Advanced Machine Learning

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Recap - Shattering

Definition (restriction of $\mathcal{H}$ to C)

Let $\mathcal{H}$ be a set hypothesis, i.e., set of functions from $\mathcal{X}$ to $\{0, 1\}$, and let C be a (finite) subset of $\mathcal{X}$, $C = \{c_{1,} c_2, \ldots, c_m\}$. The restriction of $\mathcal{H}$ to C, denoted by $\mathcal{H}_C$, is the set of functions from C to $\{0, 1\}$ that can be derived from $\mathcal{H}$. That is:

$$\mathcal{H}_C = \{h: C \rightarrow \{0, 1\} \mid h \in \mathcal{H}\}$$

Definition (Shattering)

A hypothesis class $\mathcal{H}$ *shatters* a finite set C of $\mathcal{X}$, if the restriction of $\mathcal{H}$ to C is the set of all functions from C to $\{0, 1\}$. That is $|\mathcal{H}_C| = 2^{|C|}$.

Recap - The VC-dimension

Definition (VC-dimension)

The VC - dimension of a hypothesis class $\mathcal{H}$, denoted VCdim($\mathcal{H}$), is the maximal size of a set $C \subseteq X$ that can be shattered by $\mathcal{H}$. If $\mathcal{H}$ can shatter sets of arbitrarily large size we say that $\mathcal{H}$ has infinite VC-dimension.

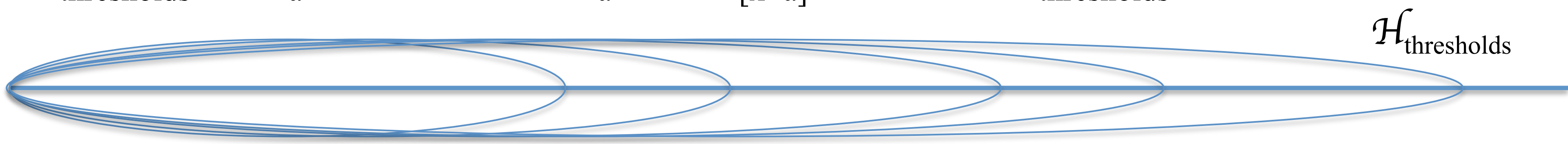
In order to show that the VC-dimension of a hypothesis class $\mathcal{H}$ is d, we need to show that:

1. **There exists a set C of size d that is shattered by $\mathcal{H}$. (VCdim($\mathcal{H}$) $\geq$ d)**
2. **Every set C of size $d + 1$ is not shattered by $\mathcal{H}$. (VCdim($\mathcal{H}$) $<$ d+1)**

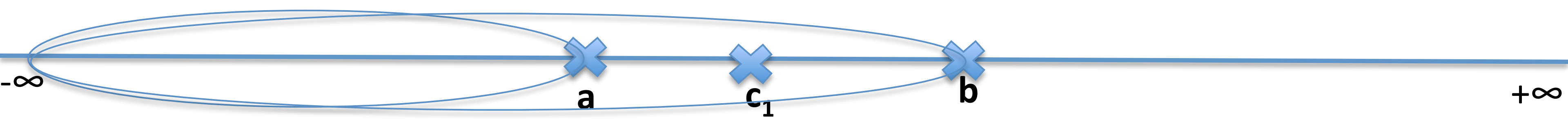
We will see in the next lecture that the converse is also true: ***a finite VC-dimension guarantees learnability***. ***Hence, the VC-dimension characterizes PAC learnability. VC-dimension is a combinatorial measure, does not imply computing probabilities.***

Recap - VCdim($\mathcal{H}_{\text{thresholds}}$)

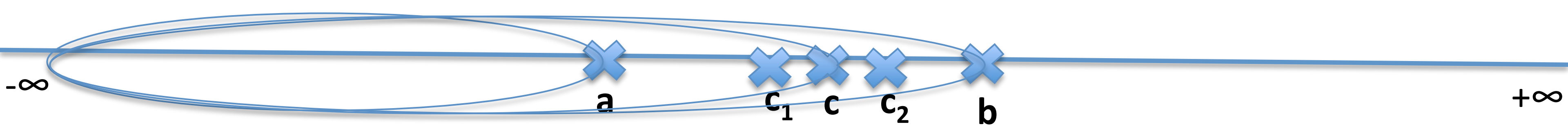
Consider $\mathcal{H} = \mathcal{H}_{\text{thersholds}}$ be the set of threshold functions over the real line.
$\mathcal{H}_{\text{thresholds}} = \{h_a\text{: R} \rightarrow \{0, 1\}, h_a(\text{x}) = \mathbf{1}_{[x<a]}, \text{a} \in \mathbf{R}\}, |\mathcal{H}_{\text{thresholds}}| = \infty.$



Consider $C = \{c_1\}$. Then $\mathcal{H}_C = \{h\text{: } C \rightarrow \{0, 1\}|\ h \in \mathcal{H}\}$ has two elements $\{h_a, h_b\}$ with $a \leq c_1$ and $b > c_1$ so $\mathcal{H}$ shatters C. $\mathcal{H}_C = \{(0),(1)\}$, $|\mathcal{H}_C| = 2^{|C|} = 2^1$



Consider $C = \{c_1, c_2|\ c_1 \leq c_2\}$. Then $\mathcal{H}_C = \{h\text{: } C \rightarrow \{0, 1\}|\ h \in \mathcal{H}\}$ has at most three elements, there is no function that realizes the labeling (0,1) and so $\mathcal{H}$ does not shatter C.

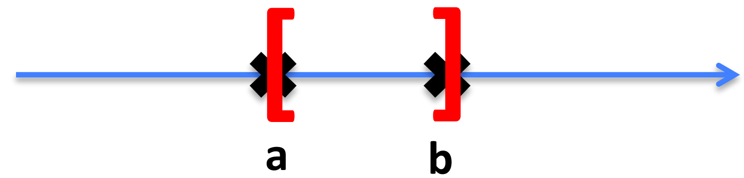


So, **VCdim($\mathcal{H}_{\text{thresholds}}$) = 1**

Recap - VCdim($\mathcal{H}_{\text{intervals}}$)

Consider $\mathcal{H} = \mathcal{H}_{\text{intervals}}$ be the set of intervals over the real line.

