Advanced Machine Learning

Bogdan Alexe,

bogdan.alexe@fmi.unibuc.ro

University of Bucharest, 2nd semester, 2020-2021

Today’s lecture: Overview

- Recap: Learnability - what do we know so far

- Shattering

- VC-dimension

Recap: Learnability - what do we know so far?

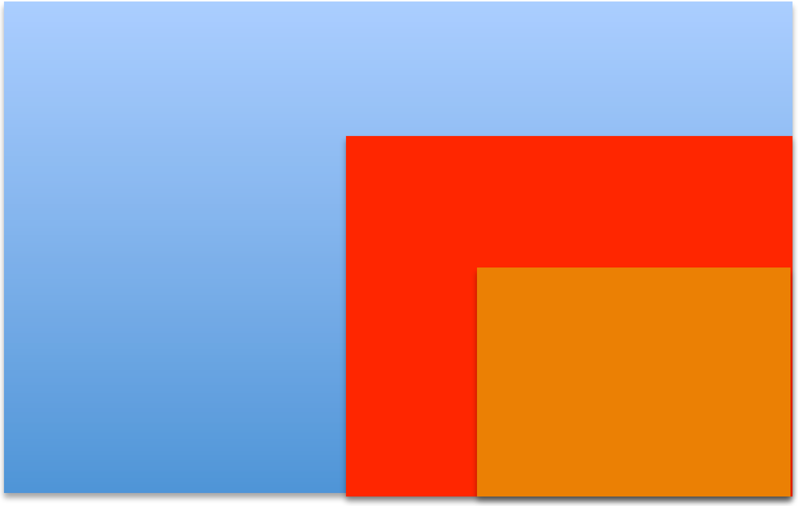
Let $\mathcal{H}$ a hypothesis class.

- is it PAC learnable?
- is it agnostic PAC learnable?
  - we do know that agnostic PAC learnability → PAC learnability

all hypothesis classes

PAC learnable

agnostic PAC learnable



Definition 3.1 (PAC Learnability). A hypothesis class $\mathcal{H}$ is PAC learnable if there exist a function $m_{\mathcal{H}} : (0,1)^2 \to \mathbb{N}$ and a learning algorithm with the following property: For every $\epsilon, \delta \in (0,1)$, for every distribution $\mathcal{D}$ over $\mathcal{X}$, and for every labeling function $f : \mathcal{X} \to \{0,1\}$, if the realizable assumption holds with respect to $\mathcal{H}, \mathcal{D}, f$, then when running the learning algorithm on $m \geq m_{\mathcal{H}}(\epsilon, \delta)$ i.i.d. examples generated by $\mathcal{D}$ and labeled by f, the algorithm returns a hypothesis h such that, with probability of at least $1-\delta$ (over the choice of the examples), $L_{(\mathcal{D},f)}(h) \leq \epsilon$.

Definition 3.4 (Agnostic PAC Learnability for General Loss Functions). A hypothesis class $\mathcal{H}$ is agnostic PAC learnable with respect to a set Z and a loss function $\ell : \mathcal{H} \times Z \to \mathbb{R}_+$, if there exist a function $m_{\mathcal{H}} : (0,1)^2 \to \mathbb{N}$ and a learning algorithm with the following property: For every $\epsilon, \delta \in (0,1)$ and for every distribution $\mathcal{D}$ over Z, when running the learning algorithm on $m \geq m_{\mathcal{H}}(\epsilon, \delta)$ i.i.d. examples generated by $\mathcal{D}$, the algorithm returns $h \in \mathcal{H}$ such that, with probability of at least $1-\delta$ (over the choice of the m training examples),

$$L_{\mathcal{D}}(h) \leq \min_{h' \in \mathcal{H}} L_{\mathcal{D}}(h') + \epsilon,$$

where $L_{\mathcal{D}}(h) = \mathbb{E}_{z \sim \mathcal{D}}[\ell(h,z)]$.

Recap: Learnability - what do we know so far?

Let $\mathcal{H}$ a hypothesis class.

- is it PAC learnable?
- is it agnostic PAC learnable?
  - we do know that agnostic PAC learnability → PAC learnability
  - we don't know that PAC learnability → agnostic PAC learnability

Size of class $\mathcal{H}$:

- finite
- infinite

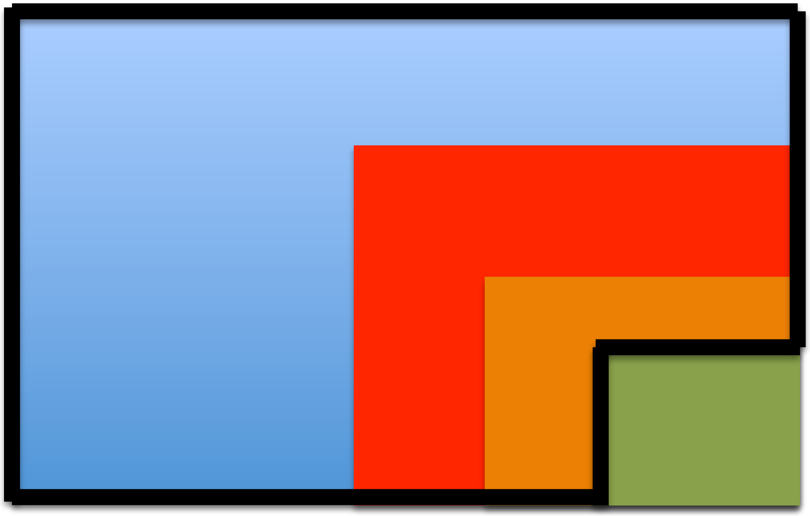
all hypothesis classes

PAC learnable

agnostic PAC learnable

finite size classes

infinite size classes



Recap: Learnability - what do we know so far?

Let $\mathcal{H}$ a hypothesis class.

- is it PAC learnable?
- is it agnostic PAC learnable?
  - we do know that agnostic PAC learnability → PAC learnability
  - we don't know that PAC learnability → agnostic PAC learnability

Size of class $\mathcal{H}$:

- finite
- infinite

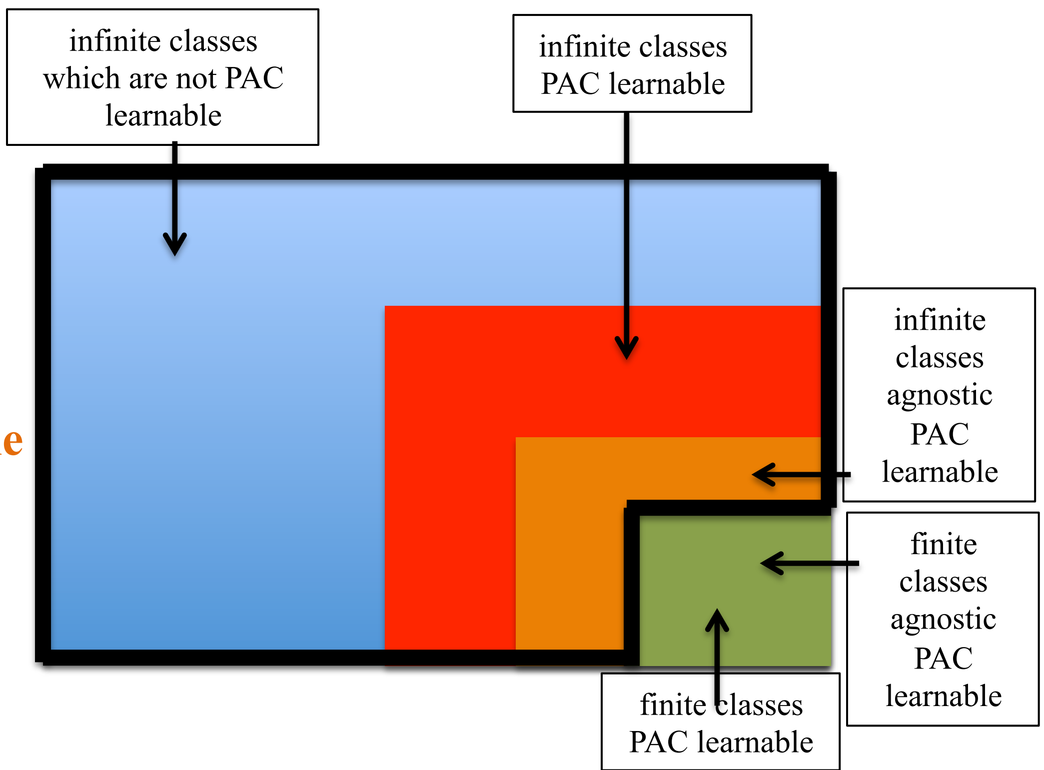
all hypothesis classes

PAC learnable

agnostic PAC learnable

finite size classes

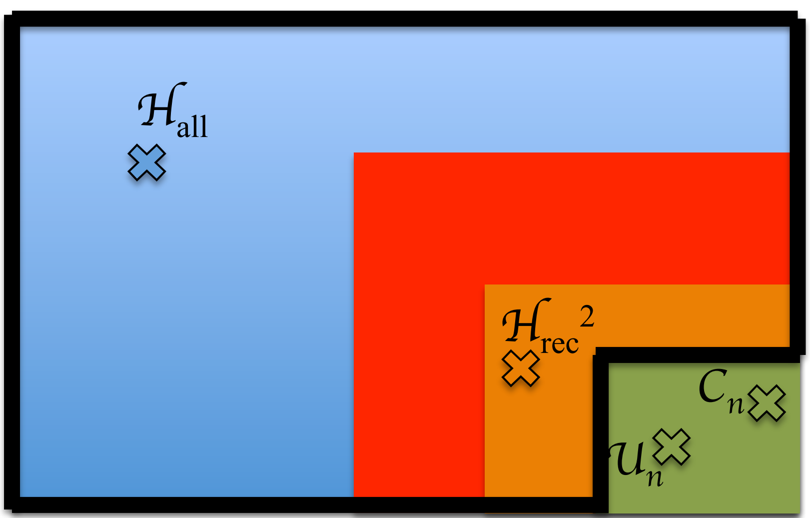
infinite size classes



Recap: Learnability - what do we know so far?

