Exercise Set 3

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1 The aim of this exercise is to show you the fundamental fact behind the real Fourier series: $(1, cosnx, sinnx)_{n\geq 1}$ is an orthogonal system with respect to the inner product.

1.1 Show that:

$(1, \sin(nx)) = 0$

$$
\begin{aligned}
(1, \sin(nx)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 * \sin(nx)dx \\
&= \frac{1}{2\pi}\left[\frac{-1}{n}\cos(nx)\right]_{-\pi}^{\pi} \\
&= \frac{-1}{2n\pi}[\cos(\pi n) - \cos(-\pi n)] \\
&= \frac{-1}{2n\pi}[(-1)^n - (-1)^n] \\
&= \frac{-1}{2n\pi}[0] \\
(1, \sin(nx)) &= \underline{\underline{0}}
\end{aligned}
$$

Proving they are orthogonal.

$(\sin(nx), \cos(mx)) = 0$

We will use the following trigonometric identity:

$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$

$$
\begin{aligned}
(\sin(nx), \cos(mx)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\cos(mx))dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \sin((n-m)x) + \sin((n+m)x)dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \sin((n-m)x)dx + \int_{-\pi}^{\pi} \sin((n+m)x)dx \\
&= \frac{-1}{4\pi}\left\{\left[\frac{1}{n-m}\cos((n-m)x)\right]_{-\pi}^{\pi} + \left[\frac{1}{n+m}\cos((n+m)x)\right]_{-\pi}^{\pi}\right\} \\
&= \frac{-1}{4\pi}\left\{\frac{1}{n-m}[(-1)^{n-m} - (-1)^{n-m}] + \frac{1}{n+m}[(-1)^{n+m} - (-1)^{n+m}]\right\} \\
&= \frac{-1}{4\pi}\{0+0\} \\
&= \underline{\underline{0}}
\end{aligned}
$$

Proving they are orthogonal.

1.2 Show that:

$(\sin(mx), \sin(nx)) = (\cos(nx), \cos(mx)) =$

$$
\begin{cases} 0 & n \neq m = 0, 1, 2, 3... \\ \frac{1}{2} & n = m = 1, 2, 3... \end{cases}
$$

We use the trigonometric identities:

$$
\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \tag{1}
$$

$$
\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)] \tag{2}
$$

$$
\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)] \tag{3}
$$

$$
\begin{aligned}
(\sin(mx), \sin(nx)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx))dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos(mx-nx) - \cos(mx+nx)dx \\
&= \frac{1}{4\pi}\left\{\frac{1}{m-n}\sin(mx-nx)\Big|_{-\pi}^{\pi} - \frac{1}{m+n}\sin(mx+nx)\Big|_{-\pi}^{\pi}\right\}
\end{aligned}
$$

First we deal with if $m \neq n$ for 0,1,2,3...:

$$\begin{aligned}
(\sin(mx), \sin(nx)) &= \frac{1}{4\pi}\left\{\frac{1}{m-n}\sin(mx-nx)\Big|_{-\pi}^{\pi} - \frac{1}{m+n}\sin(mx+nx)\Big|_{-\pi}^{\pi}\right\} \\
&= \frac{1}{4\pi}\left\{\frac{1}{m-n}(0-0) - \frac{1}{m+n}(0-0)\right\} \qquad (4) \\
&= 0
\end{aligned}$$

Now we deal with if $m = n$ for 1,2,3...:

$$\begin{aligned}
(\sin(mx), \sin(nx)) &= \frac{1}{4\pi}\left\{\frac{1}{m-n}\sin(mx-nx)\Big|_{-\pi}^{\pi} - \frac{1}{m+n}\sin(mx+nx)\Big|_{-\pi}^{\pi}\right\} \\
&= \frac{1}{4\pi}\left\{\frac{1}{m-n}(\sin((m-n)x)\Big|_{-\pi}^{\pi} - 0\right\} \qquad (5) \\
&= \frac{1}{4\pi}\left\{\frac{1}{m-n}(\sin((m-n)\pi) - \frac{1}{m-n}(\sin((m-n)(-\pi))\right\} \\
&= \frac{1}{4\pi}\left\{\frac{1}{m-n}(\sin((m-n)\pi) + \frac{1}{m-n}(\sin((m-n)(\pi))\right\} \\
&= \frac{1}{2\pi}\left\{\frac{1}{m-n}(\sin((m-n)\pi)\right\}
\end{aligned}$$