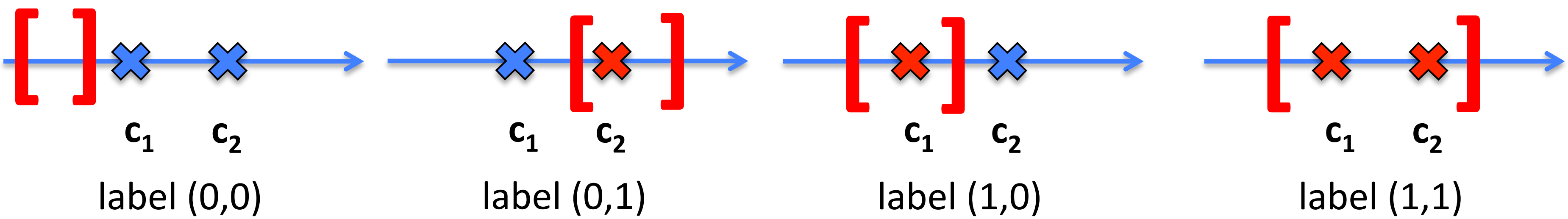
$\mathcal{H}_{\text{intervals}} = \{[a,b] \mid a \leq b,\ a, b \in \mathbf{R}\}$



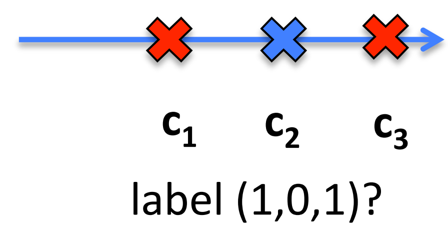
Can also view $\mathcal{H}_{\text{intervals}}$ as:

$\mathcal{H}_{\text{intervals}} = \{h_{a,b}: \mathbf{R} \rightarrow \{0, 1\},\ h_{a,b} = \mathbf{1}_{[a,b]},\ a \leq b,\ a, b \in \mathbf{R}\}$

$\mathcal{H}_{\text{intervals}}$ shatters any set A of two different points in $\mathbf{R}$.



$\mathcal{H}_{\text{intervals}}$ cannot shatter any set A of three points in $\mathbf{R}$.



So, $\color{red}{\textbf{VCdim}(\mathbf{H}_{\text{intervals}}) = 2}$

VCdim($\mathcal{H}_{\text{lines}}$), VCdim($\mathcal{H}_{\text{rec}}{}^2$)

Consider $\mathcal{H} = \mathcal{H}_{\text{lines}}$ be the set of lines in $\mathbf{R}^2$.

$\mathcal{H}_{\text{lines}} = \{h_{a,b,c}: \mathbf{R}^2 \to \{0, 1\}, h_{a,b,c}((x,y)) = \mathbf{1}_{[ax+by+c>0]}((x,y))$, a, b, c $\in \mathbf{R}\}$

$\mathcal{H}_{\text{lines}}$ shatters any set A of three non-colinnear points in $\mathbf{R}^2$.

$\mathcal{H}_{\text{lines}}$ doesn't shatter any set A of four points in $\mathbf{R}^2$ (geometric argument).

So, **VCdim($\mathbf{H}_{\text{lines}}$) = 3**

Consider $\mathcal{H} = \mathcal{H}_{\text{rec}}{}^2$ be the set of axis aligned rectangles in the $\mathbf{R}^2$.

$\mathcal{H}_{\text{rec}}{}^2 = \{[a,b] \times [c,d] |\ a \leq b,\ c \leq d,\ a, b, c, d \in \mathbf{R}\ \}$

$\mathcal{H}_{\text{rec}}{}^2$ shatters the set A of 4 points arranged as a diamond. So VCdim($\mathcal{H}_{\text{rec}}{}^2$) ≥ 4

$\mathcal{H}_{\text{rec}}{}^2$ doesn't shatter any set of A of five pints in $\mathbf{R}^2$ (geometric argument).

So, **VCdim($\mathcal{H}_{\text{rec}}{}^2$) = 4**

Some basic properties of the VCdim($\mathcal{H}$)

1. VCdim($\mathcal{H}$) $\leq \log_2|\mathcal{H}|$
2. If $\mathcal{H}_1 \subseteq \mathcal{H}_2$ then VCdim($\mathcal{H}_1$) $\leq$ VCdim($\mathcal{H}_2$)
3. If VCdim($\mathcal{H}$) $= \infty$ then $\mathcal{H}$ is not PAC learnable

Today’s lecture: Overview

- Computing the VC-dimension for some particular $\mathcal{H}$
- Assignment 1

VCdim($\mathcal{H}_{\text{sin}}$)

VCdim($\mathcal{H}_{\text{thresholds}}$) = 1, VCdim($\mathcal{H}_{\text{intervals}}$) = 2, VCdim($\mathcal{H}_{\text{lines}}$) = 3

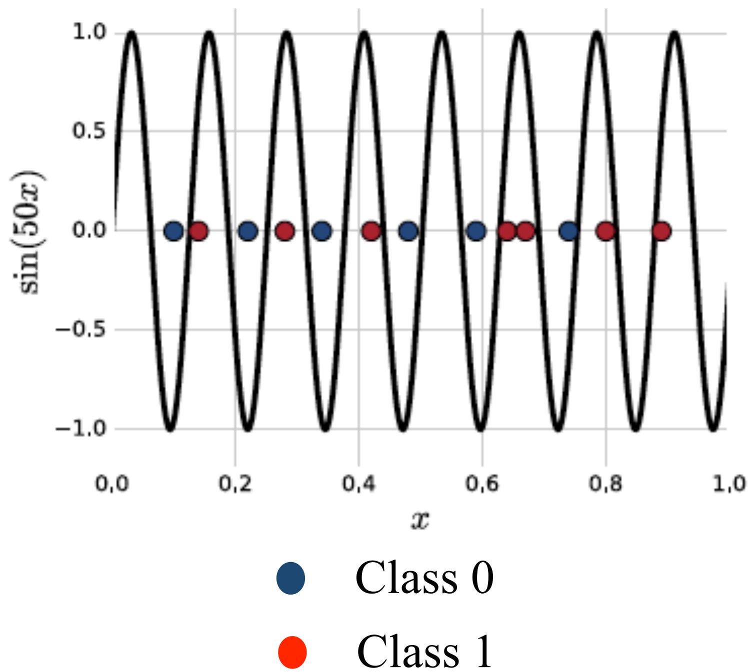
VCdim($\mathcal{H}_{\text{rec}^2}$) = 4

Consider $\mathcal{H} = \mathcal{H}_{\text{sin}}$ be the set of sin functions:

$\mathcal{H}_{\text{sin}} = \{h_\theta: \mathbf{R} \rightarrow \{0,1\} \mid h_\theta(x) = \lceil \sin(\theta x) \rceil,\ \theta \in \mathrm{R}\}$, $\lceil -1 \rceil = 0$

VCdim($\mathcal{H}_{\text{sin}}$) = ?

We will show that VCdim($\mathcal{H}_{\text{sin}}$) = ∞



VCdim($\mathcal{H}_{\sin}$)

Lemma

Let x $\in (0,1)$ and let $0.x_1x_2x_3\ldots$ be the binary representation of x. Then, for any natural number m, provided that there exist k ≥ m such that $x_k = 1$, we have:

$$\left\lceil \sin(2^m \pi x) \right\rceil = 1 - x_m$$

Example of binary representation:

$x = (0.x_1x_2x_3\ldots)_2 = x_1 \times 2^{-1} + x_2 \times 2^{-2} + x_3 \times 2^{-3} + \ldots$

$x = 0.75 = ½ + ¼ = (0.110000\ldots.)_2$

$x = 0.3 = 0 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5} + \ldots = (0.01001\ldots)_2$

VCdim($\mathcal{H}_{\sin}$)

Lemma

Let $x \in (0,1)$ and let $0.x_1x_2x_3\ldots$ be the binary representation of x. Then, for any natural number m, provided that there exist $k \geq m$ such that $x_k = 1$, we have:

$$\lceil \sin(2^m \pi x) \rceil = 1 - x_m$$

Proof

$\sin(2^m\pi x) = \sin(2^m\pi(0.x_1x_2x_3\ldots)) = \sin(2\pi * 2^{m-1}(0.x_1x_2x_3\ldots)) =$