Hypothesis classes $\mathcal{H}$ encountered until now:

- finite $\mathcal{H}$
  - C_n concept class of conjunctions of at most n Boolean literals $x_1, \ldots, x_n$
  - $\mathcal{U}_n$ universal concept class
- infinite $\mathcal{H}$
  - $\mathcal{H}_{rec}{}^2$ set of all axis-aligned rectangle lying in R^2 (with positive labels unaffected – PAC or affected by noise – agnostic PAC)
  - $\mathcal{H}_{all}$ all functions from X to {0,1}(No Free-Lunch theorem)



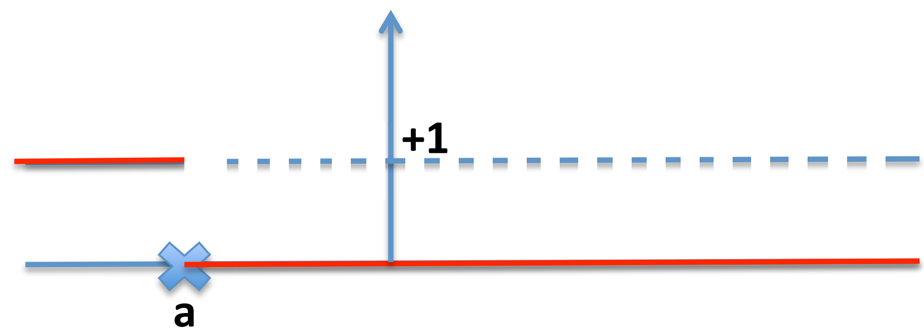
all hypothesis classes
PAC learnable
agnostic PAC learnable
finite size classes
infinite size classes

Recap: Another class example - $H_{thresholds}$

Consider $\mathcal{H}_{thresholds}$ be the set of threshold functions over the real line

$\mathcal{H}_{thresholds} = \{h_a\colon R \longrightarrow \{0, 1\},\ h_a(x) = \mathbf{1}_{[x<a]},\ a \in \mathbf{R}\},\ |\mathcal{H}_{thresholds}| = \infty$

$$\mathbf{1}_{[x<a]}(x) = \begin{cases} 1, & x < a \\ 0, & x \geq a \end{cases},$$



is the indicator function of the set $\{x \in \mathbf{R} \mid x < a\}$

Lemma

$\mathcal{H}_{thresholds}$ is PAC learnable, using the ERM learning rule, with sample complexity of

$$m_{H_{thresholds}} \leq \left\lceil \frac{1}{\varepsilon} \log \frac{1}{\delta} \right\rceil$$

Recap: No Free Lunches vs. $H_{\text{thresholds}}$

Why is $\mathcal{H}_{\text{thresholds}}$ = set of threshold classifiers not a victim of the No Free Lunch theorem? (we can PAC learn them)

The reason is simple:

- the class of threshold classifiers is so simple that an adversary has no room to create an adversarial distribution

In fact, as our discussion above shows:

- if two threshold classifiers agree on a large enough sample
- their respective thresholds will be close to each other
- there is no way you can force them to behave completely differently on unseen examples.

If that would have been possible then:

- we would have been able to create an adversarial distribution.

So, it seems necessary for PAC learnability that the general class $\mathcal{H}$ considered isn't too expressive

Shattering

How expressive is $\mathcal{H}$?

In our binary classification context, a hypothesis is a function h: $\mathcal{X} \rightarrow \{0, 1\}$

Hence the expressiveness of $\mathcal{H}$:

- is necessary a measure of how many functions $\mathcal{H}$ can express
- in the light of the No Free Lunch theorem, not only functions on $\mathcal{X}$, but also functions on (finite) subsets C of $\mathcal{X}$

Definition (restriction of $\mathcal{H}$ to C)

Let $\mathcal{H}$ be a set of hypothesis, i.e., set of functions from $\mathcal{X}$ to $\{0, 1\}$, and let C be a (finite) subset of $\mathcal{X}$, $C = \{c_1, c_2, \ldots, c_m\}$. The restriction of $\mathcal{H}$ to C, denoted by $\mathcal{H}_C$, is the set of functions from C to $\{0, 1\}$ that can be derived from $\mathcal{H}$. That is:

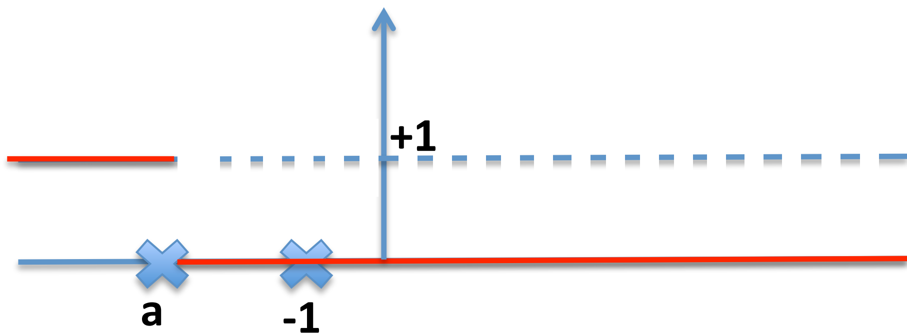
$$\mathcal{H}_C = \{h: C \rightarrow \{0, 1\} \mid h \in \mathcal{H}\}$$

Example: restriction of $\mathcal{H}$ to C

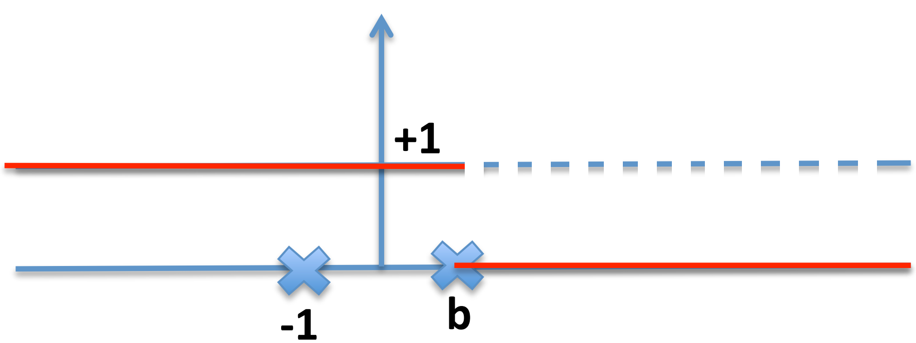
Consider $\mathcal{H} = \mathcal{H}_{\text{thresholds}}$ be the set of threshold functions over the real line $\mathcal{H}_{\text{thresholds}} = \{h_a\!: \mathrm{R} \rightarrow \{0, 1\},\ h_a(\mathrm{x}) = \mathbf{1}_{[x<a]},\ a \in \mathbf{R}\}$, $|\mathcal{H}_{\text{thresholds}}| = \infty$.

