 3. The result would be the same.

One important property of addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number c. For matrices, a similar property holds. That is, if A is an $m \times n$ matrix and O is the $m \times n$ **zero matrix** consisting entirely of zeros, then

$$A + O = A.$$

In other words, O is the **additive identity** for the set of all $m \times n$ matrices. For example, the matrices below are the additive identities for the sets of all 2×3 and 2×2 matrices.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3 \text{ zero matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2 \times 2 \text{ zero matrix}}$$

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the solutions below.

Real Numbers (Solve for x.)	**$m \times n$ Matrices (Solve for X.)**
$x + a = b$	$X + A = B$
$x + a + (-a) = b + (-a)$	$X + A + (-A) = B + (-A)$
$x + 0 = b - a$	$X + O = B - A$
$x = b - a$	$X = B - A$

EXAMPLE 6 Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

Solution Begin by solving the matrix equation for X.

$$\begin{aligned} 3X + A &= B \\ 3X &= B - A \\ X &= \tfrac{1}{3}(B - A) \end{aligned}$$

Now, substituting the matrices A and B, you have

$$X = \tfrac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right) \qquad \text{Substitute the matrices.}$$

$$= \tfrac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} \qquad \text{Subtract matrix } A \text{ from matrix } B.$$

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \text{Multiply the resulting matrix by } \tfrac{1}{3}.$$

Matrix Multiplication

Another basic matrix operation is **matrix multiplication.** At first glance, the definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

> **Definition of Matrix Multiplication**
>
> If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix given by $AB = [c_{ij}]$, where
> $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.

$$\underset{m \times n}{A} \times \underset{n \times p}{B} = \underset{m \times p}{AB}$$

Equal

Dimension of AB

The definition of matrix multiplication uses a *row by column* multiplication, where the entry in the ith row and jth column of the product AB is obtained by multiplying the entries in the ith row of A by the corresponding entries in the jth column of B and then adding the results. So, for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is, the middle two indices must be the same. The outside two indices give the dimension of the product, as shown at the left. The general pattern for matrix multiplication is shown below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = c_{ij}$$

EXAMPLE 7 Finding the Product of Two Matrices

Find the product AB, where $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

Solution To find the entries of the product, multiply each row of A by each column of B.

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-3) + 3(-4) & (-1)(2) + 3(1) \\ 4(-3) + (-2)(-4) & 4(2) + (-2)(1) \\ 5(-3) + 0(-4) & 5(2) + 0(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

EXAMPLE 8 Finding the Product of Two Matrices

Find the product AB, where $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$.

Solution Note that the dimension of A is 2×3 and the dimension of B is 3×2. So, the product AB has dimension 2×2.

$$AB = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-2) + 0(1) + 3(-1) & 1(4) + 0(0) + 3(1) \\ 2(-2) + (-1)(1) + (-2)(-1) & 2(4) + (-1)(0) + (-2)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 7 \\ -3 & 6 \end{bmatrix}$$

EXAMPLE 9 Matrix Multiplication

See LarsonPrecalculus.com for an interactive version of this type of example.

a. $\underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}\underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$

b. $\underset{3 \times 3}{\begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix}}\underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}} = \underset{3 \times 1}{\begin{bmatrix} 10 \\ -5 \\ -9 \end{bmatrix}}$

c. $\underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}}\underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} 1 \end{bmatrix}}$

d. $\underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}\underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} = \underset{3 \times 3}{\begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}}$

e. The product $\underset{3 \times 2}{\begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix}}\underset{3 \times 4}{\begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}}$ is not defined.

REMARK In Examples 9(c) and 9(d), note that the two products are different. Even when both AB and BA are defined, matrix multiplication is not, in general, commutative. That is, for most matrices, $AB \neq BA$. This is one way in which the algebra of real numbers and the algebra of matrices differ.

EXAMPLE 10 Squaring a Matrix

Find A^2, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. (*Note:* $A^2 = AA$.)

Solution

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Properties of Matrix Multiplication

Let A, B, and C be matrices and let c be a scalar.

1. $A(BC) = (AB)C$ — Associative Property of Matrix Multiplication
2. $A(B + C) = AB + AC$ — Left Distributive Property
3. $(A + B)C = AC + BC$ — Right Distributive Property
4. $c(AB) = (cA)B = A(cB)$ — Associative Property of Scalar Multiplication

Definition of the Identity Matrix

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of dimension $n \times n$** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \qquad \text{Identity matrix}$$

Note that an identity matrix must be *square*. When the dimension is understood to be $n \times n$, you can denote I_n simply by I.

If A is an $n \times n$ matrix, then the identity matrix has the property that $AI_n = A$ and $I_nA = A$. For example,

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \qquad AI = A$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}. \qquad IA = A$$

Using Matrices to Transform Vectors

In Section 6.3, you performed vector operations with vectors written in component form and with vectors written as linear combinations of the standard unit vectors **i** and **j**. Another way to perform vector operations is with the vectors written as column matrices.

EXAMPLE 11 **Vector Operations**

Let $\mathbf{v} = \langle 2, 4 \rangle$ and $\mathbf{w} = \langle 6, 2 \rangle$. Use matrices to find each vector.

a. $\mathbf{v} + \mathbf{w}$ **b.** $\mathbf{w} - 2\mathbf{v}$

Solution Begin by writing **v** and **w** as column matrices.

$$\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

a. $\mathbf{v} + \mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \langle 8, 6 \rangle$

Figure 8.1 shows a sketch of $\mathbf{v} + \mathbf{w}$.

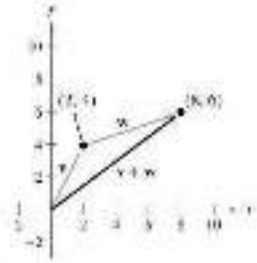


Figure 8.1

b. $\mathbf{w} - 2\mathbf{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \langle 2, -6 \rangle$

Figure 8.2 shows a sketch of $\mathbf{w} - 2\mathbf{v} = \mathbf{w} + (-2\mathbf{v})$.

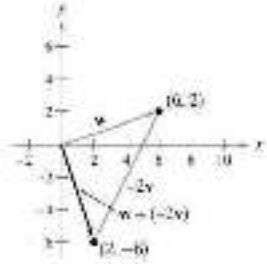


Figure 8.2

One way to transform a vector **v** is to multiply **v** by a square **transformation matrix** A to produce another vector $A\mathbf{v}$. A column matrix with two rows can represent a vector **v**, so the transformation matrix must have two columns (and also two rows) for $A\mathbf{v}$ to be defined.

EXAMPLE 12 **Describing a Vector Transformation**

Find the product $A\mathbf{v}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{v} = \langle 1, 3 \rangle$, and describe the transformation.

Solution First note that A has two columns and **v**, written as the column matrix $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, has two rows, so $A\mathbf{v}$ is defined.

$$A\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \langle 1, -3 \rangle$$

Figure 8.3 shows a sketch of the vectors **v** and $A\mathbf{v}$. The matrix A transforms **v** by reflecting **v** in the x-axis.

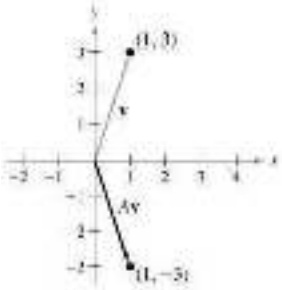


Figure 8.3

Applications

Matrix multiplication can be used to represent a system of linear equations. Note how the system below can be written as the matrix equation $AX = B$, where A is the *coefficient matrix* of the system and X and B are column matrices. The column matrix B is also called a *constant matrix*. Its entries are the constant terms in the system of equations.

System

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Matrix Equation $AX = B$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B}$$

In Example 13, $[A \;\vdots\; B]$ represents the augmented matrix formed when you *adjoin* matrix B to matrix A. Also, $[I \;\vdots\; X]$ represents the reduced row-echelon form of the augmented matrix that yields the solution of the system.

EXAMPLE 13 Solving a System of Linear Equations

For the system of linear equations, (a) write the system as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix X.

$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$$

Solution

a. In matrix form, $AX = B$, the system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}.$$

b. Form the augmented matrix by adjoining matrix B to matrix A.

$$[A \vdots B] = \begin{bmatrix} 1 & -2 & 1 & \vdots & -4 \\ 0 & 1 & 2 & \vdots & 4 \\ 2 & 3 & -2 & \vdots & 2 \end{bmatrix}$$

Using Gauss-Jordan elimination, rewrite this matrix as

$$[I \vdots X] = \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}.$$

So, the solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

EXAMPLE 14 Softball Team Expenses

Two softball teams submit equipment lists to their sponsors.

Equipment	Women's Team	Men's Team
Bats	12	15
Balls	45	38
Gloves	15	17

Each bat costs \$80, each ball costs \$4, and each glove costs \$90. Use matrices to find the total cost of equipment for each team.

Solution Write the equipment lists E and the costs per item C in matrix form as

$$E = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix}$$

and

$$C = [80 \quad 4 \quad 90].$$

The total cost of equipment for each team is the product

$$CE = [80 \quad 4 \quad 90]\begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix}$$

$$= [80(12) + 4(45) + 90(15) \quad 80(15) + 4(38) + 90(17)]$$

$$= [2490 \quad 2882].$$

So, the total cost of equipment for the women's team is \$2490, and the total cost of equipment for the men's team is \$2882.

REMARK Notice in Example 14 that it is not possible to find the total cost using the product EC because EC is not defined. That is, the number of columns of E (2 columns) does not equal the number of rows of C (1 row).

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is the **inverse** of A. The symbol A^{-1} is read as "A inverse."

EXAMPLE 1 **The Inverse of a Matrix**

Show that $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ is the inverse of $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution To show that B is the inverse of A, show that $AB = I = BA$.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, B is the inverse of A because $AB = I = BA$. This is an example of a square matrix that has an inverse. Note that not all square matrices have inverses.

Recall that it is not always true that $AB = BA$, even when both products are defined. However, if A and B are both square matrices and $AB = I_n$, then it can be shown that $BA = I_n$. So, in Example 1, you need only to check that $AB = I_2$.

Finding Inverse Matrices

If a matrix A has an inverse, then A is **invertible** (or **nonsingular**); otherwise, A is **singular.** A nonsquare matrix cannot have an inverse. To see this, note that when A is of dimension $m \times n$ and B is of dimension $n \times m$ (where $m \neq n$), the products AB and BA are of different dimensions and so cannot be equal to each other. Not all square matrices have inverses (see the matrix at the bottom of page 571). When a matrix does have an inverse, however, that inverse is unique. Example 2 shows how to use a system of equations to find the inverse of a matrix.

EXAMPLE 2 Finding the Inverse of a Matrix

Find the inverse of $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$.

Solution To find the inverse of A, solve the matrix equation $AX = I$ for X.

$$\underset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}}\underset{X}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} \underset{=}{=} \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \text{Write matrix equation.}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Multiply.}$$

Equating corresponding entries, you obtain two systems of linear equations.

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \qquad \begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$

Solve the first system using elementary row operations to determine that

$$x_{11} = -3 \quad \text{and} \quad x_{21} = 1.$$

Solve the second system to determine that

$$x_{12} = -4 \quad \text{and} \quad x_{22} = 1.$$

So, the inverse of A is

$$\begin{aligned} X &= A^{-1} \\ &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Use matrix multiplication to check this result in two ways.

Check

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark \end{aligned}$$

$$\begin{aligned} A^{-1}A &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark \end{aligned}$$

In Example 2, note that the two systems of linear equations have the *same coefficient matrix A*. Rather than solve the two systems represented by

$$\left[\begin{array}{cc|c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right]$$

separately, you can solve them *simultaneously* by *adjoining* the identity matrix to the coefficient matrix to obtain

$$\begin{array}{c} \begin{array}{cc} A & \end{array} \\ \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ \\ \end{array}$$

This "doubly augmented" matrix can be represented as

$$[A \ \vdots \ I].$$

By applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process.

$$\begin{array}{r} \\ \\ R_1 + R_2 \rightarrow \\ -4R_2 + R_1 \rightarrow \end{array} \begin{array}{c} \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \\ \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \\ \left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

So, from the "doubly augmented" matrix $[A \ \vdots \ I]$, you obtain the matrix $[I \ \vdots \ A^{-1}]$.

$$\begin{array}{c} \begin{array}{cc} A & \end{array} \\ \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \begin{array}{cc} I & \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ \\ \end{array}$$

This procedure (or algorithm) works for any square matrix that has an inverse.

Finding an Inverse Matrix

Let A be a square matrix of dimension $n \times n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain

   $$[A \ \vdots \ I].$$

2. If possible, row reduce A to I using elementary row operations on the *entire* matrix

   $$[A \ \vdots \ I].$$

   The result will be the matrix

   $$[I \ \vdots \ A^{-1}].$$

   If this is not possible, then A is not invertible.

3. Check your work by multiplying to see that

   $$AA^{-1} = I = A^{-1}A.$$

EXAMPLE 3 Finding the Inverse of a Matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}.$$

Solution Begin by adjoining the identity matrix to A to form the matrix

$$[A \ \vdots \ I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right].$$

Use elementary row operations to obtain the form $[I \ \vdots \ A^{-1}]$.

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -6R_1 + R_3 \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \rightarrow \\ \\ -4R_2 + R_3 \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \rightarrow \\ R_3 + R_2 \rightarrow \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] = [I \ \vdots \ A^{-1}]$$

So, the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}.$$

Check

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

The process shown in Example 3 applies to any $n \times n$ matrix A. When using this algorithm, if the matrix A does not reduce to the identity matrix, then A does not have an inverse. For example, the matrix below has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

To confirm that this matrix has no inverse, adjoin the identity matrix to A to form $[A \ \vdots \ I]$ and try to apply Gauss-Jordan elimination to the matrix. You will find that it is impossible to obtain the identity matrix I on the left. So, A is not invertible.

The Inverse of a 2 × 2 Matrix

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of dimension 3×3 or greater. For 2×2 matrices, however, many people prefer to use a formula for the inverse rather than Gauss-Jordan elimination. This simple formula, which works *only* for 2×2 matrices, is explained as follows. A 2×2 matrix A given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if

$$ad - bc \neq 0.$$

Moreover, if $ad - bc \neq 0$, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \qquad \text{Formula for the inverse of a } 2 \times 2 \text{ matrix}$$

The denominator

$$ad - bc$$

is the **determinant** of the 2×2 matrix A. You will study determinants in the next section.

EXAMPLE 4 Finding the Inverse of a 2 × 2 Matrix

See LarsonPrecalculus.com for an interactive version of this type of example.

If possible, find the inverse of each matrix.

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

Solution

a. The determinant of a matrix A is

$$ad - bc = 3(2) - (-1)(-2) = 4.$$

This quantity is not zero, so the matrix is invertible. The inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{Formula for the inverse of a } 2 \times 2 \text{ matrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \qquad \text{Substitute for } a, b, c, d, \text{ and the determinant.}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix} \qquad \text{Multiply by the scalar } \frac{1}{4}.$$

b. The determinant of matrix B is

$$ad - bc = 3(2) - (-1)(-6) = 0.$$

Because $ad - bc = 0$, B is not invertible.

Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution. If the coefficient matrix A of a *square* system (a system that has the same number of equations as variables) is invertible, then the system has a unique solution, which can be found using an inverse matrix as follows.

A System of Equations with a Unique Solution

If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.

EXAMPLE 5 Solving a System Using an Inverse Matrix

Use an inverse matrix to solve the system

$$\begin{cases} x + y + z = 10{,}000 \\ 0.06x + 0.075y + 0.095z = 730. \\ x - 2z = 0 \end{cases}$$

Solution Begin by writing the system in the matrix form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.075 & 0.095 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 730 \\ 0 \end{bmatrix}$$

Then, use Gauss-Jordan elimination to find A^{-1}.

$$A^{-1} = \begin{bmatrix} 15 & -200 & -2 \\ -21.5 & 300 & 3.5 \\ 7.5 & -100 & -1.5 \end{bmatrix}$$

Finally, multiply B by A^{-1} on the left to obtain the solution.

$$X = A^{-1}B = \begin{bmatrix} 15 & -200 & -2 \\ -21.5 & 300 & 3.5 \\ 7.5 & -100 & -1.5 \end{bmatrix}\begin{bmatrix} 10{,}000 \\ 730 \\ 0 \end{bmatrix} = \begin{bmatrix} 4000 \\ 4000 \\ 2000 \end{bmatrix}$$

The solution of the system is $x = 4000$, $y = 4000$, and $z = 2000$, or $(4000, 4000, 2000)$.

The Determinant of a 2 × 2 Matrix

Every *square* matrix can be associated with a real number called its **determinant.** Determinants have many uses, and several will be discussed in this section and the next section. Historically, the use of determinants arose from special number patterns that occur when systems of linear equations are solved. For example, the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Note that the denominators of the two fractions are the same. This denominator is called the *determinant* of the coefficient matrix of the system.

Coefficient Matrix	**Determinant**
$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\det(A) = a_1b_2 - a_2b_1$

The determinant of matrix A can also be denoted by vertical bars on both sides of the matrix, as shown in the definition below.

Definition of the Determinant of a 2 × 2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of A. Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended.

A convenient method for remembering the formula for the determinant of a 2×2 matrix is shown below.

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Note that the determinant is the difference of the products of the two diagonals of the matrix.

In Example 1, you will see that the determinant of a matrix can be positive, zero, or negative.

EXAMPLE 1 The Determinant of a 2 × 2 Matrix

Find the determinant of each matrix.

a. $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

b. $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

c. $C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$

Solution

a. $\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

b. $\det(B) = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

c. $\det(C) = \begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2\left(\frac{3}{2}\right) = 0 - 3 = -3$

The determinant of a matrix of dimension 1×1 is defined simply as the entry of the matrix. For example, if $A = [-2]$, then $\det(A) = -2$.

Minors and Cofactors

To define the determinant of a square matrix of dimension 3×3 or greater, it is helpful to introduce the concepts of **minors** and **cofactors.**

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & - & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & - & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & - & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

Minors and Cofactors of a Square Matrix

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

In the sign pattern for cofactors at the left, notice that *odd* positions (where $i + j$ is odd) have negative signs and *even* positions (where $i + j$ is even) have positive signs.

