ADRIAN KUKLA HW2 904056948

Qs 2.1 Activation Function

a) $g(-x) = \frac{-x}{1+|-x|} = \frac{-x}{1+|x|}$

$-g(x) = \frac{-x}{1+|x|} = g(-x)$ Hence $g(x)$ is centered at zero.

b) i) For $x>0$, $|x| = x$

$g(x) = \frac{x}{1+x}$ ~~$g'(x) = 1+x$~~

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$g'(x) = \frac{(1+x)\times 1 - x\times 1}{(1+x)^2} = \frac{1}{(1+x)^2}$

For $x<0$, $|x| = -x \Rightarrow g(x) = \frac{x}{1-x}$

$g'(x) = \frac{(1-x)\cdot 1 - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$

$$g'(x) = \begin{cases} \frac{1}{(1+x)^2}, & \text{for } x>0 \\ \frac{1}{(1-x)^2}, & \text{for } x<0 \end{cases}$$

ii) $\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} \frac{1}{(1+x)^2} = 1$

$\lim_{x\to 0^-} g'(x) = \lim_{x\to 0^-} \frac{1}{(1-x)^2} = 1$

$\Rightarrow g'(0) = 1$

iii) As $x \to +\infty$

$g'(x) = \frac{1}{(1+x)^2} \longrightarrow 0$

As $x \to -\infty$

$g'(x) = \frac{1}{(1-x)^2} \longrightarrow 0$

The gradient indicates that it is vanishing for large values of $|x|$

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Q. 3.1 Gradient Descent

Take derivative of objective function. So,

Let $D(w) = f(w^{(t)}) + \langle w - w^{(t)}, \nabla f(w^{(t)}) \rangle + \frac{\lambda}{2} \| w - w^{(t)} \|^2$

$\nabla_w D(w) = \nabla f(w^{(t)}) + \lambda (w - w^{(t)}) = 0$

as $f(w^{(t)})$ is constant, set $w = w^*$

$\Rightarrow \lambda(w^* - w^{(t)}) = -\nabla f(w^{(t)})$

$\Rightarrow w^* = w^{(t)} - \frac{1}{\lambda} \nabla f(w^{(t)})$

Matches the gradient descent formula for update

$w^{(t+1)} = w^{(t)} - \eta \nabla f(w^{(t)})$ where $\eta = \frac{1}{\lambda}$

The update rule corresponds to minimizing the first order Taylor approximation plus an L_2 proximity penalty where the gradient descent step is $\eta = \frac{1}{\lambda}$

λ (regularization term) is the reciprocal of the learning rate in gradient descent. They're inversely proportional

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Q 3.2 EC

Prove $\sum_{t=1}^{T} \langle w^{(t)} - w^*, v_t \rangle \leq \frac{\|w^*\|^2}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \|v_t\|^2$

Use telescoping

First,

$\|w^{(t+1)} - w^*\|^2 = \|w^{(t)} - w^* - \eta v_t\|^2$ by definition

$= \|w^{(t)} - w^*\|^2 - 2\eta \langle w^{(t)} - w^*, v_t \rangle + \eta^2 \|v_t\|^2$, by $\|a-b\|^2 = \|a\|^2 - 2\langle a, b\rangle + \|b\|^2$

$\Rightarrow \langle w^{(t)} - w^*, v_t \rangle = \frac{1}{2\eta} [\|w^{(t)} - w^*\|^2 - \|w^{(t+1)} - w^*\|^2] + \frac{\eta}{2} \|v_t\|^2$

$\Rightarrow \sum_{t=1}^{T} \langle w^{(t)} - w^*, v_t \rangle = \frac{1}{2\eta} \sum_{t=1}^{T} (\text{—//—}) + \frac{\eta}{2} \sum_{t=1}^{T} \|v_t\|^2$

$\Rightarrow \sum_{t=1}^{T} \langle w^{(t)} - w^*, v_t \rangle = \frac{1}{2\eta} (\|w^{(1)} - w^*\|^2 - \|w^{(T+1)} - w^*\|^2) + \frac{\eta}{2} \sum_{t=1}^{T} \|v_t\|^2$

By assumption $w^{(1)} = 0$ $\Downarrow = \|w^*\|^2$

Also $\|w^{(T+1)} - w^*\|^2 \geq 0$

Hence, $\|w^{(1)} - w^*\|^2 - \|w^{(T+1)} - w^*\|^2 \leq \|w^*\|^2$

$\Rightarrow$ Finally,

$\sum_{t=1}^{T} \langle w^{(t)} - w^*, v_t \rangle \leq \frac{1}{2\eta} \|w^*\|^2 + \frac{\eta}{2} \sum_{t=1}^{T} \|v_t\|^2$

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Q 3.3 EC

As f is convex we Jensen's inequality where

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y), \quad 0 \leq \theta \leq 1$$