Consider $C = \{-1\}$. Then $\mathcal{H}_C = \{h\!: C \rightarrow \{0, 1\} \mid h \in \mathcal{H}\} = \{h\!:\{-1\} \rightarrow \{0, 1\} \mid h \in \mathcal{H}\}$ has 2 elements $h_{a,}\ h_b$ where:

$h_a(-1) = 0$, a <= -1



$h_b(-1) = 1$, b > -1

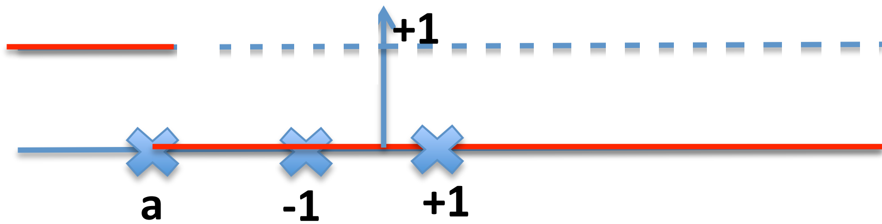


Example: restriction of $\mathcal{H}$ to C

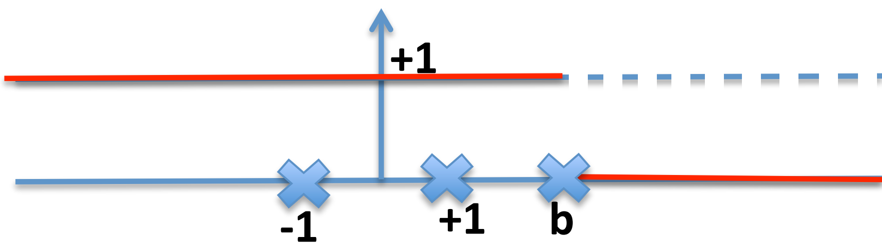
Consider $\mathcal{H} = \mathcal{H}_{\text{thresholds}}$ be the set of threshold functions over the real line $\mathcal{H}_{\text{thresholds}} = \{h_a\!: \mathrm{R} \rightarrow \{0, 1\},\ h_a(x) = \mathbf{1}_{[x<a]},\ a \in \mathbf{R}\}$, $|\mathcal{H}_{\text{thresholds}}| = \infty$.

Consider $C = \{-1, 1\}$. Then $\mathcal{H}_C = \{h\!: C \rightarrow \{0, 1\} |\ h \in \mathcal{H}\} = \{h\!: \{-1, 1\} \rightarrow \{0, 1\} |\ h \in \mathcal{H}\}$ has 3 elements h_a, h_b, h_c where:

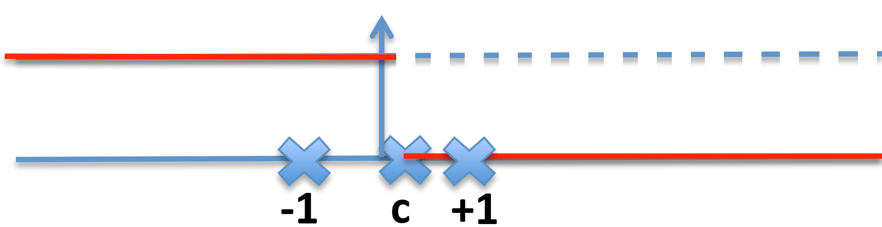
$h_a(-1) = 0,\ h_a(1) = 0,\ a < -1$



$h_b(-1) = 1,\ h_b(1) = 1,\ b \geq 1$



$h_c(-1) = 1,\ h_c(1) = 0,\ -1 \leq c < 1$



Example: restriction of $\mathcal{H}$ to C

Consider $\mathcal{H} = \mathcal{H}_{\text{thresholds}}$ be the set of threshold functions over the real line $\mathcal{H}_{\text{thresholds}} = \{h_{\text{a}}: \text{R} \rightarrow \{0, 1\}, h_{\text{a}}(\text{x}) = \mathbf{1}_{[\text{x}<\text{a}]}, \text{a} \in \mathbf{R}\}, |\mathcal{H}_{\text{thresholds}}| = \infty.$

Consider $C = \{-1, 1\}$. Then $\mathcal{H}_C = \{h: C \rightarrow \{0, 1\} | \, h \in \mathcal{H}\} = \{h: \{-1, 1\} \rightarrow \{0, 1\} | \, h \in \mathcal{H}\}$ has 3 elements $h_{\text{a},}$ h_{b}, h_{c} where:

$h_{\text{a}}(-1) = 0,\ h_{\text{a}}(1) = 0,\ \text{a} \leq -1$

$h_{\text{b}}(-1) = 1,\ h_{\text{b}}(1) = 1,\ \text{b} > 1$

$h_{\text{c}}(-1) = 1,\ h_{\text{c}}(1) = 0,\ -1 < \text{c} \leq -1$

There is no function h_{d} in $\mathcal{H}_C$ such that $h_{\text{d}}(-1) = 0$ and $h_{\text{d}}(1) = 1$ (as we work with the threshold functions)

Alternative view of functions

There is equivalence between functions from $\mathcal{X}$ to $\{0,1\}$ and subsets C of $\mathcal{X}$

- given h: $\mathcal{X} \to \{0, 1\}$, we can define $C = \{x \in \mathcal{X} |$ such that $h(x) = 1\}$
- given C subset of $\mathcal{X}$, we can define h: $\mathcal{X} \to \{0,1\}$, $h(x) = \mathbf{1}_C(x)$ indicator function of C

Can represent a subset C of $\mathcal{X}$ as a vector, put 1 for element in C, 0 otherwise.

Can represent each function from C to $\{0,1\}$ as a vector in $\{0,1\}^{|C|}$.

The restriction of $\mathcal{H}$ to C, denoted by $\mathcal{H}_C$, is the set of functions from $C = \{c_1, c_2, \ldots, c_m\}$ to $\{0, 1\}$ that can be derived from $\mathcal{H}$. It can be also seen as the set of all possible vectors that can be generated by $h \in \mathcal{H}$ with elements from C. That is:

$$\mathcal{H}_C = \{h: C \to \{0, 1\} |\ h \in \mathcal{H}\} = \{(h(c_1), h(c_2), \ldots, h(c_m)) |\ h \in \mathcal{H}\}$$

Consider $C = \{-1, 1\}$.Then $\mathcal{H}_{thresholds} = \mathcal{H}_C = \{h: C \to \{0, 1\} |\ h \in \mathcal{H}\} = \{h: \{-1, 1\} \to \{0, 1\} |\ h \in \mathcal{H}\} = \{(0,0), (1,0), (1,1)\}$. The vector (0,1) is not realizable by $\mathcal{H}_C$.

Shattering

Definition (Shattering)

A hypothesis class $\mathcal{H}$ *shatters* a finite set C of $\mathcal{X}$, if the restriction of $\mathcal{H}$ to C is the set of all functions from C to $\{0, 1\}$. That is $|\mathcal{H}_C| = 2^{|C|}$.

Examples:

Consider $\mathcal{H} = \mathcal{H}_{\text{thresholds}}$ be the set of threshold functions over the real line.

Consider $C = \{c_1\}$. Then $\mathcal{H}_C = \{h: C \rightarrow \{0, 1\} | \ h \in \mathcal{H}\}$ has two elements $\{h_a, h_b\}$ with $a < c_1$ and $b \geq c_1$ so $\mathcal{H}$ shatters C. $\mathcal{H}_C = \{(0),(1)\}$, $|\mathcal{H}_C| = 2^{|C|} = 2^1$

Consider $C = \{c_1, c_2 | \ c_1 \leq c_2\}$. Then $\mathcal{H}_C = \{h: C \rightarrow \{0, 1\} | \ h \in \mathcal{H}\}$ has at most three elements, there is no function that realizes the labeling (0,1) and so $\mathcal{H}$ does not shatter C.

Alternative view of Shattering

Definition (Shattering)

A hypothesis class $\mathcal{H}$ *shatters* a finite set C of $\mathcal{X}$, if the restriction of $\mathcal{H}$ to C is the set of all functions from C to $\{0, 1\}$. That is $|\mathcal{H}_C| = 2^{|C|}$.

Similar Definition (Shattering)

Let $\mathcal{H}$ be a collection of subsets of $\mathcal{X}$ and C a finite subset of $\mathcal{X}$.

