Advanced Machine Learning Seminar 3

Exercise 1 $\mathcal{H}$ – finite hypothesis class
$VC\dim(\mathcal{H}) \leq \lfloor \log_2(|\mathcal{H}|) \rfloor$ – upper bound

1. example of $\mathcal{H}$ infinite, $\mathcal{H}$ contains functions $h\colon [0,1] \to \{0,1\}$ and $VC\dim(\mathcal{H}) = 1$
Take $\mathcal{H}_{threshold}$ restricted to $[0,1]$

$$\mathcal{H}_{threshold,[0,1]} = \big\{h_a\colon [0,1] \to \{0,1\},\ h_a(x) = \mathbb{1}_{[x<a]},\ a \in [0,1]\big\}$$

$$h_a(x) = \begin{cases} 1, & 0 \leq x < a \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$VC\dim(\mathcal{H}_{threshold,[0,1]}) = 1$ (very similar proof with the one provided in lecture 6)

2. $\mathcal{H} = \{h_a, h_b\}$ – has only two functions h_a and h_b
Take $h_a, h_b \in \mathcal{H}_{threshold,[0,1]}$.
$h_a = h_{0.5}, h_b = h_{0.75}$
Take $A = \{0.6\}$. $\mathcal{H}$ shatters A because $\mathcal{H}_A$ has two functions $h_a(0.6) = 0$ and $h_b(0.6) = 1$.
$|\mathcal{H}_A| = 2^{|A|} = 2^1 = 2$
$\mathcal{H}$ cannot shatter any set A of min ≥ 2 points ($\mathcal{H}$ has only 2 functions).
So $VC\dim(\mathcal{H}) = 1 = \lfloor log_2(|\mathcal{H}|) \rfloor$

Exercise 2 $\mathcal{H}^d_{rec}$ – class of axis aligned rectangles in $\mathbb{R}^d$. In lecture 6 we proved that $VC\dim(\mathcal{H}^2_{rec}) = 4$. We want to show, in the general case, that $VC\dim(\mathcal{H}^d_{rec}) = 2d$.

$$\mathcal{H}^d_{rec} = \{h_{(a_1,b_1,a_2,b_2,\ldots,a_d,b_d)} \mid a_i \leq b_i, i = \overline{1,d}\}$$

$$h_{(a_1,b_1,a_2,b_2,\ldots,a_d,b_d)}(\underbrace{\underline{x}}_{\underline{x}=(x^1,x^2,\ldots,x^d)}) = \begin{cases} 1, & a_i \leq x^i \leq b_i \quad \forall i = \overline{1,d} \\ 0, & \text{otherwise} \end{cases}$$

In order to show that $VC\dim(\mathcal{H}^d_{rec}) = 2d$, we need to show that:

1) there exists a set C of $2d$ points that is shattered by $\mathcal{H}^d_{rec}$
(this will mean that $VC\dim(\mathcal{H}^d_{rec}) \geq 2d$)

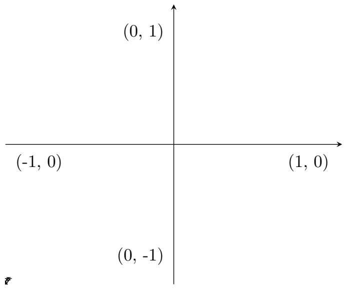
2) every set C of $2d+1$ points is not shattered by $\mathcal{H}^d_{rec}$ (this will mean that $VC\dim(\mathcal{H}^d_{rec}) < 2d+1$)

Let's prove 1).
Consider $C = \{c_1, c_2, c_3, \ldots, c_{2d-1}, c_{2d}\}$ where

$$\begin{aligned}
c_1 &= (1,0,0,\ldots,0) &&= e_1 \\
c_2 &= (0,1,0,\ldots,0) &&= e_2 \\
&\vdots \\
c_d &= (0,0,0,\ldots,1) &&= e_d \qquad c_i = e_i = -c_{i+d} \\
c_{d+1} &= (-1,0,0,\ldots,0) &&= -e_1 \qquad \forall i = \overline{1,d} \\
c_{d+2} &= (0,-1,0,\ldots,0) &&= -e_2 \\
&\vdots \\
c_{2d} &= (0,0,0,\ldots,-1) &&= -e_d
\end{aligned}$$

For $d = 2$, we will have in 2 dimensions:

$$c_1 = (1, 0) \qquad c_2 = (0, 1) \qquad c_3 = (-1, 0) \qquad c_4 = (0, -1)$$



We want to show that, for each labeling $(y_1, y_2, \ldots, y_{2d})$ of the points $(c_1, c_2, \ldots, c_{2d})$ (there are 2^{2d} possible labelings), there exists a function h in $\mathcal{H}^d_{rec}$ such that $h(c_i) = y_i \ \forall i = \overline{1, 2d}$.

