

Exercise Set 2

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1 Find the functions that have the following Laplace transforms:

a)

$$\begin{aligned}\frac{1}{s^2(s^2+1)} &= \frac{s^2+1}{s^2(s^2+1)} - \frac{s^2}{s^2(s^2+1)} \\ &= \frac{1}{s^2} - \frac{1}{s^2+1}\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \underline{\underline{t - \sin(t)}}$$

b)

$$\begin{aligned}\frac{s}{s^2+2s+1} &= \frac{s}{(s+1)^2} \\ &= \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2}\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+1}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\ &= e^{-t} - e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \\ &= e^{-t} - te^{-t}\end{aligned}$$

c)

$$\frac{2s}{(s^2 + 1)^2} = s * \frac{2 * 1^3}{(s^2 + 1^2)^2}$$

We use the formulae $\mathcal{L}^{-1}\left\{\frac{2a^3}{(s^2+a^2)^2}\right\} = \sin(at) - at \cos(at)$ and $\mathcal{L}^{-1}\{s * F(s)\} = f'(t) + f(0)$.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s}{(s^2 + 1)^2}\right\} &= \mathcal{L}^{-1}\left\{s * \frac{2 * 1^3}{(s^2 + 1^2)^2}\right\} \\ F(s) &= \frac{2 * 1^3}{(s^2 + 1^2)^2} \\ \mathcal{L}^{-1}\{s * F(s)\} &= \mathcal{L}^{-1}\left\{\frac{2 * 1^3}{(s^2 + 1^2)^2}\right\} \frac{d}{dt} + f(0) \\ &= (\sin(t) - t \cos(t)) \frac{d}{dt} + \sin(0) - t \cos(0) \\ &= \cos(t) - (\cos(t) + t \sin(t)) + 0 \\ &= t \sin(t)\end{aligned}$$

d)

$$(S - 3)^{-5}$$

$$\begin{aligned}F(S - 3) &= (S - 3)^{-5} \\ F(S) &= S^{-5} = \frac{1}{S^5} \\ \mathcal{L}^{-1}\left\{\frac{n!}{S^{n+1}}\right\} &= t^n \\ \mathcal{L}^{-1}\left\{\frac{24}{24} * F(S)\right\} &= \frac{t^4}{24} \\ \mathcal{L}^{-1}\left\{(S - 3)^{-5}\right\} &= e^{3t} \frac{t^4}{24}\end{aligned}$$

2 Find the Laplace transforms of the following functions:

$$f(t) = (u(t) - u(t - \pi)) \cos(t)$$

In this exercise we use the t-shifting theorem:

$$\mathcal{L}^{-1}\{u(t - a) * g(t - a)\} = e^{-as} * \mathcal{L}^{-1}\{g\}$$

$$\begin{aligned} f(t) &= (u(t) - u(t - \pi)) \cos(t) \\ &= u(t) \cos(t) - u(t - \pi) \cos(t) \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}\{f\} &= \mathcal{L}^{-1}\{u(t) \cos(t)\} - \mathcal{L}^{-1}\{u(t - \pi) \cos(t)\} \\ &= e^{-0s} * \frac{s}{s^2 + 1^2} - e^{-\pi s} \frac{s}{s^2 + 1^2} \\ &= \frac{s}{s^2 + 1^2} (1 - e^{-\pi s}) \end{aligned}$$

$$f(t) = u(t - 3)t^4$$

$$\begin{aligned} \mathcal{L}\{u(t - 3)t^4\} &= e^{-3s} * \mathcal{L}\{t^4\} \\ &= e^{-3s} * \frac{24}{s^5} \\ &= \frac{24e^{-3s}}{s^5} \end{aligned}$$

3 Solve:

$$y'' + 2y = \delta(t - 1), y(0) = y'(0) = 0$$

$$\begin{aligned}\mathcal{L}\{y''\} + 2 * \mathcal{L}\{y\} &= \mathcal{L}\{\delta(t - 1)\} \\ S^2 Y - Sy(0) - y'(0) + 2Y &= e^{-s} \\ S^2 Y + 2Y &= e^{-s} \\ Y &= \frac{e^{-s}}{S^2 + 2}\end{aligned}$$

$$\begin{aligned}y = \mathcal{L}^{-1}\{Y\} &= u(t - 1) * \frac{1}{\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{S^2 + (\sqrt{2})^2}\right\} \\ &= u(t - 1) * \frac{1}{\sqrt{2}} \sin(\sqrt{2}(t - 1))\end{aligned}$$

4 By definition, a function f has period T , $T > 0$, if $f(t + T) = f(t)$. The following exercise gives you a nice Laplace transform formula for T -periodic functions.

a)

Show that

$$\int_{nT}^{nT+T} e^{-st} f(t) dt = e^{-snT} \int_0^T e^{-st} f(t) dt$$

for T -periodic f .

$$\begin{aligned} \int_{nT}^{nT+T} e^{-st} f(t) dt &= \int_0^T e^{-s(t+nT)} f(t+nT) dt \\ &= \int_0^T e^{-st} e^{-snT} f(t) dt \\ &= e^{-snT} \int_0^T e^{-st} f(t) dt \end{aligned}$$

q.e.d.

We have now proved through adjusting the boundary that the initial assumption is true.

b)

Show that if f has period T then its Laplace transform F satisfies:

$$F(s) = \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t) dt$$

We start by defining the Laplace Transform of our function f :

$$\begin{aligned} F(s) &= \mathcal{L}\{f\} \\ &= \int_0^{\infty} e^{-st} f(t) dt \end{aligned}$$

We divide this integral into T -sized pieces. So we have the integral from 0 to T , T to $2T$, $2T$ to $3T$ and so on up to ∞ .

$$F(s) = \sum_{n=0}^{\infty} \int_{nT}^{nT+T} e^{-st} f(t) dt$$

We use what we found in exercise 4a:

$$\int_{nT}^{nT+T} e^{-st} f(t) dt = e^{-snT} \int_0^T e^{-st} f(t) dt$$

This results in the following:

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} \int_{nT}^{nT+T} e^{-st} f(t) dt \\ &= \sum_{n=0}^{\infty} \left(e^{-snT} \int_0^T e^{-st} f(t) dt \right) \\ &= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t) dt \end{aligned} \tag{1}$$

c)

Show that if f has period T then its Laplace transform F satisfies:

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

We start with basis in (1).

$$F(s) = \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t) dt$$

We use the summation formula given by:

$$\sum_{n=0}^N a^n = \frac{1 - a^{N+1}}{1 - a}, a \neq 1 \quad \& \quad a > 0$$

In our case $a = e^{-sT}$ and $N = \infty$:

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-snT} &= \frac{1 - (e^{-sT})^{\infty}}{1 - e^{-sT}} \\ \sum_{n=0}^{\infty} e^{-snT} &= \frac{1}{1 - e^{-sT}}, e^{-sT} \neq 1 \end{aligned}$$

This gives us all we need to prove the assumption:

$$\begin{aligned}
 F(s) &= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t) dt \\
 &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt
 \end{aligned}$$

q.e.d.