Seminar 3

① $\mathcal{H}$ – finite hypothesis class

$VC\dim(\mathcal{H}) \le \lfloor \log_2(|\mathcal{H}|) \rfloor$ – upper bound

1. example of $\mathcal{H}$ infinite, $\mathcal{H}$ contains functions $h:[0,1] \to \{0,1\}$ and $VC\dim(\mathcal{H}) = 1$

Take $\mathcal{H}$ thresholds restricted to $[0,1)$

$$\mathcal{H}_{thresholds,[0,1)} = \{ h_a : [0,1] \to \{0,1\},\ h_a(x) = \mathbb{1}_{[x<a]},\ a \in [0,1] \}$$

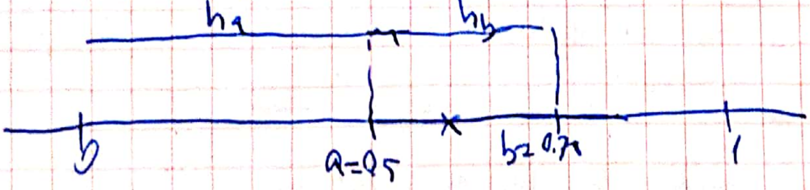
$$h_a(x) = \begin{cases} 1, & 0 \le x < a \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$VC\dim(\mathcal{H}_{threshold,[0,1)}) = 1$ (very similar proof with the one provided in lecture 5)

2. $\mathcal{H} = \{h_a, h_b\}$ – has only two function h_a and h_b

Take $h_a, h_b \in \mathcal{H}_{thresholds,[0,1)}$

$h_a = h_{0.5}$, $h_b = h_{0.75}$



Take $A = \{0.6\}$

$\mathcal{H}$ shatters A because $\mathcal{H}_A$ has two function $h_a(0.6)=1$, $h_b(0.6)=0$

$|\mathcal{H}_A| = 2^{|A|} = 2^1 = 2.$

$\mathcal{H}$ cannot shatter any set A of more ≥ 2 points ($\mathcal{H}$ has only 2 functions)

$\Rightarrow VC\dim(\mathcal{H}) = 1 = \lfloor \log_2(|\mathcal{H}|) \rfloor$

② H_{rec}^{2d} – class of axis aligned rectangles in $\mathbb{R}^d$

In lecture 5 we proved that $VCdim(H_{rec}^2) = 4$

We want to show, in the general case, that $VCdim(H_{rec}^d) = 2d$

$$H_{rec}^d = \{ h_{(a_1,b_1,a_2,b_2,\dots,a_d,b_d)} \mid a_i \le b_i \quad i=\overline{1,d} \}$$

where
$$h_{(a_1,b_1,a_2,b_2,\dots,a_d,b_d)}(\underline{x}) = \begin{cases} 1, & a_i \le x^i \le b_i \ \forall i=\overline{1,d} \\ 0, & \text{otherwise} \end{cases}$$

$$\underline{x} = (x^1, x^2, \dots, x^d)$$

In order to show that $VCdim(H_{rec}^d) = 2d$, we need to show that:

1) there exists a set C of $2d$ points that is shattered by H_{rec}^d
   (this will mean that $VCdim(H_{rec}^d) \ge 2d$)

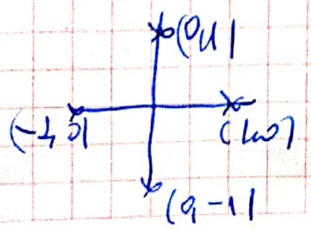
2) every set C of $2d+1$ points is not shattered by H_{rec}^d
   (this will mean that $VCdim(H_{rec}^d) < 2d+1$)

Let's prove 1)

Consider $C = \{c_1, c_2, c_3, \dots, c_{2d-1}, c_{2d}\}$

where
$c_1 = (1,0,0,\dots,0) = e_1$
$c_2 = (0,1,0,\dots,0) = e_2$ $\qquad c_i = e_i = -c_{i+d} \quad \forall i = \overline{1,d}$
$\vdots$
$c_d = (0,0,0,\dots,1) = e_d$
$c_{d+1} = (-1,0,0,\dots,0) = -e_1$
$c_{d+2} = (0,-1,0,\dots,0) = -e_2$
$\vdots$
$c_{2d} = (0,0,\dots,-1) = -e_d$

