Exercise Set 1

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1 Find the functions that have the following Laplace transforms:

a)

$$\begin{aligned}\frac{1}{s^2(s^2+1)} &= \frac{s^2+1}{s^2(s^2+1)} - \frac{s^2}{s^2(s^2+1)}\\ &= \frac{1}{s^2} - \frac{1}{s^2+1}\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= \underline{\underline{t - \sin(t)}}\end{aligned}$$

b)

$$\begin{aligned}\frac{s}{s^2+2s+1} &= \frac{s}{(s+1)^2}\\ &= \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2}\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+1}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\\ &= e^{-t} - e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\\ &= e^{-t} - te^{-t}\end{aligned}$$

c)

$$\frac{2s}{(s^2+1)^2} = s * \frac{2*1^3}{(s^2+1^2)^2}$$

We use the formulae $\mathcal{L}^{-1}\{\frac{2a^3}{(s^2+a^2)^2}\} = \sin(at) - at\cos(at)$ and $\mathcal{L}^{-1}\{s * F(s)\} = f'(t) + f(0)$.

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{s * \frac{2*1^3}{(s^2+1^2)^2}\right\}$$

$$F(s) = \frac{2*1^3}{(s^2+1^2)^2}$$

$$\begin{aligned}
\mathcal{L}^{-1}\{s * F(s)\} &= \mathcal{L}^{-1}\left\{\frac{2*1^3}{(s^2+1^2)^2}\right\}\frac{d}{dt} + f(0) \\
&= (\sin(t) - t\cos(t))\frac{d}{dt} + \sin(0) - t\cos(0) \\
&= \cos(t) - (\cos(t) + t\sin(t)) + 0 \\
&= t\sin(t)
\end{aligned}$$

d)

$$(S-3)^{-5}$$

$$\begin{aligned}
&F(S-3) = (S-3)^{-5} \\
&F(S) = S^{-5} = \frac{1}{S^5} \\
&\mathcal{L}^{-1}\left\{\frac{n!}{S^{n+1}}\right\} = t^n \\
&\mathcal{L}^{-1}\left\{\frac{24}{24} * F(S)\right\} = \frac{t^4}{24} \\
&\mathcal{L}^{-1}\left\{(S-3)^{-5}\right\} = e^{3t}\frac{t^4}{24}
\end{aligned}$$

2 Find the Laplace transforms of the following functions:

$f(t) = (u(t) - u(t-\pi))\cos(t)$

In this exercise we use the t-shifting theorem:

$$\mathcal{L}^{-1}\{u(t-a) * g(t-a)\} = e^{-as} * \mathcal{L}^{-1}\{g\}$$

$$\begin{aligned} f(t) &= (u(t) - u(t-\pi))\cos(t) \\ &= u(t)\cos(t) - u(t-\pi)\cos(t) \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}\{f\} &= \mathcal{L}^{-1}\{u(t)\cos(t)\} - \mathcal{L}^{-1}\{u(t-\pi)\cos(t)\} \\ &= e^{-0s} * \frac{s}{s^2+1^2} - e^{-\pi s}\frac{s}{s^2+1^2} \\ &= \frac{s}{s^2+1^2}(1-e^{-\pi s}) \end{aligned}$$

$f(t) = u(t-3)t^4$

$$\begin{aligned} \mathcal{L}\{u(t-3)t^4\} &= e^{-3s} * \mathcal{L}\{t^4\} \\ &= e^{-3s} * \frac{24}{s^5} \\ &= \frac{24e^{-3s}}{s^5} \end{aligned}$$

3 Solve:

$$y'' + 2y = \delta(t-1), y(0) = y'(0) = 0$$

$$\begin{aligned}
\mathcal{L}\{y''\} + 2 * \mathcal{L}\{y\} &= \mathcal{L}\{\delta(t-1)\} \\
S^2Y - Sy(0) - y'(0) + 2Y &= e^{-s} \\
S^2Y + 2Y &= e^{-s} \\
Y &= \frac{e^{-s}}{S^2+2}
\end{aligned}$$

$$\begin{aligned}
y = \mathcal{L}^{-1}\{Y\} &= u(t-1) * \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{S^2+(\sqrt{2})^2}\right\} \\
&= u(t-1) * \frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-1))
\end{aligned}$$

4 By definition, a function f has period T, $T > 0$, if $f(t+T) = f(t)$. The following exercise gives you a nice Laplace transform formula for T-periodic functions.

a)

Show that
$$\int_{nT}^{nT+T} e^{-st} f(t)dt = e^{-snT} \int_0^T e^{-st} f(t)dt$$
for T-periodic f.

$$\begin{aligned}\int_{nT}^{nT+T} e^{-st} f(t)dt &= \int_0^T e^{-s(t+nT)} f(t+nT)dt \\ &= \int_0^T e^{-st} e^{-snT} f(t)dt \\ &= e^{-snT} \int_0^T e^{-st} f(t)dt\end{aligned}$$

q.e.d.

We have now proved through adjusting the boundary that the initial assumption is true.

b)

Show that if f has period T then its Laplace transform F satisfies:
$$F(s) = \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)dt$$

We start by defining the Laplace Transform of our function f:

$$\begin{aligned}F(s) &= \mathcal{L}\{f\} \\ &= \int_0^{\infty} e^{-st} f(t)dt\end{aligned}$$

We divide this integral into T-sized pieces. So we have the integral from 0 to T, T to 2T, 2T to 3T and so on up to ∞.

$$F(s) = \sum_{n=0}^{\infty} \int_{nT}^{nT+T} e^{-st} f(t)dt$$

We use what we found in exercise 4a:

$$\int_{nT}^{nT+T} e^{-st} f(t)dt = e^{-snT} \int_0^T e^{-st} f(t)dt$$

This results in the following:

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} \int_{nT}^{nT+T} e^{-st} f(t)dt \\ &= \sum_{n=0}^{\infty} \left(e^{-snT} \int_0^T e^{-st} f(t)dt \right) \\ &= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)dt \end{aligned} \tag{1}$$

c)

Show that if f has period T then its Laplace transform F satisfies:

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)dt$$

We start with basis in (1).

$$F(s) = \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)dt$$

We use the summation formula given by:

$$\sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1 - a}, a \neq 1 \;\; \& \;\; a > 0$$

In our case $a = e^{-sT}$ and $N = \infty$:

$$\sum_{n=0}^{\infty} e^{-snT} = \frac{1 - (e^{-sT*\infty})}{1 - e^{-sT}}$$

$$\sum_{n=0}^{\infty} e^{-snT} = \frac{1}{1 - e^{-sT}}, e^{-sT} \neq 1$$

This gives us all we need to prove the assumption:

$$
\begin{aligned}
F(s) &= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t) dt \\
&= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt
\end{aligned}
$$

q.e.d.