We set $z = m - n$:

$$(\sin(mx), \sin(nx)) = \frac{1}{2\pi}\left\{\frac{\sin(z\pi)}{z}\right\}$$

Since we have $m = n$, we have $m - n = 0$. We observe our equation when we use $\lim_{z\to 0} f(z)$

$$\begin{aligned}
(\sin(mx), \sin(nx)) &= \frac{1}{2\pi}\left\{\frac{\sin(z\pi)}{z}\right\} \\
&= \lim_{z\to 0}\frac{1}{2\pi}\left\{\frac{\sin(z\pi)}{z}\right\} \\
&= \frac{1}{2\pi} * \pi \\
&= \frac{1}{2}
\end{aligned}$$

Now we prove $(\sin(mx), \sin(nx)) = (\cos(nx), \cos(mx))$

$$
\begin{aligned}
(\sin(mx), \sin(nx)) &= (\cos(nx), \cos(mx)) \\
\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx))dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx))dx \\
\int_{-\pi}^{\pi} \sin(nx)\sin(mx))dx &= \int_{-\pi}^{\pi} \cos(nx)\cos(mx))dx \\
\int_{-\pi}^{\pi} \frac{1}{2}\big[\cos((n-m)x) - \cos((n+m)x)\big]dx &= \int_{-\pi}^{\pi} \frac{1}{2}\big[\cos((n-m)x) - \cos((n+m)x)\big]dx \\
\int_{-\pi}^{\pi} \cos((n-m)x)dx - \int_{-\pi}^{\pi} \cos((n+m)x)dx &= \int_{-\pi}^{\pi} \cos((n-m)x)dx + \int_{-\pi}^{\pi} \cos((n+m)x)dx \\
-\int_{-\pi}^{\pi} \cos((n+m)x)dx &= \int_{-\pi}^{\pi} \cos((n+m)x)dx
\end{aligned}
$$

The integral of $\cos((n+m)x)$ from $-\pi$ to π is 0, and $(-0) = 0$. Therefore we know

$$(\sin(mx), \sin(nx)) = (\cos(nx), \cos(mx))$$

1.3 Prove the following:

$$a_0 = (f, 1)$$

$$\begin{aligned} a_0 = c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f * e^{-i*0*x} dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f * 1 dx \\ &= (f, 1) \end{aligned}$$

$$a_n = (f, 2 * \cos(nx))$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f * \cos(nx) dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f * 2\cos(nx) dx \\ &= (f, 2\cos(nx)) \end{aligned}$$

$$b_n = (f, 2 * \sin(nx))$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f * \sin(nx) dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f * 2\sin(nx) dx \\ &= (f, 2\sin(nx)) \end{aligned}$$

Prove the identity

$$\begin{aligned} (f, f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) f(x) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\sum_n a_n \cos(nx) + b_n \sin(nx))(\sum_m a_m \cos(mx) + b_m \sin(mx)) dx \\ &= \frac{1}{2\pi}\sum_{n=0}\sum_{m=0}\int_{-\pi}^{\pi} (a_n \cos(nx) + b_n \sin(nx))(a_m \cos(mx) + b_m \sin(mx)) dx \\ &= \frac{1}{2\pi}\sum_{n=0}\sum_{m=0}\int_{-\pi}^{\pi} (a_n a_m \cos(nx)\cos(mx) + b_n a_m \sin(nx)\cos(mx) \\ &+ a_n b_m \cos(nx)\sin(mx) + b_n b_m \sin(nx)\sin(mx)) dx \end{aligned}$$

