Seminar 4

(1) H^d_{mon} = class of monotone Boolean conjunctions over $\{0,1\}^d$

$\mathcal{H}^d_{mon} = \{ h : \{0,1\}^d \to \{0,1\},\ h(x_1, x_2, \dots, x_d) = \bigwedge_{i=1}^{d} l(x_i) \} \cup \{h^-\}$

$l(x_i) \in \{ \underbrace{x_i}_{\text{positive literal}}, \underbrace{1}_{\text{missing literal}} \}$

$h^-(x_1, \dots x_d) = 0$ always

So $|\mathcal{H}^d_{mon}| = 2^d + 1$

Examples: $d = 2$ $\quad \mathcal{H}^2_{mon} = \{ \underbrace{0}_{h^-}, \underbrace{1}_{h_{empty}}, x_1, x_2, x_1 \wedge x_2 \}$

$d = 3$ $\quad \mathcal{H}^2_{mon} = \{ 0, 1, x_1, x_2, x_3, x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3, x_1 \wedge x_2 \wedge x_3 \}$

We need to show that VC dim $(H^d_{mon}) = d$

We know that $|H^d_{mon}| = 2^d + 1$ so VC dim $(H^d_{mon}) \le \lfloor \log_2 (|H^d_{mon}|) \rfloor$

So VC dim $(H^d_{mon}) \le \lfloor \log_2 (2^d + 1) \rfloor = d$

We only need to find a set $C \subseteq \{0,1\}^d$ with d points that is shattered by H^d_{mon}.

Usually, taking $C = \{e_1, e_2, \dots e_d\}$ $e_i = (0, \dots 0, 1, 0, \dots 0)$ works, but not for this $\mathcal{H} = \mathcal{H}^d_{con}$. You cannot have a conjunction that will have $h(e_1) = h(e_2) = 1$ and $h(e_3) = \dots h(e_d) = 0$.

Instead, we chose $C = \{ (0, 1, 1, \dots 1), (1, 0, 1, 1, \dots 1), \dots, (1, 1, \dots \underset{i}{0}, 1, 1) \}$

set of vectors of the form $c_i = (1, 1, \dots 1) - e_i$ $\quad i = 1, d$.

We want to show that for each possible labeling $(y_1, y_2, \dots y_d)$ of the points $c_i = (1, 1, \dots 1) - e_i$ there exists a function $h \in \mathcal{H}^d_{mon}$ such that

$h(c_i) = y_i \quad \forall i = 1, d$

Consider $(y_1, y_2, \dots, y_d)$ a labeling and take $J = \{ j \mid y_j = 1 \}$.

If $J = \emptyset$ then h^- realizes the labeling $(0, 0, \dots 0)$

If $J = \{1, 2, \dots d\}$ then $h_{empty} = 1$ (all literals are missing) realizes the labeling $(1, 1, \dots, 1)$

If $1 \le |J| \le d-1$ then consider $h_J(x_1, x_2, ..., x_d) = \bigwedge_{j \notin J} x_j$

for example if $d=4$ and $J=\{2,3\}$ $h_J(x_1,x_2,x_3,x_4) = x_1 \wedge x_4$

$$h_J(c_1) = h_J(0,1,1,1) = 0$$
$$h_J(c_2) = h_J(1,0,1,1) = 1$$
$$h_J(c_3) = h_J(1,1,0,1) = 1$$
$$h_J(c_4) = h_J(1,1,1,0) = 0$$

We have that $h_J(c_i) = 1$ if $i \in J$ and $h_J(c_i) = 0$ if $i \notin J$

For all indices $i \in J$ c_i will have value 0 on the position i and 1 on rest, but variable x_i is not considered in the conjunction. So $h_J(c_i) = 1$

For all indices $i \notin J$ c_i will have value 0 on the position i and because the conjunction contains the literal x_i then we have that $h_J(c_i) = 0$

(2) $X = \{0,1\}^n$

$$\mathcal{H}_{n\text{-parity}} = \{ h_I \mid I \subseteq \{1,2,...,n\},\ h_I(x_1,x_2,...,x_n) = (\textstyle\sum_{i \in I} x_i) \bmod 2 \}$$

What is $VCdim(\mathcal{H}_{n\text{-parity}})$?

for each subset $I \subseteq \{1,2,...,n\}$ we have a h_I so $|\mathcal{H}_{n\text{-parity}}| = 2^n$

We know that $VCdim(\mathcal{H}_{n\text{-parity}}) \le \lfloor \log_2 2^n \rfloor = n$.

So $VCdim(\mathcal{H}_{n\text{-parity}}) \le n$.

Can we find a set C with n points that is shattered by $\mathcal{H}_{n\text{-parity}}$?