$= \sin(2\pi*(x_1x_2x_3\ldots x_{m-1}.x_m x_{m+1}\ldots))$ (left shift with m-1 position)

$= \sin(2\pi*(x_1x_2x_3\ldots x_{m-1}.x_m x_{m+1}\ldots) - 2\pi*(x_1x_2x_3\ldots x_{m-1}.0))$ (sin has period 2π)

$= \sin(2\pi*(0.x_m x_{m+1}\ldots))$

Note that $0.x_m x_{m+1}\ldots > 0$ as there exist $k \geq m$ such that $x_k = 1$

Case 1: $x_m = 0$, then $0 < 2\pi*(0.x_m x_{m+1}\ldots) < 2\pi*1/2 = \pi$. So $0 < \sin(2^m\pi x) < 1$, and from here it results that: $\lceil \sin(2^m \pi x) \rceil = 1 = 1 - 0 = 1 - x_m$

Case 2: $x_m = 1$, then $2\pi > 2\pi*(0.x_m x_{m+1}\ldots) \geq 2\pi*1/2 = \pi$. So $-1 \leq \sin(2^m\pi x) \leq 0$, and from here it results that: $\lceil \sin(2^m \pi x) \rceil = 0 = 1 - 1 = 1 - x_m$ (we consider $\lceil -1 \rceil = 0$)

VCdim($\mathcal{H}_{\text{sin}}$)

To prove VCdim($\mathcal{H}_{\text{sin}}$) = ∞, we need to pick n points which are shattered by $\mathcal{H}_{\text{sin}}$, for any n. To do so, we construct n points $x_1, x_2, \ldots, x_n \in [0, 1]$, such that the set of the m-th bits in the binary expansion, as m ranges from 1 to 2^n, ranges over all possible labelings of $x_1, x_2, \ldots, x_n$.

x_1 = 0. 0 0 0 0 ... 1 1

x_2 = 0. 0 0 0 0 ... 1 1

...

x_{n-1} = 0 0 0 1 1 ... 1 1

x_n = 0. 0 1 0 1 ... 0 1

m=1

For example, to give the labeling 1 for all instances, we just pick m=1:

$$h(x) = \lceil \sin(2^1 \pi x) \rceil = 1 - first\_bit\_of\_binary\_repres\_of\_x$$

$$\mathcal{H}_{\text{sin}} = \{h_\theta : \mathbf{R} \rightarrow \{0,1\} \mid h_\theta(x) = \lceil \sin(\theta x) \rceil, \theta \in \mathrm{R}\},$$

which returns 1 - the first bit (column) in the binary expansion.

VCdim($\mathcal{H}_{\sin}$)

To prove VCdim($\mathcal{H}_{\sin}$) = ∞, we need to pick *n* points which are shattered by $\mathcal{H}_{\sin}$, for any *n*. To do so, we construct *n* points $x_1, x_2, \ldots, x_n \in [0, 1]$, such that the set of the *m*-th bits in the binary expansion, as m ranges from 1 to 2^n, ranges over all possible labelings of $x_1, x_2, \ldots, x_n$.

$$
\begin{array}{lcl}
x_1 & = & 0.\ 0\ 0\ 0\ 0\ \ldots\ \ 1\ 1 \\
x_2 & = & 0.\ 0\ 0\ 0\ 0\ \ldots\ \ 1\ 1 \\
 & \ldots & \\
x_{n-1} & = & 0.\ 0\ 0\ 1\ 1\ \ldots\ \ 1\ 1 \\
x_n & = & 0.\ 0\ 1\ 0\ 1\ \ldots\ \ 0\ 1
\end{array}
$$

m=2

If we wish to give the labeling 1 for $x_1, x_2, \ldots, x_{n-1}$, and the labeling 0 for x_n, we pick m=2:

$$h(x) = \left\lceil \sin(2^2 \pi x) \right\rceil = 1 - second\_bit\_of\_binary\_repres\_of\_x$$

which returns 1 - the second bit (column) in the binary expansion.

VCdim($\mathcal{H}_{\sin}$)

To prove VCdim($\mathcal{H}_{\sin}$) = ∞, we need to pick *n* points which are shattered by $\mathcal{H}_{\sin}$, for any *n*. To do so, we construct *n* points $x_1, x_2, \ldots, x_n \in [0, 1]$, such that the set of the *m*-th bits in the binary expansion, as m ranges from 1 to 2^n, ranges over all possible labelings of $x_1, x_2, \ldots, x_n$.

$$
\begin{aligned}
x_1 &= 0.\,0\,0\,0\,0\ldots\;1\;1 \\
x_2 &= 0.\,0\,0\,0\,0\ldots\;1\;1 \\
&\ldots \\
x_{n-1} &= 0.\,0\,0\,1\,1\ldots\;1\;1 \\
x_n &= 0.\,0\,1\,0\,1\ldots\;0\;1
\end{aligned}
$$

$m=2^n$

If we wish to give the labeling 0 for $x_1, x_2, \ldots, x_{n-1}, x_n$ we pick $m=2^n$:

$$h(x) = \left\lceil \sin(2^{2^n} \pi x) \right\rceil$$

which returns 1 - the bit on position 2^n (column) in the binary expansion.

VCdim($\mathcal{H}_{sin}$)

To prove VCdim($\mathcal{H}_{sin}$) = ∞, we need to pick *n* points which are shattered by $\mathcal{H}_{sin}$, for any *n*. To do so, we construct *n* points $x_1, x_2, \ldots, x_n \in [0, 1]$, such that the set of the *m*-th bits in the binary expansion, as m ranges from 1 to 2^n, ranges over all possible labelings of $x_1, x_2, \ldots, x_n$.

x_1 = 0. 0 0 0 0 ... 1 1

x_2 = 0. 0 0 0 0 ... 1 1

...

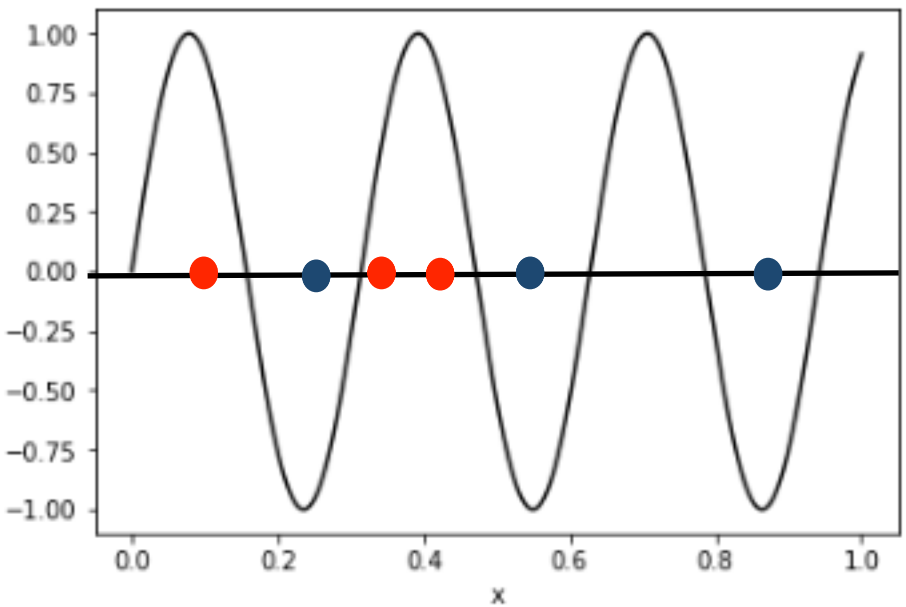
x_{n-1} = 0. 0 0 1 1 ... 1 1

x_n = 0. 0 1 0 1 ... 0 1

We conclude that $x_1, x_2, \ldots, x_{n-1}, x_n$ can be given any labeling by some $h \in \mathcal{H}_{sin}$, so it is shattered. This can be done for any *n*, so VCdim($\mathcal{H}_{sin}$) = ∞.