If we look at the n = 0 we have:

$$\frac{1}{2\pi}\sum_{m=0}\int_{-\pi}^{\pi}(a_0a_m\cos(nx)\cos(mx)+b_0a_m\sin(nx)\cos(mx)+a_0b_m\cos(nx)\sin(mx)+b_0b_m\sin(nx)\sin(mx))dx$$
$$=\frac{1}{2\pi}\sum_{m=0}\int_{-\pi}^{\pi}(a_0a_m\cos(0)\cos(mx)+0*a_m\sin(0)\cos(mx)+a_0b_m\cos(0)\sin(mx)+0*b_m\sin(0)\sin(mx))dx$$
$$=\frac{1}{2\pi}\sum_{m=0}\int_{-\pi}^{\pi}(a_0a_m\cos(0)\cos(mx)+a_0b_m\cos(0)\sin(mx))dx$$
$$=\frac{1}{2\pi}\sum_{m=1}\int_{-\pi}^{\pi}(a_0a_m\cos(0)\cos(mx)+a_0b_m\cos(0)\sin(mx))dx+\int_{-\pi}^{\pi}(a_0a_0\cos(0)\cos(0)+a_0b_0\cos(0)\sin(0))dx$$
$$=\frac{1}{2\pi}\sum_{m=1}\int_{-\pi}^{\pi}(a_0a_m\cos(0)\cos(mx)+a_0b_m\cos(0)\sin(mx))dx+\frac{1}{2\pi}\int_{-\pi}^{\pi}a_0a_0\cos(0)\cos(0)dx$$

Since we know $(\cos(nx),\cos(mx))=0$ when $n\neq m$, we find:

$$=\frac{1}{2\pi}\sum_{m=1}0+\frac{1}{2\pi}\int_{-\pi}^{\pi}(a_0a_0\cos(0)\cos(0)dx$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}a_0a_0dx$$
$$=|a_0|^2$$

We plot this back into our formulae for (f,f)

$$(f,f)=|a_0|^2+\frac{1}{2\pi}\sum_{n=1}\sum_{m=1}\int_{-\pi}^{\pi}(a_na_m\cos(nx)\cos(mx)+b_na_m\sin(nx)\cos(mx)$$
$$+a_nb_m\cos(nx)\sin(mx)+b_nb_m\sin(nx)\sin(mx))dx$$

We now use what we found in 1.1:

$$(f,f)=|a_0|^2+\frac{1}{2\pi}\sum_{n=1}\sum_{m=1}\int_{-\pi}^{\pi}(a_na_m\cos(nx)\cos(mx)+0+0+b_nb_m\sin(nx)\sin(mx))dx$$
$$=|a_0|^2+\frac{1}{2\pi}\sum_{n=1}\sum_{m=1}\int_{-\pi}^{\pi}(a_na_m\cos(nx)\cos(mx)+b_nb_m\sin(nx)\sin(mx))dx$$

From what we proved in 1.2, we know that if $m\neq n$, $a_na_m\cos(nx)\cos(mx)=0, b_nb_m\sin(nx)\sin(mx)=0$. Therefore we can reduce accordingly:

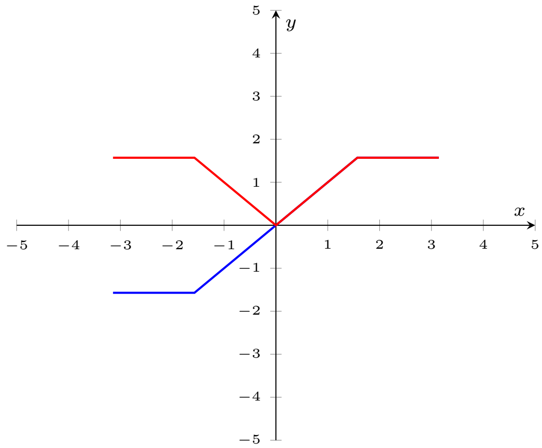
$$
\begin{aligned}
(f,f) &= |a_0|^2 + \frac{1}{2\pi}\sum_{n=1}\sum_{m=1}\int_{-\pi}^{\pi}(a_n a_m \cos(nx)\cos(mx) + b_n b_m \sin(nx)\sin(mx))dx \\
&= |a_0|^2 + \frac{1}{2\pi}\sum_{n=1}\int_{-\pi}^{\pi}(a_n a_n \cos(nx)\cos(nx) + b_n b_n \sin(nx)\sin(nx))dx
\end{aligned}
$$

