Stochastic Controlled Tomato leaf curving disease

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Abstract

1 Tomato Model

En esta sección, vamos a definir el modelo básico que trabajaremos, consideraremos que las plantas se dividen en tres tipos: plantas susceptibles, latentes e infectadas. Las moscas blancas, las cuales llamaremos vectores, se dividen en susceptibles e infectadas.

Las plantas susceptibles pasan a ser plantas latentes cuando un vector infectado se alimenta de ella a una tasa de β_p, continuando el proceso cuando las plantas latentes se convierten en plantas infectadas a una tasa de b, en cada uno de estos casos consideraremos que estaremos revisando los cultivos para el cual removeremos plantas latentes e infectadas si se detecta que dicha planta esta infectada a una tasa de r_1 y r_2 respectivamente.

Plants become latent by infected vectors, replanting latent and infected plants, latent plants become infectious plants, vectors become infected by infected plants, vectors die or depart per day, immigration from alternative hosts.

$$
\begin{aligned}
\dot{S_p} &= -\beta_p S_p \frac{I_v}{N_v} + \tilde{r_1} L_p + \tilde{r_2} I_p \\
\dot{L_p} &= \beta_p S_p \frac{I_v}{N_v} - b L_p - \tilde{r_1} L_p \\
\dot{I_p} &= b L_p - \tilde{r_2} I_p \\
\dot{S_v} &= -\beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} S_v + (1-\theta)\mu \\
\dot{I_v} &= \beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} I_v + \theta\mu
\end{aligned}
\tag{1}
$$

where

Par.	Unit	description
β_p	vector^{-1}day^{-1}	infection rate of susceptible plants by infected vectors
r_1, r_2	day^{-1}	replanting rate of latent and infected plants, respectively.
b	day^{-1}	latency rate, plant latent becomes infectious
γ	day^{-1}	vector death rate
μ	plant^{-1}day^{-1}	vector migration rates from alternative plants to crop
θ	proportion	vector migration rate
β_v	plant^{-1}day^{-1}	infection rate of susceptible vectors by an infected plant

And the plant population satisfies

$$\dot{N}_p = \dot{S}_p + \dot{L}_p + \dot{I}_p = 0,$$

this implies that $N_p(t)$ is contant for all $t \geq 0$, namely N_p. In the case of the population of vectors we do not have a constant population, but it satisfies the following property

Theorem 1 *With the same notation of SDE (1), let*

$$N_v(t) := S_v(t) + I_v(t),\ N_v^0 := S_v(0) + I_v(0),\ N_v^\infty := \frac{\mu}{\gamma}.$$

Then for any initial condition N_v^0 in $(0, N_v^\infty]$, the whole vector population satisfies

$$N_v(t) = N_v^\infty + (N_v^0 - N_v^\infty)e^{-\gamma t},\ t \geq 0.$$

We first going to adimensionality the system (1), with the following variable change:

$$x = \frac{S_p}{N_p}, y = \frac{L_p}{N_p}, z = \frac{I_p}{N_p}, v = \frac{I_p}{N_v}, w = \frac{I_v}{N_v}$$

Then, the system (1) becomes

$$\begin{aligned}
\dot{x} &= -\beta_p x w + \tilde{r_1} y + \tilde{r_2} z \\
\dot{y} &= \beta_p x w - (b + \tilde{r_1}) y \\
\dot{z} &= b y - \tilde{r_2} z \\
\dot{v} &= -\beta_v v z + (1 - \theta - v)\frac{\mu}{N_v} \\
\dot{w} &= \beta_v v z + (\theta - w)\frac{\mu}{N_v}
\end{aligned} \tag{2}$$

Following [referencia], we are interested in a model where the replanting rate of plants, and died rate of vector, r_1, r_2 and γ are now a random variables. This could be doubt to some stochastic environmental factor acts simultaneously on each plant in the crop. More precisely each replanting, died rate, makes

$$\tilde{r}_1 dt = r_1 dt + \sigma_L dB_p(t), \tag{3}$$

$$\tilde{r}_2 dt = r_2 dt + \sigma_I dB_p(t), \tag{4}$$

$$\tilde{\gamma} dt = \gamma dt + \sigma_v dB_v(t), \tag{5}$$

potentially replanting, and vector death in $[t, t+dt)$. Here $dB(t) = B(t+dt) - B(t)$is the increment of a standard Wiener process or Brownian motion. Note that $\tilde{r}_1$,$\tilde{r}_2$,$\tilde{\gamma}$ are just a random perturbations of r_1,r_2,γ, with $\mathbb{E}\left[\tilde{r}_1 dt\right] = r_1 dt$ and $\mathrm{Var}\left[\tilde{r}_1 dt\right] = \sigma_L^2 dt$, $\mathbb{E}\left[\tilde{r}_2 dt\right] = r_2 dt$ and $\mathrm{Var}\left[\tilde{r}_2 dt\right] = \sigma_I^2 dt$ and $\mathbb{E}\left[\tilde{\gamma} dt\right] = \gamma dt$ and $\mathrm{Var}\left[\tilde{\gamma} dt\right] = \sigma_v^2 dt$.