VCdim($\mathcal{H}_{\text{sin}}$)

We conclude that $x_1, x_2, \ldots, x_{n-1}, x_n$ can be given any labeling by some $h \in \mathcal{H}_{\text{sin}}$, so it is shattered. This can be done for any n, so VCdim($\mathcal{H}_{\text{sin}}$) = ∞.



Training points

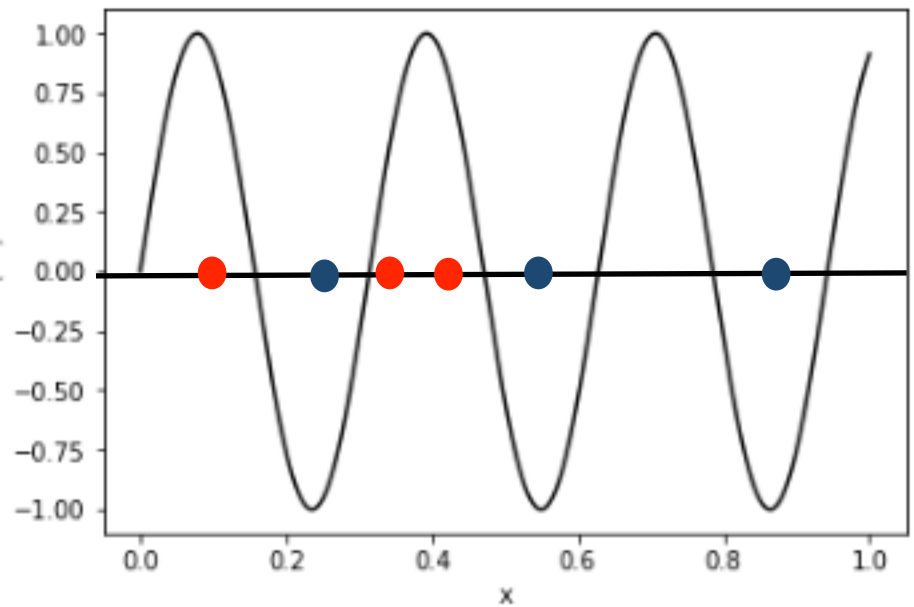
Class 0

Class 1

$$h(x) = \lceil \sin(20x) \rceil$$

VCdim($\mathcal{H}_{\text{sin}}$)

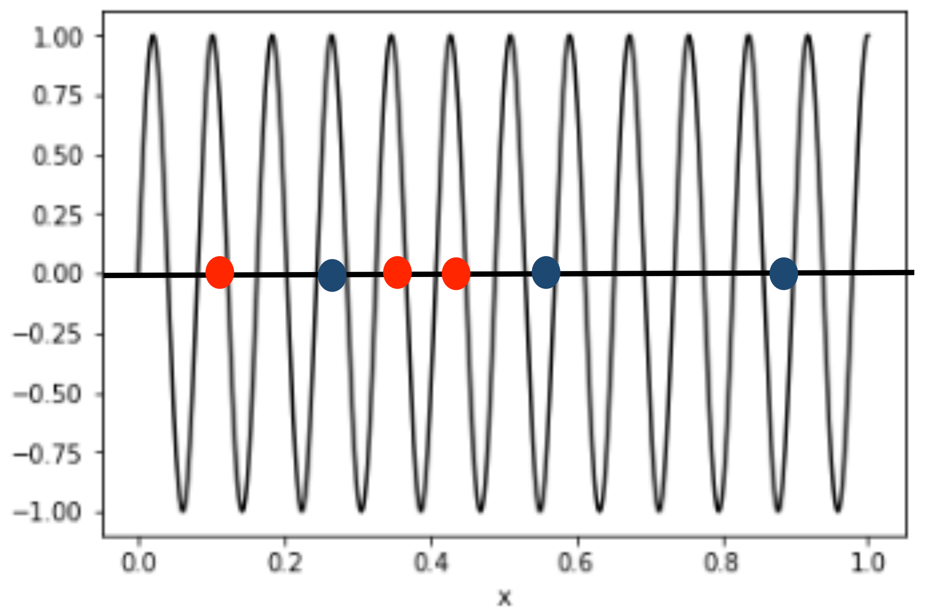
We conclude that $x_1, x_2, \ldots, x_{n-1}, x_n$ can be given any labeling by some $h \in \mathcal{H}_{\text{sin}}$, so it is shattered. This can be done for any n, so VCdim($\mathcal{H}_{\text{sin}}$) = ∞.