Consider a labeling $(y_1, y_2, \ldots, y_{2d}) \in \{0, 1\}^{2d}$.

Each point c_i has all components $= 0$, apart from component i if $i \in \{1, \ldots, d\}$ or $i - d$ if $i \in \{d + 1, \ldots, 2d\}$.

$$\begin{array}{rl|rl|c|rl} c_1 & = (\quad 1, 0, 0, \ldots, 0) & c_2 & = (0, \quad 1, 0, \ldots, 0) & & c_d & = (0, 0, 0, \ldots, \quad 1) \\ c_{d+1} & = (\ -1, 0, 0, \ldots, 0) & c_{d+2} & = (0, \ -1, 0, \ldots, 0) & \ldots & c_{2d} & = (0, 0, 0, \ldots, \ -1) \end{array}$$

We want to find $h = h_{(a_1,b_1,a_2,b_2,\ldots,a_d,b_d)}$ such that $h_{(a_1,b_1,a_2,b_2,\ldots,a_d,b_d)}(c_i) = y_i$.

The choice of the interval $[a_i, b_i]$ is influenced by the labels y_i and y_{i+d} of the points c_i and c_{i+d}. As all other points $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{i+d-1}, c_{i+d+1}, \ldots, c_{2d}$ have 0 on the i-th component, we have that $[a_i, b_i]$ should contain 0, otherwise each point will be labeled with 0.

So $[a_i, b_i]$ depends on y_i and y_{i+d}, and $[a_i, b_i]$ decides basically the label of points c_i and c_{i+d}:

$$c_i = (0, \ldots, 0, 1, 0, \ldots, 0) \qquad c_{i+d} = (0, \ldots, 0, -1, 0, \ldots, 0)$$

Possible cases:

I $\quad y_i = 0, y_{i+d} = 0$, then $[a_i, b_i] \cap \{-1, 1\} = \emptyset$
$[a_i, b_i]$ should not contain points -1 and 1.
In this case, take $a_i = -0.5, b_i = 0.5$ (many other choices are possible)

II $\quad y_i = 0, y_{i+d} = 1$, then $[a_i, b_i] \cap \{-1, 1\} = \{-1\}$
$[a_i, b_i]$ should contain only point -1 such that c_{i+d} will get label 1.
In this case, take $a_i = -2, b_i = 0.5$ (many other choices are possible)

III $\quad y_i = 1, y_{i+d} = 0$, then $[a_i, b_i] \cap \{-1, 1\} = \{1\}$
$[a_i, b_i]$ should contain only point $+1$ such that c_i will get label 1.
In this case, take $a_i = -0.5, b_i = 2$ (many other choices are possible)

IV $\quad y_i = 1, y_{i+d} = 1$, then $[a_i, b_i] \cap \{-1, 1\} = \{-1, 1\}$
$[a_i, b_i]$ should contain both points $\{-1, 1\}$ such that c_i and c_{i+d} will get label 1.
In this case, take $a_i = -2, b_i = 2$ (many other choices are possible)

In all cases, we have that $h_{(a_1,b_1,a_2,b_2,\ldots,a_d,b_d)}(c_i) = y_i, \forall i = \overline{1, 2d}$, where each interval $[a_i, b_i]$ was determined based on y_i and y_{i+d}, $i = \overline{1, d}$.

So, $VC\dim(\mathcal{H}^d_{rec}) \geq 2d$.

$\square$

2) Let C be a set of size $2d + 1$ points. We will show that C cannot be shattered by $\mathcal{H}^d_{rec}$.

Because we have $2d + 1$ points in C and there are only d dimensions, there will exist a point $x \in C$ such that, for each dimension $i = \overline{1, d}$ there will be 2 points x' and $x'' \in C$ such that $x'_i \leq x_i \leq x''_i$ (the point x_i is "inside" the convex hull determined by all other points in dimension i).

