Time Series

The correlation introduced by the sampling of adjacent points in time of experimental data, requires methematical and statistical methods known as time series analysis.

Domains of applicability:

- economy – e.g. daily stock market quotations or monthly unemployment figures;
- social sciences – e.g. birthrates, school enrollments;
- epidemiology – e.g. number of influenza cases over some period;
- medicine – e.g. blood pressure measurements over time useful for evaluating drugs.

Time Series

Two approaches – the time domain and the frequency domain.

The time domain approach views the investigation of lagged relationships as most important (e.g. how does what happens today affect what will happen tomorrow), while the frequency domain approach views the investigation of cycles as most important (e.g. what is the economic cycle through periods of expansion and recession).

Time Series

<u>Def</u>. A time series is a collection of random variables indexed according to the order they are obtained in time
$\{X_t, t \in \mathbb{Z}\}$ - a stochastic process with real values

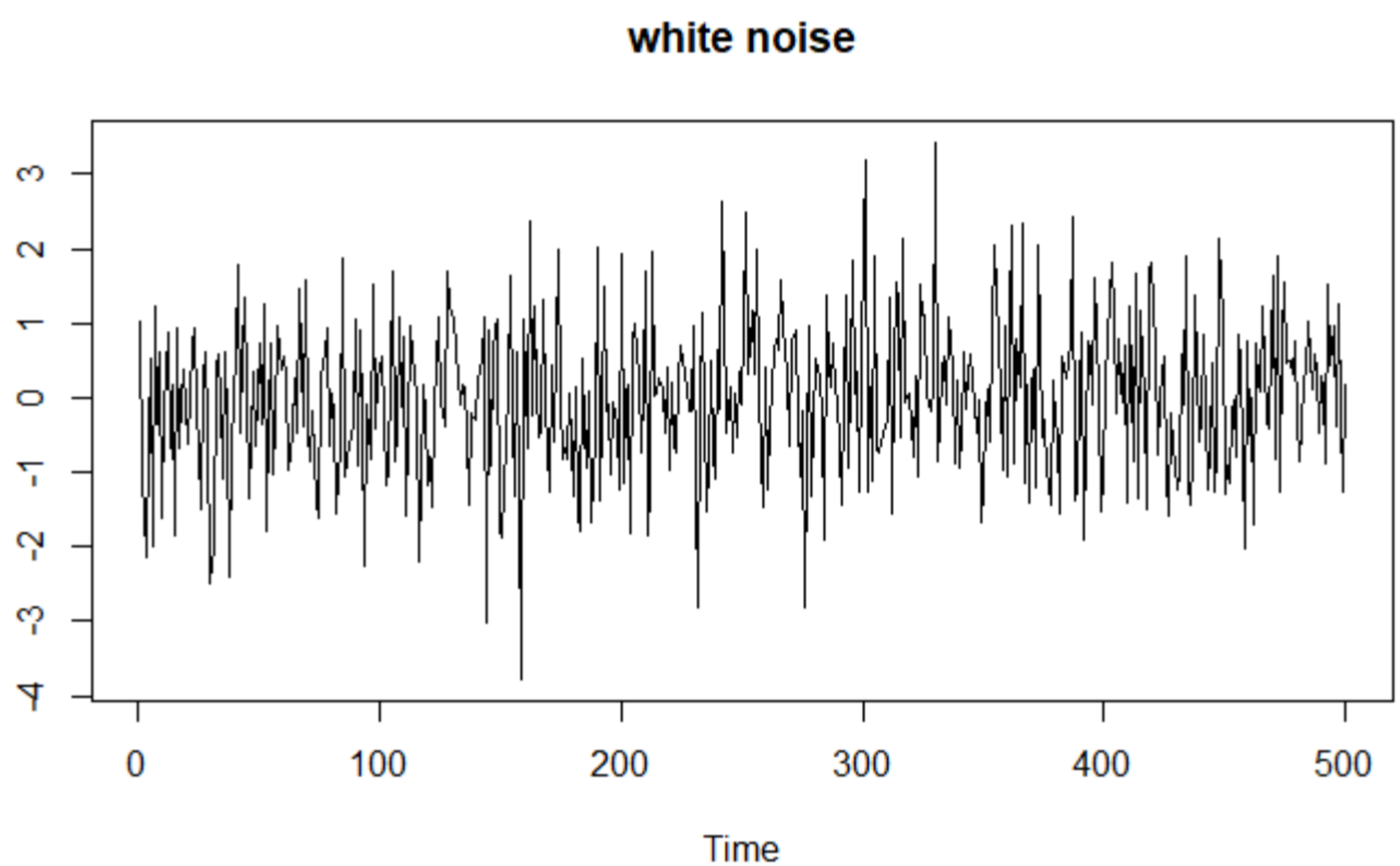
The fundamental visual characteristic of time series is their differing degrees of smoothness. One possible explanation for this smoothness is the correlation between adjacent points, so that the value at time t depends in some way on the past values $x_{t-1}, x_{t-2}, \dots$

Examples of particular time series

1) White Noise

$$w_t \sim wn(0, \sigma_w^2)$$

- many times we will require that $w_t \sim iid(0, \sigma_w^2)$
- the Gaussian white noise $w_t \sim iid\ N(0, \sigma_w^2)$