Class 0

Class 1

$$h(x) = \lceil \sin(20x) \rceil$$



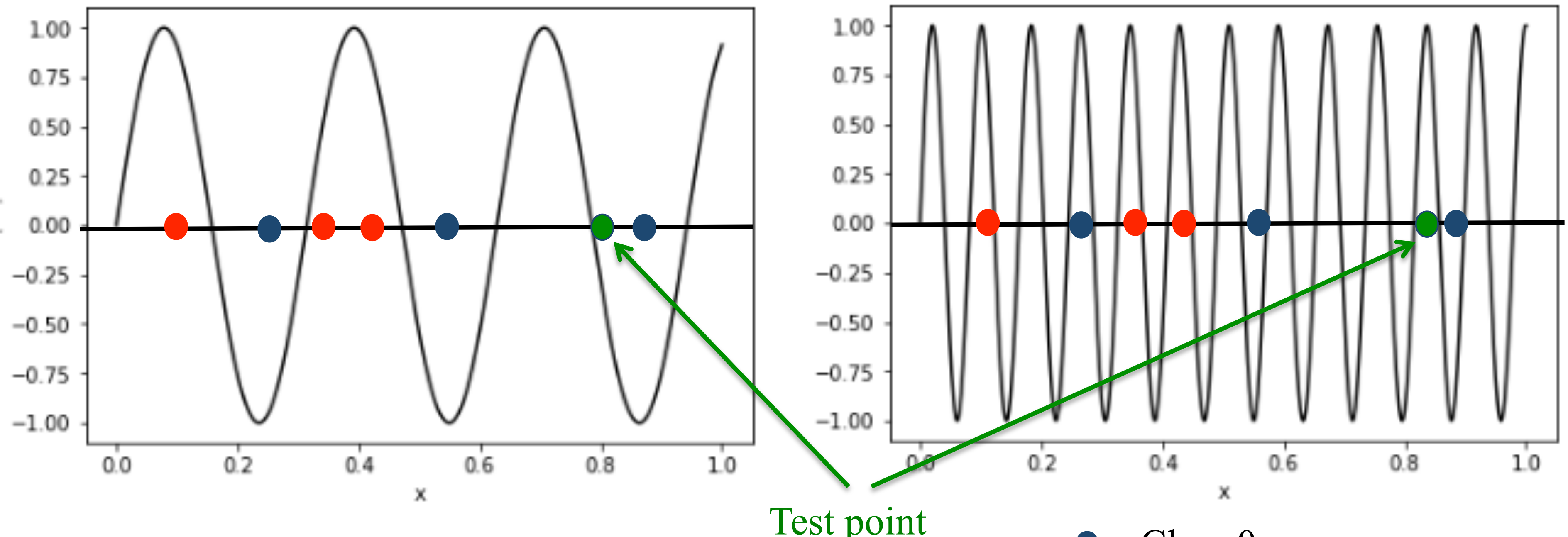
Class 0

Class 1

$$h(x) = \lceil \sin(77x) \rceil$$

VCdim($\mathcal{H}_{\sin}$)

We conclude that $x_1, x_2, \ldots, x_{n-1}, x_n$ can be given any labeling by some $h \in \mathcal{H}_{\sin}$, so it is shattered. This can be done for any n, so VCdim($\mathcal{H}_{\sin}$) = ∞.



Class 0

Class 1

$$h(x) = \left[\sin(20x)\right]$$

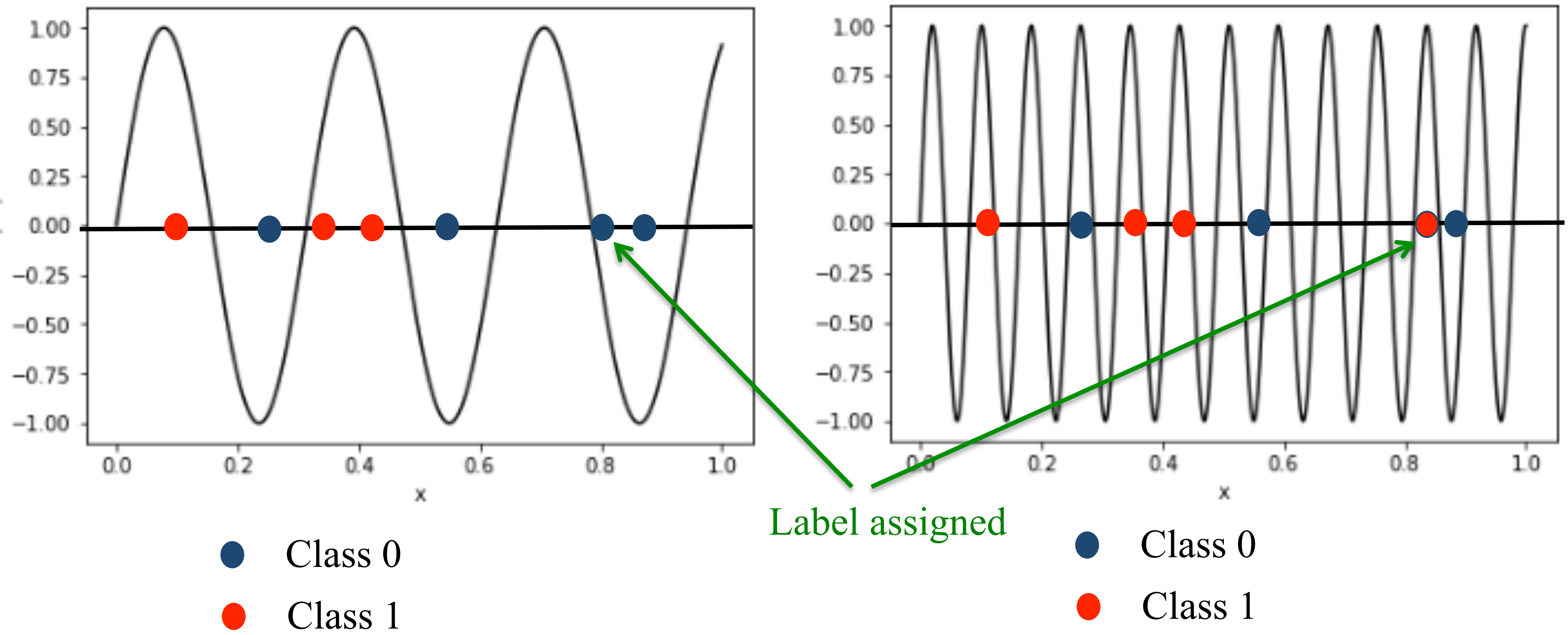
Class 0

Class 1

$$h(x) = \left[\sin(77x)\right]$$

VCdim($\mathcal{H}_{\sin}$)

We conclude that $x_1, x_2, \ldots, x_{n-1}, x_n$ can be given any labeling by some $h \in \mathcal{H}_{\sin}$, so it is shattered. This can be done for any n, so VCdim($\mathcal{H}_{\sin}$) = ∞.



$$h(x) = \left[\sin(20x)\right] \qquad h(x) = \left[\sin(77x)\right]$$

VCdim($\mathcal{HS}_0{}^{n}$)

Consider $\mathcal{H} = \mathcal{HS}^{n}$ be the set of halfspaces (linear classifiers) in R^n

$$\mathcal{H} = \mathcal{HS}^{n} = \{h_{w,b}: \mathbf{R}^n \to \{-1, 1\},\ h_{w,b}(x) = sign\left(\sum_{i=1}^{n} w_i x_i + b\right) \mid w \in \mathbf{R}^n,\ b \in \mathbf{R}\}$$

Consider label -1 to correspond to label 0, they are basically the same.

For n = 2 we have:

$$\mathcal{HS}^{2} = \mathcal{H}_{\text{lines}} = \{h_{a,b,c}: \mathbf{R}^2 \to \{0, 1\},\ h_{a,b,c}((x,y)) = \mathbf{1}_{[ax+by+c>0]}((x,y))\ ,\ a, b, c \in \mathbf{R}\}$$

Let us restrict our attention to "homogenous" linear classifiers, the ones that go through origin, b = 0.

$$\mathcal{HS}_0{}^{n} = \{h_{w,0}: \mathbf{R}^n \to \{-1, 1\},\ h_{w,0}(x) = sign\left(\sum_{i=1}^{n} w_i x_i\right) \mid w \in \mathbf{R}^n\}$$

What is the VCdim($\mathcal{HS}_0{}^{n}$)?