EXAMPLE 2 Finding the Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution To find the minor M_{11}, delete the first row and first column of A and find the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find M_{12}, delete the first row and second column.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain the minors.

$$\begin{aligned} M_{11} &= -1 & M_{12} &= -5 & M_{13} &= 4 \\ M_{21} &= 2 & M_{22} &= -4 & M_{23} &= -8 \\ M_{31} &= 5 & M_{32} &= -3 & M_{33} &= -6 \end{aligned}$$

Now, to find the cofactors, combine these minors with the checkerboard pattern of signs for a 3×3 matrix shown at the upper left.

$$\begin{aligned} C_{11} &= -1 & C_{12} &= 5 & C_{13} &= 4 \\ C_{21} &= -2 & C_{22} &= -4 & C_{23} &= 8 \\ C_{31} &= 5 & C_{32} &= 3 & C_{33} &= -6 \end{aligned}$$

The Determinant of a Square Matrix

The definition below is *inductive* because it uses determinants of matrices of dimension $(n-1) \times (n-1)$ to define determinants of matrices of dimension $n \times n$.

Determinant of a Square Matrix

If A is a square matrix (of dimension 2×2 or greater), then the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors. For example, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors.**

Verify that for a 2×2 matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

this definition of the determinant yields

$$|A| = a_1b_2 - a_2b_1$$

as previously defined.

EXAMPLE 3 **The Determinant of a 3×3 Matrix**

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

Solution Note that this is the same matrix used in Example 2. There you found that the cofactors of the entries in the first row are

$$C_{11} = -1, \quad C_{12} = 5, \quad \text{and} \quad C_{13} = 4.$$

Use the definition of the determinant of a square matrix to expand along the first row.

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First-row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14 \end{aligned}$$

In Example 3, it was efficient to expand by cofactors along the first row, but any row or column can be used. For example, expanding along the second row gives the same result.

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second-row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14 \end{aligned}$$

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero. So, the row (or column) containing the most zeros is usually the best choice for expansion by cofactors. This is demonstrated in the next example.

EXAMPLE 4 **The Determinant of a 4 × 4 Matrix**

Find the determinant of $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & 2 \end{bmatrix}$.

Solution Notice that three of the entries in the third column are zeros. So, to eliminate some of the work in the expansion, expand along the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

The cofactors C_{23}, C_{33}, and C_{43} have zero coefficients, so the only cofactor you need to find is C_{13}. Start by deleting the first row and third column of A to form the determinant that gives the minor M_{13}.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix} \qquad \text{Delete 1st row and 3rd column.}$$

$$= \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix} \qquad \text{Simplify.}$$

Now, expand by cofactors along the second row.

$$C_{13} = 0(-1)^3\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} + 2(-1)^4\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} + 3(-1)^5\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 0 + 2(1)(-8) + 3(-1)(-7)$$

$$= 5$$

So, $|A| = 3C_{13} = 3(5) = 15$.

Summary

In this week, we learned about what a matrix is, augmented/coefficient matrices, elementary row operations, Row-Echelon/Reduced Row-Echelon form, Gaussian Elimination with Back Substitution, Gauss-Jordan Elimination, matrix addition, scalar multiplication, matrix multiplication, the zero matrix and the additive identity, the identity matrix, the inverse of a matrix, the determinant of a 2 x 2 matrix, the determinant of a matrix greater than 2 x 2 and so much more.

Week 18 Algebra, Vectors, and Matrices continued Reading Note 2

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-29 4:14 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices continued**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 29, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- What does the term linear equation mean?
- What does the term system of linear equations mean?
- What operations can be performed on a system of linear equations?
- What are the three possible solutions to a linear system with two unknowns
- What are the three possible solutions to a linear system comprising of three equations and three unknowns?
- What is an augmented matrix?
- What are elementary row operations?
- What is Gaussian Elimination with Back Substitution?
- What is Row-Echelon and Reduced Row-Echelon Form?
- What are the conditions for Reduced Row-Echelon Form?
- What is a Vector Space?
- What is a linear combination?

Notes

*What does the term **linear equation** mean?*

An equation is where two mathematical expressions are defined as being equal.

A linear equation is one where all the variables such as x, y, z have index (power) of 1 or 0 only, for example

$$x + 2y + z = 5$$

is a linear equation. The following are also linear equations:

$$x = 3;\ x + 2y = 5;\ 3x + y + z + w = -8$$

The following are *not* linear equations:

1. $x^2 - 1 = 0$
2. $x + y^4 + \sqrt{z} = 9$
3. $\sin(x) - y + z = 3$

Why not?

In equation (1) the index (power) of the variable x is 2, so this is actually a quadratic equation.
In equation (2) the index of y is 4 and z is 1/2. Remember, $\sqrt{z} = z^{1/2}$.
In equation (3) the variable x is an argument of the trigonometric function sine.

Note that if an equation contains an **argument** of trigonometric, exponential, logarithmic or hyperbolic functions then the equation is not linear.

A set of linear equations is called a **linear system**.

What does the term system of linear equations mean?

Generally a finite number of linear equations with a finite number of unknowns $x, y, z, w, \ldots$ is called a **system of linear equations** or just a **linear system**.

For example, the following is a linear system of three simultaneous equations with three unknowns x, y and z:

$$\begin{aligned} x + 2y - 3z &= 3 \\ 2x - y - z &= 11 \\ 3x + 2y + z &= -5 \end{aligned}$$

In general, a linear system of m equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ is written mathematically as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \qquad (*)$$

where the coefficients a_{ij} and b_i represent real numbers. The unknowns $x_1, x_2, \ldots, x_n$ are **placeholders** for real numbers.

Example 1.1

Solve the equations about the cost of ice creams and drinks by algebraic means

$$\begin{aligned} 2x + 2y &= 3 \qquad (1) \\ 2x + y &= 2.5 \qquad (2) \end{aligned}$$

Solution

How do we solve these linear simultaneous equations, (1) and (2)?

Let's think about the information contained in these equations. The x in the first line represents the cost of an ice cream, so must have the same value as the x in the second line. Similarly, the y in the first line that represents the cost of a drink must have the same value as the y in the second line.

It follows that we can combine the two equations to see if together they offer any useful information.

How?

In this case, we subtract equation (2) from equation (1):

$$\begin{array}{rl} 2x + 2y = 3 & (1) \\ -(2x + y = 2.5) & (2) \\ \hline 0 + y = 0.5 & \end{array}$$

Note that the unknown x is eliminated in the last line which leaves $y = 0.5$.

What else do we need to find?

The other unknown x.

How?

By substituting $y = 0.5$ into equation (1):

$$2x + 2\,(0.5) = 3 \quad \text{implies that} \quad 2x + 1 = 3 \quad \text{gives} \quad x = 1$$

Hence the cost of an ice cream is £1 because $x = 1$ and the cost of a drink is £0.50 because $y = 0.5$; this is the solution to the given simultaneous equations (1) and (2).

This is also the **point of intersection**, (1, 0.5), of the graphs in Fig. 1.1. The procedure outlined in Example 1.1 is called the method of **elimination**. The values $x = 1$ and $y = 0.5$ is the solution of equations (1) and (2). In general, values which satisfy the above linear system are called the **solution** or the **solution set** of the linear system. Here is another example.

Example 1.2

Solve

$$9x + 3y = 6 \qquad (1)$$
$$2x - 7y = 9 \qquad (2)$$

Solution

We need to find the values of x and y which satisfy both equations.

How?

Taking one equation from the other doesn't help us here, but we can multiply through either or both equations by a non-zero constant.

If we multiply equation (1) by 2 and (2) by 9 then in both cases the x coefficient becomes 18. Carrying out this operation we have

$$18x + 6y = 12 \quad [\text{multiplying equation (1) by 2}]$$
$$18x - 63y = 81 \quad [\text{multiplying equation (2) by 9}]$$

How do we eliminate x from these equations?

To eliminate the unknown x we subtract these equations:

$$\begin{aligned} 18x + 6y &= 12 \\ -(18x - 63y &= 81) \\ \hline 0 + [6 - (-63)]\,y &= 12 - 81 \quad [\text{subtracting}] \\ 69y &= -69 \quad \text{which gives } y = -1 \end{aligned}$$

We have $y = -1$.

What else do we need to find?

The value of the placeholder x.

How?

By substituting $y = -1$ into the given equation $9x + 3y = 6$:

$$\begin{aligned} 9x + 3(-1) &= 6 \\ 9x - 3 &= 6 \\ 9x &= 9 \quad \text{which gives } x = 1 \end{aligned}$$

(continued...)

Hence our solution to the linear system of (1) and (2) is

$$x = 1 \text{ and } y = -1$$

We can check that this is the solution to the given system, (1) and (2), by substituting these values, $x = 1$ and $y = -1$, into the equations (1) and (2).

Note that we can carry out the following operations on a linear system of equations:

1. Interchange any pair of equations.
2. Multiply an equation by a non-zero constant.
3. Add or subtract one equation from another.

Example 1.3

Solve the linear system

$$\begin{aligned} x + 2y + 4z &= 7 && (1) \\ 3x + 7y + 2z &= -11 && (2) \\ 2x + 3y + 3z &= 1 && (3) \end{aligned}$$

Solution

What are we trying to find?

The values of x, y and z that satisfy all three equations (1), (2) and (3).

How do we find the values of x, y and z?

By elimination. To eliminate one of these unknowns, we first need to make the coefficients of x (or y or z) equal.

Which one?

There are three choices but we select so that the arithmetic is made easier, in this case it is x. Multiply equation (1) by 2 and then subtract the bottom equation (3):

$$\begin{array}{rl} 2x + 4y + 8z = 14 & [\text{multiplying (1) by 2}] \\ -(2x + 3y + 3z = 1) & (3) \\ \hline 0 + y + 5z = 13 & [\text{subtracting}] \end{array}$$

Note that we have eliminated x and have the equation $y + 5z = 13$.
How can we determine the values of y and z from this equation?
We need another equation with only y and z.
How can we get this?
Multiply equation (1) by 3 and then subtract the second equation (2):

$$\begin{array}{rl} 3x + 6y + 12z = 21 & [\text{multiplying (1) by 3}] \\ -(3x + 7y + 2z = -11) & (2) \\ \hline 0 - y + 10z = 32 & [\text{subtracting}] \end{array}$$

Again there is no x and we have the equation $-y + 10z = 32$.
How can we find y and z?
We now solve the two simultaneous equations that we have obtained

$$\begin{aligned} y + 5z &= 13 \qquad (4) \\ -y + 10z &= 32 \qquad (5) \end{aligned}$$

We add equations (4) and (5), because $y + (-y) = 0$, which eliminates y.

$$0 + 15z = 45 \text{ gives } z = \frac{45}{15} = 3$$

Hence $z = 3$, but how do we find the other two unknowns x and y?
We first determine y by substituting $z = 3$ into equation (4) $y + 5z = 13$:

$$\begin{aligned} y + (5 \times 3) &= 13 \\ y + 15 &= 13 \qquad \text{which gives } y = -2 \end{aligned}$$

We have $y = -2$ and $z = 3$. We still need to find the value of last unknown x.
How do we find the value of x?
By substituting the values we have already found, $y = -2$ and $z = 3$, into the given equation $x + 2y + 4z = 7$ (1) :

$$x + (2 \times -2) + (4 \times 3) = 7 \quad \text{gives} \quad x = -1$$

Hence the solution of the given three linear equations is $x = -1$, $y = -2$ and $z = 3$.
We can illustrate the given equations in a three-dimensional coordinate system as shown in Fig. 1.3.

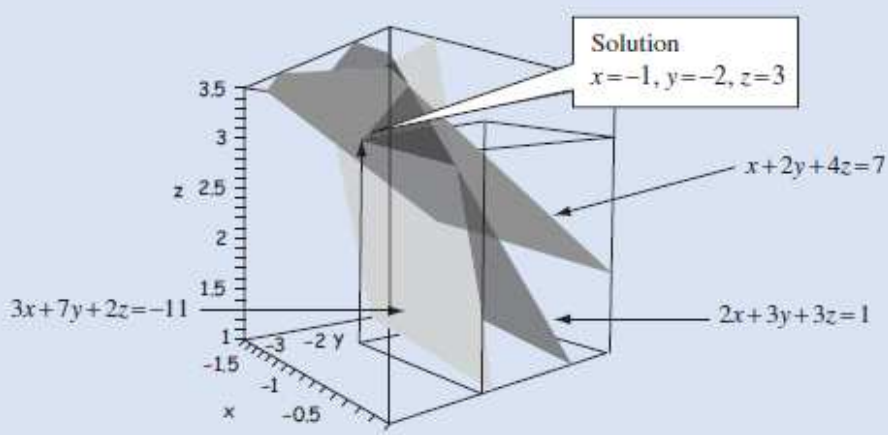


Figure 1.3

Example 1.4

Solve the linear system

$$\begin{aligned} 2x + 3y &= 6 \qquad (1) \\ 4x + 6y &= 9 \qquad (2) \end{aligned}$$

Solution

How do we solve these equations?

Multiply (1) by 2 and then subtract equation (2):

$$\begin{array}{rl} 4x + 6y = 12 & [\text{Multiplying (1) by 2}] \\ -(4x + 6y = 9) & (2) \\ \hline 0 + 0 = 3 & \end{array}$$

But how can we have $0 = 3$?

A plot of the graphs of the given equations is shown in Fig. 1.4.

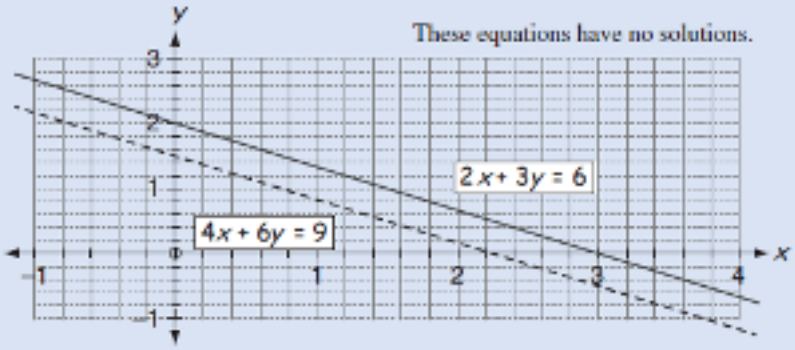


Figure 1.4

Can you see why there is no common solution to these equations?

The solution of the given equations would be the intersection of the lines shown in Fig. 1.4, but these lines are parallel so there is no intersection, therefore no solution.

By examining the given equations,

$$\begin{aligned} 2x + 3y &= 6 \qquad (1) \\ 4x + 6y &= 9 \qquad (2) \end{aligned}$$

can you see why there is no solution?

If you multiply the first equation (1) by 2 we have

$$4x + 6y = 12$$

This is a contradiction.

Why?

Because we have

$$\begin{aligned} 4x + 6y &= 12 \\ 4x + 6y &= 9 \qquad (2) \end{aligned}$$

that is, $4x + 6y$ equals both 9 and 12. This is clearly impossible. Hence the given linear system has no solution.

A system that has no solution is called **inconsistent**. If the linear system has at least one solution then we say the system is **consistent**.

Example 1.5

Graph the equations and determine the solution of this system:

$$2x + 3y = 6 \qquad (1)$$
$$4x + 6y = 12 \qquad (2)$$

Solution

The graph of the given equations is shown in Fig. 1.5.

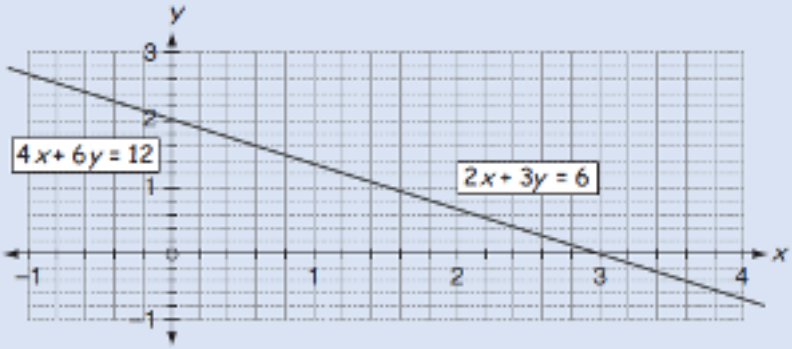


Figure 1.5

(continued...)

What do you notice?

Both the given equations produce exactly the same line; that is they coincide.

How many solutions do these equations have?

An infinite number of solutions, as you can see on the graph of Fig. 1.5. Any point on the line is a solution, and since there are an infinite number of points on the line we have an infinite number of solutions.

How can we write these solutions?

Let $x = a$ – where a is any real number – be a solution.

What then is y equal to?

Substituting $x = a$ into the given equation (1) yields

$$2a + 3y = 6 \qquad [2x + 3y = 6]$$
$$3y = 6 - 2a$$
$$y = \frac{6 - 2a}{3} = \frac{6}{3} - \frac{2}{3}a = 2 - \frac{2}{3}a$$

Hence if $x = a$ then $y = 2 - \frac{2}{3}a$.