So the label for which x has value 0 and all other $2d$ points get label 1 cannot be realized by any function $h \in \mathcal{H}^d_{rec}$ (because x is inside the rectangle) that contain all other points.

$\square$

Exercise 3 $\mathcal{H}^d_{con}$ – class of Boolean conjunctions over the variables $x_1, x_2, \ldots, x_d, d \geq 2$

$$\mathcal{H}^d_{con} = \left\{ h\colon \{0,1\}^d \to \{0,1\},\ h(x_1, x_2, \ldots, x_d) = \bigwedge_{i=\overline{1,d}} l(x_i) \right\}$$
$$l(x_i) = \text{literal of variable } x_i$$
$$l(x_i) \in \{x_i, \overline{x_i}, \underset{missing}{1}\}$$

We also consider that $h^- \in \mathcal{H}^d_{con}$, $\quad h^-(x_1, x_2, \ldots, x_d) = 0$ always.

a) So $|\mathcal{H}^d_{con}| = 3^d + 1$.

b) $VC\dim(\mathcal{H}^d_{con}) \leq \lfloor \log_2(3^d + 1) \rfloor$

c) We will show that $\mathcal{H}^d_{con}$ shatters the set of unit vectors $\{e_i, i \leq d\}$

$e_i = (0, 0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0)$

Consider $C = \{e_1, e_2, \ldots, e_d\}$. We want to prove that, for each possible labeling $(y_1, y_2, \ldots, y_d)$, there exists an $h \in \mathcal{H}^d_{con}$ such that $h(e_i) = y_i$.

Consider a labeling $(y_1, y_2, \ldots, y_d)$ and take $\mathcal{J} = \{j \mid y_j = 1\}$.

If $\mathcal{J} = \emptyset \Rightarrow h^-$ realizes the labeling $(0, 0, \ldots, 0)$.

If $\mathcal{J} = \{1, \ldots, d\} \Rightarrow h_{empty}$ (all literals are missing) $= 1\ \forall x_i$ realizes the labeling $(1, 1, \ldots, 1)$.

In all other cases, define

$$h_{\mathcal{J}} = \bigwedge_{j \notin \mathcal{J}} \overline{x_j} = \bigwedge_{j \in \{1,\ldots,d\} \setminus \mathcal{J}} \overline{x_j}$$

If $\mathcal{J} = \{1, 2, 4\}$, define $h_{\mathcal{J}} = \overline{x_3} \wedge \overline{x_5} \wedge \overline{x_6} \quad (d = 6)$.

$$\begin{aligned}
h_{\mathcal{J}}(e_1) &= h_{\mathcal{J}}(1,0,0,0,0,0) = \overline{0} \wedge \overline{0} \wedge \overline{0} = 1 \\
h_{\mathcal{J}}(e_2) &= h_{\mathcal{J}}(0,1,0,0,0,0) = \overline{0} \wedge \overline{0} \wedge \overline{0} = 1 \\
h_{\mathcal{J}}(e_4) &= h_{\mathcal{J}}(0,0,0,1,0,0) = \overline{0} \wedge \overline{0} \wedge \overline{0} = 1 \\
h_{\mathcal{J}}(e_3) &= h_{\mathcal{J}}(0,0,1,0,0,0) = \overline{1} \wedge \overline{0} \wedge \overline{0} = 0 \\
h_{\mathcal{J}}(e_5) &= h_{\mathcal{J}}(0,0,0,0,1,0) = \overline{0} \wedge \overline{1} \wedge \overline{0} = 0 \\
h_{\mathcal{J}}(e_6) &= h_{\mathcal{J}}(0,0,0,0,0,1) = \overline{0} \wedge \overline{0} \wedge \overline{1} = 0
\end{aligned}$$

So, $\quad h_{\mathcal{J}}(e_j) = 1$ if $j \in \mathcal{J}$

and $\quad h_{\mathcal{J}}(e_j) = 0$ if $j \notin \mathcal{J}$.

This proves that $\mathcal{H}^d_{con}$ shatters $C \Rightarrow VC\dim(\mathcal{H}^d_{con}) \geq d$.

d) We want to show that $VC\dim(\mathcal{H}^d_{con}) < d+1$.

