Work Sheet 3

C.1

$$\sin^2(\theta) + \cos^2(\theta) = (\sin(\theta) + i\cos(\theta))(\sin(\theta) - i\cdot\cos(\theta))$$

$$= e^{i\theta} \cdot e^{-i\cdot\theta}$$

$$= \frac{e^{i\theta}}{e^{i\theta}} = \underline{\underline{1}}$$

C.2

1. $e^{z+\pi i} = e^{z} \cdot e^{\pi i}$ | $e^{\pi i} = -1$ per Eulers formula

$$= \underline{\underline{-e^{z}}}$$

2. $\overline{e^{z}} = \overline{e^{A+Bi}} = \overline{e^{A} \cdot e^{Bi}}$

$$= \overline{e^{A}(\cos(B) + i\sin(B))}$$

$$= e^{A}(\cos(B) - i\cdot\sin(B))$$

$$= e^{A} \cdot e^{-Bi}$$

$$= e^{A-Bi} = \underline{\underline{e^{\overline{z}}}}$$

C.3.

$$e^{z} = e^{A+Bi} = e^{A}(\cos(B) + i\sin(B))$$

We now look at when $e^{z} = 0$, if ever.

$e^{A} \neq 0 \therefore \cos(B) = -i\sin(B)$

$$\tan(B) = -\frac{1}{i} = \frac{-1}{\sqrt{-1}} = \sqrt{-1} = i$$

$$B = \arctan(i)$$

$$B = \underline{i\cdot\infty}$$

This result tells us that e^{z} will approach 0 when $\lim_{B\to\infty} e^{z}$, but will never be $e^{z} = 0$.

4. e^z is not one-to-one, e.g. $e^{\pi i} = -1 \wedge e^{3\pi i} = -1$

5. $e^{-z} = e^{-A-Bi}$

$= e^{-A} \cdot (e^{-B})^i = \frac{1}{e^A} \cdot \left(\frac{1}{e^B}\right)^i$

$= \frac{1}{e^A} \cdot \left(\frac{e^0}{e^B}\right)^i$

$= \frac{1}{e^A} \cdot \frac{1}{e^{Bi}}$

$= \frac{1}{e^z}$

6.3 $e^{\frac{k\pi}{5}i}, \ k \in \mathbb{Z}$

$\sqrt[5]{1} = \left\{e^{0i}, e^{\frac{2\pi}{5}i}, e^{\frac{4\pi}{5}i}, e^{\frac{-4i}{5}}, e^{-\frac{2\pi}{5}i}\right\}$

Sum $= e^{0i} + e^{\frac{2\pi i}{5}} + e^{\frac{4\pi}{5}i} + e^{\frac{-4\pi}{5}i} + e^{-\frac{2\pi}{5}i}$

$= 1 + \cos\frac{2\pi}{5} + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) + \cos\left(\frac{-4\pi}{5}\right) + i\sin\left(\frac{-4\pi}{5}\right)$

$+ \cos\left(\frac{-2\pi}{5}\right) + i\sin\left(\frac{-2\pi}{5}\right)$

$= 1 + 2\cos\frac{2\pi}{5} + 2\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)$

$= 1 + 2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right)$

$= 1 + \frac{\sqrt{5} - 1 - \sqrt{5} - 1}{2}$

$= 0$

Product $= e^{0i} \cdot e^{\frac{2\pi}{5}i} \cdot e^{\frac{4\pi}{5}i} \cdot e^{\frac{6\pi}{5}i} \cdot e^{\frac{8\pi}{5}i}$

$= e^{0 + \frac{2\pi}{5} + \frac{4\pi}{5} + \frac{6\pi}{5} + \frac{8\pi}{5}}$

$= e^{4\pi i} = 1$