VCdim($\mathcal{H}S_0^{\,n}$)

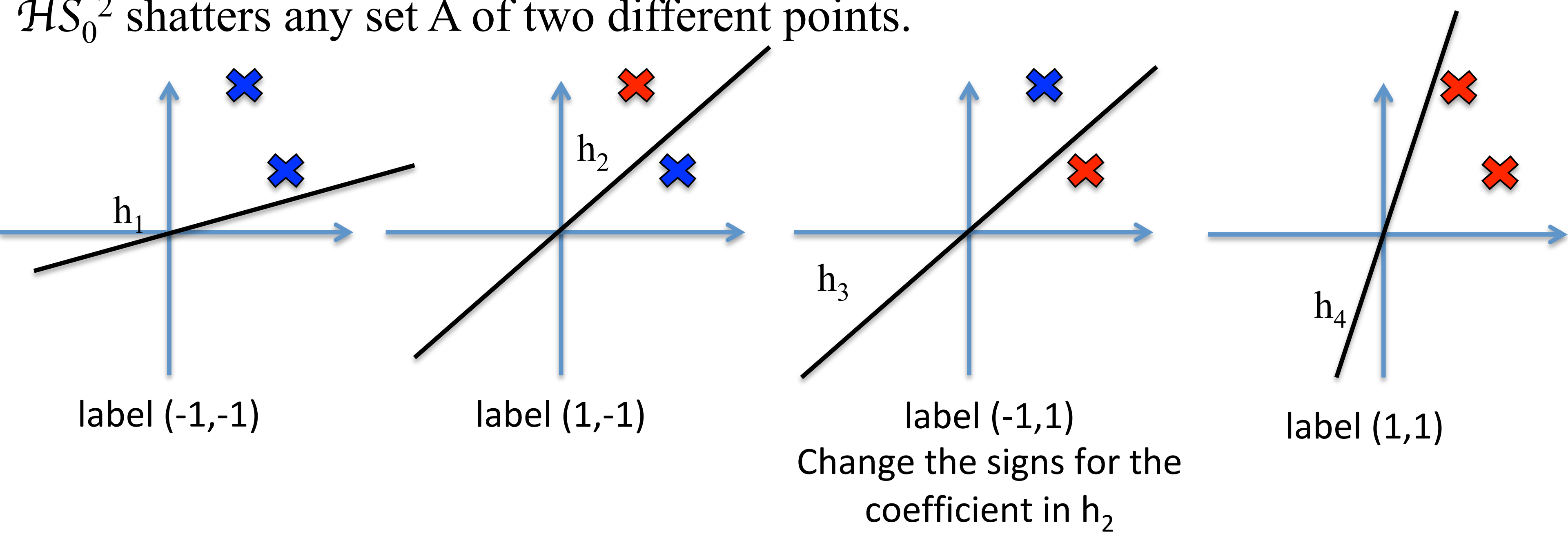
$\mathcal{H}S_0^{\,n} = \{h_{w,0}: \mathbf{R}^n \rightarrow \{-1, 1\},\ h_{w,0}(x) = sign\left(\sum_{i=1}^{n} w_i x_i\right) \mid w \in \mathbf{R}^n\}$

For n = 2 we have:

$\mathcal{H}S_0^{\,2} = \{h_{w1,w2}: \mathbf{R}^2 \rightarrow \{-1, 1\},\ h_{w1,w2}(x) = sign(w_1x_1 + w_2x_2) \mid (w_1,w_2) \in \mathbf{R}^2\}$

What is the VCdim($HS_0^{\,2}$) ?

$\mathcal{H}S_0^{\,2}$ shatters any set A of two different points.



Does $HS_0^{\,2}$ shatter a set A of three points?

Difficult to reason geometrically… choose the algebraic proof.

VCdim($\mathcal{H}S_0^n$)

We will show that VCdim($\mathcal{H}S_0^n$) = n.

Proof: 1st part

We first show that VCdim($\mathcal{H}S_0^n$) $\geq n$.

We find a set A consisting of n points in $\mathbf{R}^n$ that is shattered by $\mathcal{H}S_0^n$.

Take A = $\{e_1, e_2, \ldots, e_n\}$ to be the orthonormal basis of $\mathbf{R}^n$.

$e_1 = (1, 0, 0, \ldots, 0)$; $e_2 = (0, 1, 0, \ldots, 0)$; …. ; $e_n = (0, 0, 0, \ldots, 1)$

We want to proof that $\mathcal{H}S_0^n$ shatters A, so that VCdim($\mathcal{H}S_0^n$) $\geq n$. This is equivalent to proof that for every B $\subseteq$ A, there is a function $h_B \in \mathcal{H}S_0^n$ such that h_B gives label +1 to all elements in B and label -1 to all elements of A \ B.

Pick B subset of A, B $\subseteq$ $\{e_1, e_2, \ldots, e_n\}$. Choose $w = (w_1,w_2,\ldots,w_n)$ such that:

$$w_i = \begin{cases} 1, \text{ if } e_i \in B \\ -1, \text{ if } e_i \notin B \end{cases}$$

Then, $h_B(e_i) = \mathrm{sign}(\langle w,e_i \rangle) = w_i$ will generate the labels +1 for elements in B, -1 for elements not in B

VCdim($\mathcal{H}S_0^{\,n}$)

Proof: 2nd part

We now show that VCdim($\mathcal{H}S_0^{\,n}$) < n + 1.

We will prove that given any set A = $\{x_1, x_2, \ldots, x_{n+1}\}$ of n + 1 points in $\mathbf{R}^n$, A cannot be shattered by $\mathcal{H}S_0^{\,n}$.

The points $\{x_1, x_2, \ldots, x_{n+1}\}$ "live" in $\mathbf{R}^n$, a vector space with dimension n. So, $\{x_1, x_2, \ldots, x_{n+1}\}$ are linearly dependent and there exist coefficients $a_1, a_2, \ldots$ a_{n+1} not all of them 0 such that:

$$\sum_{i=1}^{n+1} a_i x_i = 0$$

Take P ⊆ { 1, 2, …, n +1} the set of strictly positive coefficients a_i and N ⊆ { 1, 2, …, n +1} the set of negative coefficients of a_i. So we have:

$$\sum_{i \in P} a_i x_i = \sum_{j \in N} |a_j| x_j$$

VCdim($\mathcal{H}S_0^n$)

Take $P \subseteq \{1, 2, \ldots, n+1\}$ the set of strictly positive coefficients a_i and $N \subseteq \{1, 2, \ldots, n+1\}$ the set of negative coefficients of a_i. Both P and N cannot be at the same time empty. So we have:

$$\sum_{i \in P} a_i x_i = \sum_{j \in N} |a_j| x_j$$

Assume that A is shattered by $\mathcal{H}S_0^n$ and take $B = \{x_i | \ i \in P\}$. In particular, there exist h_B such that it realizes the label consisting of +1 for all $x_i \in B$ and -1 for all $x_i \notin B$.