Assume that there exists a set $C = \{c_1, c_2, \ldots, c_{d+1}\}$ of points from $\{0,1\}^d$ that is shattered by $\mathcal{H}^d_{con}$, so $|\mathcal{H}^d_{con_C}| = |\{h \colon C \to \{0,1\}, h \in \mathcal{H}\}| = 2^{d+1}$.

$$\begin{array}{ll} c_1 \in \{0,1\}^d \Rightarrow c_1 = (c_1^1, c_1^2, \ldots, c_1^d) & \in \{0,1\}^d \\ c_2 \in \{0,1\}^d \Rightarrow c_2 = (c_2^1, c_2^2, \ldots, c_2^d) & \text{Each point } c_i \text{ has} \\ \ldots & d \text{ components from } \{0,1\} \\ c_i \in \{0,1\}^d \Rightarrow c_i = (c_i^1, c_i^2, \ldots, c_i^d) & \end{array}$$

We want to find a contradiction and show that $\mathcal{H}^d_{con}$ doesn't shatter any set C of $d+1$ points.

If $\mathcal{H}^d_{con}$ shatters C, then among the 2^{d+1} function $h \colon C \to \{0,1\}$ we will have the following $d+1$ functions (for simplicity we will denote this functions with $h_1, h_2, \ldots, h_{d+1}$):

$$\begin{array}{ll} h_1 \colon \{0,1\}^d \to \{0,1\} \text{ such that} & h_1(c_1) = 0, h_1(c_2) = 1, h_1(c_3) = 1, \ldots,\ h_1(c_{d+1}) = 1 \\ h_2 \colon \{0,1\}^d \to \{0,1\} \text{ such that} & h_2(c_1) = 1, h_2(c_2) = 0, h_2(c_3) = 1, \ldots,\ h_2(c_{d+1}) = 1 \\ h_3 \colon \{0,1\}^d \to \{0,1\} \text{ such that} & h_3(c_1) = 1, h_3(c_2) = 1, h_3(c_3) = 0, \ldots,\ h_3(c_{d+1}) = 1 \\ \vdots & \\ h_{d+1} \colon \{0,1\}^d \to \{0,1\} \text{ such that} & h_{d+1}(c_1) = 0, h_{d+1}(c_2) = 1, h_{d+1}(c_3) = 1, \ldots,\ h_{d+1}(c_{d+1}) = 0 \end{array}$$

So h_i, with $i \in \{1, \ldots, d+1\}$ realizes the labels $(1, 1, \ldots, 1, \underset{i}{0}, 1, 1, \ldots, 1)$.

So we have

$$h_i(c_j) = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \neq j \end{cases}$$

We will use the functions $h_1, h_2, \ldots, h_{d+1}$ to arrive at a contradiction. Each h_i is in $\mathcal{H}^d_{con}$, so it can be written as a conjunction of literals, where each literal from the writing of h_i can have three values for a variable x_k: $l_i(x_k) = \begin{cases} x_k, & \text{positive literal} \\ \overline{x_k}, & \text{negaitive literal} \\ 1, & \text{missing literal} \end{cases}$

For example, if we consider $d = 3$, a possible h from $\mathcal{H}^3_{con}$ could be $h = x_1 \wedge \overline{x_2}$, in this case $h = l(x_1) \wedge l(x_2) \wedge l(x_3)$ with $l(x_1) = x_1, \quad l(x_2) = \overline{x_2}, \quad l(x_3) = 1$.

In the general case, we have

$$h_i(x_1, x_2, \ldots, x_d) = \bigwedge_{k=1}^{d} l_i(x_k), \qquad l_i(x_k) \in \{x_k, \overline{x_k}, 1\}$$

Now, we go back to our $h_1, h_2, \ldots, h_{d+1}$.

$$\begin{array}{l} h_1 \text{ realizes the labels } (0, 1, 1, 1, \ldots, 1) \text{ on } \{c_1, c_2, \ldots, c_{d+1}\} = C \\ h_2 \text{ realizes the labels } (1, 0, 1, 1, \ldots, 1) \text{ on } \{c_1, c_2, \ldots, c_{d+1}\} = C \\ \vdots \\ h_{d+1} \text{ realizes the labels } (1, 1, 1, 1, \ldots, 0) \text{ on } \{c_1, c_2, \ldots, c_{d+1}\} = C \end{array}$$

We will use the labels 0 to come up with a contradiction.