Hence,

$$f(\bar{w}) \leq \frac{1}{T} \sum_{t=1}^{T} f(w^{(t)})$$

$$\Rightarrow f(\bar{w}) - f(w^*) \leq \frac{1}{T} \sum_{t=1}^{T} [f(w^{(t)}) - f(w^*)]$$

$$\Rightarrow f(\bar{w}) - f(w^*) \leq \frac{1}{T} \sum_{t=1}^{T} \langle w^{(t)} - w^*, \nabla f(w^{(t)}) \rangle$$ by first order convexity condition

Part 1 ↗

Part 2 ↘ Use 3.2 Lemma

$$\| \nabla f(w) \| \leq \rho$$

$$\| \nabla f(w^{(t)}) \|^2 \leq \rho^2$$

$$\sum_{t=1}^{T} \| \nabla f(w^{(t)}) \|^2 \leq T\rho^2$$

Moreover, $\| w^* \| \leq B \Rightarrow \| w^* \|^2 \leq B^2$

Hence,

$$\frac{1}{T} \sum_{t=1}^{T} \langle w^{(t)} - w^*, \nabla f(w^{(t)}) \rangle \leq \frac{B^2}{2\eta T} + \frac{\eta}{2} \frac{(\cancel{T}\rho^2)}{\cancel{T}}$$

Hence

$$f(\bar{w}) - f(w^*) \leq \frac{B^2}{2\eta T} + \frac{\eta \rho^2}{2}$$

Want to minimize bound by balancing both terms. Hence set

$$\frac{\eta \rho^2}{2} = \frac{B^2}{2\eta T} \Rightarrow \eta^2 = \frac{B^2}{\rho^2 T} \Rightarrow \eta = \frac{B}{\rho\sqrt{T}}$$

$$\Rightarrow \frac{B^2}{2(B/\rho\sqrt{T})T} = \frac{B \cdot \rho}{2\sqrt{T}}, \quad \left(\frac{B}{\rho\sqrt{T}}\right)\frac{\rho^2}{2} = \frac{B\rho}{2\sqrt{T}}$$

$$\Rightarrow \frac{B\rho}{2\sqrt{T}} + \frac{B\rho}{2\sqrt{T}} = \frac{B\rho}{\sqrt{T}}$$

$$\Rightarrow f(\bar{w}) - f(w^*) \leq B\rho/\sqrt{T} = O\left(\frac{1}{\sqrt{T}}\right)$$ Is the convergence rate

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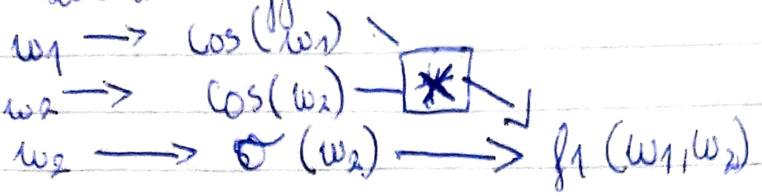
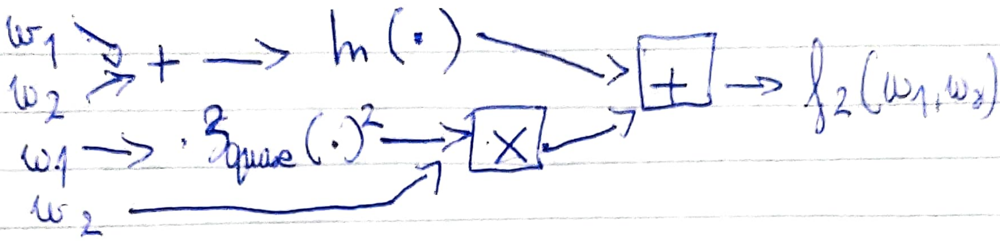
Q 4 Automatic Differentiation

a)





$f_1(1,2) \Rightarrow \cos(1)\cos(2) = 0.54 \times -0.416 = -0.225$

$\sigma(2) = \dfrac{1}{1+e^{-2}} \simeq 0.8807$

$f_1(1,2) = -0.225 + 0.8807 \simeq 0.656$

$f_2(1,2) = \ln 3 + 2 \simeq 3.0986$

$f(1,2) \simeq (0.656, 3.0986)$

b) Using Excel find

$$J(w) = \begin{pmatrix} \dfrac{\partial f_1}{\partial w_1} & \dfrac{\partial f_1}{\partial w_2} \\ \dfrac{\partial f_2}{\partial w_1} & \dfrac{\partial f_2}{\partial w_2} \end{pmatrix} \text{ where}$$

$$\frac{\partial f_i}{\partial w_j} \simeq \frac{f_i(w_1 + \Delta_j, w_2 + \Delta_j) - f_i(1,2)}{0.01}$$

$$\frac{\partial f_1}{\partial w_1} \simeq \frac{f_1(1.01, 2) - f_1(1,2)}{0.01} = 0.35$$

$$\frac{\partial f_1}{\partial w_2} \simeq \frac{f_1(1, 2.01) - f_1(1,2)}{0.01} \simeq -0.39$$

$$\frac{\partial f_2}{\partial w_1} \simeq \frac{f_2(1.01, 2) - f_2(1,2)}{0.01} \simeq 4.35$$