For $d=2$ we will have in 2 dimensions: $c_1 = (1,0)$, $c_2 = (0,1)$, $c_3 = (-1,0)$, $c_4 = (0,-1)$



We want to show that for each labeling $(y_1, y_2, \dots, y_{2d})$ of the points $(c_1, c_2, \dots, c_{2d})$ (there are 2^{2d} possible labelings) there exists a function $h \in \mathcal{H}^d_{rec}$ such that $h(c_i) = y_i \ \forall i = \overline{1, 2d}$

Consider a labeling $(y_1, y_2, \dots, y_{2d}) \in \{0,1\}^{2d}$

Each point c_i has all components $= 0$, apart from component i if $i \in \{1, \dots, d\}$ or $i-d$ if $i \in \{d+1, \dots, 2d\}$.

$c_1 = (1, 0, 0, \dots, 0)$	$c_2 = (0, 1, 0, \dots, 0)$	$\dots$	$c_d = (0, 0, \dots, 0, 1)$
$c_{d+1} = (-1, 0, 0, \dots, 0)$	$c_{d+2} = (0, -1, 0, \dots, 0)$		$c_{2d} = (0, 0, \dots, 0, -1)$

We want to find $h = h_{(a_1, b_1, a_2, b_2, \dots, a_d, b_d)}$ such that

$$h_{(a_1, b_1, a_2, b_2, \dots, a_d, b_d)}(c_i) = y_i.$$

The choice of the interval $[a_i, b_i]$ is influenced by the labels y_i and y_{i+d} of the points c_i and c_{i+d}. As all other points $c_1, c_2, \dots, c_{i-1}, c_{i+1}, \dots, c_{i+d-1}, c_{i+d+1}, \dots, c_{2d}$ have 0 on the i-th component we have that $[a_i, b_i]$ should contain 0, otherwise each point will be labelled with 0.

So $[a_i, b_i]$ depends on y_i and y_{i+d}, and $[a_i, b_i]$ decides basically the label of points c_i and c_{i+d}.

I $y_i = 0$, $y_{i+d} = 0$ then $[a_i, b_i] \cap \{-1, 1\} = \emptyset$ $c_i = (0, \dots, 0, 1, 0, \dots, 0)$
$[a_i, b_i]$ should not contain -1 and 1. $c_{i+d} = (0, \dots, 0, -1, 0, \dots, 0)$

So in this case take $a_i = -0.5$, $b_i = 0.5$ (many other choices are possible)

II $y_i = 0$, $y_{i+d} = 1$ then $[a_i, b_i] \cap \{-1, 1\} = \{-1\}$

$[a_i, b_i]$ should contain only -1 such that c_{i+d} will get label 1.

So in this case take $a_i = -2$, $b_i = 0.5$ (many other choices are possible)

III $y_i = 1$, $y_{i+d} = 0$ then $[a_i, b_i] \cap \{-1, 1\} = \{1\}$.

$[a_i, b_i]$ should contain only point $+1$, such that c_i will get label 1

So, in this case take $a_i = -0.5$, $b_i = 2$ (many other choices are possible)

IV $y_i = 1,\ y_{i+d} = 1$ then $[a_i, b_i] \cap \{-1, 1\} = \{-1, 1\}$

$[a_i, b_i]$ should contain both points $\{-1, 1\}$ such that c_i and c_{i+d} will get label 1.

So, in this case take $a_i = -2$, $b_i = 2$ (many other choices are possible)

So, in all cases we have that $h_{(a_1, b_1, a_2, b_2, \dots, a_d, b_d)}(c_i) = y_i \ \forall i = \overline{1, 2d}$

where each interval $[a_i, b_i]$ was determined based on y_i and y_{i+d}, $i = \overline{1, d}$

So $VCdim(\mathcal{H}^d_{rec}) \geq 2d$

b) Let C be a set of size $2d+1$ points. We will show that C cannot be shattered by $\mathcal{H}^d_{rec}$.