$$\text{Because } h_1(c_1)=0 \Leftrightarrow h_1(c_1^1,c_1^2,\dots,c_1^d)=\bigwedge_{k=1}^{d} l_1(c_1^k)=l_1(c_1^1)\wedge l_1(c_1^2)\wedge\cdots\wedge l_1(c_1^d)=0$$
$$\Rightarrow \exists k_1\in\{1,\dots,d\} \text{ such that } l_1(c_1^{k_1})=0$$
$$\text{Because } h_2(c_2)=0 \Leftrightarrow h_2(c_2^1,c_2^2,\dots,c_2^d)=\bigwedge_{k=1}^{d} l_2(c_2^k)=l_2(c_2^1)\wedge l_2(c_2^2)\wedge\cdots\wedge l_2(c_2^d)=0$$
$$\Rightarrow \exists k_2\in\{1,\dots,d\} \text{ such that } l_2(c_2^{k_2})=0$$

...

$$\text{Because } h_{d+1}(c_{d+1})=0 \Leftrightarrow h_{d+1}(c_{d+1}^1,c_{d+1}^2,\dots,c_{d+1}^d)=\bigwedge_{k=1}^{d} l_{d+1}(c_{d+1}^k)=$$
$$=l_{d+1}(c_{d+1}^1)\wedge l_{d+1}(c_{d+1}^2)\wedge\cdots\wedge l_{d+1}(c_{d+1}^d)=0$$
$$\Rightarrow \exists k_{d+1}\in\{1,\dots,d\} \text{ such that } l_{d+1}(c_{d+1}^{k_{d+1}})=0$$

$$\begin{aligned}
\text{So we have that } l_1(x_{k_1})&=0 \quad \text{where } x_{k_1}=c_1^{k_1} \text{ variable on position } k_1\\
l_2(x_{k_2})&=0 \quad \text{where } x_{k_2}=c_2^{k_2} \text{ variable on position } k_2\\
&\vdots\\
l_{d+1}(x_{k_{d+1}})&=0 \quad \text{where } x_{k_{d+1}}=c_{d+1}^{k_{d+1}} \text{ variable on position } k_{d+1}
\end{aligned}$$

We have $d+1$ literals that use variables $x_1,x_2,\dots,x_d$. So there are at least two literals using the same variable. Let these literals be l_i and l_j and assume that the variable they use is x_k.

$$\begin{aligned}
&\dots\\
&h_i=l_i(x_1)\wedge l_i(x_2)\wedge\cdots\wedge \underline{l_i(x_k)}\wedge\dots\\
&\dots\\
&h_j=l_j(x_1)\wedge l_j(x_2)\wedge\cdots\wedge \underline{l_j(x_k)}\wedge\dots\\
&\dots
\end{aligned}$$

We will gone use l_i and l_j to arrive at a contradiction.
We know that l_i and l_j satisfy the following conditions:
$l_i(c_i^k)=0$ (because $h_i(c_i)=0$ and the conjunction contains literal $l_i(c_i^k)$ which is 0)
$l_j(c_j^k)=0$ (because $h_j(c_j)=0$ and the conjunction contains literal $l_j(c_j^k)$ which is 0)
In general we have that $l_i(x_k)\in\{x_k,\overline{x_k},1\}$, $\quad l_j(x_k)\in\{x_k,\overline{x_k},1\}$
But $l_i(x_k)\neq 1$ because we have that $l_i(c_i^k)=0$.
Same argument goes for $l_j(x_k)\neq 1$.
So $l_i(x_k)$ can take values in $\{x_k,\overline{x_k}\}$ and $l_j(x_k)$ can take values in $\{x_k,\overline{x_k}\}$.
There are 4 possible cases.

Case 1: $l_i(x_k)=x_k$, $l_j(x_k)=x_k$

$$h_i(c_i)=l_i(c_i^1)\wedge l_i(c_i^2)\wedge\cdots\wedge \underset{\overset{\parallel}{0}}{l_i(c_i^k)}\wedge\cdots=0$$

We have that $l_i(c_i^k)=c_i^k=0$.

But we also have that $h_j(c_i)=1\Leftrightarrow l_j(c_i^1)\wedge l_j(c_i^2)\wedge\cdots\wedge l_j(c_i^k)\wedge\cdots=1$

This means that all literals are 1, including $l_j(c_i^k)$.

But $l_j(c_i^k)=c_i^k=0$. So we have a contradiction.