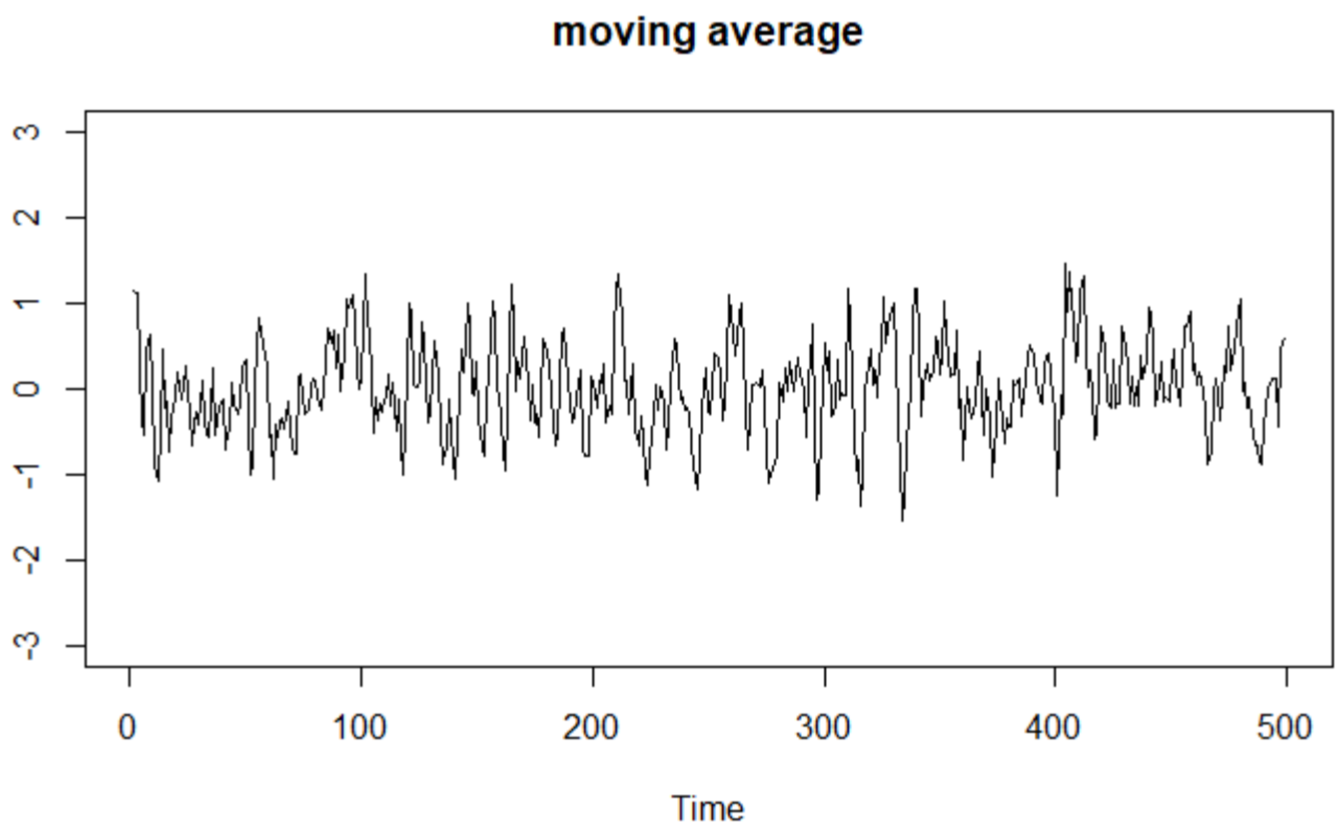


Examples of particular time series

2) Moving Averages and Filtering

w_t is replaced by a moving average that smooths the series:

$$v_t = \frac{1}{3}\left(w_{t-1} + w_t + w_{t+1}\right)$$

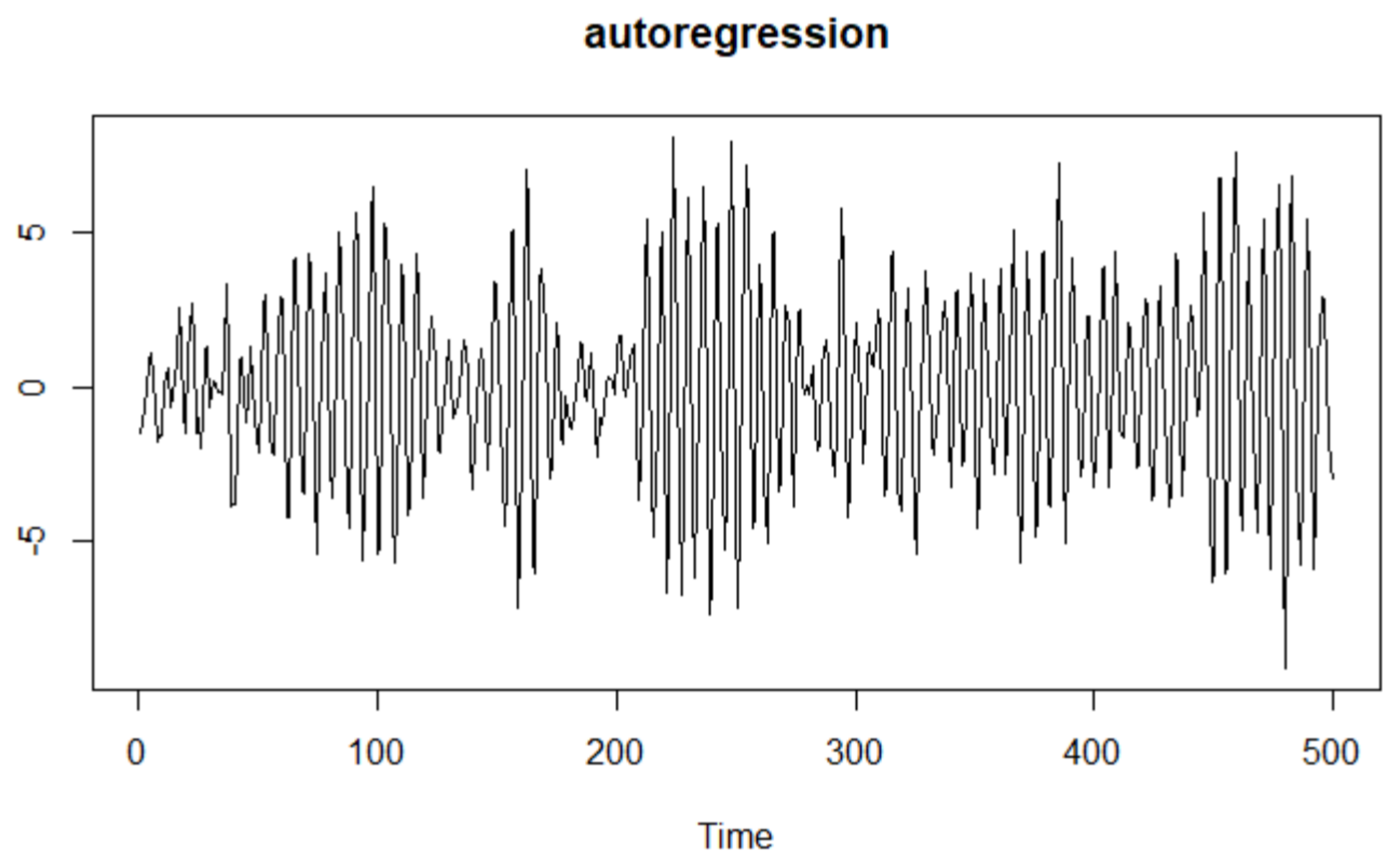


<u>Obs</u>. A linear combination of values in a time series is referred to as filtered series.