The solution of the given linear system, (1) and (2), is $x = a$ and $y = 2 - 2a/3$ where a is any real number. You can check this by substituting various values of a. For example, if $a = 1$ then

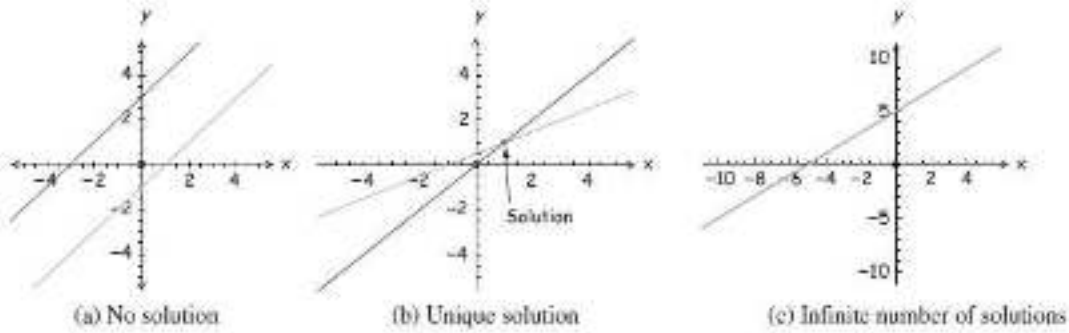
$$x = 1, \;\; y = 2 - 2(1)/3 = 4/3$$

We can check that this answer is correct by substituting these values, $x = 1$ and $y = 4/3$, into equations (1) and (2):

$$2(1) + 3(4/3) = 2 + 4 = 6 \qquad \left[\begin{array}{ll} 2x + 3y = 6 & (1) \\ 4x + 6y = 12 & (2) \end{array}\right]$$
$$4(1) + 6(4/3) = 4 + 8 = 12$$

Hence our solution works. This solution $x = a$ and $y = 2 - 2a/3$ will satisfy the given equations for any real value of a.

The graphs in Fig. 1.6 represent the three possible solutions to a linear system with two unknowns.



(a) No solution (b) Unique solution (c) Infinite number of solutions

Figure 1.6

The graphs in Fig. 1.7 illustrate solutions arising from three linear equations and three unknowns.

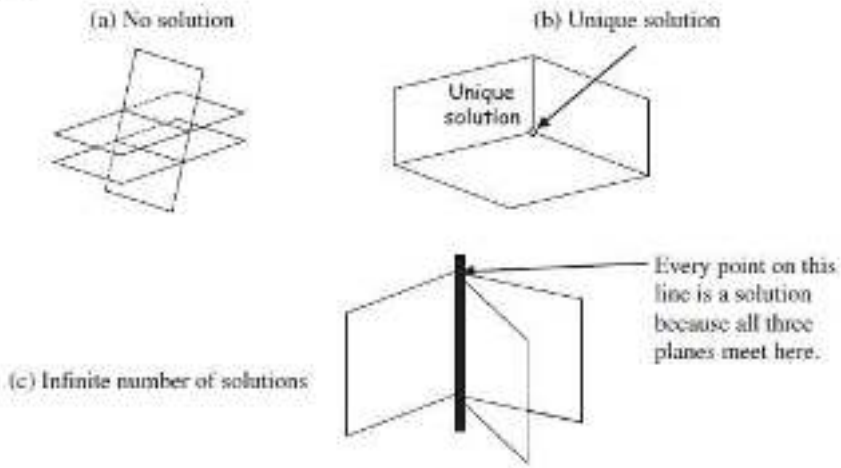


Figure 1.7

Fig. 1.7(a) shows three planes (equations) which have no point in common, hence no solution.

Fig. 1.7(b) shows the three planes (equations) with a unique point in common.

Fig. 1.7(c) shows three planes (equations) with a line in common. Every point on this line is a solution, which means we have an infinite number of solutions.

Suppose we have a general linear system of m equations with n unknowns labelled $x_1, x_2, x_3, \ldots$ and x_n given by:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the coefficients a_{ij}, b_i are real numbers and $x_1, x_2, x_3, \ldots$ and x_n are placeholders for real numbers that satisfy the equations. This general system can be stored in matrix form as

$$\left(\begin{array}{ccc|c} a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

This is an **augmented matrix**, which is a matrix containing the coefficients of the unknowns x_1, x_2, $x_3, \ldots, x_n$ and the constant values on the right hand side of the equations. In everyday English language, augmented means 'to increase'. Augmenting a matrix means adding one or more columns to the original matrix. In this case, we have added the b's column to the matrix. These are divided by a vertical line as shown above.

Example 1.6

Consider Example 1.3 section 1.1, where we had to solve the following linear system:

$$\begin{aligned} x + 2y + 4z &= 7 \\ 3x + 7y + 2z &= -11 \\ 2x + 3y + 3z &= 1 \end{aligned}$$

Write the augmented matrix of this linear system.

Solution

An augmented matrix is simply a shorthand way of representing a linear system of equations. Rather than write x, y and z after each coefficient, recognize that the first column contains the coefficients of x, the second column the coefficients of y and so on. Placing the coefficients of x, y and z on the left hand side of the vertical line in the augmented matrix and the constant values 7, −11 and 1 on the right hand side we have

$$\left(\begin{array}{ccc|r} 1 & 2 & 4 & 7 \\ 3 & 7 & 2 & -11 \\ 2 & 3 & 3 & 1 \end{array}\right)$$

Because each row of the augmented matrix corresponds to one equation of the linear system, we can carry out analogous operations such as:

1. Multiply a row by a non-zero constant.
2. Add or subtract a multiple of one row to another.
3. Interchange rows.

We refer to these operations as **elementary row operations**.

Example 1.7

Solve the following linear system by using the Gaussian elimination procedure:

$$\begin{aligned} x - 3y + 5z &= -9 \\ 2x - y - 3z &= 19 \\ 3x + y + 4z &= -13 \end{aligned}$$

Solution

What is the augmented matrix in this case?

Let R_1, R_2 and R_3 represent rows 1, 2 and 3 respectively. We have

$$\begin{aligned} x - 3y + 5z &= -9 \\ 2x - y - 3z &= 19 \\ 3x + y + 4z &= -13 \end{aligned} \quad \text{and} \quad \begin{array}{ll} \text{Row 1} & R_1 \\ \text{Row 2} & R_2 \\ \text{Row 3} & R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 & -1 & -3 & 19 \\ 3 & 1 & 4 & -13 \end{array}\right)$$

Note that each row represents an equation.

How can we find the unknowns x, y and z?

The columns in the matrix represent the x, y and z coefficients respectively. If we can transform this augmented matrix into

$$\begin{array}{c} \begin{array}{ccc} x & y & z \\ \downarrow & \downarrow & \downarrow \end{array} \\ \text{Final row} \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right) \end{array}$$

where A, B and $*$ represents any real number

then we can find z.

How?

Look at the final row.

What does this represent?

$(0 \times x) + (0 \times y) + (A \times z) = B$

$$Az = B \text{ which gives } z = \frac{B}{A} \text{ provided } A \neq 0$$

Hence we have a value for $z = B/A$.

But how do we find the other two unknowns x and y?

Now we can use a method called **back substitution**. Examine the second row of the above matrix:

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \text{Second row} \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right) \end{array}$$

By expanding the second row we get an equation in terms of y and z. From above we already know the value of $z = B/A$, so we can substitute $z = B/A$ and obtain y. Similarly from the first row we can find x by substituting the values of y and z.

We need to perform row operations on the augmented matrix to transform it from:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 & -1 & -3 & 19 \\ 3 & 1 & 4 & -13 \end{array}\right) \text{ to } \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right)$$

We need to convert this augmented matrix to an equivalent matrix with zeros in the bottom left hand corner. That is 0 in place of 2, 3 and 1.

How do we get 0 in place of 2?

Remember, we can multiply an equation by a non-zero constant, and take one equation away from another. In terms of matrices, this means that we can multiply a row and take one row away from another because each row represents an equation.

To get 0 in place of 2 we multiple row 1, R_1, by 2 and subtract the result from row 2, R_2; that is, we carry out the row operation $R_2 - 2R_1$:

$$\begin{array}{c} R_1 \\ R_2^* = R_2 - 2R_1 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2-2(1) & -1-(2\times(-3)) & -3-(2\times 5) & 19-(2\times(-9)) \\ 3 & 1 & 4 & -13 \end{array}\right)$$

We call the new middle row R_2^*. Completing the arithmetic, the middle row becomes

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 3 & 1 & 4 & -13 \end{array}\right)$$

Where else do we need a zero?

Need to get a 0 in place of 3 in the bottom row.

How?

We multiply the top row R_1 by 3 and subtract the result from the bottom row R_3; that is, we carry out the row operation, $R_3 - 3R_1$:

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^* = R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 3-3(1) & 1-(3\times(-3)) & 4-(3\times 5) & -13-(3\times(-9)) \end{array}\right)$$

We can call R_3^* the new bottom row of this matrix. Simplifying the arithmetic in the entries gives:

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^* \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 10 & -11 & 14 \end{array}\right)$$

Note that we now only need to convert the 10 into zero in the bottom row.

How do we get a zero in place of 10?

We can only make use of the bottom two rows, R_2^* and R_3^*.

Why?

Looking at the first column, it is clear that taking any multiple of R_1 away from R_3 will interfere with the zero that we have just worked to establish.

(continued...)

We execute $R_3^* - 2R_2^*$ because

$$10 - (2 \times 5) = 0 \quad \text{(gives a 0 in place of 10)}$$

Therefore we have

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^{**} = R_3^* - 2R_2^* \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0-(2\times 0) & 10-(2\times 5) & -11-[2\times(-13)] & 14-(2\times 37) \end{array} \right)$$

which simplifies to

$$\begin{array}{c} \\ R_1 \\ R_2^* \\ R_3^{**} \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 0 & 15 & -60 \end{array} \right) \end{array} \qquad (\dagger)$$

From the bottom row R_3^{**} we have

$$15z = -60 \text{ which gives } z = -\frac{60}{15} = -4$$

How do we find the other two unknowns x and y?
By expanding the middle row R_2^* of $(\dagger)$ we have:

$$5y - 13z = 37$$

We can find y by substituting $z = -4$ into this

$$5y - 13(-4) = 37 \qquad [\text{Substituting } z = -4]$$
$$5y + 52 = 37 \text{ which implies } 5y = -15 \text{ therefore } y = -\frac{15}{5} = -3$$

How can we find the last unknown x?
By expanding the first row R_1 of $(\dagger)$ we have:

$$x - 3y + 5z = -9$$

Substituting $y = -3$ and $z = -4$ into this:

$$x - (3 \times (-3)) + (5 \times (-4)) = -9$$
$$x + 9 - 20 = -9 \text{ which gives } x = 2$$

Hence our solution to the linear system is $x = 2, y = -3$ and $z = -4$.

Remember, each of the given equations can also be graphically represented by planes in a 3d coordinate system.

The solution is where all three planes (equations) meet. The equations are illustrated in Fig. 1.9.

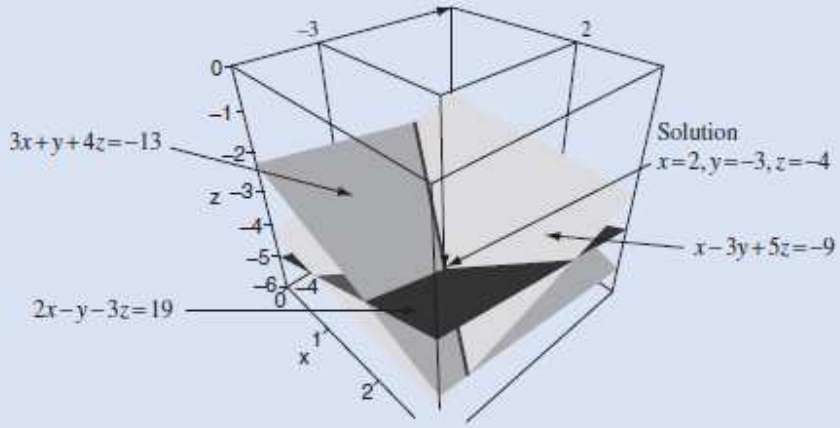


Figure 1.9

The above process is called **Gaussian elimination** with back substitution. The aim of Gaussian elimination is to produce a 'triangular' matrix with zeros in the bottom left corner of the matrix. This is achieved by the elementary row operations:

1. Multiply a row by a non-zero constant.
2. Add or subtract a multiple of one row from another.
3. Interchange rows.

We say two matrices are **row equivalent** if one matrix is derived from the other by using these three operations.

If augmented matrices of two linear systems are row equivalent then the two systems have the same solution set. You may like to check this for the above Example 1.7, to see that the solution $x = 2$, $y = -3$ and $z = -4$ satisfies the given equations.

In summary, the given linear system of equations is written in an augmented matrix, which is then transformed into a much simpler equivalent augmented matrix, which then allows us to use back substitution to find the solution of the linear system.

Example 1.8

Solve the linear system:

$$\begin{aligned} x + 3y + 2z &= 13 \\ 4x + 4y - 3z &= 3 \\ 5x + y + 2z &= 13 \end{aligned}$$

(continued...)

Solution

How can we find the unknowns x, y and z?

We use Gaussian elimination with back substitution.

The augmented matrix is:

$$\begin{array}{ll} \text{Row 1} & R_1 \\ \text{Row 2} & R_2 \\ \text{Row 3} & R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 4 & 4 & -3 & 3 \\ 5 & 1 & 2 & 13 \end{array}\right)$$

Need to convert the entries in this triangle to zeros.

Our aim is to convert this augmented matrix so that there are 0's in the bottom left hand corner, that is; the first 4 in the second row reduces to zero, and the 5 and 1 from the bottom row reduce to zero. Hence $4 \to 0, 5 \to 0$ and $1 \to 0$.

To get 0 in place of the first 4 in the middle row we multiply row 1, R_1, by 4 and take the result away from row 2, R_2, that is $R_2 - 4R_1$. To get 0 in place of 5 in the bottom row we multiply row 1, R_1, by 5 and take the result away from row 3, R_3, that is $R_3 - 5R_1$. Combining the two row operations, $R_2 - 4R_1$ and $R_3 - 5R_1$, we have

$$\begin{array}{c} R_1 \\ R_2^\dagger = R_2 - 4R_1 \\ R_3^\dagger = R_3 - 5R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 4-(4\times 1) & 4-(4\times 3) & -3-(4\times 2) & 3-(4\times 13) \\ 5-(5\times 1) & 1-(5\times 3) & 2-(5\times 2) & 13-(5\times 13) \end{array}\right)$$

We call the new row 2 and $3 - R_2^\dagger$ and $R_3^\dagger$ respectively. This simplifies to:

$$\begin{array}{c} R_1 \\ R_2^\dagger \\ R_3^\dagger \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & -14 & -8 & -52 \end{array}\right)$$

We have nearly obtained the required matrix with zeros in the bottom left hand corner. We need a 0 in place of -14 in the new bottom row, $R_3^\dagger$. We can only use the second and third rows, $R_2^\dagger$ and $R_3^\dagger$.

Why?

Because if we use first row, R_1, we will get a non-zero number in place of the zero already established in $R_3^\dagger$.

How can we obtain a 0 in place of -14?

$$R_3^\dagger - \frac{14}{8}R_2^\dagger \text{ because } -14 - \left[\frac{14}{8} \times (-8)\right] = 0 \text{ (gives 0 in place of } -14)$$

Therefore

$$\begin{array}{c} R_1 \\ R_2^\dagger \\ R_3^{\dagger\dagger} = R_3^\dagger - \frac{14}{8}R_2^\dagger \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & -14 - \left[\frac{14}{8} \times (-8)\right] & -8 - \left[\frac{14}{8} \times (-11)\right] & -52 - \left(\frac{14}{8} \times (-49)\right) \end{array}\right)$$

which simplifies to

$$\begin{array}{c} \\ R_1 \\ R_2^\dagger \\ R_3^{\dagger\dagger} \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & 45/4 & 135/4 \end{array}\right) \end{array} \quad (*)$$

This time we have called the bottom row $R_3^{\dagger\dagger}$. From this row, $R_3^{\dagger\dagger}$, we have

$$\frac{45}{4}z = \frac{135}{4} \text{ which gives } z = \frac{135}{45} = 3$$

How can we find the unknown, y?
By expanding the second row $R_2^{\dagger}$ in (*) we have

$$-8y - 11z = -49$$

We know from above that $z = 3$, therefore, substituting $z = 3$ gives

$$\begin{aligned} -8y - 11\,(3) &= -49 \\ -8y - 33 &= -49 \\ -8y &= -49 + 33 = -16 \text{ which yields } y = (-16)/(-8) = 2 \end{aligned}$$

So far we have $y = 2$ and $z = 3$.
How can we find the last unknown, x?
By expanding the first row, R_1, of (*) we have

$$x + 3y + 2z = 13$$

Substituting our values already found, $y = 2$ and $z = 3$, we have

$$\begin{aligned} x + (3 \times 2) + (2 \times 3) &= 13 \\ x + 6 + 6 &= 13 \text{ which gives } x = 1 \end{aligned}$$

Hence $x = 1$, $y = 2$ and $z = 3$ is our solution. We can illustrate these equations as shown in Fig. 1.10.

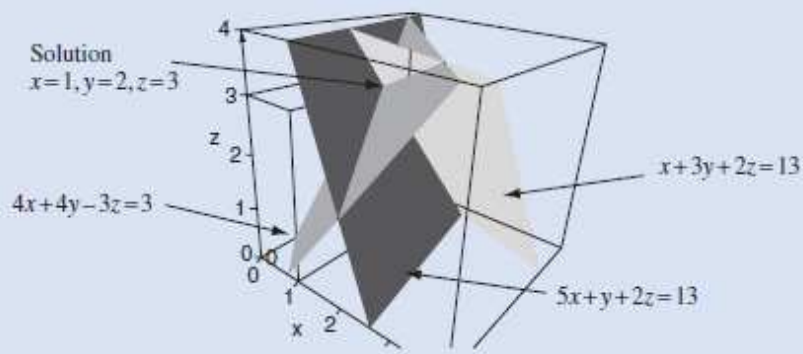


Figure 1.10

We can check that this solution is correct by substituting these $x = 1$, $y = 2$ and $z = 3$ into the given equations.

The Gaussian elimination process can be extended in the above example so that the first non-zero number in the bottom row of (*) is 1, that is

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger\dagger} \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & 45/4 & 135/4 \end{array}\right)$$

Convert this into 1

How do we convert 45/4 into 1?

Multiply the bottom row $R_3^{\dagger\dagger}$ by $\frac{4}{45}$.

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3' = \dfrac{4R_3^{\dagger\dagger}}{45} \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & \frac{4}{45}\left(\frac{45}{4}\right) & \frac{4}{45}\left(\frac{135}{4}\right) \end{array}\right)$$

which simplifies to

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

The advantage of this is that we get the z value directly. From the bottom row, R_3', we have $z = 3$. We can extend these row operations further and obtain the following matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right) \qquad (*)$$

Why would we want to achieve this sort of augmented matrix?

