Exercise Set 1

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1 Use $e^{i\pi/2} = i$ to compute $cos(n\pi/2)$ and $sin(\pi/2)$, $n = 0, 1, ...$

$$e^{i\pi/2} = i \tag{1}$$

$$(e^{i\pi/2})^n = i^n \tag{2}$$

$$e^{i*n/2*\pi} = \cos(n/2*\pi) + i*\sin(n/2*\pi) \tag{3}$$

$$i^n = \cos(n/2*\pi) + i*\sin(n/2*\pi) \tag{4}$$

We assume $n = 2*k$ for any whole number k. We do this to focus on the cosine part of the question first.

$$i^{2k} = \cos(2k/2*\pi) + i*\sin(2k/2*\pi) \tag{5}$$

$$(-1)^k = \cos(k*\pi) + i*\sin(k*\pi) \tag{6}$$

We use the fact $\sin k*\pi = 0$ for all $k \in Z$ to isolate $\cos(k*\pi)$.

$$(-1)^k = \cos(k*\pi) \tag{7}$$

We substitute back from k to n by turning the equation so that $k = n/2$. From this we get the equation:

$$\underline{\underline{\cos(n/2*\pi) = (-1)^{n/2}}} \tag{8}$$

Now we move on to the sine portion. We return to (4) and instead insert $n = 2k + 1$.

$$i^{2k+1} = \cos((2k+1)/2*\pi) + i*\sin((2k+1)/2*\pi) \tag{9}$$

$$i*(-1)^k = \cos((k+1/2)*\pi) + i*\sin((2k+1)/2*\pi) \tag{10}$$

$\cos((k + 1/2) * \pi) = 0$ for all $k \in Z$

$$i * (-1)^k = i * \sin((2k + 1)/2 * \pi) \tag{11}$$

We substitute back from k to n by turning the equation so that $k = (n - 1)/2$. From this we get the equation:

$$(-1)^{(n-1)/2} = \sin(((n - 1) + 1)/2 * \pi) \tag{12}$$

$$\underline{\underline{\sin(n/2 * \pi) = (-1)^{(n-1)/2}}} \tag{13}$$

2 Compute the Integral

$$\int_{-\pi}^{\pi} x * e^{inx} dx, n = 0, 1, 2...$$

We use integration by parts and get:

$$\int_{-\pi}^{\pi} x * e^{inx} dx = \frac{xe^{inx}}{in} - \int_{-\pi}^{\pi} \frac{e^{inx}}{in} dx$$

$$\int_{-\pi}^{\pi} x * e^{inx} dx = \frac{xe^{inx}}{in} - \frac{e^{inx}}{(in)^2}\Big|_{-\pi}^{\pi}$$

$$\int_{-\pi}^{\pi} x * e^{inx} dx = \frac{\cos(\pi * n)}{in} * (\pi - \frac{1}{in} - \pi - \frac{1}{in})$$

$$\int_{-\pi}^{\pi} x * e^{inx} dx = \frac{2\pi * \cos(\pi * n)}{in}$$

$$\int_{-\pi}^{\pi} x * e^{inx} dx = \frac{2\pi * (-1)^n}{in}$$

As this does not apply to $n = 0$, we calculate this seperately:

$$\int_{-\pi}^{\pi} x * e^{inx} dx, n = 0$$

$$\int_{-\pi}^{\pi} x dx = \frac{x^2}{2}\Big|_{-\pi}^{\pi} = \frac{\pi^2}{2} - \frac{\pi^2}{2} = 0$$

3 Compute the Laplace transforms of the following functions:

a) $f(t) = \sinh(A * t)$

$$\mathcal{L}\left\{\frac{e^{At} - e^{-At}}{2}\right\} = \frac{1}{2}\left(\frac{1}{S-A} - \frac{1}{S+A}\right) = \underline{\underline{\frac{A}{S^2 - A^2}}}$$

b) $f(t) = \cosh(A * t)$

$$\mathcal{L}\left\{\frac{e^{At} + e^{-At}}{2}\right\} = \frac{1}{2}\left(\frac{1}{S-A} + \frac{1}{S+A}\right) = \underline{\underline{\frac{S}{S^2 - A^2}}}$$

c) Piecewise function f

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < \pi \\ 1 & \text{if } t \geq \pi \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\pi e^{-st} * 0dt + \int_\pi^\infty e^{-st} * 1dt = \frac{e^{-st}}{-s}\bigg|_\pi^\infty = \frac{e^{-s\pi}}{s}$$