So, we have that $h_B(x_i) = 1$, if $x_i \in B$, meaning that $\langle w_B, x_i \rangle \geq 0$ if $x_i \in B$ and $h_B(x_i) = -1$, if $x_i \notin B$, meaning that $\langle w_B, x_i \rangle < 0$ if $x_i \notin B$

So, we have that $h_B\left(\sum_{i \in P} a_i x_i\right) = sign(\left\langle w_B, \sum_{i \in P} a_i x_i \right\rangle) = sign(\sum_{i \in P} a_i \langle w_B, x_i \rangle)$

But $a_i > 0$ (because $i \in P$) and also $\langle w_B, x_i \rangle \geq 0$ as $x_i \in B$, so we obtain that:

$$h_B\left(\sum_{i \in P} a_i x_i\right) = sign(\left\langle w_B, \sum_{i \in P} a_i x_i \right\rangle) = sign(\sum_{i \in P} a_i \langle w_B, x_i \rangle) = 1$$

VCdim($\mathcal{H}S_0^{\text{n}}$)

Take P $\subseteq$ { 1, 2, …, n +1} the set of strictly positive coefficients a_i and N $\subseteq$ { 1, 2, …, n +1}the set of negative coefficients of a_i. Both P and N cannot be at the same time empty. So we have:

$$\sum_{i\in P} a_i x_i = \sum_{j\in N} |a_j| x_j$$

Assume that A is shattered by $\mathcal{H}S_0^{\text{n}}$ and take B = $\{x_i | \ i \in P\}$. In particular, there exist h_B such that it realizes the label consisting of +1 for all $x_i \in$ B and -1 for all $x_i \notin$ B.

So, we have that $h_B(x_i) = 1$, if $x_i \in$ B, meaning that $<w_B, x_i> \geq 0$ if $x_i \in$ B and $h_B(x_i) = -1$, if $x_i \notin$ B, meaning that $<w_B, x_i> < 0$ if $x_i \notin$ B

On the other hand, we have that $h_B\left(\sum_{j\in N} |a_j| x_j\right) = sign\left(\left\langle w_B, \sum_{j\in N} |a_j| x_j\right\rangle\right) = sign\left(\sum_{j\in N} |a_j| \langle w_B, x_j\rangle\right)$

But $|a_j| > 0$ and also $<w_B, x_j> < 0$ as $x_j \notin$ B, so we obtain that:

$$h_B\left(\sum_{j\in N} |a_j| x_j\right) = sign\left(\left\langle w_B, \sum_{j\in N} |a_j| x_j\right\rangle\right) = sign\left(\sum_{j\in N} |a_j| \langle w_B, x_j\rangle\right) = -1$$

VCdim($\mathcal{H}S_0^n$)

Take $P \subseteq \{1, 2, \ldots, n+1\}$ the set of strictly positive coefficients a_i and $N \subseteq \{1, 2, \ldots, n+1\}$ the set of negative coefficients of a_i. Both P and N cannot be at the same time empty. So we have:

$$\sum_{i \in P} a_i x_i = \sum_{j \in N} |a_j| x_j$$

Assume that A is shattered by $\mathcal{H}S_0^n$ and take $B = \{x_i \mid i \in P\}$. In particular, there exist h_B such that it realizes the label consisting of +1 for all $x_i \in B$ and -1 for all $x_i \notin B$. So $h_B(x_i) = 1$, if $x_i \in B$ and $h_B(x_i) = 1$, if $x_i \notin B$

$$\sum_{i \in P} a_i x_i = \sum_{j \in N} |a_j| x_j$$

$$h_B\left(\sum_{i \in P} a_i x_i\right) = sign(\left\langle w_B, \sum_{i \in P} a_i x_i \right\rangle) = sign(\sum_{i \in P} a_i \langle w_B, x_i \rangle) = 1$$

$$h_B\left(\sum_{j \in N} |a_j| x_j\right) = sign\left(\left\langle w_B, \sum_{j \in N} |a_j| x_j \right\rangle\right) = sign\left(\sum_{j \in N} |a_j| \langle w_B, x_j \rangle\right) = -1$$

So, this is a contradiction.

VCdim($\mathcal{HS}_0^n$)

Proof:

1st part – show that VCdim($\mathcal{HS}_0^n$) $\geq$ n

A = {$e_1, e_2, \ldots, e_n$}, the orthonormal basis of $\mathbf{R}^n$ is shattered by $\mathcal{HS}_0^n$.

2nd part – show that VCdim($\mathcal{HS}_0^n$) < n + 1

Any set A = {$x_1, x_2, \ldots, x_{n+1}$} of n + 1 points in $\mathbf{R}^n$ cannot be shattered by $\mathcal{HS}_0^n$. Provide an algebraic proof, based on the fact that {$x_1, x_2, \ldots, x_{n+1}$} are linearly dependent in $\mathbf{R}^n$.

So, VCdim($\mathcal{HS}_0^n$) = n

Similarly, it can be shown that VCdim($\mathcal{HS}^n$) = n + 1

Assignment 1

Assignment 1 – good to know

- 6 problems = 5 points + 0.5 points (bonus) = 5.5 points
- deadline: in ~ 3 weeks time, Sunday, 25th of April 2021, 23:59
  - late submission policy: maximum 3 days allowed, -10% (= 0.5 points) for each day
  - upload a pdf written in a scientific editor (Word, Latex, LyX) containing your solution here: ***https://tinyurl.com/AML-2021-ASSIGNMENT1***
  - *is mandatory that you write your solution with a scientific editor, otherwise your solution would not be taken into account*
  - you can insert drawings for your proofs
- for every problem write clear explanations, proofs to justify your answer (if you write just some indications you will not get too many points)
- do not share/copy the solution with/from your colleagues: you + your colleague/s will get 0 points

Problems 1 and 2

Assignment 1

Deadline: Sunday, 25th of April 2021

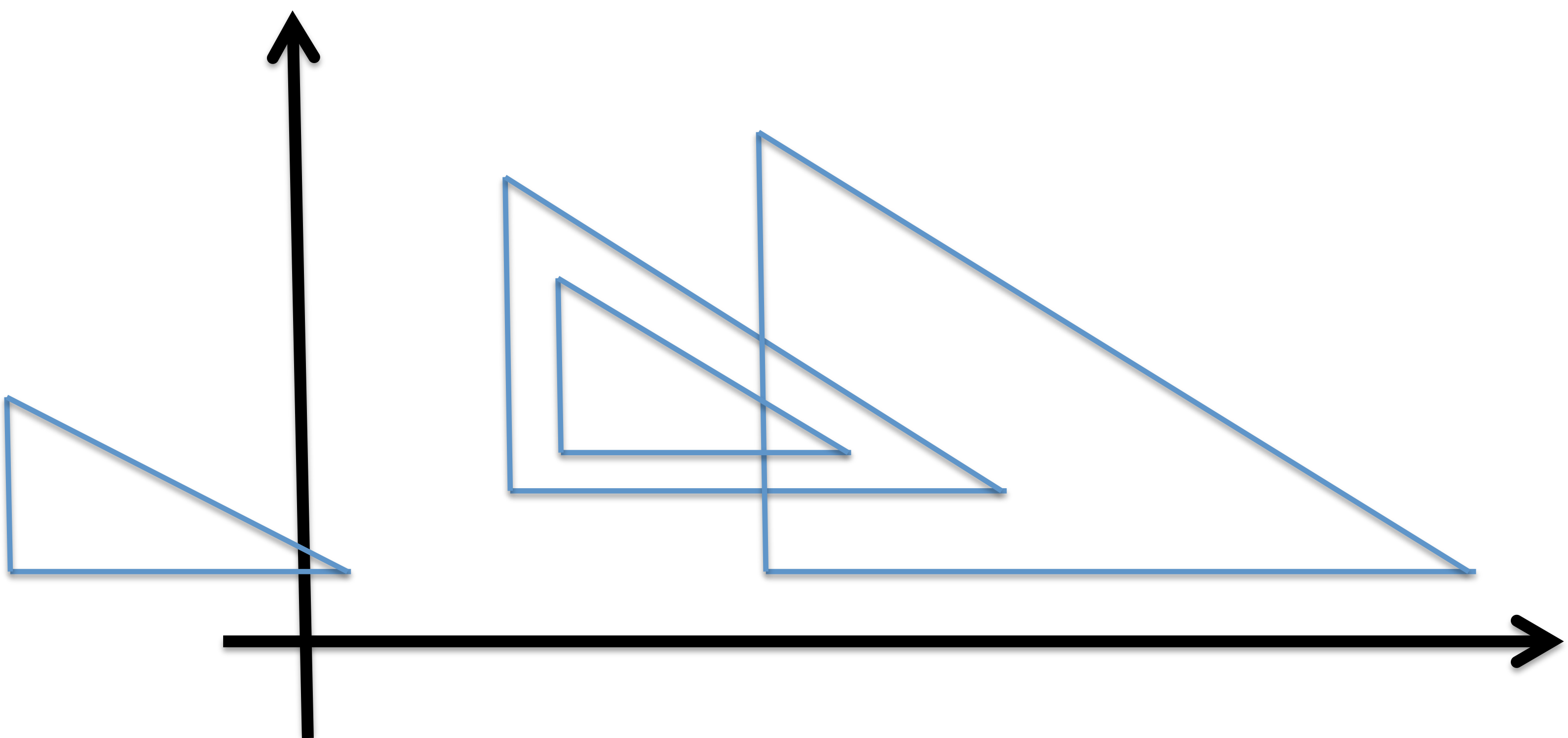
Upload your solutions at: https://tinyurl.com/AML-2021-ASSIGNMENT1