Because we can read off the x, y and z values directly from this augmented matrix. The only problem is in doing the arithmetic, because achieving this sort of matrix can be a laborious process.

This augmented matrix (*) is said to be in reduced row echelon form.

A matrix is in **reduced row echelon form**, normally abbreviated to **rref**, if it satisfies all the following conditions:

1. If there are any rows containing only zero entries then they are located in the bottom part of the matrix.
2. If a row contains non-zero entries then the first non-zero entry is a 1. This 1 is called a **leading 1**.
3. The leading 1's of two consecutive non-zero rows go strictly from top left to bottom right of the matrix.
4. The only non-zero entry in a column containing a leading 1 is the leading 1.

If condition (4) is *not* satisfied then we say that the matrix is in **row echelon form** and drop the qualification 'reduced'. In some linear algebra literature the leading 1 condition is relaxed and it is enough to say that any non-zero number is the **leading coefficient**.

For example, the following are all in reduced row echelon form:

$$\begin{pmatrix} 0 & \boxed{1} & 0 & 8 & 0 \\ 0 & 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & -6 \\ 0 & 0 & \boxed{1} & 9 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} & 5 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The following matrices are *not* in reduced row echelon form:

$$\mathbf{A} = \begin{pmatrix} 0 & \boxed{1} & 5 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ \boxed{1} & 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 5 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Why not?

In matrix **A** the third column contains a leading one but has a non-zero entry, 5.
In matrix **B** the leading ones do not go from top left to bottom right.
In matrix **C** the top row of zeros should be relegated to the bottom of the matrix as stated in condition (1) above.

However, matrix **A** is in row echelon form but not in reduced row echelon form. Matrices **B** and **C** are not in row echelon form.

The procedure which places an augmented matrix into row echelon form is called Gaussian elimination and the algorithm which places an augmented matrix into a reduced row echelon form is called **Gauss–Jordan** elimination.

- A matrix in reduced row-echelon form is unique. This means however we carry out our row operations we will always arrive at same matrix in reduced row-echelon form. However, a matrix in row-echelon form is not unique

Example 1.9

Place the augmented matrix $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 5 & -15 \end{array}\right) \end{array}$ into reduced row echelon form.

Solution

Why should we want to place this matrix into reduced row echelon form?

In a nutshell, it's to avoid back substitution. If we look at the bottom row of the given augmented matrix we have $5z = -15$.

We need to divide by 5 in order to find the z value.

The reduced row echelon form, rref, gives us the values of the unknowns directly, and we do not need to carry out further manipulation or elimination.

What does reduced row echelon form mean in this case?

It means convert the given augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 5 & -15 \end{array}\right) \text{ into something like } \left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

This means that we need to get 0 in place of the 5 in the second row, and 0's in place of the 5 and -3 in the first row. We also need a 1 in place of -13 in the middle row and 1 in place of the 5 in the bottom row.

How do we convert the 5 in the bottom row into 1?

Divide the last row by 5 (remember, this is the same as multiplying by $1/5$):

$$\begin{array}{c} R_1 \\ R_2 \\ R_3' = R_3/5 \end{array} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

How do we get 0 in place of -3 in the first row and the 5 in the second row?

We execute the row operations $R_1 + 3R_3'$ and $R_2 - 5R_3'$:

$$\begin{array}{c} R_1^* = R_1 + 3R_3' \\ R_2^* = R_2 - 5R_3' \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1+3(0) & 5+3(0) & -3+3(1) & -9+3(-3) \\ 0-5(0) & -13-5(0) & 5-5(1) & 37-5(-3) \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1^* \\ R_2^* \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1 & 5 & 0 & -18 \\ 0 & -13 & 0 & 52 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

How do we get a 1 in place of -13?

Divide the middle row by -13 (remember, this is the same as multiplying by $-1/13$):

$$\begin{array}{c} R_1^* \\ R_2^{**} = R_2^*/(-13) \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1 & 5 & 0 & -18 \\ 0 & -13/(-13) & 0 & 52/(-13) \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Simplifying the second row gives

$$\begin{array}{c} R_1^* \\ R_2^{**} \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1 & 5 & 0 & -18 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

This matrix is now in row echelon form but not in reduced row echelon form. We need to convert the 5 in the top row into 0 to get it into reduced row echelon form.

How?

We carry out the row operation, $R_1^* - 5R_2^{**}$:

$$\begin{array}{c} R_1^{**} = R_1^* - 5R_2^{**} \\ R_2^{**} \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1-5(0) & 5-5(1) & 0-5(0) & -18-5(-4) \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Simplifying the top row entries gives

$$\begin{array}{c} \\ R_1^{**} \\ R_2^{**} \\ R_3' \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right) \end{array}$$

Hence we have placed the given augmented matrix into reduced row echelon form.

3.1.1 Vector space

Let V be a non-empty set of elements called vectors. We define two operations on the set V– **vector addition** and **scalar multiplication**. Scalars are real numbers.

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in the set V. The set V is called a **vector space** if it satisfies the following 10 axioms.

1. The vector addition $\mathbf{u} + \mathbf{v}$ is also in the vector space V. Generally in mathematics we say that we have **closure** under vector addition if this property holds.
2. Commutative law: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. Associative law: $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. Neutral element. There is a vector called the **zero** vector in V denoted by $\mathbf{O}$ which satisfies

$$\mathbf{u}+\mathbf{O}=\mathbf{u} \text{ for every vector } \mathbf{u} \text{ in } V$$

5. Additive inverse. For every vector $\mathbf{u}$ there is a vector $-\mathbf{u}$ (minus $\mathbf{u}$) which satisfies the following:

$$\mathbf{u}+(-\mathbf{u})=\mathbf{O}$$

6. Let k be a real scalar then $k\mathbf{u}$ is also in V. We say that we have **closure** under scalar multiplication if this axiom is satisfied.
7. Associative law for scalar multiplication. Let k and c be real scalars then

$$k(c\mathbf{u})=(kc)\mathbf{u}$$

8. Distributive law for vectors. Let k be a real scalar then

$$k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\,\mathbf{v}$$

9. Distributive law for scalars. Let k and c be real scalars then

$$(k+c)\mathbf{u}=k\mathbf{u}+c\mathbf{u}$$

10. Identity element. For every vector $\mathbf{u}$ in V we have

$$1\mathbf{u}=\mathbf{u}$$

We say that if the elements of the set V satisfy the above 10 axioms then V is called a vector space and the elements are known as vectors.

3.2.2 Revision of linear combination

Linear combination combines the two fundamental operations of linear algebra – vector addition and scalar multiplication.

In the last chapter we introduced linear combination in $\mathbb{R}^n$. For example, we had

$$\mathbf{u} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \cdots + k_n\mathbf{e}_n \qquad (k\text{'s are scalars})$$

which is a **linear combination** of the standard unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, ... and $\mathbf{e}_n$.

Similarly for general vector spaces we define linear combination as:

Definition (3.6). Let $\mathbf{v}_1$, $\mathbf{v}_2, \ldots$ and $\mathbf{v}_n$ be vectors in a vector space. If a vector $\mathbf{x}$ can be expressed as

$$\mathbf{x} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \text{ (where } k\text{'s are scalars)}$$

then we say $\mathbf{x}$ is a **linear combination** of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, ... and $\mathbf{v}_n$.

Example 3.8

Let P_2 be the set of all polynomials of degree less than or equal to 2.

Let $\mathbf{v}_1 = t^2 - 1$, $\mathbf{v}_2 = t^2 + 3t - 5$ and $\mathbf{v}_3 = t$ be vectors in P_2.

Show that the quadratic polynomial

$$\mathbf{x} = 7t^2 - 15$$

is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution

How do we show $\mathbf{x}$ is a linear combination of vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$?

We need to find the values of the scalars k_1, k_2 and k_3 which satisfy

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{x} \quad (*)$$

How can we determine these scalars?

By substituting $\mathbf{v}_1 = t^2 - 1$, $\mathbf{v}_2 = t^2 + 3t - 5$, $\mathbf{v}_3 = t$ and $\mathbf{x} = 7t^2 - 15$ into (*):

$$\begin{aligned}
k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 &= k_1(t^2 - 1) + k_2(t^2 + 3t - 5) + k_3 t \\
&= k_1t^2 - k_1 + k_2t^2 + 3k_2t - 5k_2 + k_3t && \text{[expanding]} \\
&= (k_1 + k_2)t^2 + (3k_2 + k_3)t - (k_1 + 5k_2) && \text{[factorizing]} \\
&= 7t^2 - 15 && \text{[remember } \mathbf{x} = 7t^2 - 15\text{]}
\end{aligned}$$

By equating coefficients of the last two lines

$$(k_1 + k_2)t^2 + (3k_2 + k_3)t - (k_1 + 5k_2) = 7t^2 - 15$$

gives

$$\begin{aligned}
k_1 + k_2 &= 7 && \text{[equating } t^2\text{]} \\
3k_2 + k_3 &= 0 && \text{[equating } t\text{]} \\
k_1 + 5k_2 &= 15 && \text{[equating constants]}
\end{aligned}$$

Solving these equations gives the values of the scalars: $k_1 = 5, k_2 = 2$ and $k_3 = -6$. Substituting these into $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{x}$:

$$5\mathbf{v}_1 + 2\mathbf{v}_2 - 6\mathbf{v}_3 = \mathbf{x}$$

This means that adding 5 lots of $\mathbf{v}_1$, 2 lots of $\mathbf{v}_2$ and -6 lots of $\mathbf{v}_3$ gives the vector $\mathbf{x}$:

$$5(t^2 - 1) + 2(t^2 + 3t - 5) - 6t = 7t^2 - 15$$

You may like to check the algebra.
We conclude that $\mathbf{x}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The next proposition allows us to check that S is a subspace, by carrying out the test for scalar multiplication and vector addition in a single calculation.

Proposition (3.7). A non-empty subset S containing vectors $\mathbf{u}$ and $\mathbf{v}$ is a subspace of a vector space $V \Leftrightarrow$ any linear combination $k\mathbf{u} + c\,\mathbf{v}$ is also in S (k and c are scalars).

How do we prove this proposition?
Since it is a '$\Leftrightarrow$' statement we need to prove it both ways.

Proof.
($\Rightarrow$). Let S be a subspace of V, then S is a vector space. If $\mathbf{u}$ and $\mathbf{v}$ are in S then $k\mathbf{u} + c\mathbf{v}$ is also in S because the vector space S is closed under scalar multiplication and vector addition.
($\Leftarrow$). Assume $k\mathbf{u} + c\mathbf{v}$ is in S.

Substituting $k = c = 1$ into $k\mathbf{u} + c\mathbf{v}$ we have $\mathbf{u} + \mathbf{v}$ is also in S. Similarly for $c = 0$ we have $k\mathbf{u} + c\mathbf{v} = k\mathbf{u}$ is in S.

Hence we have closure under vector addition and scalar multiplication. By Proposition (3.5): S is subspace of $V \Leftrightarrow S$ is closed under vector addition and scalar multiplication.

We conclude that S is a subspace.

■

Proposition (3.7) is another test for a subspace. Hence a subspace of a vector space V is a non-empty subset S of V, such that for all vectors $\mathbf{u}$ and $\mathbf{v}$ in S and all scalars k and c we have $k\mathbf{u} + c\mathbf{v}$ is also in S.

Example 3.9

Let S be the subset of vectors of the form $(x \quad y \quad 0)^T$ in the vector space $\mathbb{R}^3$. Show that S is a subspace of $\mathbb{R}^3$.

(continued...)

Solution

How do we show that S is a subspace of $\mathbb{R}^3$?

We can use the above Proposition (3.7), which means we need to show that any linear combination $k\mathbf{u} + c\mathbf{v}$ is in S for any vectors $\mathbf{u}$ and $\mathbf{v}$ in S.

Let $\mathbf{u} = (a \quad b \quad 0)^T$ and $\mathbf{v} = (c \quad d \quad 0)^T$ be in S. Then for real scalars k_1 and k_2 we have

$$k_1\mathbf{u} + k_2\mathbf{v} = k_1\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} + k_2\begin{pmatrix} c \\ d \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} k_1 a \\ k_1 b \\ 0 \end{pmatrix} + \begin{pmatrix} k_2 c \\ k_2 d \\ 0 \end{pmatrix} = \begin{pmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \\ 0 \end{pmatrix}$$

Hence $k_1\mathbf{u} + k_2\mathbf{v}$ is also in S.

By the above Proposition (3.7):

S is subspace of $V \Leftrightarrow$ any linear combination $k\mathbf{u} + c\mathbf{v}$ is also in S.

We conclude that the given set S is a subspace of the vector space $\mathbb{R}^3$.

You might find it easier to use this test (3.7) rather than (3.5) because you only need to recall that the linear combination is closed in S.

Note that the given set S in the above example describes the xy plane in three-dimensional space $\mathbb{R}^3$ as shown in Fig. 3.10:

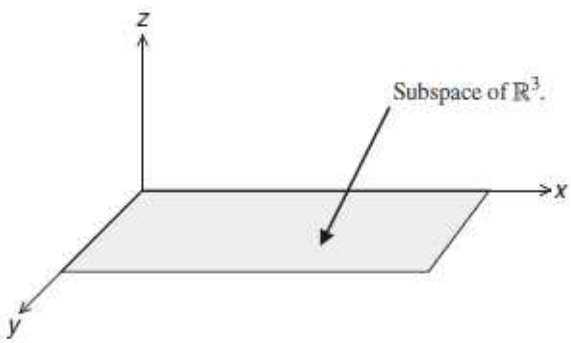


Figure 3.10

Sometimes we write $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ as the set $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; z = 0 \right\}$ where the vertical line in the set means 'such that'. That is, S contains the set of vectors in $\mathbb{R}^3$ such that the last entry is zero.

Summary

In this week, we learned about a linear equation means, what a system of linear equations is, possible solutions to a system of linear equations, what an augmented matrix is, what elementary row operations are, what Gaussian Elimination with Back Substitution is, what row-echelon and reduced row-echelon forms of a matrix are, the conditions for reduced row-echelon form, what a vector space is and finally what a linear combination is in terms of a vector space.

Week 18 Algebra, Vectors, and Matrices continued Reading Note 3

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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices continued**	Course: BSc Computer Science
		Class: Computational Mathematics[Reading]
		Date: July 29, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- What is a matrix?
- What is a square matrix?
- What is a diagonal matrix?
- What is an identity matrix?
- How is the addition and subtraction of matrices performed?
- How do we multiply or divide a matrix by a number?
- How do we multiply two matrices together?
- What is the inverse of a 2 x 2 matrix?
- What is the determinant of a 2 x 2 matrix?
- How can matrices be used to solve simultaneous equations?

Notes

A **matrix** is a set of numbers arranged in the form of a rectangle and enclosed in curved brackets. The plural of matrix is **matrices**. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 2 \\ -3 & 4 & 5 \\ \frac{1}{2} & 2 & 1 \end{pmatrix}$$

are all matrices. Each number in a matrix is known as an **element**. To refer to a particular matrix we label it with a capital letter, so that we could write

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 4 & 5 \\ \frac{1}{2} & 2 & 1 \end{pmatrix}$$

We refer to the **size** of a matrix by giving its number of rows and number of columns, in that order. So matrix A above has two rows and three columns – we say it is a 'two by three' matrix, and write this as '2×3'. Similarly B has size 3×1 and C has size 3×3. Some particular types of matrix occur so frequently that they have been given special names.

A **square** matrix has the same number of rows as columns. So

$$\begin{pmatrix} 2 & -7 \\ -1 & 6 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & -1 \\ 9 & 8 & 5 \\ 6 & -7 & 2 \end{pmatrix}$$

are both square matrices.

A **diagonal** matrix is a square matrix in which all the elements are 0 except those on the diagonal from the top left to the bottom right. This diagonal is called the **leading diagonal**. On the leading diagonal the elements can take any value including zero. So

$$\begin{pmatrix} 7 & 0 \\ 0 & 9 \end{pmatrix}, \quad \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -17 \end{pmatrix}$$

are all diagonal matrices.

An **identity** matrix is a diagonal matrix, all the diagonal entries of which are equal to 1. So

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are all identity matrices.

Matrices of the same size can be added to, or subtracted from, one another. All matrices can be multiplied or divided by a scalar, that is a number. Under certain conditions, a matrix can be multiplied by another matrix. A matrix can never be divided by another matrix.

Addition and subtraction

Two matrices having the same size can be added or subtracted by simply adding or subtracting the corresponding elements. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+2 \\ 3+1 & 4+0 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 4 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 7 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 3 & 9 \\ 7 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-0 & 2-1 & 3-3 & 4-9 \\ 2-7 & 1-0 & 1-0 & 7-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & -5 \\ -5 & 1 & 1 & 6 \end{pmatrix}$$

The matrices

$$\begin{pmatrix} 1 & 2 & 9 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix}$$

can be neither added to nor subtracted from one another because they have different sizes.

Multiplying or dividing a matrix by a number

A matrix is multiplied (divided) by a number by multiplying (dividing) each element by that number. For example,

$$4\begin{pmatrix} 1 & 2 \\ 3 & -9 \end{pmatrix} = \begin{pmatrix} 4\times 1 & 4\times 2 \\ 4\times 3 & 4\times -9 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 12 & -36 \end{pmatrix}$$

and

$$\frac{1}{4}\begin{pmatrix} 16 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\times 16 \\ \frac{1}{4}\times 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

WORKED EXAMPLE

27.1 If

$$A = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} \text{ and } B = \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix}$$

find (a) $A + B$ (b) $A - B$ (c) $7A$ (d) $-\frac{1}{2}B$ (e) $3A + 2B$

Solution (a)

$$A + B = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} + \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 3 \end{pmatrix}$$

(b)

$$A - B = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 15 \end{pmatrix}$$

(c)

$$7A = 7\begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 35 \\ 7 \\ 63 \end{pmatrix}$$

(d)

$$-\frac{1}{2}B = -\frac{1}{2}\begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix}$$

(e)

$$3A + 2B = 3\begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} + 2\begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 15 \\ 3 \\ 27 \end{pmatrix} + \begin{pmatrix} 16 \\ 4 \\ -12 \end{pmatrix} = \begin{pmatrix} 31 \\ 7 \\ 15 \end{pmatrix}$$

Multiplying two matrices together

Two matrices can only be multiplied together if the number of columns in the first is the same as the number of rows in the second. The product of two such matrices is a matrix that has the same number of rows as the first matrix and the same number of columns as the second. In symbols, this states that if A has size $p \times q$ and B has size $q \times s$, then AB has size $p \times s$.