Examples of particular time series

3) Autoregressions

$$x_t = x_{t-1} - 0.9\, x_{t-2} + w_t, \text{ with the initial values } x_0 \text{ and } x_{-1}$$



Examples of particular time series

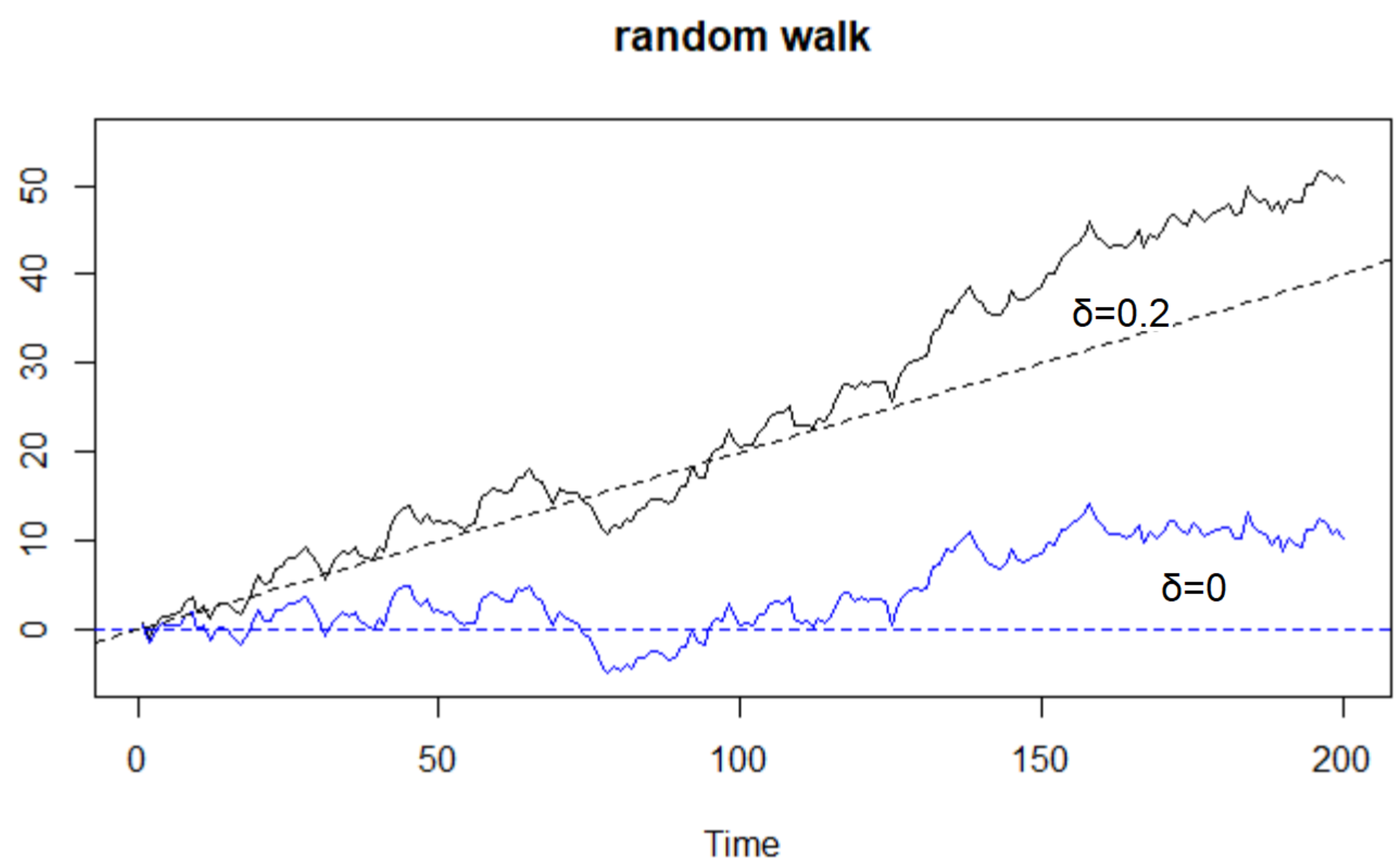
4) Random Walk with Drift

$$x_t = \delta + x_{t-1} + w_t, \text{ with the initial value } x_0 = 0$$

w_t is white noise

δ is called drift

It may be rewritten as $x_t = \delta t + \sum_{j=1}^{t} w_j$



Examples of particular time series

5) Signal in noise

Many realistic models assume an underlying signal with some consistent periodic variation, "contaminated" by an additional random noise:

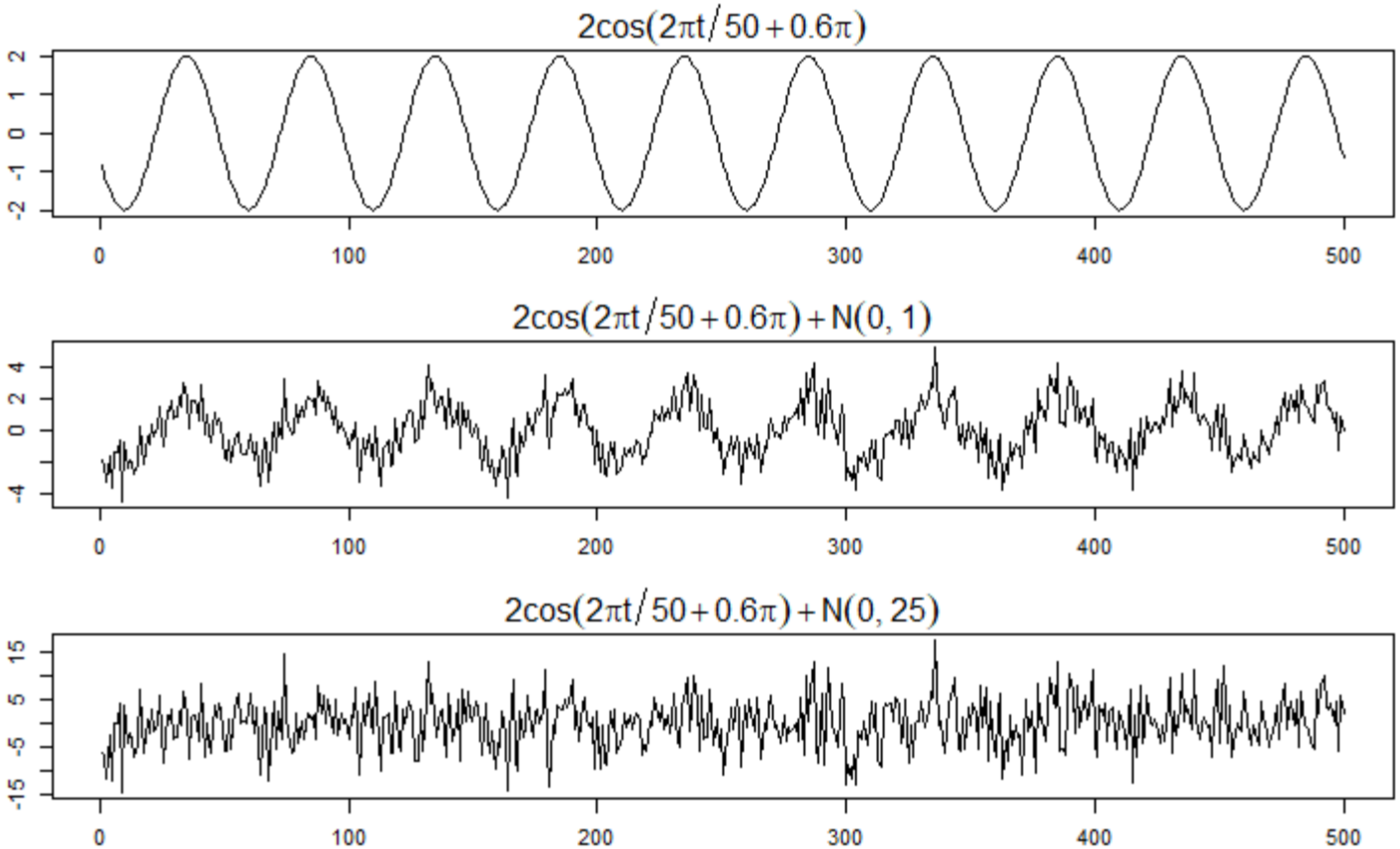
$$x_t = 2\cos\left(2\pi \frac{t+15}{50}\right) + w_t$$

In general, a sinusoidal waveform can be written as

$A\cos(2\pi\omega t + \phi)$, where A is the amplitude, ω is the frequency and ϕ is the phase shift.

Examples of particular time series

Signal in noise



Definitions

- The joint distribution function provides a complete description of a time series, observed as a collection of n random variables at arbitrary time points $t_1, \dots, t_n$.

$$F_{t_1,\dots,t_n}(x_1,\dots,x_n) = \Pr(X_{t_1} \le x_1, \dots, X_{t_n} \le x_n)$$

These multidimensional distribution functions cannot be easily written unless the random variables are jointly normal.

- The marginal distribution functions and the corresponding marginal density functions

$$F_t(x) = \Pr(X_t \le x)$$

$$f_t(x) = \frac{\partial F_t(x)}{\partial x}$$

Definitions

- The mean function

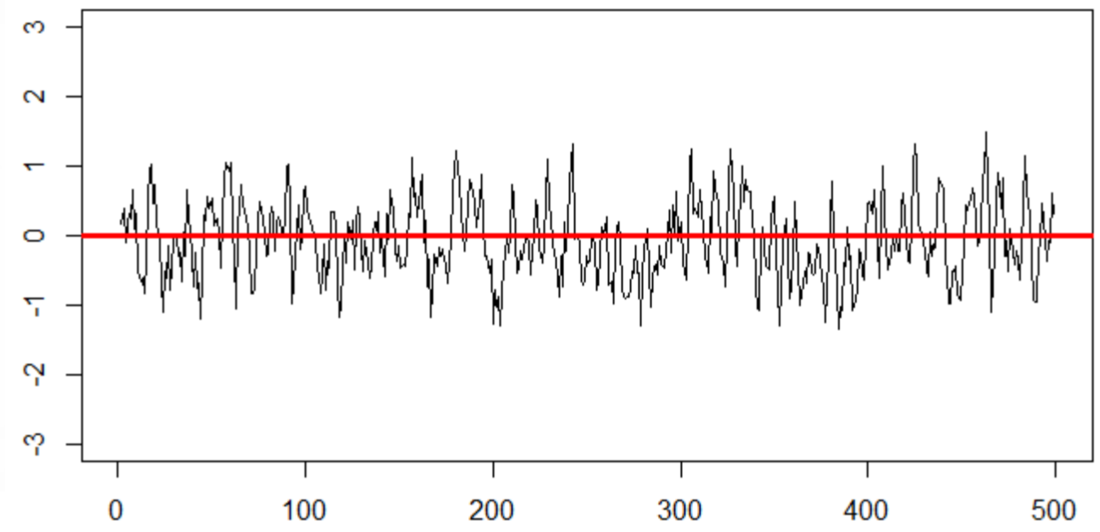
$$\mu_x(t) = E(X_t) = \int_{-\infty}^{\infty} x\, f_t(x)\, dx$$

- for a Moving Average series

$$v_t = \frac{1}{3}\left(w_{t-1} + w_t + w_{t+1}\right)$$

$$w_t \sim wn(0, \sigma_w^2)$$

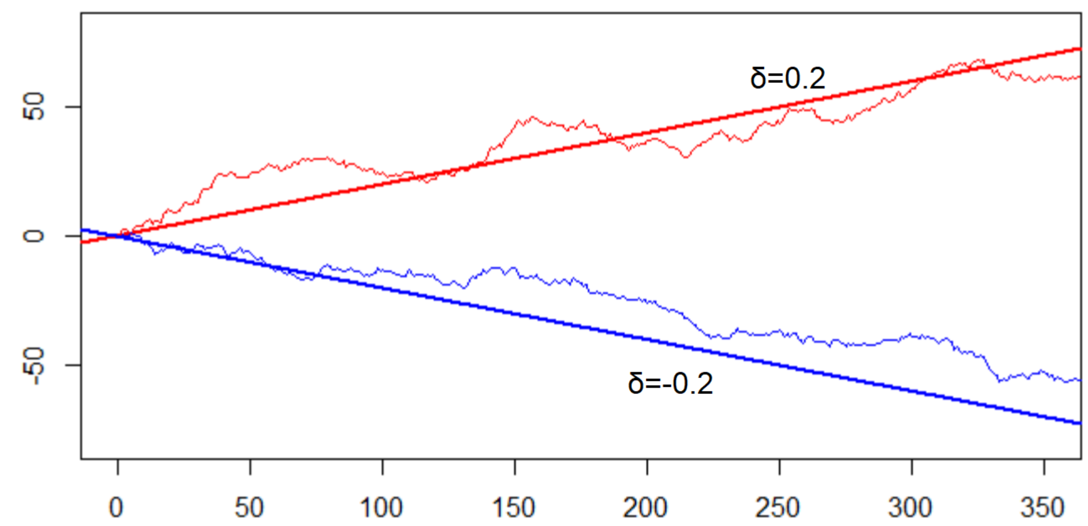
$$\mu_v(t) = \frac{1}{3}\left[E(w_{t-1}) + E(w_t) + E(w_{t+1})\right] = 0$$