1. **(0.5 points)** Give an example of a finite hypothesis class $\mathcal{H}$ with VCdim($\mathcal{H}$) = 2021. Justify your choice.

2. **(0.5 points)** Consider $\mathcal{H}_{\text{balls}}$ to be the set of all balls in $\mathbf{R}^2$:
   $\mathcal{H}_{\text{balls}} = \{B(x,r),\ x \in \mathbb{R}^2,\ r \geq 0\}$, where $B(x,r) = \{y \in \mathbb{R}^2 \mid \|y - x\|_2 \leq r\}$
   As mentioned in the lecture, we can also view $\mathcal{H}_{\text{balls}}$ as the set of indicator functions of the balls B(x,r) in the plane: $\mathcal{H}_{\text{balls}} = \{ h_{x,r}: \mathbb{R}^2 \to \{0,1\}, h_{x,r} = \mathbf{1}_{B(x,r)}, x \in \mathbb{R}^2, r > 0\}$.
   Can you give an example of a set A in $\mathbf{R}^2$ of size 4 that is shattered by $\mathcal{H}_{\text{balls}}$? Give such an example or justify why you cannot find a set A of size 4 shattered by $\mathcal{H}_{\text{balls}}$.

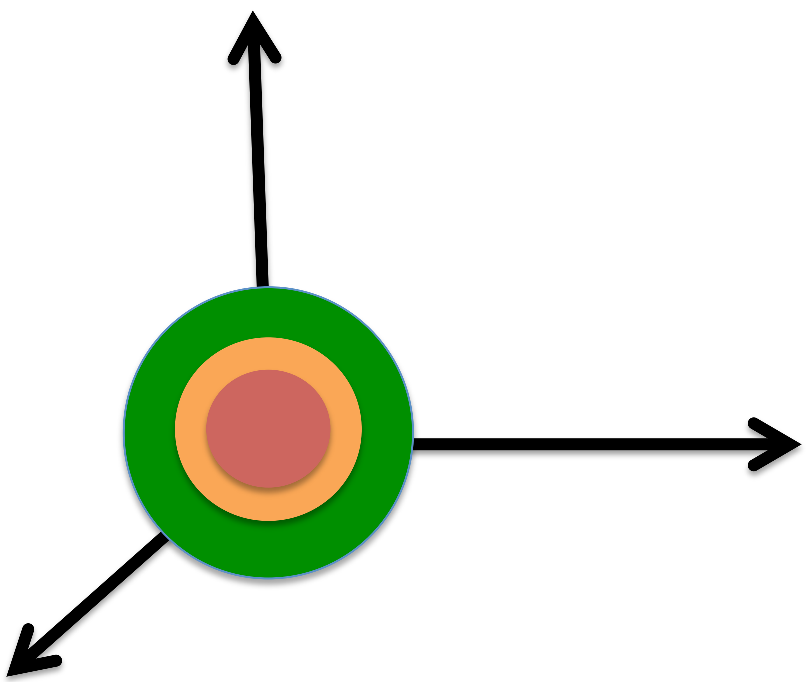
Problem 3

3. **(1 point)** Let $\mathcal{X} = \mathbf{R}^2$ and consider $\mathcal{H}_\alpha$ the set of concepts defined by the area inside a right triangle ABC with the two catheti AB and AC parallel to the axes (Ox and Oy) and with AB/AC = α (fixed constant > 0). Consider the realizability assumption. Show that the class $\mathcal{H}_\alpha$ can be (ϵ, δ) – PAC learned by giving an algorithm A and determining an upper bound on the sample complexity $m_H(\epsilon, \delta)$ such that the definition of PAC-learnability is satisfied.



Problem 4

4. **(1 point)** Consider $\mathcal{H}$ to be the class of all centered in origin sphere classifiers in the 3D space. A centered in origin sphere classifier in the 3D space is a classifier h_r that assigns the value 1 to a point if and only if it is inside the sphere with radius r > 0 and center given by the origin **O**(0,0,0). Consider the realizability assumption.
   a. show that the class $\mathcal{H}$ can be (ϵ, δ) – PAC learned by giving an algorithm A and determining an upper bound on the sample complexity $m_H(\epsilon, \delta)$ such that the definition of PAC-learnability is satisfied. **(0.5 points)**
   b. compute VCdim($\mathcal{H}$). **(0.5 points)**



Problems 5 and 6

5. **(1 point)** Let $\mathcal{H} = \{h_\theta : \mathbb{R} \to \{0,1\}, h_\theta(x) = \mathbf{1}_{[\theta,\theta+1]\cup[\theta+2,+\infty)}(x), \theta \in \mathbb{R}\}$. Compute $\mathrm{VCdim}(\mathcal{H})$.

6. **(1 point)** Let $\mathcal{X}$ be an instance space and consider $\mathcal{H} \subseteq \{0,1\}^{\mathcal{X}}$ a hypothesis space with finite VC dimension. For each $x \in \mathcal{X}$, we consider the function z_x: $\mathcal{H} \to \{0,1\}$ such that $z_x(h) = h(x)$ for each $h \in \mathcal{H}$. Let $Z = \{z_x : \mathcal{H} \to \{0,1\}, x \in \mathcal{X}\}$. Prove that $\mathrm{VCdim}(Z) < 2^{\mathrm{VCdim(H)}+1}$.