WORKED EXAMPLES

27.2 Find AB where

$$A = \begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Solution The size of A is 2×2. The size of B is 2×1. Therefore the number of columns in the first matrix, A, equals the number of rows in the second matrix B. We can therefore find the product AB. The result will be a 2×1 matrix. Let us call it C. The working is as follows:

$$C = \begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 4 \times 5 \\ 6 \times 2 + 3 \times 5 \end{pmatrix} = \begin{pmatrix} 22 \\ 27 \end{pmatrix}$$

The first row of A multiplies the first column of B to give

$$(1 \times 2) + (4 \times 5) = 2 + 20 = 22$$

This is the element in row 1, column 1 of C. The second row of A then multiplies the first column of B to give

$$(6 \times 2) + (3 \times 5) = 12 + 15 = 27$$

This is the element in row 2, column 1 of C.

27.3 Using the same A and B as the previous example, is it possible to find BA?

Solution Recall that B has size 2×1 and A has size 2×2. When written in the order BA, the number of columns in the first matrix is 1 whereas the number of rows in the second is 2. It is not therefore possible to find BA.

27.4 Find, if possible,

$$\begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 9 \\ 8 & 7 \\ -7 & 3 \end{pmatrix}$$

Solution The number of columns in the first matrix is 3 and this is the same as the number of rows in the second. We can therefore perform the multiplication and the answer will have size 2×2.

$$\begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 9 \\ 8 & 7 \\ -7 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 1 + 4 \times 8 + 9 \times -7 & 1 \times 9 + 4 \times 7 + 9 \times 3 \\ 2 \times 1 + 0 \times 8 + 1 \times -7 & 2 \times 9 + 0 \times 7 + 1 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} -30 & 64 \\ -5 & 21 \end{pmatrix}$$

27.5 Find $(1 \quad 5)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution

$$(1 \quad 5)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = (1 \times 2 + 5 \times 1) = (7)$$

Note that in this example the result is a 1×1 matrix, that is a single number.

27.6 Find $\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Solution

$$\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ 5x + 3y \end{pmatrix}$$

27.7 Find IX where I is the 2×2 identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} a \\ b \end{pmatrix}$$

Solution

$$IX = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \times a + 0 \times b \\ 0 \times a + 1 \times b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Note that the effect of multiplying the matrix X by the identity matrix is to leave X unchanged. This is a very important and useful property of identity matrices, which we shall require later. It should be remembered. This property of identity matrices should remind you of the fact that multiplying a number by 1 leaves the number unchanged: for example, $7 \times 1 = 7$, $1 \times -9 = -9$. An identity matrix plays the same role for matrices as the number 1 does when dealing with ordinary arithmetic.

Consider the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

An important matrix that is related to A is known as the **inverse** of A and is given the symbol A^{-1}. Here, the superscript '-1' should not be read as a power, but is meant purely as a notation for the inverse matrix. A^{-1} can be found from the following formula:

Key point

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This formula states that:

- the elements on the leading diagonal are interchanged
- the remaining elements change sign
- the resulting matrix is multiplied by $\dfrac{1}{ad - bc}$

The inverse matrix has the property that:

Key point

$$AA^{-1} = A^{-1}A = I$$

That is, when a 2×2 matrix and its inverse are multiplied together the result is the identity matrix.

WORKED EXAMPLE

27.8 Find the inverse of the matrix

$$A = \begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}$$

and verify that $AA^{-1} = A^{-1}A = I$.

Solution Using the formula for the inverse we find

$$A^{-1} = \frac{1}{(6)(2)-(5)(2)}\begin{pmatrix} 2 & -5 \\ -2 & 6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -5 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{pmatrix}$$

The inverse of $\begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}$

is therefore $\begin{pmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{pmatrix}$

Evaluating AA^{-1} we find

$$\begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6\times 1 + 5\times -1 & 6\times(-\frac{5}{2}) + 5\times 3 \\ 2\times 1 + 2\times -1 & 2\times(-\frac{5}{2}) + 2\times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the 2×2 identity matrix. Also evaluating $A^{-1}A$ we find

$$\begin{pmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1\times 6 + (-\frac{5}{2})\times 2 & 1\times 5 + (-\frac{5}{2})\times 2 \\ -1\times 6 + 3\times 2 & -1\times 5 + 3\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the 2×2 identity matrix. We have shown that $AA^{-1} = A^{-1}A = I$.

The quantity $ad - bc$ in the formula for the inverse is known as the **determinant** of the matrix A. We often write it as $|A|$, the vertical bars indicating that we mean the determinant of A and not the matrix itself.

Key point

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then its determinant is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

WORKED EXAMPLES

27.9 Find the value of the determinant

$$\begin{vmatrix} 8 & 7 \\ 9 & 2 \end{vmatrix}$$

Solution The determinant is given by $(8)(2) - (7)(9) = 16 - 63 = -47$.

27.10 Find the determinant of each of the following matrices:

(a) $C = \begin{pmatrix} 7 & 2 \\ 4 & 9 \end{pmatrix}$ (b) $D = \begin{pmatrix} 4 & -2 \\ 9 & -1 \end{pmatrix}$ (c) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $E = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

Solution (a) $|C| = (7)(9) - (2)(4) = 63 - 8 = 55$. Note that the determinant is always a single number.

(b) $|D| = (4)(-1) - (-2)(9) = -4 + 18 = 14$.

(c) $|I| = (1)(1) - (0)(0) = 1 - 0 = 1$. It is always true that the determinant of an identity matrix is 1.

(d) $|E| = (4)(1) - (2)(2) = 4 - 4 = 0$. In this example the determinant of the matrix E is zero.

As we have seen in the previous example, on some occasions a matrix may be such that $ad - bc = 0$, that is its determinant is zero. Such a matrix is called **singular**. When a matrix is singular it cannot have an inverse. This is because it is impossible to evaluate the quantity $1/(ad - bc)$, which appears in the formula for the inverse; the quantity $1/0$ has no meaning in mathematics.

WORKED EXAMPLE

27.11 Show that the matrix

$$P = \begin{pmatrix} 6 & -2 \\ -24 & 8 \end{pmatrix}$$

has no inverse.

Solution We first find the determinant of P:

$$|P| = (6)(8) - (-2)(-24) = 48 - 48 = 0$$

The determinant is zero and so the matrix P is singular. It does not have an inverse.

Matrices can be used to solve simultaneous equations. Suppose we wish to solve the simultaneous equations

$$3x + 2y = -3$$

$$5x + 3y = -4$$

First of all we rewrite them using matrices as

$$\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

Writing

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

we have $AX = B$. Note that we are trying to find x and y; in other words we must find X. Now, if

$$AX = B$$

then, provided A^{-1} exists, we multiply both sides by A^{-1} to obtain

$$A^{-1}AX = A^{-1}B$$

But $A^{-1}A = I$, so this becomes

$$IX = A^{-1}B$$

However, multiplying X by the identity matrix leaves X unaltered, so we can write

$$X = A^{-1}B$$

In other words, if we multiply B by the inverse of A we will have X as required. Now, given

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}$$

then

$$A^{-1} = \frac{1}{(3)(3) - (2)(5)}\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$$

Finally,

$$X = A^{-1}B = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Therefore the solution of the simultaneous equations is $x = 1$ and $y = -3$.

We note that the crucial step is

Key point

$$AX = B$$

$$X = A^{-1}B \quad \text{provided } A^{-1} \text{ exists}$$

WORKED EXAMPLE

27.12 Solve the simultaneous equations

$$x + 2y = 13$$
$$2x - 5y = 8$$

Solution First of all we rewrite the equations using matrices as

$$\begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

Writing

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

we have $AX = B$. We must now find the matrix X. If $AX = B$ then multiplying both sides by A^{-1} gives $A^{-1}AX = A^{-1}B$. But $A^{-1}A = I$, so this becomes $IX = A^{-1}B$. It follows that

$$X = A^{-1}B$$

We must multiply B by the inverse of A. The inverse of A is

$$A^{-1} = \frac{1}{(1)(-5) - (2)(2)}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{-9}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix}$$

Finally,

$$X = A^{-1}B = \frac{1}{-9}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 13 \\ 8 \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -81 \\ -18 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

Therefore the solution of the simultaneous equations is $x = 9$ and $y = 2$.

Summary

In this week, we learned about what a matrix is, how addition/subtraction on matrices is performed, how to multiply two matrices together, the inverse/determinant of a 2 x 2 matrix and finally how matrices can be used to solve simultaneous equations.

Week 19 Combinatorics and Probabilities Lecture Note

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Cornell Notes	**Topic:** **Combinatorics and Probabilities**	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 30, 2020

Essential Question:

What is the probability of an event and how can we use the principles of counting to evaluate such probabilities?

Questions/Cues:

- What is the probability of an event?
- What are the properties of probability?
- What is the joint probability of two events?
- What are permutations?
- What are combinations?

Notes

Probability: definition

$$P(\text{event}) = \frac{\text{N of favourable outcomes}}{\text{N of total outcomes}}$$

Examples coin toss

What is the probability of having head (H)?

N of total outcomes 2

Favourable outcomes 1

➡ P(H)=1/2

Probability: definition

$$P(\text{event}) = \frac{\text{N of favourable outcomes}}{\text{N of total outcomes}}$$

Example: three coin tosses

What is the probability of having 2 heads and one tail (no order)

HHH
HHT
HTH
HTT
TTT
TTH
THT
THH

N total outcomes: $8=2^3= 2x2x2$
Favourable outcomes: 3

➡ P(2H1T)=3/8 ➡ P(HHT)=1/8

Probability: properties

$$P(\text{event}) = \frac{\text{N of favourable outcomes}}{\text{N of total outcomes}} \quad \Rightarrow \quad 0 \leqslant P(\text{event}) \leqslant 1$$

Probability for the occurrence of either of two incompatible events:

Given two incompatible (i.e. cannot occur simultaneously) events A and B

P(A or B)= P(A) +P(B)

Ex: one coin toss A=H B=T (incompatible)

➡ P(H or T)=1/2 +1/2=1

Probability: properties

In general for events compatible (can occur simultaneously)

P(A or B)= P(A) +P(B) –P(A and B)

Example: die roll
What's the probability of getting either 4 (A) or an even number (B)?

Favourable outcomes
A: 4 → n=1
B: 2,4,6. → n=3
A and B: 4 → n=1
Total outcomes: n=6

P(A)= 1/6 ; P(B)=3/6=1/2; P(A and B)=1/6

→P(A or B)=1/6+1/2-1/6=1/2

Joint Probability of two events

We have seen P(A or B)(= $P(A \cup B)$)
let us consider P(A and B) (also P(A,B) and $P(A \cap B)$)

Case 1) <u>Independent events:</u>
The outcome of one event does not affect the other

➡ P(A and B) =P(A) P(B)

Examples: two coin flips

P(H in flip1 and T in flip2)=P(H)P(T)=1/2x1/2=1/4

HH
HT
TH
TT
Total cases 4
Favourable cases 1 → P(A and B)=1/4

Case 2) <u>dependent events:</u>
The outcome of one event affects the other

P(A and B)=P(A)P(B|A) (=P(B)P(A|B))

P(A|B) conditional probability
probability that, given the occurrence of B, A occurs as well

Example: card deck
A= clubs B= king
A and B= king of clubs

P(A)=13/53 ; P(B)=4/53
P(A and B)=n favour./ n total =1/53 ≠ P(A)P(B)

P(B|A)=1/13
P(A|B)=1/4

P(A and B)= P(A)P(B|A) =(13/53) (1/13) =1/53
=P(B)P(A|B) =(4/53) (1/4)=1/53

Permutations (ordering important)

Let's see other situation three socks RBY from a drawer
What are the possible outcomes when you pick randomly (order important)?

RBY
RYB
YRB
YBR
BRY
BYR

Total number of outcomes: 6.
It's like the counting principle but you start from a set of 3 elements
So for the first element you have 3 possibility, the second 2, the last one 1
So it's 3x2x1

In general for the permutation n elements it's n!=n(n-1)(n-2)….1 (0!=1)

P(n)=n! P(3)=3x2x1=6

Permutations and Combinations

You want also to calculate the permutation of only two (r) elements from a set of three (n).

RB
BR
RY
YR
BY
YB

Formula for this case is P(r,n)=n!/(n-r)!= 3!/(3-2)!= 6/1=6

If we don't want to count more than once the groups (couples)
that are different only for by the order
we must divide by r! (2!=2)

$C(n,r)=\frac{n!}{(n-r)!r!} = \binom{n}{r} = 6/(1x2)=3$

These are called combinations, total n. of groups of r elements in
a set of n total elements in which order does not matter.

11 SOCKS

R	G	B
2	4	5

1) $P(R)$?

$N_{TOT} = 11$

$N(R) = 2$

$\Rightarrow P(R) = \frac{2}{11} \approx 18\%$

2) $P(RR)$

RR
RR
RB
BR
BB
BB

$N_{TOT} = P_2^{11} = \frac{11!}{(11-2)!} = \frac{\cancel{9!} \times 10 \times 11}{\cancel{9!}} = 110$

$N_{TOT} = 11 \times 10 = 110$

$N(RR) = 2 = P_2^2 = \frac{2!}{(2-2)!} = 2 \Rightarrow P(RR) = \frac{2}{110} \approx 1.8\%$

3) $P(GGG)$?

$N_{TOT} = C_3^{11} = \frac{11!}{(11-3)!\,3!} = \frac{11!}{8!\,3!} =$

$= \frac{\cancel{8!}\,\cancel{3}^{3} \times \cancel{10}^{5} \times 11}{\cancel{8!}\,\cancel{2} \times \cancel{3}} = 165$

$N(GGG) = C_3^4 = \frac{4!}{(4-3)!\,3!} = 4$

$P(GGG) = \frac{4}{165} = 2.4\%$

4) $P(XXX)$

R	G	B
2	4	5

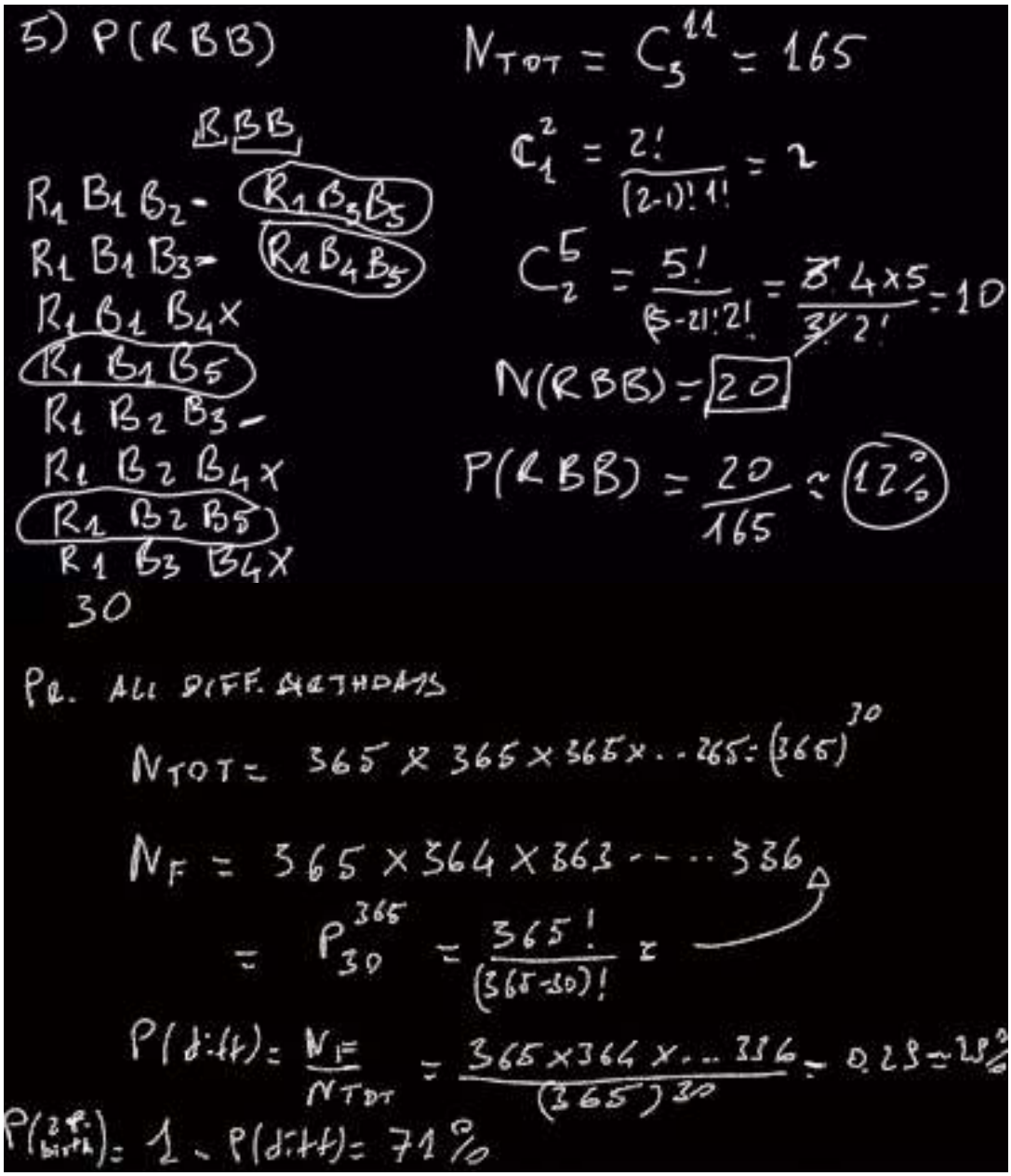
$P(XXX) = P(GGG) + \underline{P(BBB)}$

$N_{TOT} = C_3^{11} = 165$

$N(BBB) = C_3^5 = \frac{5!}{(5-3)!\,3!} = \frac{4 \times 5}{2!} = 10$

$P(BBB) = 10/165 = 6\%$

$P(XXX) = P(GGG) + P(BBB) = (2.4 + 6)\% = 8.4\%$



Summary

In this week, we learned about what the probability of an event is, the properties of probability, the joint probability of two events, what permutations are and what combinations are.