- for a Random Walk with Drift

$$x_t = \delta t + \sum_{j=1}^{t} w_j$$

$$\mu_x(t) = \delta t + \sum_{j=1}^{t} E(w_j) = \delta t$$



Definitions

The lack of independence between two adjacent values can be assessed numerically using the notions of covariance and correlation.

- The autocovariance function

$$\gamma_X(s,t) = \mathrm{cov}(X_s, X_t) = E\left((X_s - \mu_X(s))(X_t - \mu_X(t)\right), \ \forall s,t$$

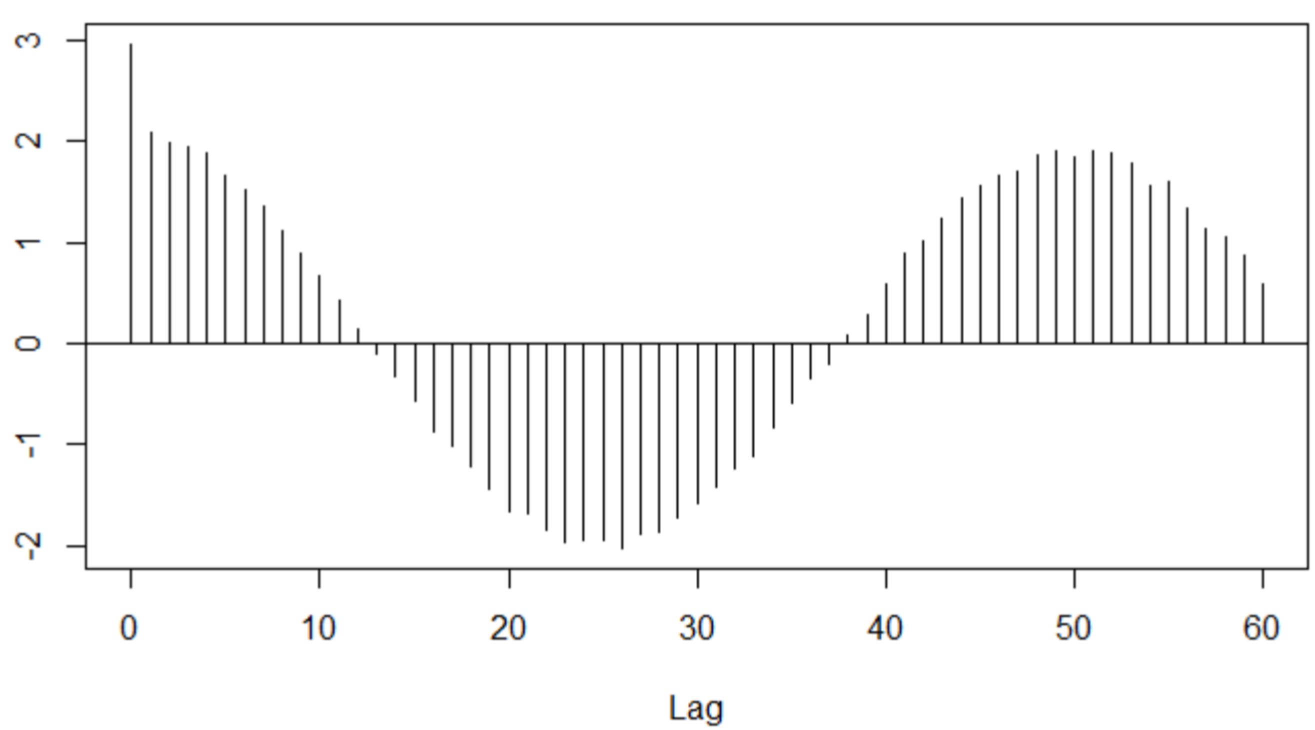
if $s=t$ then $\gamma_X(t,t) = \mathrm{var}(X_t)$

Definitions

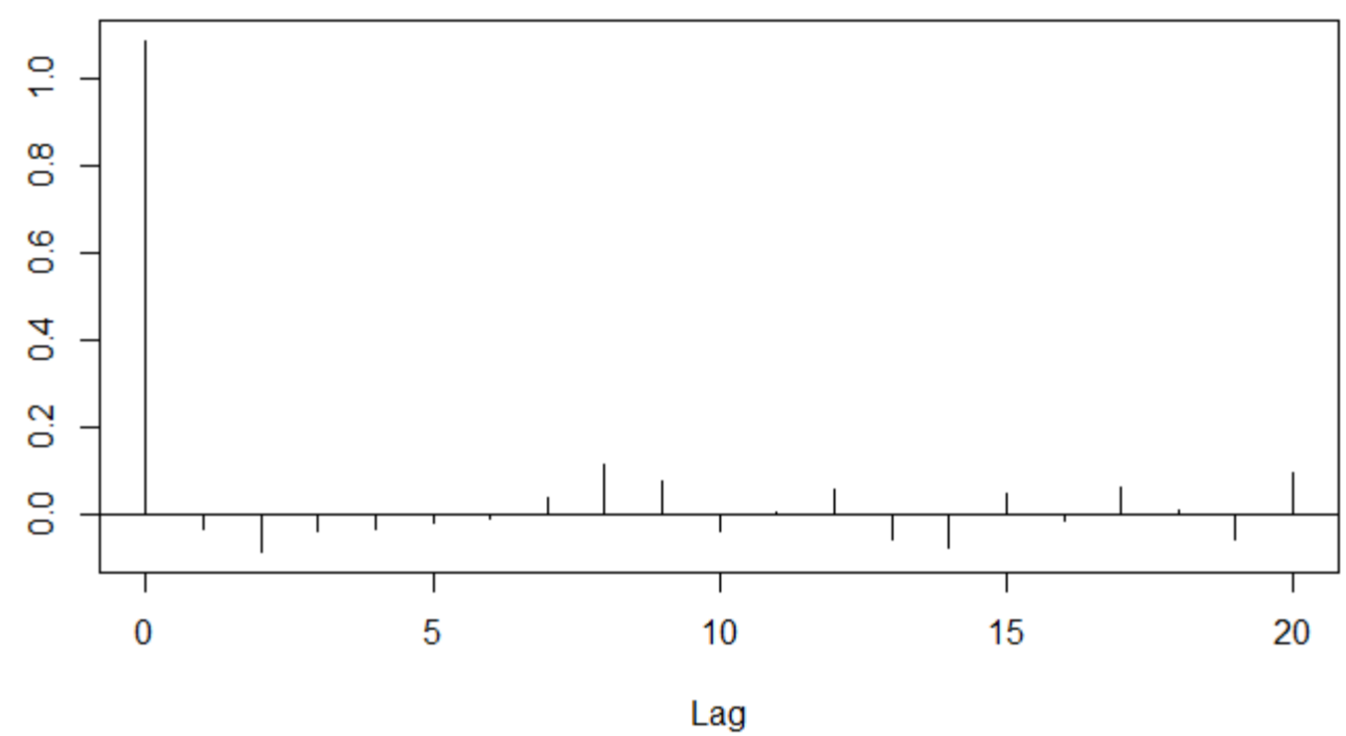
The autocovariance measures the linear dependence between two points at different times.

