Exercise Set 8

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Consider the data points:

x_i	-2	1	6
y_i	1	2	3

Set up the table of divided differences, and write down the second order interpolation polynomial in the Newton form.

$$\begin{array}{c|cccc}
-2 & 1 & & \\
 & & \frac{1}{5} & \\
1 & 2 & & \frac{-1}{60} \\
 & & \frac{1}{5} & \\
6 & 3 & &
\end{array}$$

$$P_2(x) = -\frac{1}{60}(x-1)(x+2) + \frac{1}{3}(x+2) + 1$$

$$P_2(x) = \underline{\underline{-\frac{1}{60}x^2 + \frac{19}{60}x + \frac{26}{15}}}$$

Set up the table of divided differences, and write down the second order interpolation polynomial in the Newton form.

$$\begin{array}{c|ccccc}
-2 & 1 & & & \\
 & & \frac{2}{5} & & \\
-\frac{3}{4} & \frac{3}{2} & & \frac{-4}{105} & \\
 & & \frac{2}{7} & & \frac{11}{4200} \\
1 & 2 & & \frac{-3}{175} & \\
 & & \frac{1}{5} & & \\
6 & 3 & & &
\end{array}$$

$$P_3(x) = \frac{11}{4200}(x-1)(x+\frac{3}{4})(x+2) - \frac{4}{105}(x+\frac{3}{4})(x+2) + \frac{2}{5}(x+2) + 1$$

$$P_3(x) = \underline{\underline{\frac{44x^3 - 563x^2 + 4905x + 29214}{16800}}}$$

Use the inverse interpolation to find an approximation to the solution of f(x) = 0. How close to the exact solution is the approximation?

$$f(x) = x^2 - 3, [a,b] = [1,3], x_0 = 1, x_1 = 2, x_2 = 3$$

We evaluate the function in the three points:

x_i	1	2	3
y_i	-2	1	6

We inverse the table:

y_i	-2	1	6
x_i	1	2	3

We find the Newton's divided difference interpolating polynomial:

$$\begin{array}{r|llll}
-2 & 1 & & \\
 & & \frac{1}{5} & \\
1 & 2 & & \frac{-1}{60} \\
 & & \frac{1}{5} & \\
6 & 3 & &
\end{array}$$

$$P_2(x) = -\frac{1}{60}x^2 + \frac{19}{60}x + \frac{26}{15}$$
$$P_2(0) = -\frac{1}{60}0^2 + \frac{19}{60}0 + \frac{26}{15}$$
$$P_2(0) = \frac{26}{15}$$

The error is:

$$\sqrt{3} - P_2(0) = \sqrt{3} - \frac{26}{15} = \underline{\underline{1.283 * 10^{-3}}}$$

We now use (1) this to approximate $f(0)$:

$$f(0) \approx P_3(0) = -1$$

Consider the integral:

$$\int_1^3 e^{-x} dx$$

Find numerical approximations to the integral using Simpson's method over 1 and 2 intervals, that is $S_1(1,3)$ and $S_2(1,3)$. Find an error estimate for $S_2(1,3)$, and compare with the real error.

$$\begin{aligned} S_1(1,3) &= \frac{1}{3}(e^{-1} + 4e^{-2} + e^{-3}) \\ &= \underline{\underline{0.31967}} \\ S_2(1,3) &= \frac{1}{6}(e^{-1} + 4e^{-3/2} + 2e^{-2} + 4e^{-5/2} + e^{-3}) \\ &= \underline{\underline{0.31820}} \end{aligned}$$

$$\begin{aligned} C &= \frac{-f^{(4)}(x)}{2880} \\ H &= b - a \\ I(1,3) - S_1(1,3) &\approx CH^5 \\ I(1,3) - S_2(1,3) \approx 2C(\frac{H}{2})^5 &= C\frac{H^5}{16} \end{aligned}$$

$$\begin{aligned} S_2 - S_! = \frac{15}{16}CH^5 &\implies CH^5 = \frac{16}{15}(S_2 - S_1) \\ \xi_2 = \left|\frac{1}{16}CH^5\right| &= \left|\frac{1}{15}(S_2 - S_1)\right| = 0.000098 \end{aligned}$$

