Exercise Set 7

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Oktober 2019

Consider the data points:

x_i	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1
$f(x_i)$	$-\frac{1}{2}$	$-\frac{5}{4}$	$\frac{1}{4}$	$\frac{5}{2}$

Task 1: Use Lagrange interpolatation to find the polynomial of minimal degree interpolating these points. Use this to find an approximation to $f(0)$.

$$\begin{aligned} l_0 &= \frac{x+\frac{1}{2}}{-1+\frac{1}{2}} * \frac{x-\frac{1}{2}}{-1-\frac{1}{2}} * \frac{x-1}{-1-1} \\ &= \frac{x^3 - x^2 - \frac{x}{4} + \frac{1}{4}}{-\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} l_1 &= \frac{x+1}{-\frac{1}{2}+1} * \frac{x-\frac{1}{2}}{-\frac{1}{2}-\frac{1}{2}} * \frac{x-1}{-\frac{1}{2}-1} \\ &= \frac{x^3 - \frac{x^2}{2} - x + \frac{1}{2}}{\frac{3}{4}} \end{aligned}$$

$$\begin{aligned} l_2 &= \frac{x+1}{\frac{1}{2}+1} * \frac{x+\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}} * \frac{x-1}{\frac{1}{2}-1} \\ &= \frac{x^3 + \frac{x^2}{2} - x - \frac{1}{2}}{-\frac{3}{4}} \end{aligned}$$

$$\begin{aligned} l_3 &= \frac{x+1}{1+1} * \frac{x+\frac{1}{2}}{1+\frac{1}{2}} * \frac{x-\frac{1}{2}}{1-\frac{1}{2}} \\ &= \frac{x^3 + x^2 - \frac{x}{4} - \frac{1}{4}}{\frac{3}{2}} \end{aligned}$$

$$
\begin{aligned}
P_3(x) &= -\frac{1}{2}l_0 - \frac{5}{4}l_1 + \frac{1}{4}l_2 + \frac{5}{2}l_3 \\
&= x^3(\frac{1}{3} - \frac{5}{3} - \frac{1}{3} + \frac{5}{3}) + x^2(\frac{1}{3} + \frac{5}{6} - \frac{1}{6} - \frac{5}{3}) + x(-\frac{1}{12} + \frac{5}{3} + \frac{1}{3} - \frac{5}{12}) + (\frac{1}{12} - \frac{5}{6} + \frac{1}{6} - \frac{5}{12}) \\
&= \underline{\underline{2x^2 + \frac{3}{2}x - 1}} \qquad (1)
\end{aligned}
$$

We now use (1) this to approximate $f(0)$:

$$f(0) \approx P_3(0) = -1$$

Consider the function $f(x) = x^2\cos(x)$

Find the polynomial of degree 3 that interpolates $f(x)$ at four points:

Equally distributed nodes

x_i	-1	0	1	2
$f(x_i)$	$\cos(1)$	1	$\cos(1)$	$4\cos(2)$

$$\begin{aligned} l_0 &= \frac{x}{-1} * \frac{x-1}{-2} * \frac{x-2}{-3} \\ &= \frac{x^3 - 3x^2 - 2x}{-6} \end{aligned}$$

$$\begin{aligned} l_1 &= \frac{x+1}{1} * \frac{x-1}{-1} * \frac{x-2}{-2} \\ &= \frac{x^3 - 2x^2 - x + 2}{2} \end{aligned}$$

$$\begin{aligned} l_2 &= \frac{x+1}{2} * \frac{x}{1} * \frac{x-2}{-1} \\ &= \frac{x^3 - x^2 - 2x}{-2} \end{aligned}$$

$$\begin{aligned} l_3 &= \frac{x+1}{3} * \frac{x}{2} * \frac{x-1}{1} \\ &= \frac{x^3 - x}{6} \end{aligned}$$

$$\begin{aligned} P_3(x) = x^3(&\frac{-\cos(1)}{6} + \frac{1}{2} + \frac{-\cos(1)}{2} + \frac{4\cos(2)}{6}) + x^2(\frac{\cos(1)}{2} + \frac{\cos(1)}{2} - 1) \\ &+ x(\frac{-\cos(1)}{3} - \frac{1}{2} + \cos(1) - \frac{4\cos(2)}{6}) + 1 \\ &= x^3(\frac{-2\cos(1)}{3} + \frac{1}{2} + \frac{2\cos(2)}{3}) + x^2(\cos(1) - 1) + x(\frac{2\cos(1)}{3} - \frac{1}{2} - \frac{2\cos(2)}{3}) + 1 \\ &\approx -0.1376x^3 - 0.4597x^2 + 0.1376x + 1 \end{aligned}$$

subsection*Find by hand a bound for the maximal interpolation error in that interval in these two cases.

Equally distributed nodes

The bound for the maximum interpolation error is given by

$$|e(x)| \leq \frac{1^4}{4(3+1)} M$$

where

$$a = -1, b = 2, h = 1, n = 3$$

and

$$M = \max_{x \in [a,b]} |f^{n+1}(x)| = 2$$

giving:

$$|e(x)| \leq \frac{1}{8}$$

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Chebyshev nodes

The bound for the maximum interpolation error is given by

$$|e(x)| \leq \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} M$$

where

$$a = -1, b = 2, n = 3$$

and

$$M = \max_{x \in [a,b]} |f^{n+1}(x)| = 2$$

giving:

$$|e(x)| \leq \frac{27}{512} \approx 0.0527$$