- very smooth series → autocovariance functions that stay large when t and s are far apart
- choppy series → autocovariance functions that are nearly zero for large separations between t and s.

Autocovariance for a sinusoidal waveform



Autocovariance for white noise



Definitions

Obs. if X_s and X_t are bivariate normal, then $\gamma_X(s,t)=0$ iff X_s and X_t are independent

- for White Noise

$$\gamma_w(s,t) = \text{cov}(w_s, w_t) = \begin{cases} \sigma_w^2, & s=t \\ 0, & s\neq t \end{cases}$$

Property Covariance of Linear Combinations

$$U = \sum_{j=1}^{n} a_j X_j \qquad V = \sum_{k=1}^{m} b_k Y_k$$

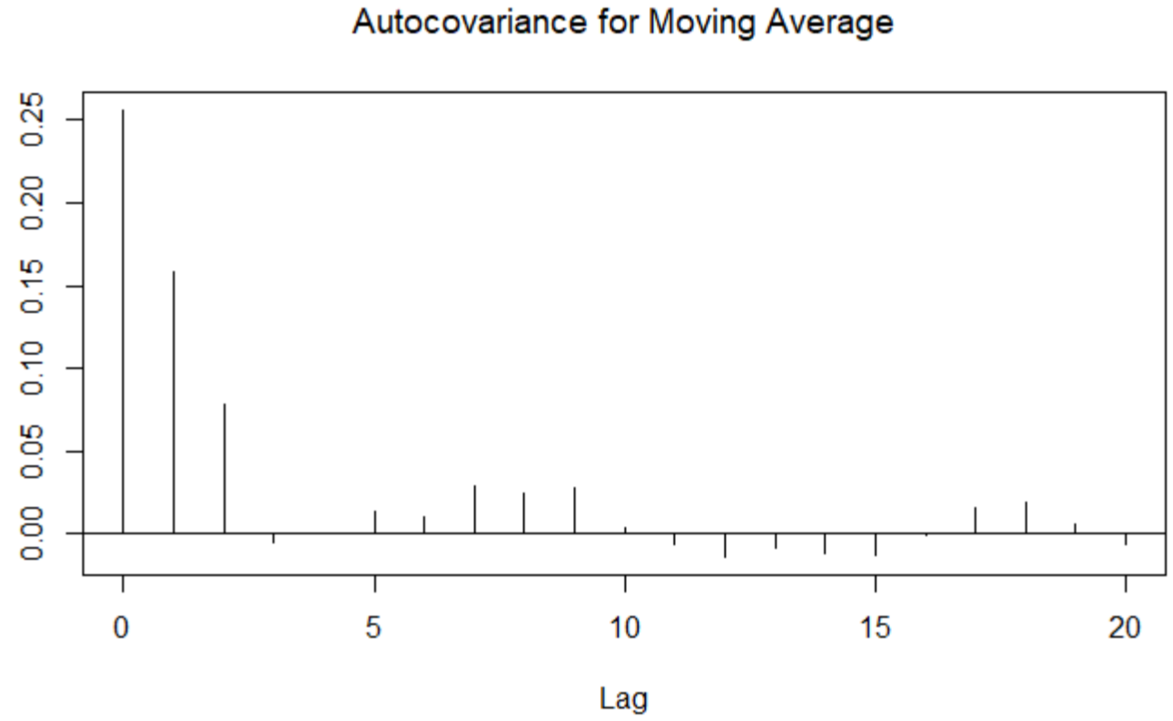
$$\text{cov}(U,V) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k \,\text{cov}(X_j, Y_k)$$

Definitions

- autocovariance for a Moving Average

$$\gamma_V(s,t) = \mathrm{cov}(V_s, V_t) = \mathrm{cov}\left(\frac{1}{3}(w_{s-1} + w_s + w_{s+1}), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right)$$

$$\gamma_V(s,t) = \begin{cases} \frac{3}{9}\sigma_w^2, & s=t \\ \frac{2}{9}\sigma_w^2, & |s-t|=1 \\ \frac{1}{9}\sigma_w^2, & |s-t|=2 \\ 0, & |s-t|>2 \end{cases}$$



- autocovariance for a Random Walk

$$X_t = \sum_{j=1}^{t} w_j$$

$$\gamma_X(s,t) = \mathrm{cov}\left(\sum_{j=1}^{t} w_j, \sum_{k=1}^{s} w_k\right) = \min\{s,t\} \cdot \sigma_w^2$$

$\mathrm{Var}(X_t) = t\,\sigma_w^2$ – increases as time increases

Definitions

- The autocorrelation function (ACF)

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\cdot\gamma(t,t)}}$$

Obs. $-1 \leq \rho(s,t) \leq 1$

The ACF measures the linear predictability of the series at time t (i.e. x_t) using only the value of x_s.

If we predict x_t from x_s through a linear relationship

$$X_t = \beta_0 + \beta_1 X_s, \text{ then } \rho(s,t) = \begin{cases} 1 & \text{when } \beta_1 > 0 \\ -1 & \text{when } \beta_1 < 0 \end{cases}$$

The ACF is a rough measure of the ability to forecast the series at time t from the value at time s.