Week 19 Combinatorics and Probabilities Reading Note 1

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Cornell Notes	**Topic:** **Combinatorics and Probabilities**	Course: BSc Computer Science
		Class: Computational Mathematics[Reading]
		Date: July 30, 2020

Essential Question:

What is the probability of an event and how can we use the principles of counting to evaluate such probabilities?

Questions/Cues:

- What is an average?
- What is the arithmetic mean?
- What is the median?
- What is the mode?
- What is the variance?
- What is standard deviation?
- What is the probability scale and complementary events?
- What is meant by unbiased?
- What is a theoretical probability?
- What is an experimental probability?
- What are independent events?

Notes

- Average = To represent a whole set of data by a single number that in some way represents the set

The arithmetic mean

The **arithmetic mean**, or simply the **mean**, of a set of values is found by adding up all the values and dividing the result by the total number of values in the set. It is given by the following formula:

Key point

$$\text{mean} = \frac{\text{sum of the values}}{\text{total number of values}}$$

WORKED EXAMPLES

30.1 Eight students sit a mathematics test and their marks out of 10 are 4, 6, 6, 7, 7, 7, 8 and 10. Find the mean mark.

Solution The sum of the marks is $4+6+6+7+7+7+8+10=55$. The total number of values is 8. Therefore,

$$\text{mean} = \frac{\text{sum of the values}}{\text{total number of values}} = \frac{55}{8} = 6.875$$

The examiner can quote 6.875 out of 10 as the 'average mark' of the group of students.

30.2 In a hospital a patient's body temperature is recorded every hour for six hours. Find the mean temperature over the six-hour period if the six temperatures, in °C, were

36.5 36.8 36.9 36.9 36.9 37.0

Solution To find the mean temperature the sum of all six values is found and the result is divided by 6. That is,

$$\text{mean} = \frac{36.5+36.8+36.9+36.9+36.9+37.0}{6}$$

$$= \frac{221.0}{6} = 36.83\ ^\circ\text{C}$$

In more advanced work we make use of a formula for the mean that requires knowledge of some special notation. Suppose we have n values and we call these x_1, x_2, x_3 and so on up to x_n. The mean of these values is given the symbol $\bar{x}$, pronounced 'x bar'. To calculate the mean we must add up these values and divide by n, that is

$$\text{mean} = \bar{x} = \frac{x_1+x_2+x_3+\cdots+x_n}{n}$$

A notation is often used to shorten this formula. In mathematics, the Greek letter sigma, written $\sum$, stands for a sum. The sum x_1+x_2 is written

$$\sum_{i=1}^{2} x_i$$

and the sum $x_1+x_2+x_3+x_4$ is written

$$\sum_{i=1}^{4} x_i$$

Note that i runs through all integer values from 1 to n.

where the values below and above the sigma sign give the first and last values in the sum. Similarly $x_1+x_2+x_3+\cdots+x_n$ is written $\sum_{i=1}^{n} x_i$. Using this notation, the formula for the mean can be written in the following way:

Key point

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

WORKED EXAMPLES

30.3 Express the following in sigma notation:

(a) $x_1 + x_2 + x_3 + \cdots + x_8 + x_9$ (b) $x_{10} + x_{11} + \cdots + x_{100}$

Solution (a) $x_1 + x_2 + x_3 + \cdots + x_8 + x_9 = \sum_{i=1}^{9} x_i$

(b) $x_{10} + x_{11} + \cdots + x_{100} = \sum_{i=10}^{100} x_i$

30.4 Find the mean of the values $x_1 = 5$, $x_2 = 7$, $x_3 = 13$, $x_4 = 21$ and $x_5 = 29$.

Solution The number of values equals 5, so we let $n = 5$. The sum of the values is

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5 = 5 + 7 + 13 + 21 + 29 = 75$$

The mean is

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{5} x_i}{n} = \frac{75}{5} = 15$$

When the data is presented in the form of a frequency distribution the mean is found by first multiplying each data value by its frequency. The results are added. This is equivalent to adding up all the data values. The mean is found by dividing this sum by the sum of all the frequencies. Note that the sum of the frequencies is equal to the total number of values. Consider the following example.

WORKED EXAMPLE

30.5 Thirty-eight students sit a mathematics test and their marks out of 10 are shown in Table 30.1. Find the mean mark.

Solution Each data value, in this case the mark, is multiplied by its frequency, and the results are added. This is equivalent to adding up all the 38 individual marks. This is shown in Table 30.2.

Table 30.1
Marks of 38 students in a test

Mark, m	Frequency, f
0	0
1	0
2	1
3	0
4	1
5	7
6	16
7	8
8	3
9	1
10	1
Total	38

Table 30.2
Marks of 38 students multiplied by frequency

Mark, m	Frequency, f	$m \times f$
0	0	0
1	0	0
2	1	2
3	0	0
4	1	4
5	7	35
6	16	96
7	8	56
8	3	24
9	1	9
10	1	10
Totals	38	236

Note that the sum of all the frequencies is equal to the number of students taking the test. The number 236 is equal to the sum of all the individual marks. The mean is found by dividing this sum by the sum of all the frequencies:

$$\text{mean} = \frac{236}{38} = 6.21$$

The mean mark is 6.21 out of 10.

Using the sigma notation the formula for the mean mark of a frequency distribution with N classes, where the frequency of value x_i is f_i, becomes

Key point

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{N} f_i \times x_i}{\sum_{i=1}^{N} f_i}$$

Note that $\sum_{i=1}^{N} f_i = n$; that is, the sum of all the frequencies equals the total number of values.

When the data is in the form of a grouped distribution the class midpoint is used to calculate the mean. Consider the following example.

WORKED EXAMPLE

30.6 The heights, to the nearest centimetre, of 100 students are given in Table 30.3. Find the mean height.

Solution Because the actual heights of students in each class are not known we use the midpoint of the class as an estimate. The midpoint of the class 164–165 is 164.5. Other midpoints and the calculation of the mean are shown in Table 30.4.

Then

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{N} f_i \times x_i}{\sum_{i=1}^{N} f_i} = \frac{17150}{100} = 171.5$$

The mean height is 171.5 cm.

Table 30.3
Heights of 100 students

Height (cm)	Frequency
164–165	4
166–167	8
168–169	10
170–171	27
172–173	30
174–175	10
176–177	6
178–179	5
Total	100

Table 30.4
Heights of 100 students with midpoints multiplied by frequency

Height (cm)	Frequency f_i	Midpoint x_i	$f_i \times x_i$
164–165	4	164.5	658.0
166–167	8	166.5	1332.0
168–169	10	168.5	1685.0
170–171	27	170.5	4603.5
172–173	30	172.5	5175.0
174–175	10	174.5	1745.0
176–177	6	176.5	1059.0
178–179	5	178.5	892.5
Total	100		17150.0

The median

A second average that also typifies a set of data is the **median**.

Key point

The **median** of a set of numbers is found by listing the numbers in ascending order and then selecting the value that lies halfway along the list.

WORKED EXAMPLE

30.7 Find the median of the numbers

1 2 6 7 9 11 11 11 14

Solution The set of numbers is already given in order. The number halfway along the list is 9, because there are four numbers before it and four numbers after it in the list. Hence the median is 9.

When there is an even number of values, the median is found by taking the mean of the two middle values.

WORKED EXAMPLE

30.8 Find the median of the following salaries: £24,000, £12,000, £16,000, £22,000, £10,000 and £25,000.

Solution The numbers are first arranged in order as £10,000, £12,000, £16,000, £22,000, £24,000 and £25,000. Because there is an even number of values there are two middle figures: £16,000 and £22,000. The mean of these is

$$\frac{16{,}000 + 22{,}000}{2} = 19{,}000$$

The median salary is therefore £19,000.

The mode

A third average is the **mode**.

Key point The **mode** of a set of values is that value that occurs most often.

WORKED EXAMPLE

30.9 Find the mode of the set of numbers

1 1 4 4 5 6 8 8 8 9

Solution The number that occurs most often is 8, which occurs three times. Therefore 8 is the mode. Usually a mode is quoted when we want to represent the most popular value in a set.

Sometimes a set of data may have more than one mode.

WORKED EXAMPLE

30.10 Find the mode of the set of numbers

20 20 21 21 21 48 48 49 49 49

Solution In this example there is no single value that occurs most frequently. The number 21 occurs three times, but so does the number 49. This set has two modes. The data is said to be **bimodal**.

Suppose we have n values x_1, x_2, x_3 up to x_n. Their mean is $\bar{x}$ given by

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The variance is found from the following formula:

Key point

$$\text{variance} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

If you study this carefully you will see that to calculate the variance we must:

- calculate the mean value $\bar{x}$
- subtract the mean from each value in turn, that is find $x_i - \bar{x}$
- square each answer to get $(x_i - \bar{x})^2$
- add up all these squared quantities to get $\sum_{i=1}^{n} (x_i - \bar{x})^2$
- divide the result by n to get

$$\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

which is the variance

The standard deviation is found by taking the square root of the variance:

Key point

$$\text{standard deviation} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

WORKED EXAMPLE

30.11 Find the variance and standard deviation of 10, 15.8, 19.2 and 8.7.

Solution First the mean is found:

$$\bar{x} = \frac{10 + 15.8 + 19.2 + 8.7}{4} = \frac{53.7}{4} = 13.425$$

The calculation to find the variance is given in Table 30.7.

$$\text{variance} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} = \frac{73.049}{4} = 18.262$$

The standard deviation is the square root of the variance:

$$\sqrt{18.262} = 4.273$$

Table 30.7

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
10	$10 - 13.425 = -3.425$	$(-3.425)^2 = 11.731$
15.8	$15.8 - 13.425 = 2.375$	$2.375^2 = 5.641$
19.2	$19.2 - 13.425 = 5.775$	$5.775^2 = 33.351$
8.7	$8.7 - 13.425 = -4.725$	$(-4.725)^2 = 22.326$
Total		73.049

When dealing with a grouped frequency distribution with N classes the formula for the variance becomes

Key point

$$\text{variance} = \frac{\sum_{i=1}^{N} f_i(x_i - \bar{x})^2}{\sum_{i=1}^{N} f_i}$$

As before, the standard deviation is the square root of the variance.

WORKED EXAMPLE

30.12 In a period of 30 consecutive days in July the temperature in °C was recorded as follows:

18	19	20	23	24	24	21	18	17	16
16	17	17	17	18	19	20	20	22	23
24	24	25	23	21	21	20	19	19	18

(a) Produce a grouped frequency distribution showing data grouped from 16 to 17 °C, 18 to 19 °C and so on.
(b) Find the mean temperature of the grouped data.
(c) Find the standard deviation of the grouped data.

Solution (a) The data is grouped using a tally chart as in Table 30.8.

Temperature range (°C)	Tally	Frequency
16–17	~~////~~ /	6
18–19	~~////~~ ///	8
20–21	~~////~~ //	7
22–23	////	4
24–25	~~////~~	5
Total		30

(b) When calculating the mean temperature from the grouped data we do not know the actual temperatures. We do know the frequency of each class. The best we can do is use the midpoint of each class as an estimate of the values in that class. The class midpoints and the calculation to determine the mean are shown in Table 30.9. The mean is then

$$\bar{x} = \frac{\sum_{i=1}^{N} f_i \times x_i}{\sum_{i=1}^{N} f_i} = \frac{603}{30} = 20.1\ ^\circ\text{C}$$

Temperature range (°C)	Frequency f_i	Class midpoint x_i	$f_i \times x_i$
16–17	6	16.5	99.0
18–19	8	18.5	148.0
20–21	7	20.5	143.5
22–23	4	22.5	90.0
24–25	5	24.5	122.5
Total	30		603

(c) To find the variance and hence the standard deviation we must subtract the mean, 20.1, from each value, square and then add the results. Finally this sum is divided by 30. The complete calculation is shown in Table 30.10.

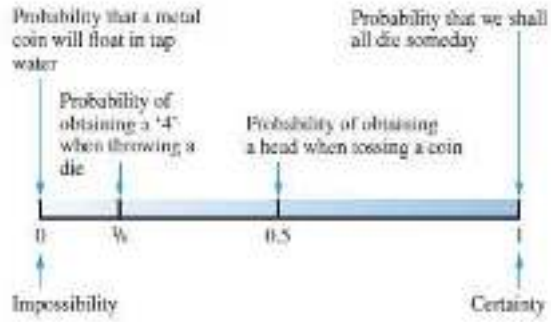
Finally the variance is given by

Temperature range (°C)	Frequency f_i	Class midpoint x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
16–17	6	16.5	−3.60	12.96	77.76
18–19	8	18.5	−1.60	2.56	20.48
20–21	7	20.5	0.40	0.16	1.12
22–23	4	22.5	2.40	5.76	23.04
24–25	5	24.5	4.40	19.36	96.80
Total	30				219.20

$$\text{variance} = \frac{\sum_{i=1}^{N} f_i(x_i - \bar{x})^2}{\sum_{i=1}^{N} f_i} = \frac{219.20}{30} = 7.307$$

The standard deviation is the square root of the variance, that is $\sqrt{7.307} = 2.703$.

Figure 31.1
The probability scale



probability scale in Figure 31.1. It is important to note that no probability can lie outside the range 0 to 1.

Key point All probabilities lie in the range [0, 1].

Complementary events

Consider the following situation. A light bulb is tested. Clearly it either works or it does not work. Here we have two events: the first is that the light bulb works and the second is that the light bulb does not work. When the bulb is tested, one or other of these events must occur. Furthermore, each event excludes the other. In such a situation we say the two events are **complementary**. In general, two events are complementary if one of them must happen and, when it does, the other event cannot. The sum of the probabilities of the two complementary events must always equal 1. This is known as **total probability**. We shall see how this result can be used to calculate probabilities shortly.

- Unbiased = the outcome or probability has the same chance of occurrence as any other outcome or probability
- Theoretical Probability = A probability that is quite unlikely, although not impossible, when all events are equally likely it is calculated from the following formula:

Key point When all events are equally likely

$$P\begin{pmatrix}\text{obtaining our}\\ \text{chosen event}\end{pmatrix} = \frac{\text{number of ways the chosen event can occur}}{\text{total number of possibilities}}$$

WORKED EXAMPLES

31.1 A fair die is thrown. What is the probability of obtaining an even score?

Solution The chosen event is throwing an even score, that is a '2', '4' or '6'. There are therefore three ways that this chosen event can occur out of a total of six, equally likely, ways. So

$$P(\text{even score}) = \frac{3}{6} = \frac{1}{2}$$

31.2 A fair coin is tossed. What is the probability that it will land with its head uppermost?

Solution There are two equally likely ways the coin can land: head uppermost or tail uppermost. The chosen event, that the coin lands with its head uppermost, is just one of these ways. The chance of getting a head is the same as the chance of getting a tail. Therefore,

$$P(\text{head}) = \frac{1}{2}$$

Note from the previous example that $P(\text{tail}) = \frac{1}{2}$ and also that

$$P(\text{tail}) + P(\text{head}) = 1$$

Note that the two events, getting a head and getting a tail, are complementary because one of them must happen and either one excludes the other. Therefore the sum of the two probabilities, the total probability, equals 1.

WORKED EXAMPLE

31.3 Two coins are tossed.

(a) Write down all the possible outcomes.

(b) What is the probability of obtaining two tails?

Solution (a) Letting H stand for head and T for tail, the possible outcomes are

$$H, H \qquad H, T \qquad T, H \qquad T, T$$

There are four possible outcomes, each one equally likely to occur.

(b) Obtaining two tails is just one of the four possible outcomes. Therefore, the probability of obtaining two tails is $\frac{1}{4}$.

In some circumstances we do not have sufficient information to calculate a theoretical probability. We know that if a coin is unbiased the probability of obtaining a head is $\frac{1}{2}$. But suppose the coin is biased so that it is more likely to land with its tail uppermost. We can experiment by tossing the coin a large number of times and counting the number of tails obtained. Suppose we toss the coin 100 times and obtain 65 tails. We can then estimate the probability of obtaining a tail as $\frac{65}{100}$ or 0.65. Such a probability is known as an **experimental probability** and is accurate only if a very large number of experiments are performed. Generally, we can calculate an experimental probability from the following formula:

Key point

$$P\begin{pmatrix}\text{chosen event}\\ \text{occurs}\end{pmatrix} = \frac{\text{number of ways the chosen event occurs}}{\text{total number of times the experiment is repeated}}$$

WORKED EXAMPLES

31.4 A biased die is thrown 1000 times and a score of '6' is obtained on 200 occasions. If the die is now thrown again what is the probability of obtaining a score of '6'?

Solution Using the formula for the experimental probability we find

$$P(\text{throwing a '6'}) = \frac{200}{1000} = 0.2$$

If the die were unbiased the theoretical probability of throwing a '6' would be $\frac{1}{6} = 0.167$, so the die has been biased in favour of throwing a '6'.

31.5 A manufacturer produces microwave ovens. It is known from experience that the probability that a microwave oven is of an acceptable standard is 0.92. Find the probability that an oven selected at random is not of an acceptable standard.