We also know from 1.2 that $\sin(nx)\sin(nx) = \frac{1}{2}, \cos(nx)\cos(nx) = \frac{1}{2}$

$$
\begin{aligned}
(f,f) &= |a_0|^2 + \frac{1}{2\pi}\sum_{n=1}\int_{-\pi}^{\pi}(\frac{1}{2}a_n a_n + \frac{1}{2}b_n b_n)dx \\
&= \underline{\underline{|a_0|^2 + \sum_{n=1}\frac{1}{2}|a_n|^2 + \frac{1}{2}|a_n|^2}}
\end{aligned}
$$

2 Draw the graph of the odd extension and the even extension.

In red, the even extension, in blue, the odd extension.



2.1 The function of the even expansion is given by:

f(x) =

$$\begin{cases} \frac{\pi}{2} & x \geq \frac{\pi}{2} \\ x & 0 < x < \frac{\pi}{2} \\ -x & \frac{-\pi}{2} < x < 0 \\ \frac{\pi}{2} & x \leq \frac{\pi}{2} \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\left\{\int_{\pi/2}^{\pi} \frac{\pi}{2} + \int_0^{\pi/2} x + \int_{-\pi/2}^{0} -x + \int_{-\pi}^{-\pi/2} \frac{\pi}{2}\right\} \\ &= \frac{1}{2\pi}\left\{\frac{\pi^2}{2} - \frac{\pi^2}{4} + \frac{\pi^2}{8} - 0 + 0 + \frac{\pi^2}{8} - \frac{\pi^2}{4} + \frac{\pi^2}{2}\right\} \\ &= \underline{\underline{\frac{3\pi}{8}}} \end{aligned} \tag{6}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\left\{\int_0^{\pi} f(x) * \cos(\frac{n\pi x}{\pi})\right\} \\ &= \frac{2}{\pi}\left\{\int_{\pi/2}^{\pi} \frac{\pi}{2} * \cos(nx) + \int_{\pi/2}^{\pi} x * \cos(nx)\right\} \\ &= \int_{\pi/2}^{\pi} \cos(nx) + \frac{2}{\pi}\int_{\pi/2}^{\pi} x * \cos(nx) \\ &= \frac{\sin(nx)}{n}\bigg|_{\pi/2}^{\pi} + \frac{2x\sin(nx)}{n\pi}\bigg|_0^{\pi/2} + \frac{2\cos(nx)}{n^2\pi}\bigg|_0^{\pi/2} \\ &= \left[\frac{\sin(n*\pi)}{n} - \frac{\sin(n\pi/2)}{n}\right] + \left[\frac{2\pi\sin(n\pi/2)}{2n\pi} - 0\right] + \left[\frac{2\cos(n\pi/2)}{n^2\pi} - \frac{2\cos(0)}{n^2\pi}\right] \\ &= -\frac{\sin(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n} + \frac{2}{n^2\pi}(\cos(n\pi/2) - 1) \\ a_n &= \frac{2}{n^2\pi}(\cos(n\pi/2) - 1) \end{aligned}$$

This gives us the two cases of odd and even numbers for n:
When n is odd:

$$\begin{aligned} a_n &= \frac{2}{n^2\pi}(\cos(n\pi/2) - 1) \\ &= \frac{2}{n^2\pi}(0 - 1) \\ &= -\frac{2}{n^2\pi} \end{aligned}$$

when n is even:

$$\begin{aligned} a_n &= \frac{2}{n^2\pi}(\cos(n\pi/2) - 1) \\ &= \frac{2}{n^2\pi}((-1)^{\frac{n}{2}} - 1) \end{aligned}$$