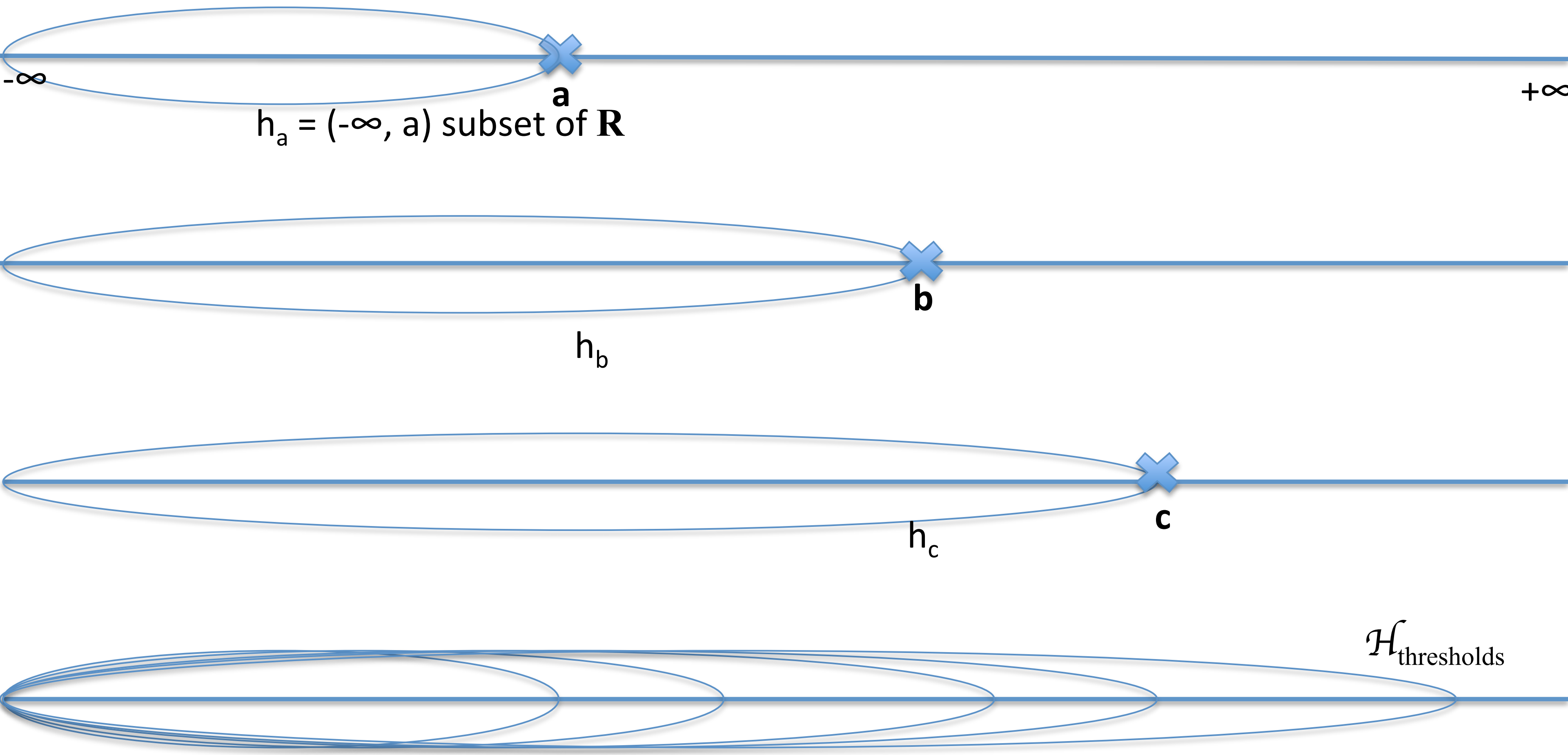
$\mathcal{H}$ shatters C if for every subset B of C there exist some subset $h_B \in \mathcal{H}$ such that $B = h_b \cap C$

In other words, using the elements of $\mathcal{H}$, we can cut C in every possible way.

Shattering – graphical representation

Examples:

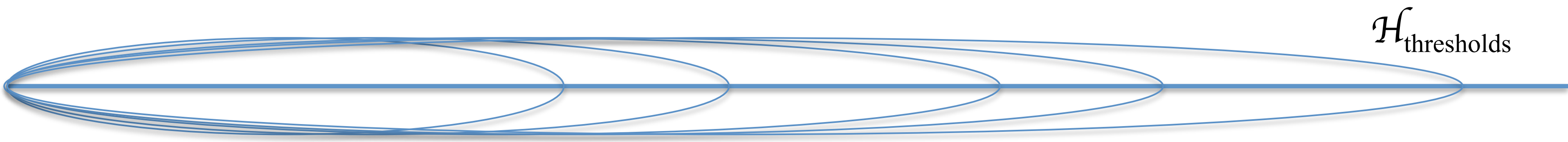
Consider $\mathcal{H} = \mathcal{H}_{\text{thresholds}}$ be the set of threshold functions over the real line.



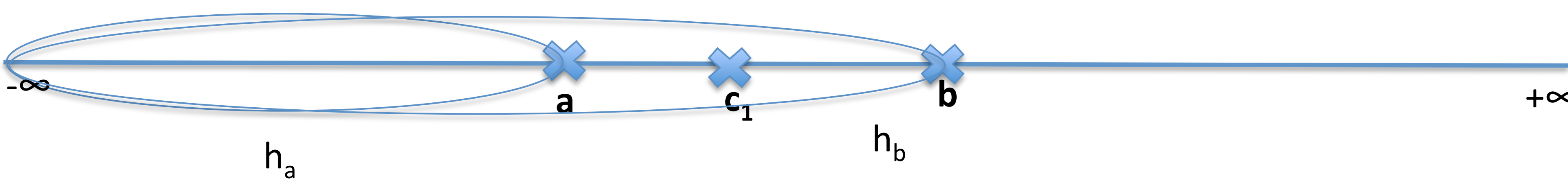
Shattering – graphical representation

Examples:

Consider $\mathcal{H} = \mathcal{H}_{\text{thersholds}}$ be the set of threshold functions over the real line.



Consider $C = \{c_1\}$. Then $\mathcal{H}_C = \{h: C \rightarrow \{0, 1\} \mid h \in \mathcal{H}\}$ has two elements $\{h_a, h_b\}$ with $a < c_1$ and $b \geq c_1$ so $\mathcal{H}$ shatters C. $\mathcal{H}_C = \{(0),(1)\}$, $|\mathcal{H}_C| = 2^{|C|} = 2^1$



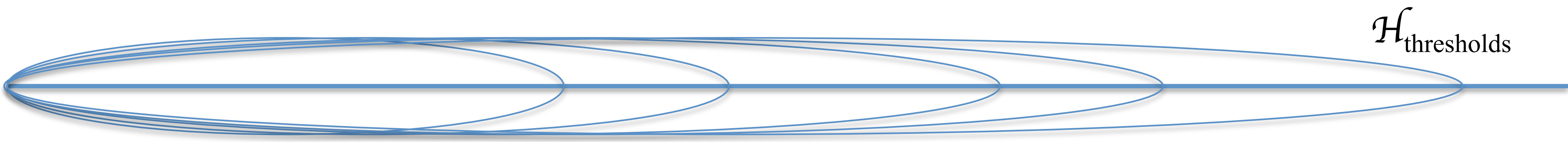
h_a generates label 0

h_b generates label 1

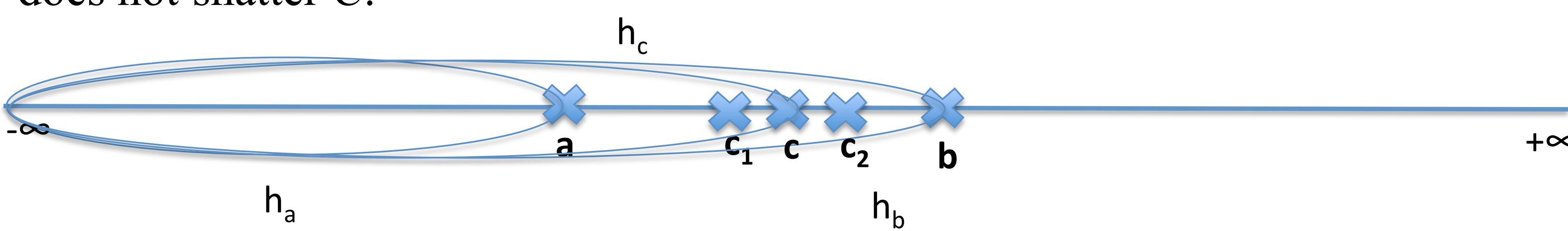
Shattering – graphical representation

Examples:

Consider $\mathcal{H} = \mathcal{H}_{\text{thersholds}}$ be the set of threshold functions over the real line.



Consider $C = \{c_1, c_2 | c_1 \leq c_2\}$. Then $\mathcal{H}_C = \{h: C \rightarrow \{0, 1\} | h \in \mathcal{H}\}$ has at most three elements, there is no function that realizes the labeling (0,1) and so $\mathcal{H}$ does not shatter C.



h_a generates label (0,0)

h_b generates label (1,1)

h_c generates label (1,0)

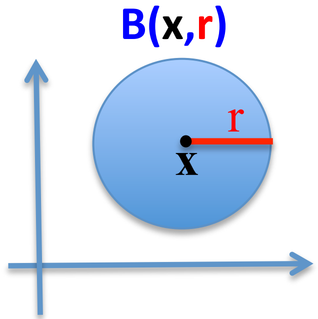
cannot generate the label (0,1)

Shattering – example $\mathcal{H}_{\text{balls}}$

Consider $\mathcal{H} = \mathcal{H}_{\text{balls}}$ be the set of all balls in $\mathbf{R}^2$:

$\mathcal{H}_{\text{balls}} = \{B(x,r),\ x \in \mathbf{R}^2,\ r \geq 0\ \}$,

$B(x,r) = \{y \in \mathbf{R}^2 \mid \|\ y - x\ \|_2 \leq r\}$

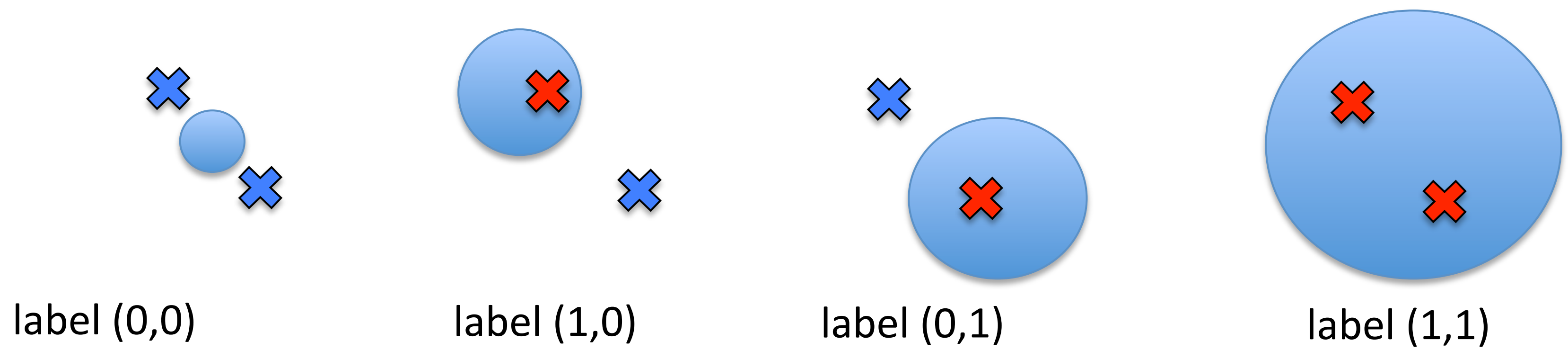


Can also view $\mathcal{H}_{\text{balls}}$ as:

$\mathcal{H}_{\text{balls}} = \{h_{x,r} \colon \mathbf{R}^2 \rightarrow \{0, 1\},\ h_{x,r} = \mathbf{1}_{B(x,r)},\ x \in \mathbf{R}^2,\ r \geq 0\ \}$

Is there a set A in $\mathbf{R}^2$ of size 2 shattered by $\mathcal{H}_{\text{balls}}$?

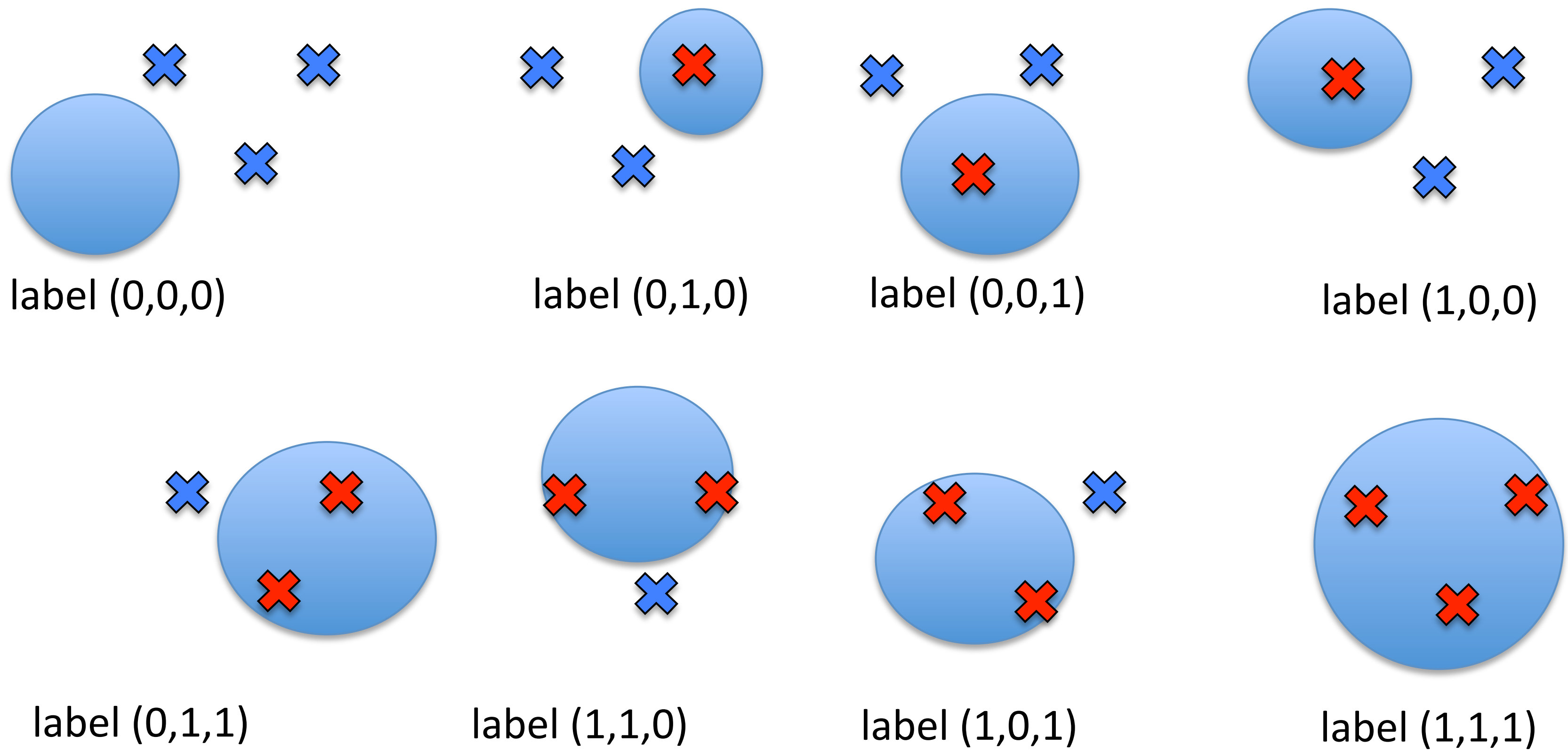
Any set A of two distinct points in $\mathbf{R}^2$ is shattered by $\mathcal{H}_{\text{balls}}$



Shattering – example $\mathcal{H}_{\text{balls}}$

Is there a set A in $\mathbf{R}^2$ of size 3 shattered by $\mathcal{H}_{\text{balls}}$?

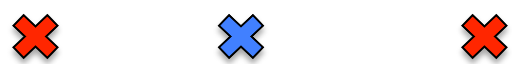
Any set A of three distinct points in $\mathbf{R}^2$ that are not collinear is shattered by $\mathcal{H}_{\text{balls}}$



Shattering – example $\mathcal{H}_{\text{balls}}$

Is there a set A in $\mathbf{R}^2$ of size 3 shattered by $\mathcal{H}_{\text{balls}}$?

Any set A of three distinct points in $\mathbf{R}^2$ that are collinear is not shattered by $\mathcal{H}_{\text{balls}}$



cannot realize the label (1, 0, 1)

What are the conditions for which a set A in $\mathbf{R}^2$ of size 4 is shattered by $\mathcal{H}_{\text{balls}}$?

No Free lunches - revisited

- from the proof of the No Free Lunch theorem, we saw that we can create an adversarial distribution if $\mathcal{H}$ shatters a too large class.

- let $\mathcal{H}$ be a hypothesis class of functions h: $\mathcal{X} \rightarrow \{0, 1\}$ and S a training set of size m. If there exists a set $C \subseteq \mathcal{X}$ of size $2m$ that is shattered by $\mathcal{H}$, then for any learning algorithms A who receives S there exists a distribution $\mathcal{D}$ over $\mathcal{X} \times \{0, 1\}$ such that:
  - there exists a function $f: \mathcal{X} \rightarrow \{0,1\}$ with $L_{\mathcal{D}}(f) = 0$
  - with probability of at least 1/7 over the choice of $S \sim \mathcal{D}^m$ we have that $L_{\mathcal{D}}(A(S)) \geq 1/8$

- the labels of the m instances give us no information about the labels of the rest of the instances in C – every possible labeling of the rest of the instances can be explained by some hypothesis in $\mathcal{H}$.

- *"If someone can explain every phenomenon, his explanations are worthless"*

- shattering is good, but don't shatter too much.

VC-dimension

The VC-dimension

Definition (VC-dimension)

The VC - dimension of a hypothesis class $\mathcal{H}$, denoted VCdim($\mathcal{H}$), is the maximal size of a set $C \subseteq \mathcal{X}$ that can be shattered by $\mathcal{H}$. If $\mathcal{H}$ can shatter sets of arbitrarily large size we say that $\mathcal{H}$ has infinite VC-dimension.

Theorem

Let $\mathcal{H}$ be a class of infinite VC-dimension. Then, $\mathcal{H}$ is not PAC learnable.

Proof: Since $\mathcal{H}$ has an infinite VC-dimension, for any training set S of size *m*, there exists a shattered set of size *2m*, and the claim follows for the No Free Lunch theorem.