d) Piecewise function g

$$g(t) = \begin{cases} 0 & \text{if } 0 < t < \pi \\ \cos(t) & \text{if } t \geq \pi \end{cases}$$

$$\mathcal{L}\{g(t)\} = \int_0^\pi e^{-st} * 0dt + \int_\pi^\infty e^{-st} * \cos(t)dt$$

$$\int_\pi^\infty e^{-st} * \cos(t)dt = e^{-st} * \sin(t) + s * \int_\pi^\infty e^{-st} * \sin(t)dt$$

$$\int_\pi^\infty e^{-st} * \cos(t)dt = e^{-st} * \sin(t) + s * (-e^{-st} * \cos(t) - s^2 * \int_\pi^\infty e^{-st} * \cos(t)dt$$

$$(s^2 + 1)\int_\pi^\infty e^{-st} * \cos(t)dt = e^{-st} * \sin(t) + s * (-e^{-st} * \cos(t)$$

$$\int_\pi^\infty e^{-st} * \cos(t)dt = \frac{e^{-st} * \sin(t) + s * (-e^{-st} * \cos(t))}{s^2 + 1}\bigg|_\pi^\infty \tag{14}$$

$$\underline{\underline{\mathcal{L}\{g(t)\} = \frac{s * e^{-s*\pi}}{s^2 + 1}}}$$

e) $h(t) = t^2 e^t$

$$\mathcal{L}\{h\} = \int_0^\infty e^{-st} * t^2 * e^t dt = \int_0^\infty e^{-(s-1)t} * t^2$$

$$F(s-1) = \int_0^\infty e^{-(s-1)t} * t^2$$

$$F(s) = \int_0^\infty e^{-st} * t^2$$

$$F(s) = \mathcal{L}\{t^2\} = \frac{1}{s} * \mathcal{L}\{2t\} + 0^2 = \frac{1}{s^2} * \mathcal{L}\{2\} + 0^2 = \frac{2}{s^3}$$

$$\underline{\underline{F(s-1) = \frac{2}{(s-1)^3} = \mathcal{L}\{t^2 e^t\}}}$$

g) $j(t) = e^t \cos(t)$

$$\mathcal{L}\{j\} = \int_0^\infty e^{-st} * \cos(t) * e^t dt = \int_0^\infty e^{-(s-1)t} * \cos(t)$$

$$F(s-1) = \int_0^\infty e^{-(s-1)t} * \cos(t)$$

$$F(s) = \int_0^\infty e^{-st} * \cos(t)$$

We reuse what we found in (14)

$$\int_\pi^\infty e^{-st} * \cos(t) dt = \frac{e^{-st} * \sin(t) + s * (-e^{-st} * \cos(t))}{s^2+1}\Bigg|_0^\infty$$

$$F(s) = \frac{s}{s^2+1}$$

$$\mathcal{L}\{j\} = F(s-1) = \frac{s-1}{(s-1)^2+1}$$

f) $i(t) = e^t \sin(t)$

$$\mathcal{L}\{i\} = \int_0^\infty e^{-st} * \sin(t) * e^t dt = \int_0^\infty e^{-(s-1)t} * \sin(t)$$

$$F(s-1) = \int_0^\infty e^{-(s-1)t} * \sin(t)$$

$$F(s) = \int_0^\infty e^{-st} * \sin(t)$$

$$\int_0^\infty e^{-st} * \sin(t) = -\cos(t)e^{-st} - s\int_0^\infty e^{-st}\cos(t)$$

$$\int_0^\infty e^{-st} * \sin(t) = -\cos(t)e^{-st} - s(e^{-st} * sin(t) + \int_0^\infty se^{-st}\sin(t))$$

$$\int_0^\infty e^{-st} * \sin(t) = -\cos(t)e^{-st} - se^{-st} * sin(t) - s^2\int_0^\infty e^{-st}\sin(t)$$

$$(s^2+1)\int_0^\infty e^{-st}\sin(t) = -\cos(t)e^{-st} - se^{-st}\sin(t)$$

$$\int_0^\infty e^{-st}\sin(t) = \left.\frac{-\cos(t)e^{-st} - se^{-st}\sin(t)}{s^2+1}\right|_0^\infty$$