This means when $n\%4 = 0$, $a_n = 0$, and when $n\%4 = 2$, $a_n = -\frac{4}{n^2\pi}$
Hence we have:

$$a_1 = -\frac{2}{1^2\pi} \tag{7}$$

$$a_2 = -\frac{4}{2^2\pi} \tag{8}$$

$$a_3 = -\frac{2}{3^2\pi} \tag{9}$$

$$a_4 = 0 \tag{10}$$

$$a_5 = -\frac{2}{5^2\pi} \tag{11}$$

This gives us:

$$\begin{aligned} f_e(x) &= a_0 + a_1 * cos(x) + a_2 * cos(2x) + a_3 * cos(3x) + a_4 * cos(4x) + a_5 * cos(5x) + ... \\ &= \frac{3\pi}{8} - \frac{2}{\pi} * cos(x) - \frac{4}{2^2\pi} * cos(2x) - \frac{2}{3^2\pi} * cos(3x) - 0 * cos(4x) - \frac{2}{5^2\pi} * cos(5x) - ... \\ &= \underline{\underline{\frac{3\pi}{8} + \frac{2}{\pi}\left(- cos(x) - \frac{2cos(2x)}{2^2} - \frac{cos(3x)}{3^2} - \frac{cos(5x)}{5^2} - ...\right)}} \end{aligned}$$

2.2 The function of the even expansion is given by:

f(x) =

$$\begin{cases} \frac{\pi}{2} & x \geq \frac{\pi}{2} \\ x & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ -\frac{\pi}{2} & x \leq \frac{\pi}{2} \end{cases}$$

First we find b_n by using the coefficient equation:

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\bigg\{\int_0^{\pi} f(x) * \sin(\frac{n\pi x}{\pi})\bigg\} \\
&= \frac{2}{\pi}\bigg\{\int_0^{\pi/2} x\sin(nx) + \int_{\pi/2}^{\pi} \frac{\pi}{2}\sin(nx)\bigg\} \\
&= \frac{2}{\pi}\bigg\{\frac{\sin(nx)}{n^2}\bigg|_0^{\pi/2} - \frac{x\cos(nx)}{n}\bigg|_0^{\pi/2} - \frac{\pi}{2n}\cos(nx)\bigg|_{\pi/2}^{\pi}\bigg\} \\
&= \frac{2}{\pi}\bigg\{\frac{\sin(\frac{n\pi}{2})}{n^2} - \frac{\pi\cos(\frac{n\pi}{2})}{2n} + \frac{\pi}{2n}\cos(\frac{n\pi}{2}) - \frac{\pi}{2n}\cos(n\pi)\bigg\} \\
&= \frac{2}{\pi}\frac{\sin(\frac{n\pi}{2})}{n^2} - \frac{(-1)^n}{n}
\end{aligned}
$$

Hence we have:

$$
\begin{aligned}
b_1 &= \frac{2}{\pi} + 1 \\
b_2 &= \frac{2*0}{\pi*4} - \frac{1}{2} = -\frac{1}{2} \\
b_3 &= -\frac{2}{3^2\pi} + \frac{1}{3} \\
b_4 &= \frac{2*0}{\pi*16} - \frac{1}{4} = -\frac{1}{4} \\
b_5 &= \frac{2}{5^2\pi} - \frac{-1}{5} = \frac{2}{5^2\pi} + \frac{1}{5}
\end{aligned}
$$

$$
\begin{aligned}
f_o(x) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \\
&= \sum_{n=1}^{\infty} b_n \sin(nx) \\
&= \underline{\underline{(\frac{2}{\pi}+1)\sin(x) - \frac{1}{2}\sin(2x) + (-\frac{2}{3^2\pi}+\frac{1}{3})\sin(3x) - \frac{1}{4}\sin(4x) + (\frac{2}{5^2\pi}+\frac{1}{5})\sin(5x) + ...}}
\end{aligned}
$$