Find the number of intervals m that guarantees that the approximation of the integral by the composite Simpsons method is less than 10^{-8}

$$\begin{aligned} e_m &= \frac{(b-a)h^4}{180} f^{(4)}(\xi) \\ &= \frac{(b-a)^5}{180(2m)^4} f^{(4)}(\xi) \end{aligned}$$

$$\max_{\xi \in [1,3]} f^{(4)}(\xi) = e^{-1}$$

$$\begin{aligned} e_m &\leq \frac{(3-1)^5}{180(2m)^4} e^{-1} \\ &\leq \frac{32}{180(2m)^4 e} \leq 10^{-8} \end{aligned}$$

$$\begin{aligned} \frac{32}{180(2m)^4 e} &\leq 10^{-8} \\ \frac{32 * 10^8}{180 * 2^4 e} &\leq m^4 \\ \sqrt[4]{\frac{32 * 10^8}{180 * 2^4 e}} &\leq m \\ \underline{\underline{25.285}} &\leq m \end{aligned}$$

Gauss

Find the Gauss–Legendre quadrature over the interval $[-1, 1]$ with $m = 3$

Find nodes:

We find the nodes as they are defined as the roots of the Legendre polynomial for n:

$$L_3 = \frac{d^3}{dt^3}(t^2 - 1)^3$$

$$x_0 = -\sqrt{\frac{3}{5}}, x_1 = 0, x_2 = \sqrt{\frac{3}{5}}$$

Find cardinals:

$$\ell_0(x) = \frac{x}{-\sqrt{\frac{3}{5}}} * \frac{x - \sqrt{\frac{3}{5}}}{-2\sqrt{\frac{3}{5}}} = \frac{x^2 - x\sqrt{\frac{3}{5}}}{\frac{6}{5}}$$

$$\ell_1(x) = \frac{x + \sqrt{\frac{3}{5}}}{\sqrt{\frac{3}{5}}} * \frac{x - \sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}}} = \frac{\frac{3}{5} - x^2}{\frac{3}{5}}$$

$$\ell_2(x) = \frac{x}{\sqrt{\frac{3}{5}}} * \frac{x + \sqrt{\frac{3}{5}}}{2\sqrt{\frac{3}{5}}} = \frac{x^2 + x\sqrt{\frac{3}{5}}}{\frac{6}{5}}$$

Find weight functions:

$$w_0 = \int_{-1}^{1} \ell_0(t)dt = \frac{5}{9}$$

$$w_1 = \int_{-1}^{1} \ell_1(t)dt = \frac{8}{9}$$

$$w_2 = \int_{-1}^{1} \ell_2(t)dt = \frac{5}{9}$$

Which gives the polynomial:

$$\int_{-1}^{1} f(t)dt \approx \int_{-1}^{1} p_2(t)dt = \sum_{i=0}^{2} w_i f(t_i) = \frac{1}{9}\left[\ 5f(-\sqrt{\frac{3}{5}}) + 8f(0) + 5f(-\sqrt{\frac{3}{5}})\ \right]$$

$$\begin{aligned}\int_{-1}^{1} f(t)dt &\approx \int_{-1}^{1} p_2(t)dt = \sum_{i=0}^{2} w_i f(t_i) = \frac{1}{9}\left[\ 5f(-\sqrt{\frac{3}{5}}) + 8f(0) + 5f(-\sqrt{\frac{3}{5}})\ \right] \\ &\approx \frac{1}{9}\left[\ 5e^{\sqrt{\frac{3}{5}}} + 8 + 5e^{-\sqrt{\frac{3}{5}}}\ \right] \\ &\approx 2.35034\end{aligned}$$

Confirm that the quadrature has degree of precision 5.