Because we have $2d+1$ points in C and there are only d dimensions there will exists a point $x \in C$ such that for each dimension $i = \overline{1, d}$ there will be 2 points x' and $x'' \in C$ such that

$$x'_i \leq x_i \leq x''_i$$

(point x_i is "inside" the convex hull determined by all other points in dimension i)

So the label when x has value 0 and all other $2d$ points get label 1 cannot be realized by any function $h \in \mathcal{H}^d_{rec}$ (because x is inside the rectangle that contain all other points)

③ $\mathcal{H}^d_{con}$ – class of Boolean conjunctions over the variables $x_1, x_2, \dots, x_d$, $d \geq 2$

$$\mathcal{H}^d_{con} = \{ h : \{0,1\}^d \to \{0,1\},\ h(x_1, x_2, \dots, x_d) = \bigwedge_{i=1}^{d} l(x_i) \}$$

$l(x_i)$ = literal of variable x_i

$l(x_i) \in \{ x_i, \overline{x_i}, \underset{\text{missing}}{1} \}$

We also consider that $h^- \in \mathcal{H}^d_{con}$, $h^-(x_1, x_2, \dots, x_d) = 0$ always.

a) So $|\mathcal{H}^d_{con}| = 3^d + 1$.

b) $VCdim(\mathcal{H}^d_{con}) \leq \lfloor \log_2(3^d+1) \rfloor$

c) We will show that $\mathcal{H}^d_{con}$ shatters the set of unit vectors $\{e_i; i \leq d\}$

$$e_i = (0, 0, \dots, 0, \underset{i}{1}, 0, \dots, 0).$$

Consider $C = \{e_1, e_2, \dots, e_d\}$. We want to prove that for each possible labelling $(y_1, y_2, \dots, y_d)$ there exist an $h \in \mathcal{H}^d_{con}$ s.t. $h(e_i) = y_i$

Consider a labelling $(y_1, y_2, \dots, y_d)$ and take

$$J = \{ j \mid y_j = 1 \}$$

If $J = \emptyset \Rightarrow h^-$ realizes the labelling $(0, 0, \dots, 0)$

If $J = \{1, \dots, d\} \Rightarrow h_{\text{empty}}$ (all literals are missing) $= 1$ $\forall$ realizes the labelling $(1, 1, \dots, 1)$.

In all other cases define $h_J = \bigwedge_{j \notin J} \overline{x_j} = \bigwedge_{j \in \{1, \dots, d\} \setminus J} \overline{x_j}$

if $J = \{1, 2, 4\}$ define $h_J = \overline{x_3} \wedge \overline{x_5} \wedge \overline{x_6}$

$d = 6$

$h_J(e_1) = h_J(1,0,0,0,0,0) = \overline{0} \wedge \overline{0} \wedge \overline{0} = 1$.

$h_J(e_2) = h_J(0,1,0,0,0,0) = \overline{0} \wedge \overline{0} \wedge \overline{0} = 1$

$h_J(e_4) = h_J(0,0,0,1,0,0) = \overline{0} \wedge \overline{0} \wedge \overline{0} = 1$

$h_J(e_3) = h_J(0,0,1,0,0,0) = \overline{1} \wedge \overline{0} \wedge \overline{0} = 0$

$h_J(e_5) = h_J(0,0,0,0,1,0) = \overline{0} \wedge \overline{1} \wedge \overline{0} = 0$

$h_J(e_6) = h_J(0,0,0,0,0,1) = \overline{0} \wedge \overline{0} \wedge \overline{1} = 0$

So, $h_J(l_j)=1$ if $j\in J$
and $h_J(l_j)=0$ if $j\notin J$
This proves that $\mathcal{H}^d_{con}$ shatters set $C \Rightarrow VCdim(H^d_{con})\geq d$

d) We want to show that $VCdim(H^d_{con})<d+1$
Assume that there exists a set $C=\{c_1,c_2,c_3,\dots,c_{d+1}\}$ of points from $\{0,1\}^d$ that is shattered by H^d_{con}, so $|H^d_{con_C}|=|\{h:C\to\{0,1\},$
$h\in\mathcal{H}\}|=2^{d+1}$