$$F(s) = \int_0^\infty e^{-st}\sin(t) = \left.\frac{1}{s^2+1}\right|_0^\infty$$

$$\mathcal{L}\{i\} = F(s-1) = \frac{1}{(s-1)^2+1}$$

4 Solve the following initial value problems using Laplace transforms:

4.1 a)

$$\begin{cases} y'' - 2y' + 2y = 6e^{-t} \\ y(0) = 0, y'(0) = 1; \end{cases}$$

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{6e^{-t}\} \tag{15}$$

$$s^2 * Y - sy(0) - y'(0) - 2(sY - y(0)) + 2Y = \frac{6}{s+1} \tag{16}$$

$$s^2 * Y - 1 - 2sY + 2Y = \frac{6}{s+1} \tag{17}$$

$$Y = \frac{7+s}{(s+1)(s^2-2s+2)} \tag{18}$$

$$\frac{As+B}{s^2-2s+2} + \frac{C}{s+1} = \frac{s+7}{(s^2-2s+2)(s+1)} \tag{19}$$

$$\begin{aligned} A + C &= 0 \\ B + 2C &= 7 \\ A + B - 2C &= 1 \end{aligned}$$

$$\begin{aligned} A &= -6/5 \\ B &= 23/5 \\ C &= 6/5 \end{aligned}$$

$$Y = \frac{23-6s}{5(s^2-2s+2)} + \frac{6}{5(s+1)}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{23-6s}{5(s^2-2s+2)} + \frac{6}{5(s+1)}\}$$

$$y = \mathcal{L}^{-1}\{\frac{23-6s}{5(s^2-2s+2)}\} + \frac{6}{5}\mathcal{L}^{-1}\{\frac{1}{(s+1)}\}$$

$$y = \mathcal{L}^{-1}\{\frac{\frac{23-6s}{5}}{(s-1)^2+1}\} + \frac{6}{5}e^{-t}$$

$$y = \frac{1}{5}\left(\mathcal{L}^{-1}\{\frac{23-6s}{(s-1)^2+1}\} + 6e^{-t}\right)$$

$$y = \frac{1}{5}\left(23 * \mathcal{L}^{-1}\{\frac{1}{(s-1)^2+1}\} - 6 * \mathcal{L}^{-1}\{\frac{s}{(s-1)^2+1}\} + 6e^{-t}\right)$$

We know from exercise 3f that:

$$\mathcal{L}^{-1}\{\frac{1}{(s-1)^2+1}\} = e^t \sin(t)$$

$$y = \frac{1}{5}\left(23 * e^t \sin(t) - 6 * \mathcal{L}^{-1}\{\frac{s}{(s-1)^2+1}\} + 6e^{-t}\right)$$

We use what we found in Exercise 3:

$$\mathcal{L}^{-1}\{\frac{s}{(s-1)^2+1}\} = \mathcal{L}^{-1}\{\frac{1}{(s-1)^2+1}\} + \mathcal{L}^{-1}\{\frac{s-1}{(s-1)^2+1}\} = e^t \sin t + e^t \cos t$$

$$y = \frac{23}{5}e^t \sin(t) - \frac{6}{5} * e^t \sin t - \frac{6}{5}e^t \cos t + \frac{6}{5}e^{-t}$$

$$y = \frac{17}{5}e^t \sin(t) - \frac{6}{5}e^t \cos(t) + \frac{6}{5}e^{-t}$$

4.2 b)

$$\begin{cases} y'' + y = f(t) \\ y(0) = y'(0) = 0; \end{cases}$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < \pi \\ 1 & \text{if } t \geq \pi \end{cases}$$

$$s^2 Y - sy(0) - y'(0) + Y = \mathcal{L}\{f(t)\}$$

We have solved this $\mathcal{L}\{f(t)\}$ before, so we use the previously aquired solution for this.

$$(s^2+1)Y = \frac{e^{-s\pi}}{s}$$

$$Y = \frac{e^{-s\pi}}{(s^2+1)s}$$

$$\frac{As+B}{s^2+1} + \frac{C}{s} = \frac{1}{(s^2+1)s}$$

$$\begin{aligned} A + C &= 0 \\ C &= 1 \\ B &= 0 \end{aligned}$$

$$\begin{aligned} A &= -1 \\ B &= 0 \\ C &= 1 \end{aligned}$$

$$Y = \frac{e^{-s\pi}}{(s^2+1)s} = e^{-s\pi}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$y = u(t-\pi) * (1 - \cos(t))$$