Checking for degree 0:

$$I[1](-1,1) = \int_{-1}^{1} 1 = 2$$
$$G[1](-1,1) = \frac{1}{9}[5 + 8 + 5] = 2$$

Checking for degree 1:

$$I[t](-1,1) = \int_{-1}^{1} t = 0$$
$$G[t](-1,1) = \frac{1}{9}[5\sqrt{\frac{3}{5}} + 8 * 0 - 5\sqrt{\frac{3}{5}}] = 0$$

Checking for degree 2:

$$I[t^2](-1,1) = \int_{-1}^{1} t^2 = \frac{2}{3}$$
$$G[t^2](-1,1) = \frac{1}{9}\left[5\sqrt{\frac{3}{5}}^2 + 8 * 0^2 + 5\sqrt{\frac{3}{5}}^2\right] = \frac{1}{9}\left[3 + 3\right] = \frac{2}{3}$$

Checking for degree 3:

$$I[t^3](-1,1) = \int_{-1}^{1} t^3 = 0$$
$$G[t^3](-1,1) = \frac{1}{9}\left[5\sqrt{\frac{3}{5}}^3 + 8 * 0^3 - 5\sqrt{\frac{3}{5}}^3\right] = 0$$

Checking for degree 4:

$$I[t^4](-1,1) = \int_{-1}^{1} t^4 = \frac{2}{5}$$
$$G[t^4](-1,1) = \frac{1}{9}\left[5\sqrt{\frac{3}{5}}^4 + 8 * 0^2 + 5\sqrt{\frac{3}{5}}^4\right] = \frac{1}{9}\left[\frac{3^2}{5} + \frac{3^2}{5}\right] = \frac{2}{5}$$

Checking for degree 5:

$$I[t^5](-1,1) = \int_{-1}^{1} t^5 = 0$$
$$G[t^5](-1,1) = \frac{1}{9}\left[5\sqrt{\frac{3}{5}}^5 + 8 * 0^5 - 5\sqrt{\frac{3}{5}}^5\right] = 0$$

Checking for degree 6:

$$I[t^6](-1,1) = \int_{-1}^{1} t^6 = \frac{2}{7}$$

$$G[t^6](-1,1) = \frac{1}{9}\left[5\sqrt{\frac{3}{5}}^6 + 8 * 0^6 + 5\sqrt{\frac{3}{5}}^6\right] = \frac{1}{9}\left[\frac{3^3}{5^2} + \frac{3^3}{5^2}\right] = \frac{3}{25}$$

This does not hold and therefore our degree of precision is 5.

Transfer the quadrature over to some arbitrary interval [a, b]. Use it to find an approximation to $\int_1^3 e^{-x}dx$. What is the error?

Transfering the quadrature over to some arbitrary interval [a,b]:

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{1}{9}[5f(a) + 8f(c) + 5f(b)] * \frac{b-a}{2}\\ &\approx \underline{\underline{\frac{h}{9}[5f(c-h) + 8f(c) + 5f(c+h)]}}\end{aligned}$$

Approximating $\int_1^3 e^{-x}dx$:

$$\begin{aligned}\int_1^3 e^{-x}dx &\approx \frac{h}{9}[5f(c-h) + 8f(c) + 5f(c+h)]\\ &\approx \frac{1}{9}[5f(1) + 8f(2) + 5f(3)]\\ &\approx \frac{1}{9}[5e^{-1} + 8e^{-2} + 5e^{-3}]\\ &\approx \frac{1}{9}[5e^{-1} + 8e^{-2} + 5e^{-3}]\\ &\approx \underline{\underline{0.3523}}\end{aligned}$$

Making the error:

$$\frac{1}{9}[5e^{-1} + 8e^{-2} + 5e^{-3}] - \int_1^3 e^{-x}dx = \underline{\underline{0.03424}}$$

Find an error expression for the composite Gauss–Legendre quadrature:

$$
\begin{aligned}
E(Q) &= \sum_{k=0}^{m-1} E(a+kH, a+(k+1)H) \\
&= \sum_{k=0}^{m-1} \int_{a+kH}^{a+(k+1)H} f(x)dx - Q(a+kH, a+(k+1)H) \\
&= \sum_{k=0}^{m-1} \frac{((a+kH+H)-(a+kH))}{2016000} f^{(6)}(\eta) \\
&= \sum_{k=0}^{m-1} \frac{(H)}{2016000} f^{(6)}(\eta) \\
&= \sum_{k=0}^{m-1} \frac{(b-a)}{2016000 * m} f^{(6)}(\eta) \\
&= \frac{m(b-a)}{2016000 * m} f^{(6)}(\eta) \\
&= \underline{\underline{\frac{(b-a)}{2016000} f^{(6)}(\eta)}}
\end{aligned}
$$