$c_1\in\{0,1\}^d \Rightarrow c_1=(c_1^1,c_1^2,\dots,c_1^d)\in\{0,1\}^d$
$c_2\in\{0,1\}^d \Rightarrow c_2=(c_2^1,c_2^2,\dots,c_2^d)$ Each point c_i has
... d components from $\{0,1\}$
$c_i\in\{0,1\}^d \Rightarrow c_i=(c_i^1,c_i^2,\dots,c_i^d)$

We want to find a contradiction and show that $\mathcal{H}^d_{con}$ doesn't shatter any set C of d+1 points.
If H^d_{con} shatters C then among the 2^{d+1} functions $h:C\to\{0,1\}$ we will have the following d+1 ~~...~~ functions:

$h_1:\{0,1\}^d\to\{0,1\}$ s.t. $h_1(c_1)=0,\ h_1(c_2)=1,\ h_1(c_3)=1,\dots,h_1(c_{d+1})=1$
$h_2:\{0,1\}^d\to\{0,1\}$ s.t. $h_2(c_1)=1,\ h_2(c_2)=0,\ h_2(c_3)=1,\dots h_2(c_{d+1})=1$
$h_3:\{0,1\}^d\to\{0,1\}$ s.t. $h_3(c_1)=1,\ h_3(c_2)=1,\ h_3(c_3)=0,\dots h_3(c_{d+1})=1$
$\vdots$
$h_{d+1}:\{0,1\}^d\to\{0,1\}$ s.t. $h_{d+1}(c_1)=1,\ h_{d+1}(c_2)=1,\ h_{d+1}(c_3)=1,\dots h_{d+1}(c_{d+1})=0.$

So h_i, with $i\in\{1,\dots,d+1\}$ realizes the labels $(1,1,\dots,1,\underset{i}{0},1,1,\dots 1)$
So we have $h_i(c_j)=\begin{cases}0, & \text{if } i=j\\ 1, & \text{if } i\neq j\end{cases}$

We will use the functions $h_1,h_2,\dots h_{d+1}$ to arrive at a contradiction.
Each h_i is in $\mathcal{H}_d^{con}$, so it can be written as a conjunction of literals, where each literal from the writing of h_0 can have three values ~~...~~ for a variable x_k: $l_i(x_k)=\begin{cases}x_k, & \text{positive literal}\\ \bar{x}_k, & \text{negative literal}\\ 1, & \text{missing literal}\end{cases}$

For example if we consider $d=3$ a possible h from H^3_{con} could be

$$h_1 = x_1 \wedge \bar{x}_2$$

in this case $h = l(x_1) \wedge l(x_2) \wedge l(x_3)$

$$l(x_1) = x_1, \quad l(x_2) = \bar{x}_2, \quad l(x_3) = 1.$$

In the general case we have

$$h_i(x_1, x_2, \dots, x_d) = \bigwedge_{k=1}^{d} l_i(x_k), \quad l_i(x_k) \in \{x_k, \bar{x}_k, 1\}$$

Now, we go back to our $h_1, h_2, \dots, h_{d+1}$.

h_1 realizes the labels $(0,1,1,\dots,1)$ on $\{c_1, c_2, \dots, c_{d+1}\} = C$

h_2 realizes the labels $(1,0,1,\dots,1)$ on $\{c_1, c_2, \dots, c_{d+1}\} = C$

$\vdots$

h_{d+1} realizes the labels $(1,1,1,\dots,0)$ on $\{c_1, c_2, \dots, c_{d+1}\} = C$

We will the labels 0 to come up with a contradiction

Because $h_1(c_1) = 0 \Leftrightarrow h_1(c_1^1, c_1^2, \dots, c_1^d) = \bigwedge_{k=1}^{d} l_1(c_1^k) =$

$$= l_1(c_1^1) \wedge l_1(c_1^2) \wedge \dots \wedge l_1(c_1^d) = 0$$

$$\Rightarrow \exists k_1 \in \{1, \dots, d\} \text{ such that } l_1(c_1^{k_1}) = 0$$

Because $h_2(c_2) = 0 \Rightarrow$ —— (same argument as above) $\Rightarrow$

$$\exists k_2 \in \{1, \dots, d\} \text{ such that } l_2(c_2^{k_2}) = 0$$

...