We will see in the next lecture that the converse is also true: ***a finite VC-dimension guarantees learnability***. ***Hence, the VC-dimension characterizes PAC learnability. VC-dimension is a combinatorial measure, does not imply computing probabilities.***

Determining the VC-dimension of $\mathcal{H}$

Definition (VC-dimension)

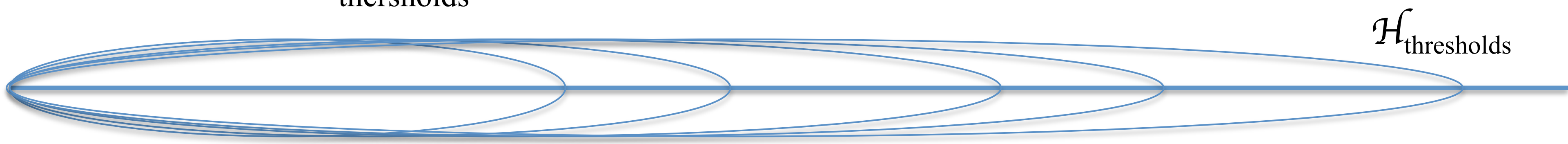
The VC - dimension of a hypothesis class $\mathcal{H}$, denoted VCdim($\mathcal{H}$), is the maximal size of a set $C \subseteq X$ that can be shattered by $\mathcal{H}$. If $\mathcal{H}$ can shatter sets of arbitrarily large size we say that $\mathcal{H}$ has infinite VC-dimension.

In order to show that the VC-dimension of a hypothesis class $\mathcal{H}$ is d, we need to show that:

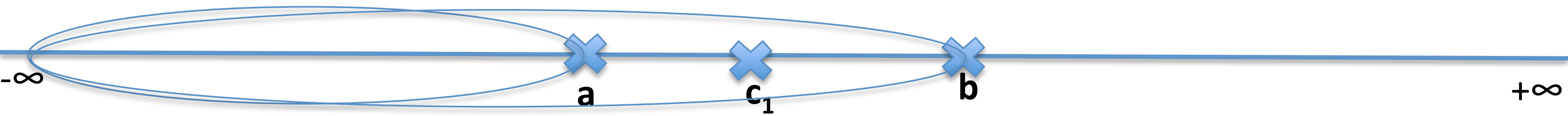
1. **There exists a set C of size d that is shattered by $\mathcal{H}$. (VCdim($\mathcal{H}$) $\geq$ d)**
2. **Every set C of size $d + 1$ is not shattered by $\mathcal{H}$. (VCdim($\mathcal{H}$) < d+1)**

VCdim($\mathcal{H}_{\text{thresholds}}$)

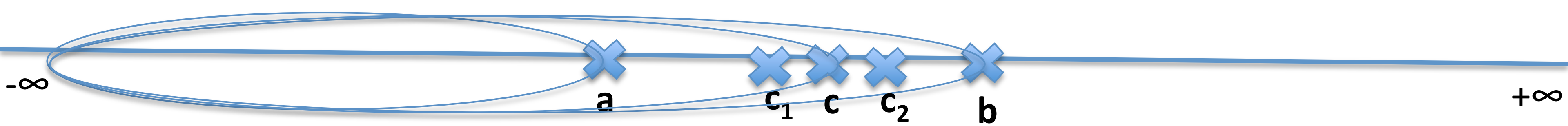
Consider $\mathcal{H} = \mathcal{H}_{\text{thersholds}}$ be the set of threshold functions over the real line.



Consider $C = \{c_1\}$. Then $\mathcal{H}_C = \{h\colon C \to \{0, 1\} \mid h \in \mathcal{H}\}$ has two elements $\{h_a, h_b\}$ with $a \le c_1$ and $b > c_1$ so $\mathcal{H}$ shatters C. $\mathcal{H}_C = \{(0),(1)\}$, $|\mathcal{H}_C| = 2^{|C|} = 2^1$



Consider $C = \{c_1, c_2 \mid c_1 \le c_2\}$. Then $\mathcal{H}_C = \{h\colon C \to \{0, 1\} \mid h \in \mathcal{H}\}$ has at most three elements, there is no function that realizes the labeling (0,1) and so $\mathcal{H}$ does not shatter C.

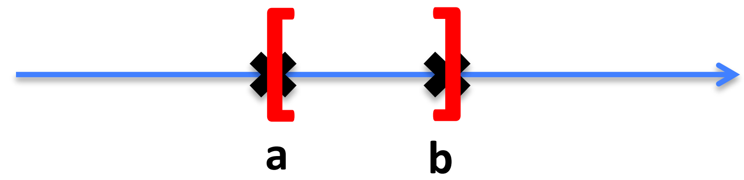


So, **VCdim($\mathcal{H}_{\text{thresholds}}$) = 1**

VCdim($\mathcal{H}_{\text{intervals}}$)

Consider $\mathcal{H} = \mathcal{H}_{\text{intervals}}$ be the set of intervals over the real line.

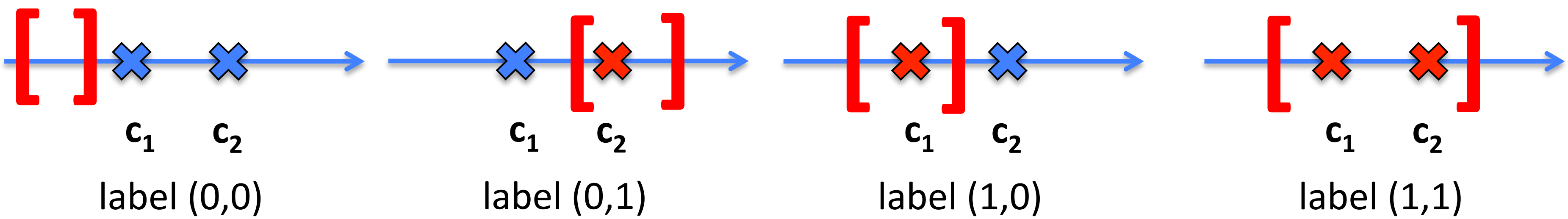
$\mathcal{H}_{\text{intervals}} = \{[a,b] \mid a \leq b,\ a, b \in \mathbf{R}\}$



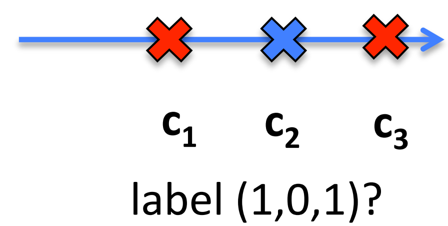
Can also view $\mathcal{H}_{\text{intervals}}$ as:

$\mathcal{H}_{\text{intervals}} = \{h_{a,b}: \mathbf{R} \to \{0, 1\},\ h_{a,b} = \mathbf{1}_{[a,b]},\ a \leq b,\ a, b \in \mathbf{R}\}$

$\mathcal{H}_{\text{intervals}}$ shatters any set A of two different points in **R**.



$\mathcal{H}_{\text{intervals}}$ cannot shatter any set A of three points in **R**.



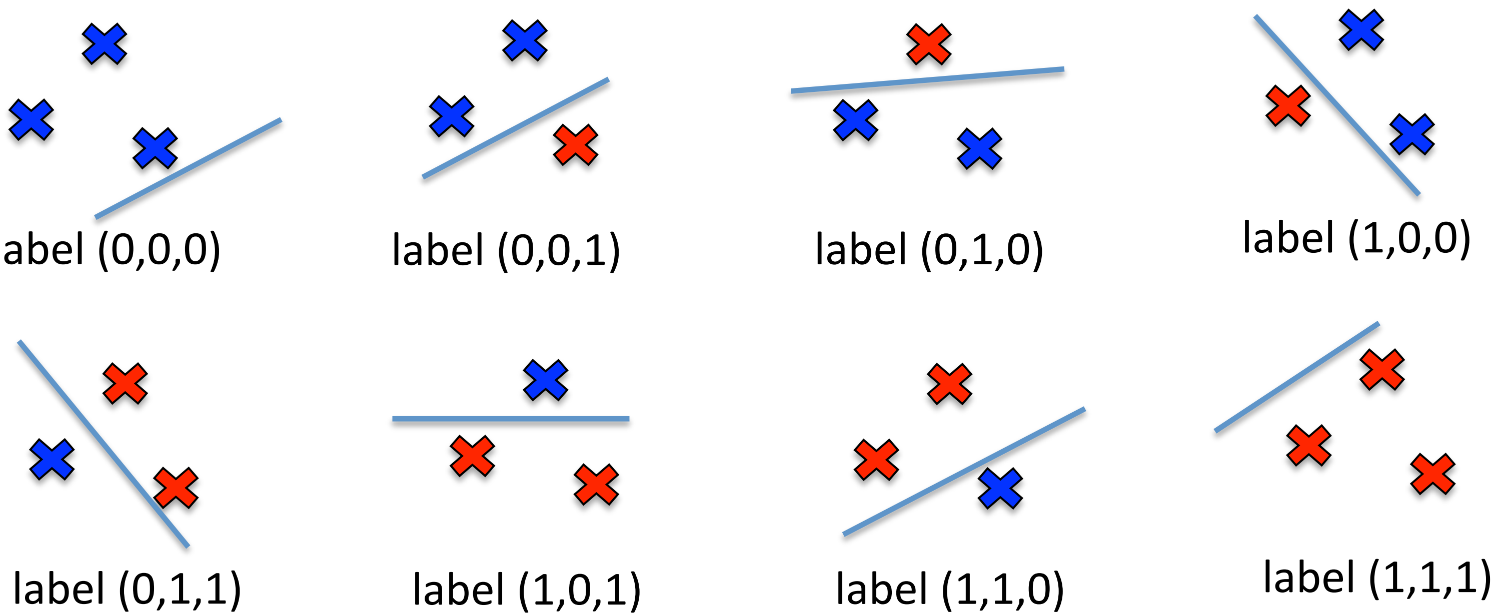
So, $\mathbf{VCdim(H_{intervals}) = 2}$

VCdim($\mathcal{H}_{\text{lines}}$)

Consider $\mathcal{H} = \mathcal{H}_{\text{lines}}$ be the set of lines in $\mathbf{R}^2$.