Solution When an oven is tested either it is of an acceptable standard or it is not. An oven cannot be both acceptable and unacceptable. The two events, that the oven is acceptable or that the oven is unacceptable, are therefore complementary. Recall that the sum of the probabilities of complementary events is 1 and so

$$P(\text{oven is not acceptable}) = 1 - 0.92 = 0.08$$

Two events are **independent** if the occurrence of either one in no way affects the occurrence of the other. For example, if an unbiased die is thrown twice the score on the second throw is in no way affected by the score on the first. The two scores are independent. The **multiplication law** for independent events states the following:

Key point

If events A and B are independent, then the probability of obtaining A and B is given by

$$P(A \text{ and } B) = P(A) \times P(B)$$

WORKED EXAMPLE

31.6 A die is thrown and a coin is tossed. What is the probability of obtaining a '6' and a head?

Solution These events are independent since the score on the die in no way affects the result of tossing the coin, and vice versa. Therefore

$$P(\text{throwing a '6' and tossing a head}) = P(\text{throwing a '6'}) \times P(\text{tossing a head})$$

$$= \frac{1}{6} \times \frac{1}{2}$$

$$= \frac{1}{12}$$

When several events are independent of each other the multiplication law becomes

$$P(A \text{ and } B \text{ and } C \text{ and } D \ldots) = P(A) \times P(B) \times P(C) \times P(D) \ldots$$

WORKED EXAMPLE

31.7 A coin is tossed three times. What is the probability of obtaining three heads?

Solution The three tosses are all independent events since the result of any one has no effect on the others. Therefore

$$P(3 \text{ heads}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Summary

In this week, we learned about what an average is, what mean, median, and mode are, what variance and standard deviation is, what complementary and independent events are, and what a theoretical and experimental probability is.

Week 19 Combinatorics and Probabilities Reading Note 2

Notebook:	Computational Mathematics		
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Cornell Notes	**Topic:** **Combinatorics and Probabilities**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 30, 2020

Essential Question:

What is the probability of an event and how can we use the principles of counting to evaluate such probabilities?

Questions/Cues:

- What is the fundamental counting principle?
- What is a permutation?
    - What is the number of permutations of n elements?
    - What is the permutations of n elements taken r at a time?
- What are distinguishable permutations?
- What are combinations?
- What is an experiment, its outcomes, the sample space of the experiment and the related event?
- What is the probability of an event?
- What are mutually exclusive events?
- What are independent events?
- What is the complement of an event?

Notes

EXAMPLE 1 Selecting Pairs of Numbers at Random

You place eight pieces of paper, numbered from 1 to 8, in a box. You draw one piece of paper at random from the box, record its number, and *replace* the paper in the box. Then, you draw a second piece of paper at random from the box and record its number. Finally, you add the two numbers. How many different ways can you obtain a sum of 12?

Solution To solve this problem, count the different ways to obtain a sum of 12 using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

So, a sum of 12 can occur in five different ways.

EXAMPLE 2 Selecting Pairs of Numbers at Random

You place eight pieces of paper, numbered from 1 to 8, in a box. You draw one piece of paper at random from the box, record its number, and *do not* replace the paper in the box. Then, you draw a second piece of paper at random from the box and record its number. Finally, you add the two numbers. How many different ways can you obtain a sum of 12?

Solution To solve this problem, count the different ways to obtain a sum of 12 using two *different* numbers from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can occur in four different ways.

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For example, the number of ways that three events E_1, E_2, and E_3 can occur is

$$m_1 \cdot m_2 \cdot m_3.$$

EXAMPLE 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

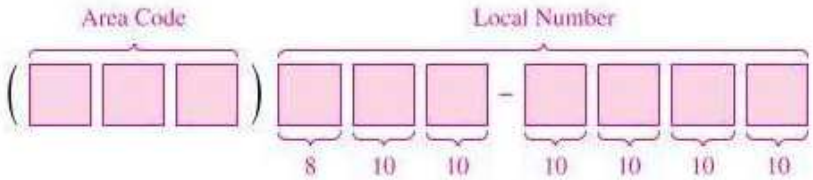
Solution There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter. The English alphabet contains 26 letters, so it follows that the number of two-letter pairs is

$$26 \cdot 26 = 676.$$

EXAMPLE 4 Using the Fundamental Counting Principle

Telephone numbers in the United States have 10 digits. The first three digits are the *area code* and the next seven digits are the *local telephone number*. How many different telephone numbers are possible within each area code? (Note that a local telephone number cannot begin with 0 or 1.)

Solution The first digit of a local telephone number cannot be 0 or 1, so there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of telephone numbers that are possible within each area code is

$$8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000.$$

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of a Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

EXAMPLE 5 Finding the Number of Permutations

How many permutations of the letters

A, B, C, D, E, and F

are possible?

Solution Consider the reasoning below.

First position: Any of the *six* letters

Second position: Any of the remaining *five* letters

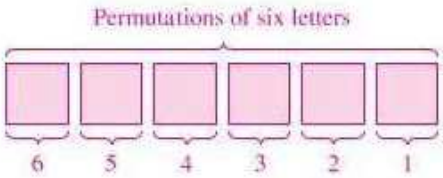
Third position: Any of the remaining *four* letters

Fourth position: Any of the remaining *three* letters

Fifth position: Either of the remaining *two* letters

Sixth position: The *one* remaining letter

So, the numbers of choices for the six positions are as shown in the figure.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

Generalizing the result in Example 5, the number of permutations of n different elements is $n!$.

Number of Permutations of n Elements

The number of permutations of n elements is

$$n \cdot (n-1) \cdot \cdots \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways of ordering n elements.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you may want to order r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time.** The next example demonstrates this ordering.

EXAMPLE 6 Counting Horse Race Finishes

Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

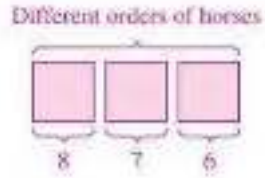
Solution Here are the different possibilities.

Win (first position): *Eight* choices

Place (second position): *Seven* choices

Show (third position): *Six* choices

The numbers of choices for the three positions are as shown in the figure.



So, using the Fundamental Counting Principle, there are

$$8 \cdot 7 \cdot 6 = 336$$

different ways in which the eight horses can come in first, second, and third.

Generalizing the result in Example 6 gives the formula below.

Permutations of n Elements Taken r at a Time

The number of permutations of n elements taken r at a time is

$${}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1).$$

Using this formula, rework Example 6 to find that the number of permutations of eight horses taken three at a time is

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} \\ &= 336 \end{aligned}$$

which is the same answer obtained in the example.

Remember that for permutations, order is important. For example, to find the possible permutations of the letters A, B, C, and D taken three at a time, count A, B, D and B, A, D as different because the *order* of the elements is different.

Consider, however, the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters is ${}_4P_4 = 4!$. However, not all of these arrangements are *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, use the formula below.

Distinguishable Permutations

Consider a set of n objects that has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdots + n_k.$$

The number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!}.$$

EXAMPLE 7 Distinguishable Permutations

See LarsonPrecalculus.com for an interactive version of this type of example.

In how many distinguishable ways can the letters in BANANA be written?

Solution This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways the letters can be written is

$$\begin{aligned}\frac{n!}{n_1! \cdot n_2! \cdot n_3!} &= \frac{6!}{3! \cdot 2! \cdot 1!}\\ &= \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!}\\ &= 60.\end{aligned}$$

The 60 different distinguishable permutations are as listed below.

AAABNN	AAANBN	AAANNB	AABANN
AABNAN	AABNNA	AANABN	AANANB
AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN
ABNANA	ABNNAA	ANAABN	ANAANB
ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB
ANNABA	ANNBAA	BAAANN	BAANAN
BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA
NAAABN	NAAANB	NAABAN	NAABNA
NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA
NBAAAN	NBAANA	NBANAA	NBNAAA
NNAAAB	NNAABA	NNABAA	NNBAAA

Combinations

When you count the number of possible permutations of a set of elements, order is important. As a final topic in this section, you will look at a method of selecting subsets of a larger set in which order is *not* important. Such subsets are called **combinations of *n* elements taken *r* at a time.** For example, the combinations

$$\{A, B, C\} \quad \text{and} \quad \{B, A, C\}$$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important. So, you would count only one of the two sets. Another example of how a combination occurs is in a card game in which players are free to reorder the cards after they have been dealt.

EXAMPLE 8 **Combinations of *n* Elements Taken *r* at a Time**

In how many different ways can three letters be chosen from the letters

A, B, C, D, and E?

(The order of the three letters is not important.)

Solution The subsets listed below represent the different combinations of three letters that can be chosen from the five letters.

{A, B, C}	{A, B, D}
{A, B, E}	{A, C, D}
{A, C, E}	{A, D, E}
{B, C, D}	{B, C, E}
{B, D, E}	{C, D, E}

So, when order is not important, there are 10 different ways that three letters can be chosen from five letters.

Combinations of *n* Elements Taken *r* at a Time

The number of combinations of n elements taken r at a time is

$$_nC_r = \frac{n!}{(n-r)!r!}$$

which is equivalent to $_nC_r = \dfrac{_nP_r}{r!}$.

To see how to use this formula, rework the counting problem in Example 8. In that problem, you want to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$$_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10$$

which is the same answer obtained in the example.

A♥	A♦	A♣	A♠
2♥	2♦	2♣	2♠
3♥	3♦	3♣	3♠
4♥	4♦	4♣	4♠
5♥	5♦	5♣	5♠
6♥	6♦	6♣	6♠
7♥	7♦	7♣	7♠
8♥	8♦	8♣	8♠
9♥	9♦	9♣	9♠
10♥	10♦	10♣	10♠
J♥	J♦	J♣	J♠
Q♥	Q♦	Q♣	Q♠
K♥	K♦	K♣	K♠

Ranks and suits in a standard deck of playing cards
Figure 9.2

EXAMPLE 9 **Counting Card Hands**

A standard poker hand consists of five cards dealt from a deck of 52 (see Figure 9.2). How many different poker hands are possible? (Order is not important.)

Solution To determine the number of different poker hands, find the number of combinations of 52 elements taken five at a time.

$$
\begin{aligned}
{}_{52}C_5 &= \frac{52!}{(52-5)!5!} \\
&= \frac{52!}{47!5!} \\
&= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= 2{,}598{,}960
\end{aligned}
$$

EXAMPLE 10 **Forming a Team**

You are forming a 12-member swim team from 10 girls and 15 boys. The team must consist of five girls and seven boys. How many different 12-member teams are possible?

Solution There are ${}_{10}C_5$ ways of choosing five girls. There are ${}_{15}C_7$ ways of choosing seven boys. By the Fundamental Counting Principle, there are ${}_{10}C_5 \cdot {}_{15}C_7$ ways of choosing five girls and seven boys.

$$
{}_{10}C_5 \cdot {}_{15}C_7 = \frac{10!}{5! \cdot 5!} \cdot \frac{15!}{8! \cdot 7!} = 252 \cdot 6435 = 1{,}621{,}620
$$

So, there are 1,621,620 12-member swim teams possible.

The Probability of an Event

Any happening for which the result is uncertain is an **experiment.** The possible results of the experiment are **outcomes,** the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.**

For example, when you toss a six-sided die, the numbers 1 through 6 can represent the sample space. For this experiment, each of the outcomes is *equally likely.*

To describe sample spaces in such a way that each outcome is equally likely, you must sometimes distinguish between or among various outcomes in ways that appear artificial. Example 1 illustrates such a situation.

EXAMPLE 1 **Finding a Sample Space**

Find the sample space for each experiment.

a. You toss one coin.

b. You toss two coins.

c. You toss three coins.

Solution

a. The coin will land either heads up (denoted by H) or tails up (denoted by T), so the sample space is

$$S = \{H, T\}.$$

b. Either coin can land heads up or tails up, so the possible outcomes are as follows.

HH = heads up on both coins

HT = heads up on the first coin and tails up on the second coin

TH = tails up on the first coin and heads up on the second coin

TT = tails up on both coins

So, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Note that this list distinguishes between the two cases HT and TH, even though these two outcomes appear to be similar.

c. Using notation similar to that used in part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Note that this list distinguishes among the cases HHT, HTH, and THH, and among the cases HTT, THT, and TTH.

To find the probability of an event, count the number of outcomes in the event and in the sample space. The *number of equally likely outcomes* in event E is denoted by $n(E)$, and the number of equally likely outcomes in the sample space S is denoted by $n(S)$. The probability that event E will occur is given by $n(E)/n(S)$.

The Probability of an Event

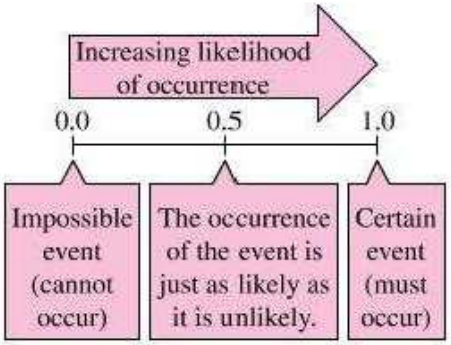
If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is

$$P(E) = \frac{n(E)}{n(S)}.$$

The number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, so the probability of an event must be a number between 0 and 1, inclusive. That is,

$$0 \le P(E) \le 1$$

as shown in the figure. If $P(E) = 0$, then event E *cannot occur,* and E is an **impossible event.** If $P(E) = 1$, then event E *must occur,* and E is a **certain event.**



EXAMPLE 2 Finding the Probability of an Event

See LarsonPrecalculus.com for an interactive version of this type of example.

a. You toss two coins. What is the probability that both land heads up?

b. You draw one card at random from a standard deck of 52 playing cards. What is the probability that it is an ace?

Solution

a. Using the results of Example 1(b), let

$$E = \{HH\} \quad \text{and} \quad S = \{HH, HT, TH, TT\}.$$

The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

b. The deck has four aces (one in each suit), so the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

EXAMPLE 3 Finding the Probability of an Event

You toss two six-sided dice. What is the probability that the total of the two dice is 7?

Solution There are six possible outcomes on each die, so by the Fundamental Counting Principle, there are $6 \cdot 6$ or 36 different outcomes when you toss two dice. To find the probability of rolling a total of 7, you must first count the number of ways in which this can occur.

First Die	Second Die
1	6
2	5
3	4
4	3
5	2
6	1

So, a total of 7 can be rolled in six ways, which means that the probability of rolling a total of 7 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

EXAMPLE 4 **Finding the Probability of an Event**

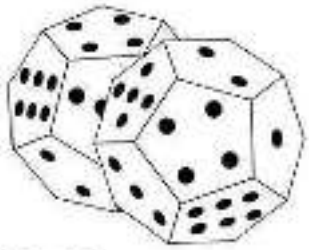


Figure 9.3

Twelve-sided dice, as shown in Figure 9.3, can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 occurs twice on each die. Show that these dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of the various events.

Solution For an ordinary six-sided die, each of the numbers

1, 2, 3, 4, 5, and 6

occurs once, so the probability of rolling any one of these numbers is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For one of the 12-sided dice, each number occurs twice, so the probability of rolling each number is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

EXAMPLE 5 Random Selection

The figure shows the numbers of degree-granting postsecondary institutions in various regions of the United States in 2015. What is the probability that an institution selected at random is in one of the three southern regions? *(Source: National Center for Education Statistics)*

Solution From the figure, the total number of institutions is 4622. There are $858 + 302 + 451 = 1611$ institutions in the three southern regions, so the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1611}{4622} \approx 0.349.$$

EXAMPLE 6 **Finding the Probability of Winning a Lottery**

In Arizona's The Pick game, a player chooses six different numbers from 1 to 44. If these six numbers match the six numbers drawn (in any order), the player wins (or shares) the top prize. What is the probability of winning the top prize when the player buys one ticket?

Solution To find the number of outcomes in the sample space, use the formula for the number of combinations of 44 numbers taken six at a time.

$$\begin{aligned} n(S) &= {}_{44}C_6 \\ &= \frac{44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 \cdot 39}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 7{,}059{,}052 \end{aligned}$$

When a player buys one ticket, the probability of winning is

$$P(E) = \frac{1}{7{,}059{,}052}.$$

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** when A and B have no outcomes in common. In the terminology of sets, the intersection of A and B is the empty set, which implies that

$$P(A \cap B) = 0.$$

For example, when you toss two dice, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, then the probability of A or B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

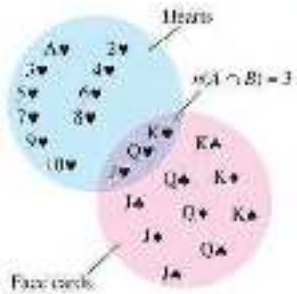


Figure 9.4

EXAMPLE 7 **Probability of a Union of Events**

You draw one card at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

Solution The deck has 13 hearts, so the probability of drawing a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, the deck has 12 face cards, so the probability of drawing a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Three of the cards are hearts and face cards (see Figure 9.4), so it follows that

$$P(A \cap B) = \frac{3}{52}.$$

Finally, applying the formula for the probability of the union of two events, the probability of drawing either a heart or a face card is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &\approx 0.423. \end{aligned}$$

EXAMPLE 8 Probability of Mutually Exclusive Events

The human resources department of a company has compiled data showing the number of years of service for each employee. The table shows the results.

DATA

Spreadsheet at LarsonPrecalculus.com

Years of Service	Number of Employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	35
35–39	21
40–44	8
45 or more	2

a. What is the probability that an employee chosen at random has 4 or fewer years of service?

b. What is the probability that an employee chosen at random has 9 or fewer years of service?

Solution

a. To begin, add the number of employees to find that the total is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. Then the probability of choosing an employee who has 4 or fewer years of service is

$$\begin{aligned} P(A) &= \frac{157}{529} \\ &\approx 0.297. \end{aligned}$$

b. Let event B represent choosing an employee with 5 to 9 years of service. Then

$$P(B) = \frac{89}{529}.$$

Event A from part (a) and event B have no outcomes in common, so these two events are mutually exclusive and

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.465. \end{aligned}$$

So, the probability of choosing an employee who has 9 or fewer years of service is about 0.465.

Independent Events

Two events are **independent** when the occurrence of one has no effect on the occurrence of the other. For example, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. To find the probability that two independent events will occur, *multiply* the probabilities of each.

Probability of Independent Events

If A and B are independent events, then the probability that both A and B will occur is

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

This rule can be extended to any number of independent events.