Because $h_{d+1}(c_{d+1}) = 0 \Rightarrow \dots$ (same argument as above) $\Rightarrow$

$$\exists k_{d+1} \in \{1, \dots, d\} \text{ such that } l_{d+1}(c_{d+1}^{k_{d+1}}) = 0$$

So we have that $l_1(x_{k_1}) = 0$ when $x_{k_1} = c_1^{k_1}$ – variable on position k_1

$l_2(x_{k_2}) = 0$ when $x_{k_2} = c_2^{k_2}$ – variable on position k_2

$\vdots$

$l_{d+1}(x_{k_{d+1}}) = 0$ when $x_{k_{d+1}} = c_{d+1}^{k_{d+1}}$ – variable on position k_{d+1}

We have $d+1$ literals that use variables $x_1, x_2, \dots, x_d$. So there are at least two literals using the same variable. Let these literals be l_i and l_j and assume that the variable they use is x_k.

$- - - \text{———} - - -$

$h_i = l_i(x_1) \wedge l_i(x_2) \wedge \text{—} \wedge \underline{l_i(x_k)} \wedge \text{——}$

$\text{———————} -$

$h_j = l_j(x_1) \wedge l_j(x_2) \wedge \text{——} \wedge \underline{l_j(x_k)} \wedge \text{——}$

$\cdot \cdot \cdot \cdot -$

We will grow use l_i and l_j to arrive at a contradiction

We know that l_i and l_j satisfy the following condition

$l_i(c_i^k) = 0$ (because $h_i(c_i) = 0$ and the conjunction contains literal $l_i(c_i^k)$ which is 0)

$l_j(c_j^k) = 0$ (because $h_j(c_j) = 0$ and the conjunction contains literal $l_j(c_j^k)$ which is 0).

In general we have that $l_i(x_k) \in \{x_k, \bar{x}_k, 1\}$

$l_j(x_k) \in \{x_k, \bar{x}_k, 1\}$

But $l_i(x_k) \neq 1$ because we have that $l_i(c_i^k) = 0$.

Same argument goes for $l_j(x_k) \neq 1$.

So $l_i(x_k)$ can take values in $\{x_k, \bar{x}_k\}$

and $l_j(x_k)$ can take values in $\{x_k, \bar{x}_k\}$

There are 4 possible cases

<u>Case 1</u> $l_i(x_k) = x_k$, $l_j(x_k) = x_k$

$h_i(c_0) = l_i(c_i^1) \wedge l_i(c_i^2) \wedge \text{—} \wedge \underbrace{l_i(c_i^k)}_{0} \wedge \text{——} = 0$

we have that $l_i(c_0^k) = c_0^k = 0$.

But we also have that

$h_j(c_0) = 1 \Leftrightarrow l_j(c_0^1) \wedge l_j(c_0^2) \wedge \text{—} \wedge l_j(c_0^k) \wedge \text{—} = 1$

This means that all literals are 1, including $l_j(c_0^k)$

But $l_j(c_i^k) = c_i^k = 0$. So we have a contradiction

Case 2 $\quad l_i(x_k) = \overline{x_k}\,,\ l_j(x_k) = \overline{x_k}$

$$h_i(c_i) = \ldots \wedge \underset{\substack{\| \\ 0}}{l_i(c_i^k)} \wedge \ldots = 0$$

$$l_i(c_i^k) = \overline{c_i^k} = 1 - c_i^k = 0 \Rightarrow c_i^k = 1$$

But we have that

$$h_j(c_i) = 1 \Leftrightarrow \ldots \wedge l_j(c_i^k) \wedge \ldots = 1$$

$$\Rightarrow l_j(c_i^k) = 1$$

But $l_j(c_i^k) = 1 - c_i^k = 0$. Contradiction

Case 3 $\quad l_i(x_k) = x_k\,,\ l_j(x_k^m) = \overline{x_k}$

Take another point c_m that is different than c_i and c_j

$$m \neq i,\ m \neq j \text{ and } 1 \leq m \leq d+1$$

So we have $h_i(c_m) = h_j(c_m) = 1$.