$$\frac{\partial f_2}{\partial w_2} \simeq \frac{f_2(1, 2.01) - f_2(1,2)}{0.01} \simeq 1.37$$

$$J(w) = \begin{bmatrix} 0.35 & -0.39 \\ 4.35 & 1.33 \end{bmatrix}$$

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Q4 c)

$$\frac{\partial f_1}{\partial w_1} = -\sin(w_1)\cos(w_2) = 0.35$$

$$\frac{\partial f_1}{\partial w_2} = -\cos(w_1)\sin(w_2) \cancel{+} \sigma(w_2)(1-\sigma(w_2)) = -0.39$$

$$\frac{\partial f_2}{\partial w_1} = \frac{1}{w_1+w_2} + 2w_1w_2 = 4.33$$

$$\frac{\partial f_2}{\partial w_2} = \frac{1}{w_1+w_2} + w_1^2 = 1.33$$

$$J(w) = \begin{bmatrix} 0.35 & -0.39 \\ 4.33 & 1.33 \end{bmatrix}$$

d) Backward mode yields the same result as the forward mode for a function $f: \mathbb{R}^2 \to \mathbb{R}^2$. It uses the chain rule to differentiate w.r.t. intermediate functions to derive same result

$$J(w) = \begin{bmatrix} 0.35 & -0.39 \\ 4.33 & 1.33 \end{bmatrix}$$

e) Yes I love it - automatic differentiation eliminates manual derivative calculations.

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Q5 Convolutions

a) Need to show $SC = CS$. Show by inspection

$$SC = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & \ddots & & & 0 \\ & \ddots & \ddots & & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & & a_2 \\ a_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & & a_0 \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_{n-1} \\ \vdots \\ a_{n-1} \end{pmatrix}$$

Diagonals multiply a set of 1s and a_{n-1}

$$CS = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & & a_2 \\ a_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & & a_0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_{n-1} \\ \vdots \\ a_{n-1} \end{pmatrix}$$

Moreover $C_a(Sx) = S(C_a)x \ \forall x \Rightarrow C_a S = SC_a$

Shifting input and then convolving is equivalent to convolving & then shifting

b) Forward direction $\Rightarrow$ if an operation is a circular convolution then it is represented by a circulant matrix C_a which commutes with S i.e. $SC = CS$

Reverse direction $\Rightarrow$ Let's say L is shift equivariant so

$$L(Sx) = S(Lx)$$

Because L is linear it implies it is a matrix M (linearity equations)

Since L is shift equivariant then $MS = SM$
L is $= M$

However, $M = a_0 I + \alpha_1 S + \alpha_2 S^2 + \dots + \alpha_{n-1} S^{n-1}$

Since a matrix that commutes with S is a set of polynomials of S and polynomials of S are circulant matrices.

Hence M is a circulant matrix s.th.

$$M(Sx) = S(Mx) \quad \forall x$$

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Q5

c) Means that CNNs are very good at analysing computer vision related tasks. So convolutions are shift equivariant and hence this means that convolutional layers can detect features independent of their position in spatial or time space. This avoids redundant parameter learning across positions.

Equivariance ensures robustness to translations which improves performance on tasks where the alignment of inputs varies like motion detection in videos or object recognition.

Also CNNs can reduce number of parameters compared to just a fully connected layer thanks to the shift equivariance property.

For 2D images 2D convolutions are used

For 3D videos 3D convolutions are used.

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Q6

$A_1(w) = \frac{1}{2}(w-2)^2 \Rightarrow \nabla A_1(w) = w-2$

$A_2(w) = \frac{1}{2}(w+1)^2 \Rightarrow \nabla A_2(w) = w+1$

$w_{new} = w_{old} - \eta \nabla$ [sampled term]

$A_1(w)$ w_{new} moves in direction $-(w-2)$

$A_2(w)$ w_{new} moves in direction $-(w+1)$

Select $w=0$

A_1 at $w=0$ $\nabla A_1(0) = -2$

$\Rightarrow w_{new} = 0 - \eta(-2) = 2\eta$

Evaluate $f(2\eta) < f(0)$

$f(w) = \frac{1}{2}[w^2 - 4w + 4 + w^2 + 2w + 1] = w^2 - w + 2.5$

$f(2\eta) = 4\eta^2 - 2\eta + 2.5$ $f(0) = 2.5$

if $\eta < 1/2$

$f(2\eta) - f(0) = 4\eta^2 - 2\eta < 0 \Rightarrow$

Hence $f(2\eta) < f(0)$ hence f decreases

A_2 at $w=0$ $\nabla A_2(0) = 1$

$\Rightarrow w_{new} = 0 - \eta(1) = -\eta$

$f(-\eta) = \eta^2 + \eta + 2.5$

$f(-\eta) - f(0) = \eta^2 + \eta + 2.5 - 2.5 = \eta^2 + \eta > 0$

so f increases

Hence for $w=0$ if A_2 is sampled, the next iteration will increase the overall function f.

Hence.

SGD is **NOT** guaranteed to decrease overall loss function.