Exercise Set 3

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1 Try to verify the following computations:

a)

$f(t) = (u(t) - u(t-a))t$

$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}$

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{(u(t) - u(t-a))t\} \\
&= \mathcal{L}\{u(t)t\} - \mathcal{L}\{u(t-a)t\} \\
&= \frac{e^{-0s}}{s^2} - \mathcal{L}\{u(t-a)(t-a+a)\} \\
&= \frac{1}{s^2} - (\mathcal{L}\{u(t-a)(t-a) + \mathcal{L}\{u(t-a)a\}) \\
&= \frac{1}{s^2} - \mathcal{L}\{u(t-a)(t-a)\} - \mathcal{L}\{u(t-a)a\} \\
&= \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}
\end{aligned}$$

b)

The Laplace tranform of $f(t) = u(t-\pi)sin(t) = -\frac{e^{-\pi s}}{s^2+1}$

$$\begin{aligned}
f(t) &= u(t-\pi)sin(t) \\
&= u(t-\pi)sin(t-\pi+\pi) \\
&= -u(t-\pi)sin(t-\pi)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{-u(t-\pi)sin(t-\pi)\} \\
&= -e^{-\pi s} * \mathcal{L}\{sin(t-\pi)\} \\
&= -e^{-\pi s} * \mathcal{L}\{-sin(t)\} \\
&= e^{-\pi s} * \mathcal{L}\{sin(t)\} \\
&= \frac{e^{-\pi s}}{s^2+1}
\end{aligned}$$

c)

$$\begin{aligned}
i' + 2i + \int_0^t i(\tau)d\tau &= \delta(t-1), i(0) = 0 \\
\mathcal{L}\{i'\} + \mathcal{L}\{2i\} + \mathcal{L}\{\int_0^t i(\tau)d\tau\} &= \mathcal{L}\{\delta(t-1)\} \\
sI - i(0) + 2I + \frac{I}{s} &= e^{-s} \\
I &= \frac{e^{-s}}{s+2+\frac{1}{s}} \\
I &= e^{-s}\frac{s}{s^2+2s+1}
\end{aligned}$$

$$\begin{aligned}
i = \mathcal{L}^{-1}\{I\} &= \mathcal{L}\{e^{-s}\frac{s}{s^2+2s+1}\} \\
&= u(t-1)\mathcal{L}\{\frac{s}{s^2+2s+1}\}
\end{aligned}$$

$$\begin{aligned}
\frac{s}{s^2+2s+1} &= \frac{A}{s+1} + \frac{B}{(s+1)^2} \\
s &= A(s+1) + B \\
\\
A &= 1 \\
B &= -1
\end{aligned}$$

$$\begin{aligned}
i &= u(t-1)\mathcal{L}\{\frac{s}{s^2+2s+1}\} \\
&= u(t-1)\mathcal{L}\{\frac{1}{s+1} - \frac{1}{(s+1)^2}\} \\
&= u(t-1)\mathcal{L}\{\frac{1}{s+1}\} - u(t-1)\mathcal{L}\{\frac{1}{(s+1)^2}\} \\
&= u(t-1)\big(e^{-(t-1)} - e^{-(t-1)}(t-1)\big)
\end{aligned}$$

2 Use Laplace transform to solve this convolution equation:

$$y - y \circledast t = t$$

$$\begin{aligned}
\mathcal{L}\{y - y \circledast t\} &= \mathcal{L}\{t\} \\
\mathcal{L}\{y\} - \mathcal{L}\{y\} * \mathcal{L}\{t\} &= \mathcal{L}\{t\} \\
Y - Y\frac{1}{s^2} &= \frac{1}{s^2} \\
Y &= \frac{1}{s^2 - 1} \\
y = \mathcal{L}\{Y\} &= \mathcal{L}\{\frac{1}{s^2 - 1}\} \\
y &= \underline{\underline{\cosh(t)}}
\end{aligned}$$

3 Solve the following system of equations:

$$\begin{cases} x' = 2x - y \\ y' = 3x - 2y \end{cases}$$

$$x(0) = 0, y(0) = 1$$

$$\begin{aligned} x' &= 2x - y \\ \mathcal{L}\{x'\} &= \mathcal{L}\{2x - y\} \\ sX - x(0) &= 2X - Y \\ X(s-2) &= -Y \\ X &= -\frac{Y}{s-2} \end{aligned} \tag{1}$$

$$\begin{aligned} y' &= 3x - 2y \\ \mathcal{L}\{y'\} &= \mathcal{L}\{3x - 2y\} \\ sY - y(0) &= 3X - 2Y \\ Y(s+2) &= 3X + 1 \\ Y &= \frac{3X+1}{s+2} \end{aligned} \tag{2}$$

We combine (1) and (2):

$$\begin{aligned} X &= -\frac{Y}{s-2} \\ &= -\frac{\frac{3X+1}{s+2}}{s-2} \\ &= -\frac{3X+1}{(s-2)(s+2)} \\ X + \frac{3X}{s^2-4} &= -\frac{1}{s^2-4} \\ X\frac{s^2-1}{s^2-4} &= -\frac{1}{s^2-4} \\ X &= -\frac{1}{s^2-1} \end{aligned} \tag{3}$$