Let's try the "usual" set $C = \{e_1, e_2, ..., e_n\}$ $e_i = (0, ..., 0, \underset{i}{1}, 0, ..., 0)$
set of unit vectors

We want to show that for each possible labelling $(y_1, y_2, ..., y_n)$ of $(e_1, e_2, ..., e_n)$ you can find a corresponding h such that $h(e_i) = y_i$ $\forall i = 1..n$

Consider $(y_1, y_2, ..., y_n)$ such a labeling and take $I = \{ i \mid y_i = 1 \}$

Then $h_I(e_i) = (\sum_{i \in I} x_i) \bmod 2 = \begin{cases} 1, & \text{if } i \in I \\ 0, & \text{otherwise} \end{cases}$

So $VCdim(\mathcal{H}_{n\text{-parity}}) \ge n$

③ $\mathcal{X}$ - finite domain, $|\mathcal{X}| < \infty$

$k \leq |\mathcal{X}|$

3.1. $\mathcal{H}^{\mathcal{X}}_{=k} = \{ h \in \{0,1\}^{\mathcal{X}} \mid |\{x : h(x) = 1\}| = k \}$ = set of all functions that assign the value 1 to exactly k elements of $\mathcal{X}$.

$VCdim(\mathcal{H}^{\mathcal{X}}_{=k}) = ?$

if $k=0 \to \mathcal{H}^{\mathcal{X}}_{=0} = \{h^-\}$, all points get the value 0

if $k=1 \to \mathcal{H}^{\mathcal{X}}_{=1}$ has $|\mathcal{X}|$ functions = n functions

$\mathcal{X} = \{x_1, x_2, \ldots x_n\}$, $n = |\mathcal{X}|$

$h_i : \{x_1, \ldots, x_n\} \to \{0,1\}$ $h_i(x_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j. \end{cases}$

if $k=2 \to \mathcal{H}^{\mathcal{X}}_{=2}$ has C_n^2 elements

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if $k = n-1 \to \mathcal{H}^{\mathcal{X}}_{=n-1}$ has n elements

if $k = n \to \mathcal{H}^{\mathcal{X}}_{=n}$ has 1 element $h^+(x_i) = 1 \ \forall i = \overline{1,n}$.

We first show that $VCdim(\mathcal{H}^{\mathcal{X}}_{=k}) \leq \min(k, n-k)$

$VCdim(\mathcal{H})$

Case 1) if $n \geq 2k$, in this case $\min(k, n-k) = k$

$k \leq \frac{n}{2}$.

$\mathcal{H}$ will consists of functions h that label exactly k elements of $\mathcal{X}$ with label 1. So any set C with more than k points cannot be shattered because the labeling with all 1's $(1,1,1,\ldots 1)$ cannot be realised by any $h \in \mathcal{H}$

Case 2) if $n < 2k$, in this case $\max(k, n-k) = n-k$

$k > \frac{n}{2}$

$\mathcal{H}$ will consists of functions h that labels k element of $\mathcal{X}$ with label 1, and n-k points of $\mathcal{X}$ with label 0. So any set with more than n-k+1 points cannot be shattered by $\mathcal{H}$ as the labelling with all 0's $(0,0,\ldots 0)$ cannot be realiz by any $h \in \mathcal{H}$

So we have that $VCdim(\mathcal{H}) \leq \min(k, n-k)$.

We will prove that $VCdim(\mathcal{H}) = \min(k, n-k)$

Consider $k' = \min(k, n-k)$. We need to show that there exists a set of points $A = \{x_{i_1}, x_{i_2}, \dots, x_{i_{k'}}\} \subseteq X$ that is shattered by $\mathcal{H}$

This means that for each subset $B \subseteq A$ we can find $h_B \in \mathcal{H}$ such that $h_B(x) = \begin{cases} 1, x \in B \\ 0, x \in A \setminus B \end{cases}$

We choose a set of $k - |B|$ points $B' = \{b_1, b_2, \dots, b_{k-|B|}\} \subseteq X \setminus A$.

We can make the choice since $k - |B| \leq |X \setminus A|$

$$k - |B| \leq n - k'$$

$$k' - |B| \leq n - k$$

$$k' - |B| \leq k' \leq n - k \text{ (this is true)}$$

So $B \subseteq A$ has $|B|$ elements

$B' \subseteq X \setminus A$ has $k - |B|$ elements

So $|B \cup B'| = |B| + |B'|$ (as $B \cap B' = \phi$) $= k$

So if we consider the characteristic function of the set $B \cup B'$ we have $\mathbb{1}_{B \cup B'}(x) = \begin{cases} 1, x \in B \cup B' \\ 0, \text{otherwise} \end{cases}$

What is more important $\mathbb{1}_{B \cup B'}$ takes value 1 for exactly k points, so is a member of $\mathcal{H}$.