Stationary time series

The notion of regularity of a time series is introduced by the concept called stationarity.

Def. A strictly stationary time series is one for which
$(X_{t_1}, \dots, X_{t_k})'$ and $(X_{t_1+h}, \dots, X_{t_k+h})'$ have the same distribution, for all $k = 1, 2, \dots$, all time points $t_1, \dots, t_k$ and all time shifts $h = 0, \pm 1, \pm 2 \dots$

That is,

$$F_{t_1,\dots,t_n}(x_1,\dots,x_n) = \Pr\left(X_{t_1} \le x_1, \dots, X_{t_n} \le x_n\right) =$$
$$= F_{t_1+h,\dots t_n+h}(x_1,\dots,x_n)$$

Stationary time series

When $k=1$, $F_t(x) = F_s(x) \quad \forall\, t, s$

it implies that, if the mean function exists then

$$\mu(s) = \mu(t) \quad \forall\, s, t$$

When $k=2$, $F_{t,s}(x_1, x_2) = F_{t+h, s+h}(x_1, x_2) \quad \forall\, t, s, \; \forall h$

if the variance function of the series exists, then

$$\gamma(s,t) = \gamma(s+h, t+h) \quad \forall\, t, s, \; \forall h$$

(it does not depend on the actual time)

Stationary time series

<u>Def</u>. A weakly stationary time series is a process such that

i) $\text{var}(X_t) < \infty \quad \forall t \in \mathbb{Z}$

ii) $\mu(t) = \mu \quad \forall t \in \mathbb{Z}$

iii) $\gamma(s,t) = \gamma(s+h, t+h) \quad \forall s,t \quad \forall h$

<u>Obs</u>. We will use the term stationary to mean weakly stationary.

Stationarity requires regularity in the mean and autocorrelation functions.

Stationary time series

<u>Def</u>. A weakly stationary time series is a process such that

i) $\mathrm{var}(X_t) < \infty \quad \forall t \in \mathbb{Z}$

ii) $\mu(t) = \mu \quad \forall t \in \mathbb{Z}$

iii) $\gamma(s,t) = \gamma(s+h, t+h) \quad \forall s,t \quad \forall h$

<u>Obs</u>. We will use the term stationary to mean weakly stationary.

Stationarity requires regularity in the mean and autocorrelation functions.

<u>Obs</u>. Strict stationarity implies stationarity. The converse is not true. An important case where stationarity implies strict stationarity is if the time series is Gaussian (i.e. all finite distributions of the series are Gaussian).

Stationary time series

Obs. For a stationary time series

$$\gamma(t+h, t) = \gamma(h, 0) \stackrel{\text{not}}{=} \gamma(h)$$ h is time shift or lag

The ACF will be written

$$\rho(h) = \frac{\gamma(t+h, t)}{\sqrt{\gamma(t+h, t+h) \cdot \gamma(t, t)}} = \frac{\gamma(h)}{\gamma(0)}$$

with the property $-1 \leq \rho(h) \leq 1$

Stationary time series

- Stationarity of White Noise

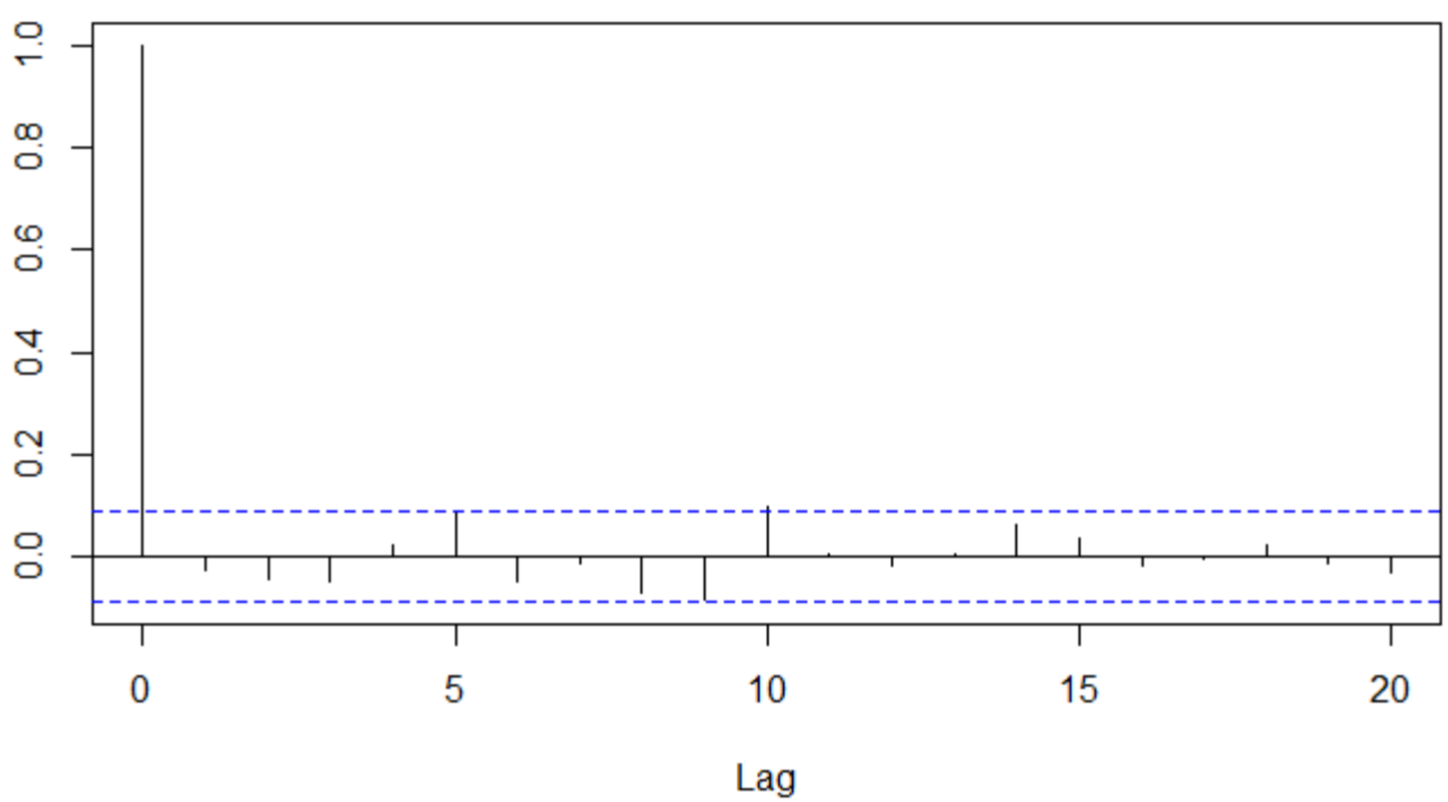
$$w_t \sim iid(0, \sigma_w^2)$$

$$\gamma_w(h) = cov(w_{t+h}, w_t) = \begin{cases} \sigma_w^2 & h=0 \\ 0 & h \neq 0 \end{cases}$$