We use (3) in (2):

$$
\begin{aligned}
Y &= \frac{3X+1}{s+2} \\
&= \frac{3(-\frac{1}{s^2-1})+1}{s+2} \\
&= \frac{\frac{-3+s^2-1}{s^2-1}}{s+2} \\
&= \frac{s^2-4}{(s+2)(s^2-1)} \\
&= \frac{s-2}{s^2-1} \\
&= \frac{s-1}{s^2-1} - \frac{1}{s^2-1} \\
&= \frac{1}{s+1} - \frac{1}{s^2-1}
\end{aligned} \tag{4}
$$

From (3) we find the solution for x:

$$
\begin{aligned}
x = \mathcal{L}^{-1}\{X\} &= \mathcal{L}^{-1}\{-\frac{1}{s^2-1}\} \\
&= \underline{\underline{-\sinh(t)}}
\end{aligned}
$$

From (4) we find the solution for y:

$$
\begin{aligned}
y = \mathcal{L}^{-1}\{Y\} &= \mathcal{L}^{-1}\{\frac{1}{s+1} - \frac{1}{s^2-1}\} \\
&= \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{1}{s^2-1}\} \\
&= \underline{\underline{e^{-t} - \sinh(t)}}
\end{aligned}
$$

4 Prove the following formulas for complex Fourier series expansion:

We are given the following formula in the exercise text:

$$c_n := \frac{1}{2\pi}\int_{\pi}^{\pi} f(x)e^{-inx}dx$$

and it is stated the formulae for the fourier series of a functin f is given by:

$$\sum_{n\in\mathcal{Z}} c_n e^{inx}$$

a)

$x = \sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx}$ when $-\pi < x < \pi$

We use the given formulae and find c_n first.

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{\pi}^{\pi} xe^{-inx}dx \\
&= \frac{1}{2\pi}\left(\frac{-xe^{-inx}}{in} - \frac{e^{-inx}}{in}\Bigg|_{\pi}^{\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{(1-\pi-\pi-1)e^{in\pi}}{in}\right) \\
&= \frac{1}{2\pi}\left(\frac{(-2\pi)e^{in\pi}}{in}\right) \\
&= \frac{(-1)e^{in\pi}}{in} \\
c_n &= \underline{\underline{\frac{i(-1)^n}{n}}}
\end{aligned}$$

We plot into the given formulae:

$$\sum_{n\in\mathcal{Z}} c_n e^{inx}$$

$$c_n = \frac{i(-1)^n}{n}$$

$$\sum_{n\in\mathcal{Z}} \frac{i(-1)^n}{n} e^{inx}$$

Since $f(x) = x$ is a odd function, we know that $a_n = 0$. Therefore we get:

$$\sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx}$$

b)

$x(2\pi - x) = -\frac{\pi^2}{3} + \sum_{n\neq 0} \left(\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2} \right) e^{inx}$ when $-\pi < x < \pi$

We use the given formulae and find c_n first.

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{\pi}^{\pi} x(2\pi - x)e^{-inx} dx \\
c_n &= \frac{1}{2\pi} \left(2\pi \int_{\pi}^{\pi} xe^{-inx} dx - \int_{\pi}^{\pi} x^2 e^{-inx} dx \right) \\
2\pi \int_{\pi}^{\pi} xe^{-inx} dx &= 2\pi \frac{2\pi i(-1)^n}{n} \\
&= \frac{(2\pi)^2 i(-1)^n}{n} \\
\int_{\pi}^{\pi} x^2 e^{-inx} dx &= \frac{x^2 e^{-inx}}{-in} \Bigg|_{-\pi}^{\pi} - \frac{2}{-in} \int_{\pi}^{\pi} xe^{-inx} dx \\
&= 0 + \frac{2}{in} \frac{2\pi i(-1)^n}{n} \\
&= \frac{4\pi(-1)^n}{n^2} \\
c_n &= \frac{1}{2\pi} \left(2\pi \int_{\pi}^{\pi} xe^{-inx} dx - \int_{\pi}^{\pi} x^2 e^{-inx} dx \right) \\
c_n &= \frac{1}{2\pi} \left(\frac{(2\pi)^2 i(-1)^n}{n} - \frac{4\pi(-1)^n}{n^2} \right) \\
c_n &= \frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2} \\
c_0 &= \frac{1}{2\pi} \int_{\pi}^{\pi} x(2\pi - x) dx \\
&= \frac{1}{2\pi} \int_{\pi}^{\pi} 2\pi x dx - \int_{\pi}^{\pi} x^2 dx \\
&= \frac{-2\pi^3}{6\pi} \\
&= \frac{-\pi^2}{3}
\end{aligned}$$

This all concludes in the fourier series expansion of:

$$x(2\pi - x) = \frac{-\pi^2}{3} + \sum_{n\neq 0} \left(\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2} \right) e^{inx}$$