So in this case we take $h_B = \mathbb{1}_{B \cup B'}$

h_B will have the desired property that $h_B(x) = \begin{cases} 1, x \in B \\ 0, x \in A \setminus B \end{cases}$

So any set A of $k' = \min(k, n-k)$ can be shattered by $\mathcal{H}$

So $VCdim(\mathcal{H}) = k' = \min(k, n-k)$

3.2 $\mathcal{H}_{at\text{-}most\text{-}k} = \{ h \in \{0,1\}^{\mathcal{X}} : |\{x : h(x) = 1\}| \leq k$ or $|\{x : h(x) = 0\}| \leq k \}$

if $k=0$, $\mathcal{H}_{at\text{-}most\text{-}0} = \{h^-, h^+\}$

$|\{x : h^-(x) = 1\}| \leq 0$ $\quad$ $|\{x : h^+(x) = 0\}| \leq 0$.

if $k=1$ $\mathcal{H}_{at\text{-}most\text{-}1} = \{h^-, h^+\} \cup \{$functions h which labels just one point with label $1\} \cup \{$functions h wich label just 1 point with label $0\}$.

$\rightarrow \mathcal{H}^{\mathcal{X}}_{\leq 1}$

Case 1, if $n = |\mathcal{X}| \leq 2k + 1$ than we have that

$\mathcal{H}_{at\text{-}most\text{-}k} = \{0,1\}^{\mathcal{X}} = \{h : \mathcal{X} \to \{0,1\}\}$

This is true because any function $h, \mathcal{X} \to \{0,1\}$ will have either at most k point label wiht 0 or at most point label with 1

$\Downarrow$ $h \in \mathcal{H}_{at\text{-}most\text{-}k}$ $\quad$ $\Downarrow$ $h \in \mathcal{H}_{at\text{-}most\text{-}k}$

Example: Take $n=7$, $k=4$. $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$

	x	x_1	x_2	x_3	x_4	x_5	x_6	x_7
at most 4 1's, 4 0's $\leftarrow$	$h(x)$	1	1	0	0	1	0	1
at most 4 1's $\leftarrow$	$h(x)$	0	0	1	0	0	0	0
at most 4 1's. $\leftarrow$	$h(x)$	0	1	0	0	1	0	0

In this case $\text{VCdim}(\mathcal{H}_{at\text{-}most\text{-}k}) = \text{VCdim}(\{0,1\}^{\mathcal{X}}) = n = |\mathcal{X}|$.

Case 2 if $n = |\mathcal{X}| \geq 2k+2$

We first show that $\text{VCdim}(\underbrace{\mathcal{H}_{at\text{-}most\text{-}k}}_{\mathcal{H}}) = \text{VCdim}(\mathcal{H}) \geq 2k+1$

Consider any set A of $2k+1$ points in $\mathcal{X}$, $A = \{a_1, a_2, \dots, a_{2k+1}\}$

We will show that A is shattered by $\mathcal{H}$. It's enough to show that for each possible labeling $(y_1, y_2, \dots, y_{2k+1})$ of the points $(a_1, \dots, a_{2k+1})$ we can find an $h \in \mathcal{H}$ such that $h(y_i) = a_i$

~~[illegible]~~

Take $J = \{ j \mid y_j = 1 \}$, and take $B_y = \{ a_j \in A \mid y_j = 1, \ j \in J \}$. Then

if $|J| \le k$ we know that $\mathbb{1}_{B_y} \in \mathcal{H}_{at-most-k}$ so we take

$$h = \mathbb{1}_{B_y} \qquad h(a_j) = \begin{cases} 1, & y_j = 1 \\ 0, & \text{otherwise} \end{cases}$$

If $|J| > k$, take $C_y = \{ a_j \in A \mid y_j = 0, \ j \notin J \}$. has at most k elements

So we take in this case

$$h = \mathbb{1}_{A \setminus C_y} \qquad h(a_j) = \begin{cases} 1, & y_j = 1 \\ 0, & y_j = 0 \end{cases}$$

So, we have that $VCdim(\mathcal{H}) \ge 2k+1$

We show now that $VCdim(\mathcal{H}) < 2k+2$

Consider any set A of $2k+2$ points $A = \{ a_1, a_2, \dots, a_{2k+2} \}$

There is no $h \in \mathcal{H}$ that will label the first $k+1$ points with 1 and the last $k+1$ points with 0.

So, in conclusion $VCdim(\mathcal{H}_{at-most-k}) = \min(|X|, 2k+1)$