EXAMPLE 9 Probability of Independent Events

A random number generator selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

Solution Let event A represent selecting a number from 1 to 5. Then the probability of selecting a number from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three numbers are less than or equal to 5 is

$$\begin{aligned} P(A) \cdot P(A) \cdot P(A) &= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \\ &= \frac{1}{64}. \end{aligned}$$

EXAMPLE 10 **Probability of Independent Events**

In 2015, approximately 65% of Americans expected much of the workforce to be automated within 50 years. In a survey, researchers selected 10 people at random from the population. What is the probability that all 10 people expected much of the workforce to be automated within 50 years? *(Source: Pew Research Center)*

Solution Let event A represent selecting a person who expected much of the workforce to be automated within 50 years. The probability of event A is 0.65. Each of the 10 occurrences of event A is an independent event, so the probability that all 10 people expected much of the workforce to be automated within 50 years is

$$[P(A)]^{10} = (0.65)^{10}$$
$$\approx 0.013.$$

The Complement of an Event

The **complement of an event** A is the collection of all outcomes in the sample space that are *not* in A. The complement of event A is denoted by A'. Because $P(A \text{ or } A') = 1$ and A and A' are mutually exclusive, it follows that $P(A) + P(A') = 1$. So, the probability of A' is

$$P(A') = 1 - P(A).$$

Probability of a Complement

Let A be an event and let A' be its complement. If the probability of A is $P(A)$, then the probability of the complement is

$$P(A') = 1 - P(A).$$

For example, if the probability of *winning* a game is $P(A) = \frac{1}{4}$, then the probability of *losing* the game is $P(A') = 1 - \frac{1}{4} = \frac{3}{4}$.

EXAMPLE 11 **Probability of a Complement**

A manufacturer has determined that a machine averages one faulty unit for every 1000 it produces. What is the probability that an order of 200 units will have one or more faulty units?

Solution To solve this problem as stated, you would need to find the probabilities of having exactly one faulty unit, exactly two faulty units, exactly three faulty units, and so on. However, using complements, it is much less tedious to find the probability that all units are perfect and then subtract this value from 1. The probability that any given unit is perfect is 999/1000, so the probability that all 200 units are perfect is

$$P(A) = \left(\frac{999}{1000}\right)^{200} \approx 0.819$$

and the probability that at least one unit is faulty is

$$P(A') = 1 - P(A) \approx 1 - 0.819 = 0.181.$$

Summary

In this week, we learned about what the fundamental counting principle is, what a permutation is, what distinguishable permutations are, what combinations are, what an experiment, its sample space and outcomes are, what the probability of an event is, what

mutually exclusive events are, what independent events are, and what the complement of an event is.

Week 20 Combinatorics and Probabilities continued Lecture Note

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-31 4:06 PM
Author:	SUKHJIT MANN		
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Cornell Notes	**Topic:** **Combinatorics and Probabilities continued**	Course: BSc Computer Science
		Class: Computational Mathematics[Lecture]
		Date: July 31, 2020

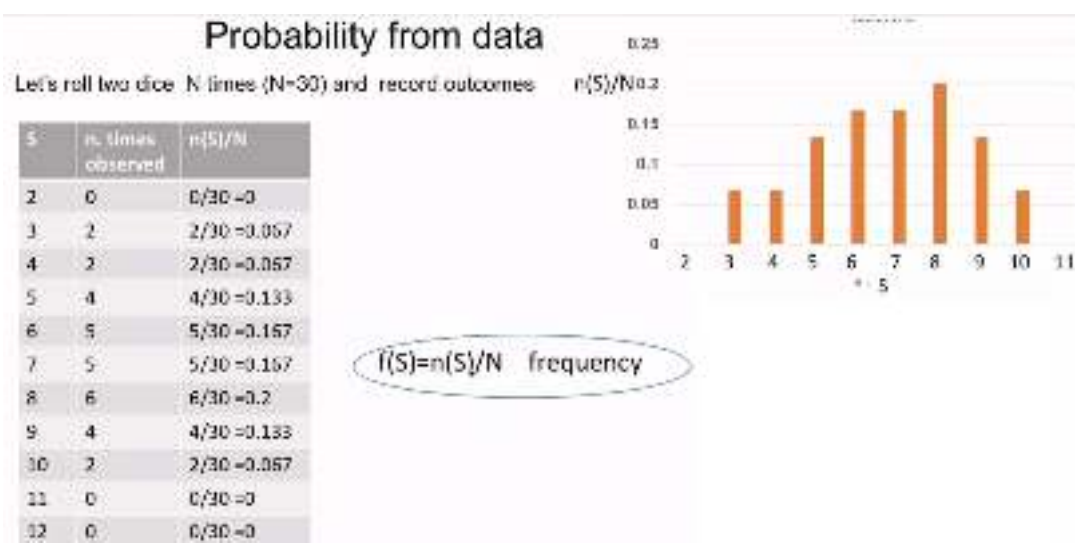
Essential Question:

What is the probability of an event and how can we use the principles of counting to evaluate such probabilities?

Questions/Cues:

- How can we calculate the probability of an event from data and what is frequency?
- What is the experimental definition of probability?
- What is the mean, median and mode?
- What is variance and standard deviation?
- What is normal distribution?
- What is Chebyshev's theorem?

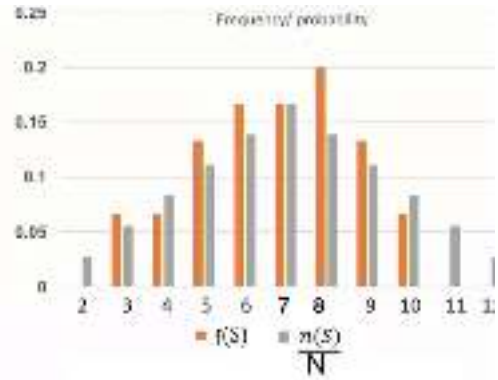
Notes



S	n. times observed	n(S)/N
2	0	0/30 =0
3	2	2/30 =0.067
4	2	2/30 =0.067
5	4	4/30 =0.133
6	5	5/30 =0.167
7	5	5/30 =0.167
8	6	6/30 =0.2
9	4	4/30 =0.133
10	2	2/30 =0.067
11	0	0/30 =0
12	0	0/30 =0

Probability data vs theory

S	n, times observed	n(S)/N	P(S)
2	0	0/30	1/36
3	2	2/30	2/36
4	3	2/30	3/36
5	4	4/30	4/36
6	5	5/30	5/36
7	5	5/30	6/36
8	6	6/30	5/36
9	4	4/30	4/36
10	3	2/30	3/36
11	0	0/30	2/36
12	0	0/30	1/36



Experimental def of probability:

$P(S) = \lim_{N\to\infty} f(S) = \lim_{N\to\infty} \frac{n(S)}{N}$

Statistical analysis of data

Let's say we roll two dice N times and we record outcomes

$S_1\ S_2\ S_3 \ldots\ S_N$

Mean

$m=(S_1+S_2+S_3 \ldots\ S_N)/N = \frac{1}{N}\sum_{i=1}^{N} S_i$

→if only M different results $m = \sum_{i=1}^{M} S_i\, n(S_i)/N$

Median

S_m that separates your ordered data set in two halves

$S_1\ S_2\ S_3 \ldots S_m \ldots S_{N-2} S_{N-1} S_N$

If N odd median=$S_{(N+1)/2}$
If N even median=$(S_{N/2}+S_{N/2+1})/2$

Mode or most probable value
Outcome of maximal probability, that happens most frequently

Mean,median and mode

Let's roll 2 dice N times (N=30) and record outcome

S_i	n. times observed	n(S)/N
2	0	0/30
3	2	2/30
4	2	2/30
5	4	4/30
6	5	5/30
7	5	5/30
8	6	6/30
9	4	4/30
10	2	2/30
11	0	0/30
12	0	0/30

$$m=\frac{1}{N}\sum_{i=1}^{N} S_i = \frac{2x0+3x2+4x2+5x4+6x5+7x5+8x6+9x4+10x2+11x0+12x0}{30} =$$

=203/30=6.8

Median=7

3,3,4,4,5,5,5,5,6,6,6,6,6,7,7,7,7,7,8,8,8,8,8,8,9,9,9, 9

14 14

Mode=8

Variance and standard deviation

Variance

$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(S_i - m)^2$ also $\sum_{i=1}^{M}(S_i - m)^2 n(S_i)/N$

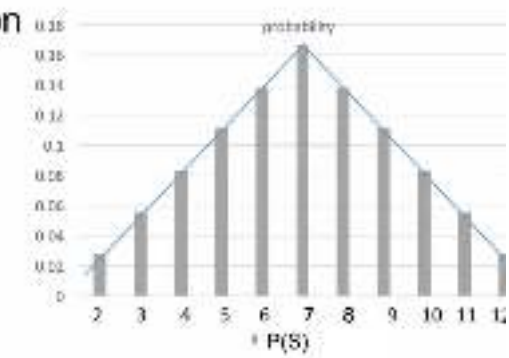
on data $\rightarrow$ $\frac{1}{N-1}\sum_{i=1}^{N}(S_i - m)^2$

standard deviation

$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(S_i - m)^2}$ on data $\rightarrow$ $\sqrt{\frac{1}{(N-1)}\sum_{i=1}^{N}(S_i - m)^2}$

Probability distribution

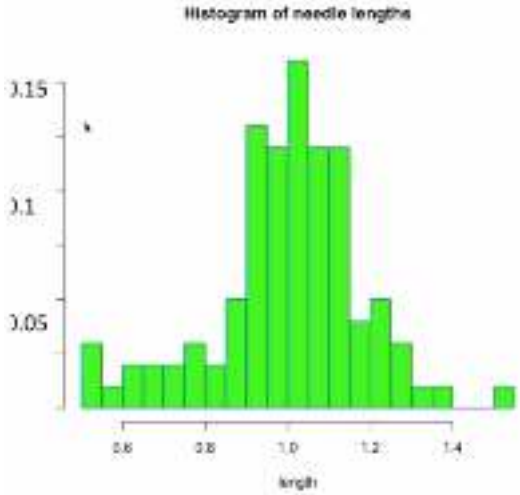
S	P(S)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36



P(S)= (S-1)/36 for 2<=S<=7
P(S)=(13-S)/36 for 7<S<=12

Verify $m=\sum_{i=1}^{11} S_i P(S_i)=7$, $\sigma^2=\sum_{i=1}^{11}(S_i-m)^2 P(S_i)=5.8$, mode=7

Continuous variables: Normal Distribution



100 Needles lengths

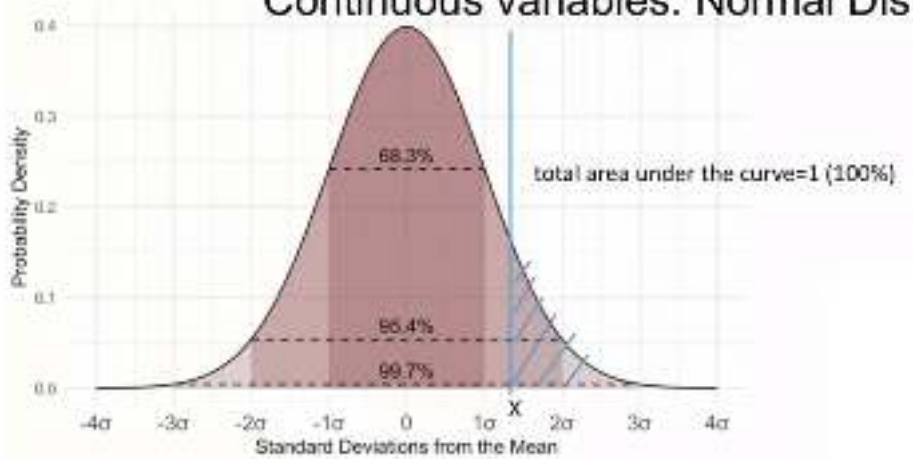
N.	length
1	1.2 cm
2	0.95 cm
3	1.1 cm
4	1.3 cm
5	0.9 cm
6	1.15 cm
7	1.0 cm
...	....
100	0.92 cm

From data: $m=1$ $\sigma^2=0.2$

dx = width of histograms='bin'

Frequency=(n. of observed values in bin)/100(total measurements)

Continuous variables: Normal Distribution



$$p(x)=\frac{e^{-\frac{(x-m)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

- p(X)dx = probability that length falls in interval dx around value X
- p(x>X) = Area marked by blue lines

Chebyshev's theorem

$$p(|x-m|\geq k\sigma) \leq 1/k^2 \quad \text{with } k>0$$

for a wide class of probability distributions $k>1$

⇒ you can take the mean value m as an estimate of the outcome and the variance σ as an estimate of the uncertainty in the outcome

Summary

In this week, we learned about how to calculate probability from data and what frequency is,

what the experimental definition of probability is, what mean, median and mode are, what variance and standard deviation are, what normal distribution is, and what Chebyshev's theorem is.

Binary:

2^0	1	2^4	16	2^8	256
2^1	2	2^5	32	2^9	512
2^2	4	2^6	64	2^{10}	1024
2^3	8	2^7	128		

Octal:

8^0	1
8^1	8
8^2	64
8^3	512
8^4	4096
8^5	32768

Hexadecimal:

Decimal	Hexadecimal
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
10	A
11	B
12	C
13	D
14	E
15	F

16^0	1
16^1	16
16^2	256
16^3	4096
16^4	65536

Non-integers

↳ binary to decimal

$\overset{3\ 2\ 1\ 0\ \ -1\ -2\ -3}{1101.101} = 2^3+2^2+2^0+2^{-1}+2^{-3} = 13.625_{10}$

↳ decimal to binary

17.375

$\rightarrow 10001_2$

0.375×2=0.75=0+0.75
0.75×2= 1.5 = 1+0.5
0.5 ×2= 1.0 = 1+0
0 → STOP ➡ $0.375_{10}=0.011_2$

10001.011_2

Decimal to Octal	Remainder
3726 ÷ 8 = 465 r 6	6
465 ÷ 8 = 58 r 1	1
58 ÷ 8 = 7 r 2	2
7 ÷ 8 = 0 r 7	7

read ↑

7216

Operations:

$$\begin{array}{r} 101 \\ +\ 111 \\ \hline 1100 \end{array} \qquad \begin{array}{r} 110 \\ -\ 101 \\ \hline 001 \end{array} \qquad \begin{array}{r} 1100 \\ \times\ 1111 \\ \hline 1100 \\ 1100 \\ 1100 \\ +\ 1100 \\ \hline 10110100 \end{array}$$

$11100110_2 \div 110_2 = ?$

```
110 | 11100110=
    -110
      10
     -110 won't go   0
      100
     -110 won't go   0
      1001
      -110           1
       0111
       -110          1
         10          0
Remainder→
```

➡ $11100110_2 \div 110_2 = 100110_2$

With remainder 10
in decimal 230/6=38.333

Sequence

$a: N \rightarrow X$

convergent, if $\lim_{n\to\infty} a_n = L < \infty$

else → divergent (goes to ∞ or oscillates)

defined
↳ explicit ex: $a_n = 2n+1$
↳ by recursion

↳ **Arithmetic** ex: $a_n = \underset{\text{common difference}}{d} + a_{n-1} = a_0 + (n-1)d$

Series: $S_n = \sum_{i=0}^{n} a_i = \frac{n}{2}(2a_0 + (n-1)d)$

↳ **Geometric** $a_n = \underset{\text{common ratio}}{r} \times a_{n-1} = a_0 r^{n-1}$

$S_n = \sum_{i=0}^{n} a_i = \frac{a_0(1-r^n)}{1-r}$

Congruence

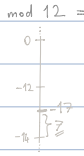
a and b are congruent mod k: $a \equiv b \pmod k$ → maps integers to $Min_k = \{0, 1, \dots, k-1\}$

if they have the same remainder when divided by k

$a \bmod k = b \bmod k$

$a \underset{\times}{\overset{+}{-}} b \bmod k = (a \bmod k \underset{\times}{\overset{+}{-}} b \bmod k) \bmod k$

for negatives: $-17 \bmod 12 = 7$ (not −5)



for division ⇒ Multiplicative Inverse: $(m \times m^{-1}) \bmod k = 1$

ex:
- $\frac{1}{2} \bmod 7 = ?$ $= 2^{-1} \bmod 7$

try (2×0)
(2×1)
⋮
$(2\times\boxed{4}) \bmod 7 = 1$

$2^{-1} \bmod 7 = 4$

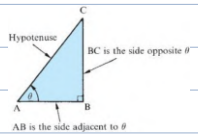
- $\frac{6}{2} \bmod 7 = ?$

$= (6\times2^{-1}) \bmod 7 = (6\times4) \bmod 7 = 24 \bmod 7 = 3$

Trigonometry

Degree Radian

$180° = \pi$



$\sin\theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}} = \frac{BC}{AC}$

$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{AB}{AC}$

$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta} = \frac{BC}{AB}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

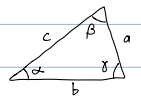
$\sin^2\theta + \cos^2\theta = 1$

$\cos(0°) = \sin(90°) = 1$

$\sin(0°) = \cos(90°) = 0$

$h^2 = a^2 + b^2$

for general triangle



sine rule:
$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

cosine rule:
$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$

$\alpha + \beta + \gamma = 180°$

Functions

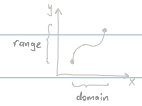
$f(x) = y$

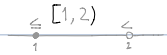
independent

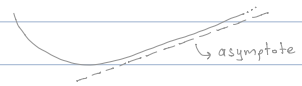
dependent

domain: possible values of x

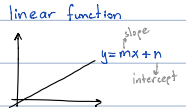
range: " " " y



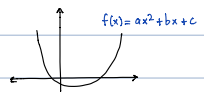


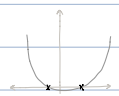


linear function

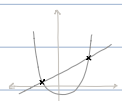


quadratic function





solutions for y=0



solutions for simultaneous equations

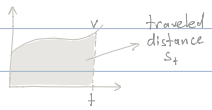
$$x^2 + px + q = 0 \qquad ax^2 + bx + c = 0$$

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Motion

Velocity $v = \frac{\Delta s}{\Delta t}$

Acceleration $a = \frac{\Delta v}{\Delta t}$



$$v_t = v_0 + at$$

$$s_t = v_0 t + \frac{1}{2} a t^2$$

$$s_t = \frac{1}{2}(v_0 + v_t)\, t$$

$$v_t^2 = v_0^2 + 2 a s_t$$