$\mathcal{H}_{\text{lines}} = \{h_{a,b,c}\colon \mathbf{R}^2 \rightarrow \{0, 1\},\ h_{a,b,c}((x,y)) = \mathbf{1}_{[ax+by+c>0]}((x,y))\ ,\ a, b, c \in \mathbf{R}\}$

$\mathcal{H}_{\text{lines}}$ shatters any set A of three non-colinnear points in $\mathbf{R}^2$.



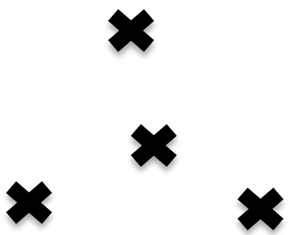
VCdim($\mathcal{H}_{\text{lines}}$)

Consider $\mathcal{H} = \mathcal{H}_{\text{lines}}$ be the set of lines in $\mathbf{R}^2$.

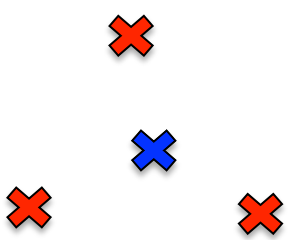
$\mathcal{H}_{\text{lines}} = \{h_{a,b,c}: \mathbf{R}^2 \to \{0, 1\}, h_{a,b,c}((x,y)) = \mathbf{1}_{[ax+by+c>0]}((x,y)) \text{ , a, b, c} \in \mathbf{R}\}$

$\mathcal{H}_{\text{lines}}$ doesn't shatter any set A of four points in $\mathbf{R}^2$.

Case a: one point is interior to the convex hull of the other 3

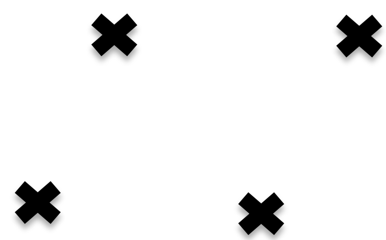


label (1,1,1,0) is un-realizable

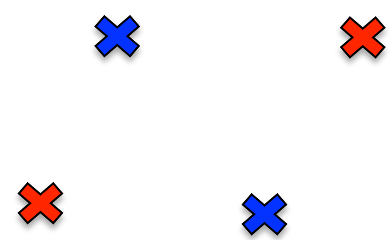


So, **VCdim($\mathrm{H}_{\text{lines}}$) = 3**

Case b: no point is interior to the convex hull of the other 3



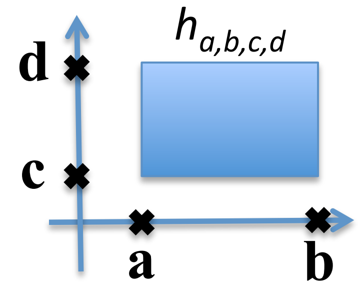
label (1,0,1,0) is un-realizable



VCdim($\mathcal{H}_{rec}{}^2$)

Consider $\mathcal{H} = \mathcal{H}_{rec}{}^2$ be the set of axis aligned rectangles in the $\mathbf{R}^2$.

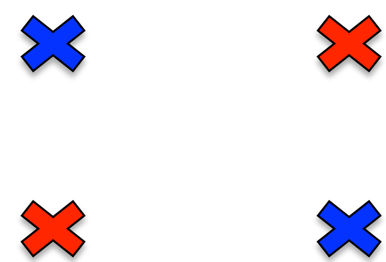
$\mathcal{H}_{rec}{}^2 = \{[a,b] \times [c,d] |\ a \leq b,\ c \leq d,\ a,\ b,\ c,\ d \in \mathbf{R}\ \}$



Can also view $\mathcal{H}_{rec}{}^2$ as:

$\mathcal{H}_{rec}{}^2 = \{h_{a,b,c,d}\colon \mathbf{R}^2 \to \{0,\ 1\},\ h_{a,b,c,d} = \mathbf{1}_{[a,b] \times [c,d]},\ a \leq b,\ c \leq d,\ a,\ b,\ c,\ d \in \mathbf{R}\ \}$

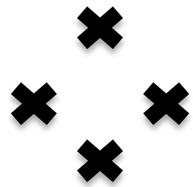
Does $\mathcal{H}_{rec}{}^2$ shatters any set A of four different points in $\mathbf{R}$? Take A the set of 4 vertices of a rectangle with axis aligned in $\mathbf{R}^2$. Then $\mathcal{H}_{rec}{}^2$ doesn't shatter A (the (0,1,0,1) labeling is not realizable).

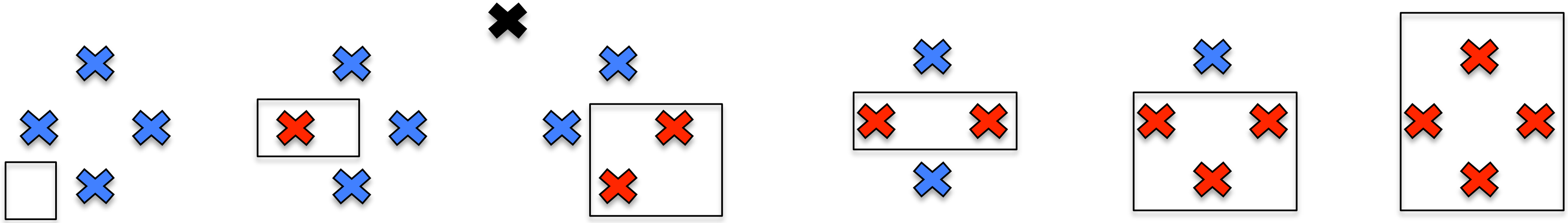


Does this mean that VCdim($\mathcal{H}_{rec}{}^2$) < 4?

No! Either we show that all sets A of size 4 are not shattered by $\mathcal{H}_{rec}{}^2$ (VCdim($\mathcal{H}_{rec}{}^2$) < 4) or find a set A of size 4 that is shattered (VCdim($\mathcal{H}_{rec}{}^2$) $\geq$ 4) .

VCdim($\mathcal{H}_{rec}{}^2$)

Consider the set A:  Then A is shattered by $\mathcal{H}_{rec}{}^2$.

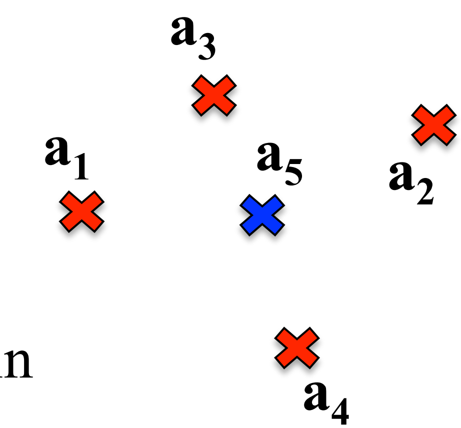


label (0,0,0,0) | label (1,0,0,0) | label (0,1,1,0) | label (1,0,1,0) | label (1,1,1,0) | label (1,1,1,1)

Can realize all 16 possible labels. So VCdim($\mathcal{H}_{rec}{}^2$) ≥ 4

Show now that all sets A of 5 points are not shattered by $\mathcal{H}_{rec}{}^2$.

