Øving 4

1 $f(x) = \int_0^x e^{t^2}\,dt$ $\qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \int_0^x t^{2n}$ $\qquad e^{t^2} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}$

$$\underline{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1}}$$

Taylor om $x=0$:

$n=0$: $\frac{1}{1} \cdot \frac{x}{1} = x$

$n=1$: $\frac{-1}{1} \cdot \frac{x^3}{3} = -\frac{x^3}{3}$

$n=2$: $\frac{1}{2} \cdot \frac{x^5}{5} = \frac{x^5}{10}$

$n=3$: $-\frac{1}{6} \cdot \frac{x^7}{7} = -\frac{x^7}{42}$

Taylor om $x=0$: $x - \frac{x^3}{3} + \frac{x^5}{10} - 0x^7$

b) Vi bruker feilestimat for alternerende rekker:

$n=0, x=1$: $\frac{1}{1} \cdot \frac{1}{1} = 1$

$n=1, x=1$: $\frac{-1}{1} \cdot \frac{1}{3} = -\frac{1}{3}$

$n=2, x=1$: $\frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$

$n=3, x=1$: $-\frac{1}{6} \cdot \frac{1}{7} = -\frac{1}{42}$

$n=4, x=1$: $\frac{1}{24} \cdot \frac{1}{9} = \frac{1}{216}$

$n=5, x=1$: $\frac{1}{120} \cdot \frac{1}{11} = -\frac{1}{1320}$

$n=6, x=1$: $\frac{1}{720} \cdot \frac{1}{13} = \frac{1}{9360}$

Feilestimater er alltid mindre eller lik til den neste i rekken. Dermed vil feilestimatet ved $n=5$:

$|f(1) - T_5(1,0)| \leq |T_6(1,0)|$

$\leq \frac{1}{9360}$

Vi må ha 6 ledd, $n=0$ til $n=5$.

$\boxed{2}$

$$\sum_{n=1}^{\infty} \frac{n}{(n-1)!} x^n$$

Vi bruker forholdstesten:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\ x^{n+1} \cdot (n-1)!}{n \cdot x^n \cdot n!}$$

$$= \frac{n+1}{n} \cdot x \cdot \frac{1}{n}$$

$$= \frac{x \cdot (n+1)}{n^2}$$

$$\rho = \lim_{n \to \infty} \frac{x\ (n+1)}{n^2} = 0$$

Siden n går mot uendelig og $x \in \mathbb{R}$, vil rekken konvergere for alle $x \in \mathbb{R}$

b)

$$\sum_{n=1}^{\infty} \frac{n}{(n-1)!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)}{n!} x^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{n x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$= x^2 \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} + x \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= x^2 e^x + x e^x$$

$$= \underline{\underline{e^x \cdot x\,(x+1)}}$$

$\boxed{3}$

$$y' = k \cdot (y-20)$$

$$\frac{y'}{y-20} = k$$

$$\int \frac{1}{y-20}\,dy = \int k$$

$$\ln(y-20) = kx + C$$

$$y-20 = e^{kx+C}$$

$$y-20 = C_1 e^{kx} \qquad e^C = C_1$$

$$\underline{y = C_1 e^{kx} + 20}$$

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$$y(0) = 25$$

$$25 = C_1 e^{k \cdot 0} + 20$$

$$5 = C_1$$

$$y = 5e^{kx} + 20$$

b)

$$y(3) = 22$$

$$22 = 5e^{k3} + 20$$

$$\frac{2}{5} = e^{3k}$$

$$\underline{\frac{1}{3}\ln\left(\frac{2}{5}\right) = k}$$

$$21 = 5e^{kt} + 20$$

$$\frac{1}{5} = e^{kt}$$

$$\ln\frac{1}{5} = k \cdot t$$

$$\frac{\ln\frac{1}{5}}{\frac{\ln\left(\frac{2}{5}\right)}{3}} = t \qquad t = \frac{3\ln\left(\frac{1}{5}\right)}{\ln\left(\frac{2}{5}\right)} = \underline{\underline{5{,}27}} \text{ timer}$$

$\boxed{4}$

$b = 3$
$a = 1$
$n = 4$

$$h = \frac{b-a}{n} = \frac{1}{2}$$

$$\int_1^3 x \ln x \, dx \approx S_n = \frac{h}{3}\left(f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + f(3)\right)$$

$$= \frac{1}{6}\left(0 + 4\cdot\frac{3}{2}\cdot\ln\frac{3}{2} + 2\cdot 2\cdot\ln(2) + 4\cdot\frac{5}{2}\cdot\ln\left(\frac{5}{2}\right) + 3\cdot\ln(3)\right)$$

$$= \frac{1}{6}\left(6\cdot\ln\left(\frac{3}{2}\right) + 4\ln 2 + 10\ln\frac{5}{2} + 3\ln 3\right)$$

$$= \frac{1}{6}\left(6\ln 3 - 6\ln 2 + 4\ln 2 + 10\ln 5 - 10\ln 2 + 3\ln 3\right)$$

$$= \frac{1}{6}\left(9\ln 3 - 12\ln 2 + 10\ln 5\right)$$

$$= \frac{3}{2}\ln 3 - 2\ln 2 + \frac{10}{6}\ln 5$$

$$\approx \underline{\underline{2{,}944}}$$

b) $$\left|\int_a^b f(x)\,dx - S_n\right| \leq \frac{K(b-a)^5}{180 n^4}$$

$$10^{-4} \leq \frac{K(2)^5}{180 n^4}$$

Finner K:

$f(x) = x \ln x$ $\quad f'(x) = \ln x + 1$ $\quad f''(x) = \frac{1}{x}$ $\quad f'''(x) = -\frac{1}{x^2}$

$f''''(x) = \frac{2}{x^3}$

Øvre grensa til $f^{(4)}(x)$ er 2 $\Rightarrow \boxed{K = 2}$

$$n^4 \geq \frac{2^6}{180}\cdot 10^4$$

$$n \geq \sqrt[4]{\frac{2^6}{180}}\cdot 10 \qquad n \geq 7{,}7$$

n må være 8 for at feilen skal være mindre enn 10^{-4}.