$$h_i(c_m) = \ldots \wedge l_i(c_m^k) \wedge \ldots = 1$$

$$\Rightarrow l_i(c_m^k) = c_m^k = 1$$

$$h_j(c_m) = \ldots \wedge l_j(c_m^k) \wedge \ldots = 1$$

$$\Rightarrow l_j(c_m^k) = 1 - c_m^k = 1 \Rightarrow c_m^k = 0. \text{ Contradiction}$$

Case 4 $\quad l_i(x_k) = \overline{x_k}\,,\ l_j(x_k) = x_k$

Same as case 3, you will ~~see~~ see that $l_i(c_m^k) = 1 - c_m^k = 1 \Rightarrow c_m^k = 0$

$$l_j(c_m^k) = \underline{c_m^k} = 1. \text{ Contradiction.}$$

Exercise (4) $\mathcal{H} = \{ h_{a,b,s} : a \leq b,\ s \in \{-1,1\} \},\ h_{a,b,s}(x) = \begin{cases} s, & x \in [a,b] \\ -s, & x \notin [a,b] \end{cases}$

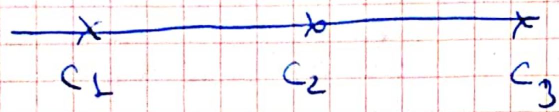
see label 0 as label -1.

$VCdim(\mathcal{H}) = ?$

$\mathcal{H}$ contains functions parametrized by 3 params (a, b, s).

Intuition tells us that $VCdim(\mathcal{H}) = 3$ (not always, but usually).

Lets consider $C = \{c_1, c_2, c_3\}$ a set of 3 distinctive points with $c_1 \leq c_2 \leq c_3$ (for example take $c_1 = 0$, $c_2 = 1$, $c_3 = 2$)



to obtain labels $(0,0,0)$ $(-1,-1,-1)$ take $a = b = c_1 - 1$, $s = 1$ ✓

or $a = c_1$, $b = c_3$, $s = -1$

to obtain labels $(1,1,1)$ take $a = c_1$, $b = c_3$, $S = 1$ ✓

to obtain labels $(1,-1,-1)$ take $a = c_1$, $b = \frac{c_1+c_2}{2}$, $S = 1$

to obtain labels $(-1,1,1)$ take $a = c_1$, $b = \frac{c_1+c_2}{2}$, $S = -1$

to obtain labels $(-1,1,-1)$ take $a = \frac{c_1+c_2}{2}$, $b = \frac{c_2+c_3}{2}$, $S = 1$

to obtain labels $(1,-1,1)$ take $a = \frac{c_1+c_2}{2}$, $b = \frac{c_2+c_3}{2}$, $S = -1$

to obtain labels $(-1,-1,1)$ take $a = \frac{c_2+c_3}{2}$, $b = c_3 + 1$, $S = 1$

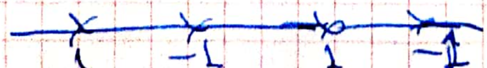
to obtain labels $(1,1,-1)$ take $a = \frac{c_2+c_3}{2}$, $b = c_3 + 1$, $S = -1$

So $\mathcal{H}$ shatter C, so $VCdim(\mathcal{H}) \geq 3$.

Now, take C a set of 4 points, $C = \{c_1, c_2, c_3, c_4\}$

$$c_1 \leq c_2 \leq c_3 \leq c_4$$

Then $\mathcal{H}$ cannot realize the labels $(1,-1,1,-1)$.



This happen for any C. So $VCdim(\mathcal{H}) < 4$. $\boxed{\text{So } VCdim(\mathcal{H}) = 3}$