A = $\{a_1, a_2, a_3, a_4, a_5\}$. Consider a_1 – the leftmost point (smaller x) , a_2 – the rightmost point (larger x), a_3 – the lowest point (smaller y), a_4 – the highest point (larger y). Then every rectangle containing a_1, a_2, a_3, a_4 will also contain a_5, so (1,1,1,1,0) is not realizable. So, **VCdim($\mathcal{H}_{rec}{}^2$) = 4**



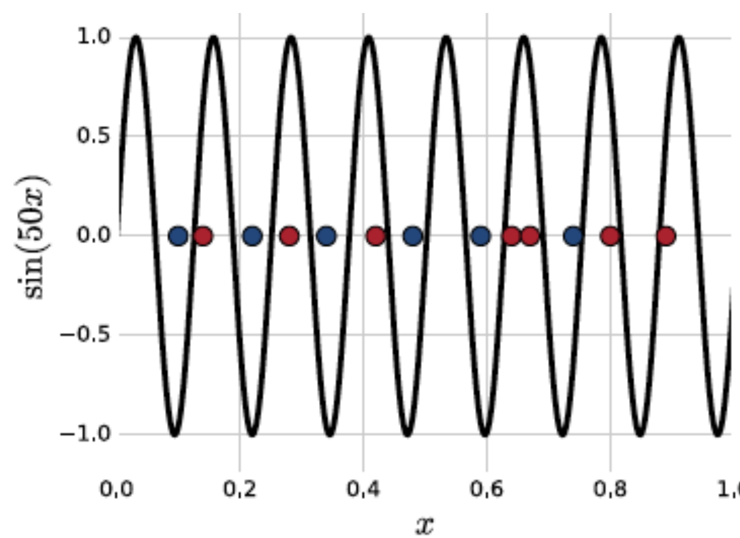
VCdim($\mathcal{H}_{\text{sin}}$)

VCdim($\mathcal{H}_{\text{thresholds}}$) = 1, VCdim($\mathcal{H}_{\text{intervals}}$) = 2, VCdim($\mathcal{H}_{\text{lines}}$) = 3
VCdim($\mathcal{H}_{\text{rec}^2}$) = 4

Consider $\mathcal{H} = \mathcal{H}_{\text{sin}}$ be the set of sin functions:
$\mathcal{H}_{\text{sin}} = \{h_\theta : \mathbf{R} \rightarrow \{0,1\} \mid h_\theta(x) = \lceil \sin(\theta x) \rceil,\ \theta \in \mathrm{R}\}$, $\lceil -1 \rceil = 0$

VCdim($\mathcal{H}_{\text{sin}}$) = ?

It is possible to prove that VCdim($\mathcal{H}$)=∞, namely, for every d, one can find d points that are shattered by $\mathcal{H}$.



VCdim($\mathcal{H}_{\sin}$)

Lemma

Let $x \in (0,1)$ and let $0.x_1x_2x_3\ldots$ be the binary representation of x. Then, for any natural number m, provided that there exist $k \geq m$ such that $x_k = 1$, we have:

$$\lceil \sin(2^m \pi x) \rceil = 1 - x_m$$

Example of binary representation:

$x = (0.x_1x_2x_3\ldots)_2 = x_1 \times 2^{-1} + x_2 \times 2^{-2} + x_3 \times 2^{-3} + \ldots$

$x = 0.75 = ½ + ¼ = (0.110000\ldots)_2$

$x = 0.3 = 0 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5} + \ldots = (0.01001\ldots)_2$

VCdim($\mathcal{H}_{\sin}$)

Lemma

Let x $\in (0, 1)$ and let $0.x_1x_2x_3\ldots$ be the binary representation of x. Then, for any natural number m, provided that there exist $k \geq m$ such that $x_k = 1$, we have:

$$\lceil \sin(2^m \pi x) \rceil = 1 - x_m$$

Proof

$\sin(2^m\pi x) = \sin(2^m\pi(0.x_1x_2x_3\ldots)) = \sin(2\pi * 2^{m-1}(0.x_1x_2x_3\ldots)) =$

$= \sin(2\pi*(x_1x_2x_3\ldots x_{m-1}.x_m\, x_{m+1}\ldots))$ (left shift with m-1 position)

$= \sin(2\pi*(x_1x_2x_3\ldots x_{m-1}.x_m\, x_{m+1}\ldots) - 2\pi*(x_1x_2x_3\ldots x_{m-1}.0))$ (sin has period 2π)

$= \sin(2\pi*(0.x_mx_{m+1}\ldots))$

Note that $0.x_mx_{m+1}\ldots > 0$ as there exist $k \geq m$ such that $x_k = 1$

Case 1: $x_m = 0$, then $0 < 2\pi*(0.x_mx_{m+1}\ldots) < 2\pi*1/2 = \pi$. So $0 < \sin(2^m\pi x) < 1$, and from here it results that: $\lceil \sin(2^m \pi x) \rceil = 1 = 1 - 0 = 1 - x_m$

Case 2: $x_m = 1$, then $2\pi > 2\pi*(0.x_mx_{m+1}\ldots) \geq 2\pi*1/2 = \pi$. So $-1 \leq \sin(2^m\pi x) \leq 0$, and from here it results that: $\lceil \sin(2^m \pi x) \rceil = 0 = 1 - 1 = 1 - x_m$ (we consider $\lceil -1 \rceil = 0$)

VCdim($\mathcal{H}_{\sin}$)

To prove VCdim($\mathcal{H}_{\sin}$) = ∞, we need to pick *n* points which are shattered by $\mathcal{H}_{\sin}$, for any *n*. To do so, we construct *n* points $x_1, x_2, \ldots, x_n \in [0, 1]$, such that the set of the *m*-th bits in the binary expansion, as m ranges from 1 to 2^n, ranges over all possible labelings of $x_1, x_2, \ldots, x_n$.

x_1 = 0. 0 0 0 0 ... 1 1

x_2 = 0. 0 0 0 0 ... 1 1

...

x_{n-1} = 0 0 0 1 1 ... 1 1

x_n = 0. 0 1 0 1 ... 0 1

m=1

For example, to give the labeling 1 for all instances, we just pick m=1:

$$h(x) = \lceil \sin(2^1 \pi x) \rceil$$

which returns 1 - the first bit (column) in the binary expansion.

VCdim($\mathcal{H}_{\sin}$)

To prove VCdim($\mathcal{H}_{\sin}$) = ∞, we need to pick n points which are shattered by $\mathcal{H}_{\sin}$, for any n. To do so, we construct n points $x_1, x_2, \ldots, x_n \in [0, 1]$, such that the set of the m-th bits in the binary expansion, as m ranges from 1 to 2^n, ranges over all possible labelings of $x_1, x_2, \ldots, x_n$.

$$
\begin{array}{lcl}
x_1 & = & 0.\ 0\ 0\ 0\ 0\ \ldots\ \ 1\ 1 \\
x_2 & = & 0.\ 0\ 0\ 0\ 0\ \ldots\ \ 1\ 1 \\
 & \ldots & \\
x_{n-1} & = & 0.\ 0\ 0\ 1\ 1\ \ldots\ \ 1\ 1 \\
x_n & = & 0.\ 0\ 1\ 0\ 1\ \ldots\ \ 0\ 1
\end{array}
$$

m=2

If we wish to give the labeling 1 for $x_1, x_2, \ldots, x_{n-1}$, and the labeling 0 for x_n, we pick m=2:

$$h(x) = \lceil \sin(2^2 \pi x) \rceil$$

which returns 1 - the second bit (column) in the binary expansion.

VCdim($\mathcal{H}_{\sin}$)

To prove VCdim($\mathcal{H}_{\sin}$) = ∞, we need to pick n points which are shattered by $\mathcal{H}_{\sin}$, for any n. To do so, we construct n points $x_1, x_2, \ldots, x_n \in [0, 1]$, such that the set of the m-th bits in the binary expansion, as m ranges from 1 to 2^n, ranges over all possible labelings of $x_1, x_2, \ldots, x_n$.

x_1 = 0. 0 0 0 0 ... 1 1

x_2 = 0. 0 0 0 0 ... 1 1

...

x_{n-1} = 0. 0 0 1 1 ... 1 1

x_n = 0. 0 1 0 1 ... 0 1

$m=2^n$

If we wish to give the labeling 0 for $x_1, x_2, \ldots, x_{n-1}, x_n$ we pick $m=2^n$:

$$h(x) = \left\lceil \sin(2^{2^n} \pi x) \right\rceil$$

which returns 1 - the bit on position 2^n (column) in the binary expansion.

VCdim($\mathcal{H}_{\sin}$)

To prove VCdim($\mathcal{H}_{\sin}$) = ∞, we need to pick *n* points which are shattered by $\mathcal{H}_{\sin}$, for any *n*. To do so, we construct *n* points $x_1, x_2, \ldots, x_n \in [0, 1]$, such that the set of the *m*-th bits in the binary expansion, as m ranges from 1 to 2^n, ranges over all possible labelings of $x_1, x_2, \ldots, x_n$.

x_1 = 0. 0 0 0 0 ... 1 1

x_2 = 0. 0 0 0 0 ... 1 1

...

x_{n-1} = 0. 0 0 1 1 ... 1 1

x_n = 0. 0 1 0 1 ... 0 1

We conclude that $x_1, x_2, \ldots, x_{n-1}, x_n$ can be given any labeling by some $h \in \mathcal{H}_{\sin}$, so it is shattered. This can be done for any *n*, so VCdim($\mathcal{H}_{\sin}$) = ∞.

Some basic properties of the VCdim($\mathcal{H}$)

1. VCdim($\mathcal{H}$) $\leq \log_2|\mathcal{H}|$

2. If $\mathcal{H}_1 \subseteq \mathcal{H}_2$ then VCdim($\mathcal{H}_1$) $\leq$ VCdim($\mathcal{H}_2$)

3. If VCdim($\mathcal{H}$) $= \infty$ then $\mathcal{H}$ is not PAC learnable