The ACF is given by $\rho_w(0) = 1$

$\rho_w(h) = 0$ for $h \neq 0$

Autocorrelation for white noise



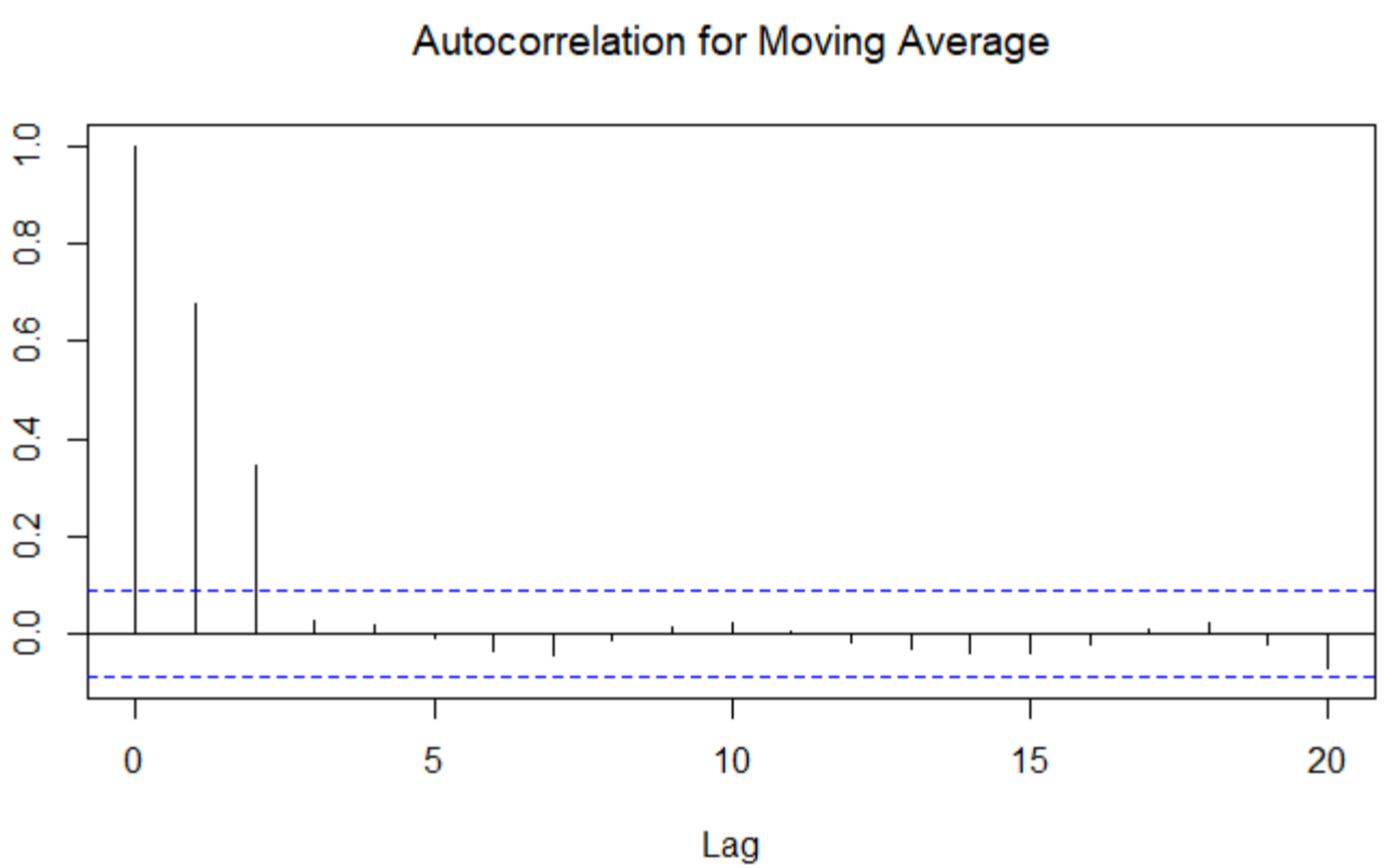
Stationary time series

- Stationarity of a Moving Average

$$v_t = \frac{1}{3}\left(w_{t-1} + w_t + w_{t+1}\right)$$

$$\gamma_v(h) = \begin{cases} \frac{3}{9}\sigma_w^2 & h=0 \\ \frac{2}{9}\sigma_w^2 & h=\pm 1 \\ \frac{1}{9}\sigma_w^2 & h=\pm 2 \\ 0 & |h|>2 \end{cases}$$

$$\rho_v(h) = \begin{cases} 1, & h=0 \\ \frac{2}{3}, & h=\pm 1 \\ \frac{1}{3}, & h=\pm 2 \\ 0, & |h|>2 \end{cases}$$



Stationary time series

- A Random Walk is not Stationary

$$X_t = \sum_{j=1}^{t} w_j$$

$$\gamma(s,t) = \min\{s,t\} \cdot \sigma_w^2 \quad \text{depends on time}$$

To do:

Check the stationarity for the following time series:

1) X_t – independent random variables

$$X_t = \begin{cases} Exp(1), & t \text{ odd} \\ N(1,1), & t \text{ even} \end{cases}$$

2) $X_t = A\cos(\theta t) + B\sin(\theta t)$, $\theta \in [-\pi, \pi]$

A, B – two independent random variables

$E(A) = E(B) = 0$

$Var(A) = Var(B) = 1$

3) $w_t \sim iid(0, \sigma_w^2)$

$$X_t = w_t + \theta \cdot w_{t-1}$$

4) $X_t = \begin{cases} Y_t, & t \text{ even} \\ Y_t + 1, & t \text{ odd} \end{cases}$, where $\{Y_t\}$ is a stationary time series.