Case 2: $l_i(x_k) = \overline{x_k}$, $l_j(x_k) = \overline{x_k}$

$$h_i(c_i) = l_i(c_i^1) \wedge l_i(c_i^2) \wedge \cdots \wedge \underset{\substack{\parallel \\ 0}}{l_i(c_i^k)} \wedge \cdots = 0$$

We have that $l_i(c_i^k) = \overline{c_i^k} = 1 - c_i^k = 0 \Rightarrow c_i^k = 1$.

But we have that $h_j(c_i) = 1 \Leftrightarrow l_j(c_i^1) \wedge l_j(c_i^2) \wedge \cdots \wedge l_j(c_i^k) \wedge \cdots = 1 \Rightarrow l_j(c_i^k) = 1$.

But $l_j(c_i^k) = 1 - c_i^k = 0$. Contradiction.

Case 3: $l_i(x_k) = x_k, l_j(x_k) = \overline{x_k}$

Take another point c_m that is different than c_i and c_j, $m \neq i, m \neq j$ and $1 \leq m \leq d+1$.

So we have $h_i(c_m) = h_j(c_m) = 1$

$$h_i(c_m) = \cdots \wedge l_i(c_m^k) \wedge \cdots = 1 \Rightarrow l_i(c_m^k) = c_m^k = 1$$
$$h_j(c_m) = \cdots \wedge l_j(c_m^k) \wedge \cdots = 1 \Rightarrow l_j(c_m^k) = 1 - c_m^k = 1 \Rightarrow c_m^k = 0. \text{ Contradiction.}$$

Case 4: $l_i(x_k) = \overline{x_k}, l_j(x_k) = x_k$

Same as case 3, you will see that

$l_i(c_m^k) = 1 - c_m^k = 1 \Rightarrow c_m^k = 0$

$l_j(c_m^k) = c_m^k = 1$. Contradiction.

Exercise 4

$$\mathcal{H} = \left\{ \begin{array}{l} h_{a,b,s} \colon a \leq b,\ s \in \{-1, 1\}, \\ h_{a,b,s}(x) = \begin{cases} s, & x \in [a, b] \\ -s, & x \notin [a, b] \end{cases} \end{array} \right\}$$

See label 0 as label -1.

$VC\dim(\mathcal{H}) = ?$

$\mathcal{H}$ contains functions parametrized by 3 params (a, b, s). Intuition tells us that $VC\dim(\mathcal{H}) = 3$ (not always, but usually).

Let's consider $C = \{c_1, c_2, c_3\}$ a set of 3 distinct points with $c_1 < c_2 < c_3$ (for example, take $c_1 = 0, c_2 = 1, c_3 = 2$).

To obtain labels $(0,0,0)((-1,-1,-1))$, take $a = b = c_1 - 1$, $s = 1$ or $a = c_1$, $b = c_3$, $s = -1$ ✓

To obtain labels $(1,1,1)$, take $a = c_1$, $b = c_3$, $s = 1$ ✓

To obtain labels $(1,-1,-1)$, take $a = c_1$, $b = \frac{c_1 + c_2}{2}$, $s = 1$

To obtain labels $(-1,1,1)$, take $a = c_1$, $b = \frac{c_1 + c_2}{2}$, $s = -1$

To obtain labels $(-1,1,-1)$, take $a = \frac{c_1 + c_2}{2}$, $b = \frac{c_2 + c_3}{2}$, $s = 1$

To obtain labels $(1,-1,1)$, take $a = \frac{c_1 + c_2}{2}$, $b = \frac{c_2 + c_3}{2}$, $s = -1$

To obtain labels $(-1,-1,1)$, take $a = \frac{c_2 + c_3}{2}$, $b = c_3 + 1$, $s = 1$

To obtain labels $(1,1,-1)$, take $a = \frac{c_2 + c_3}{2}$, $b = c_3 + 1$, $s = -1$

So $\mathcal{H}$ shatters C, so $VC\dim(\mathcal{H}) \geq 3$.

Now, take C, a set of 4 points, $C = \{c_1, c_2, c_3, c_4\}$, $c_1 \leq c_2 \leq c_3 \leq c_4$.

Then $\mathcal{H}$ cannot realize the labels (1, -1, 1, -1).

This happens for any C. So $VC\dim(\mathcal{H}) < 4$. $\boxed{\text{So } VC\dim(\mathcal{H}) = 3}$