Thus, the stochastic tomato model is given by the following system of coupled Ito's SDE's

$$\begin{aligned}
dS_p &= \left(-\beta_p S_p \frac{I_v}{N_v} + r_1 L_p + r_2 I_p\right) dt + (\sigma_L L_p + \sigma_I I_p) dB_p(t)\\
dL_p &= \left(\beta_p S_p \frac{I_v}{N_v} - b L_p - r_1 L_p\right) dt - \sigma_L L_p dB_p(t)\\
dI_p &= \left(b L_p - r_2 I_p\right) dt - \sigma_I I_p dB_p(t)\\
dS_v &= \left(-\beta_v S_v \frac{I_p}{N_p} - \gamma S_v + (1-\theta)\mu\right) dt - \sigma_v S_v dB_v(t)\\
dI_v &= \left(\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta\mu\right) dt - \sigma_v I_v dB_v(t)
\end{aligned} \tag{6}$$

and the corresponding change of variable, the system (6) can be reewriting as

$$\begin{aligned}
dx(t) &= (-\beta_p x w + r_1 y + r_2 z) dt + (\sigma_L y + \sigma_I z) dB_p(t)\\
dy(t) &= (\beta_p x w - (b + r_1) y) dt - \sigma_L y dB_p(t)\\
dz(t) &= (b y - r_2 z) dt - \sigma_I z dB_p(t)\\
dv(t) &= \left(-\beta_v v z + (1-\theta-v)\frac{\mu}{N_v}\right) dt\\
dw(t) &= \left(\beta_v v z + (\theta - w)\frac{\mu}{N_v}\right) dt
\end{aligned} \tag{7}$$

2 Existence of unique positive solution

Thereom *.* of [Mao Book] assures ths existence of unique solution of (6) in a compact interval. Since we study asymptotic behaviour, we have to assure the existence of unique positive invariant solution to SDE (6). To this end, let $\mathbb{R}_+^n$ the first octant of $\mathbb{R}^n$ and consider

$$\mathbf{E} := \left\{ (S_p, L_p, I_p, S_v, I_v)^\top \in \mathbb{R}_+^5 : \quad 0 \leq S_p + L_p + I_p \leq N_p, \quad S_v + I_v \leq \frac{\mu}{\gamma} \right\},$$

the following result prove that this set is positive invariant.

Theorem 2 *For any initial values* $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0)) \in \mathbf{E}$*, exists unique invariant global positive solution to SDE* (6) $(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top$ *with probability one, that is,*

$$\mathbb{P}\left[(L_p(t), I_p(t), S_v(t), I_v(t)) \in \mathbf{E}, \quad \forall t \geq 0\right] = 1.$$

Proof. Since the system (6) are $\mathbb{R}$, its coefficients are locally Lipschitz. We know [Mao Ref.] That for any initial condition $(S_{p_0}, S_{v_0}) \in (0, N_p) \times (0, N_v)$ there is a unique maximal local solution $(L_p(t), I_p(t), I_v(t))$ at $t \in [0, \tau_e)$, where τ_e is the explosion time. Let $k_0 > 0$ be sufficiently large for $\frac{1}{k_0} < L_{p_0} < N_p - \frac{1}{k_0}$, $\frac{1}{k_0} < I_{p_0} < N_p - \frac{1}{k_0}$, and $\frac{1}{k_0} < I_{v_0} < N_v - \frac{1}{k_0}$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : (L_p(t), I_p(t), I_v(t)) \notin D_{k_0}\right\},$$

where $D_{k_0} := \left(\frac{1}{k_0}, N_p - \frac{1}{k_0}\right) \times \left(\frac{1}{k_0}, N_p - \frac{1}{k_0}\right) \times \left(\frac{1}{k_0}, N_v - \frac{1}{k_0}\right)$, and we set $\inf \emptyset = \infty$. Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(L_p(t), I_p(t), I_v(t)) \in (0, N_p) \times (0, N_p) \times (0, N_v)$ a.s. for all $t \geq 0$. In other words, to complete the proof all wee need to show is that $\tau_\infty = \infty$ a.s.Suppose the above statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}[\tau_\infty \leq T] > \epsilon.$$

Hence there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}[\tau_k \leq T] > \epsilon. \; \forall k \geq k_1. \tag{8}$$

Define a function $V : (0, N) \to \mathbb{R}_+$ by

$$V(x) := \frac{1}{x} + \frac{1}{N - x},$$

where N, x can be N_p, N_v and L_p, I_p, I_v, respectively. By Itô formula, we have, for any $t \in [0, T]$ and $k \geq k_1$

$$\mathbb{E}V(L_p(t \wedge \tau_k)) = V(L_{p_0}) + \mathbb{E}\int_0^{t\wedge\tau_k} LV(L_p(s))ds, \tag{9}$$

where $LV : (0, N_p) \to \mathbb{R}$ is defined by

$$LV(L_p) = \left[-\frac{1}{L_p^2} + \frac{1}{(N - L_p)^2}\right]\left[\beta_p S_p \frac{I_v}{N_v} - (b + r_1)L_p\right] + \frac{1}{2}\left[\frac{2}{L_p^3} + \frac{1}{(N - L_p)^3}\right]\sigma_L L_p^2.$$

It's not hard to see that

$$LV(L_p) \leq \frac{\beta_p N_p}{N_p - L_p} + \frac{b + r_1}{L_p} + \sigma_L^2\left[\frac{1}{x} + \frac{1}{N_p - L_p}\right] \leq C_1 V(L_p),$$

where $C_1 = (\beta_p N_p) \vee (b + r_1) + \sigma_L^2$. Sustituting this into (9), we get

$$\begin{aligned}\mathbb{E}V(L_p(t \wedge \tau_k)) &\leq V(L_{p_0}) + \mathbb{E}\int_0^{t\wedge\tau_k} C_1 V(L_p(s))ds \\ &\leq V(L_{p_0}) + C_1 \int_0^t \mathbb{E}V(L_p(s \wedge \tau_k))ds.\end{aligned}$$

The Gronwall inequality yields that

$$\mathbb{E}V(L_p(T \wedge \tau_k)) \leq V(L_{p_0})e^{C_1 T} \tag{10}$$

By simiular arguments we can see that

$$LV(I_p) = \left[-\frac{1}{I_p^2} + \frac{1}{(N - I_p)^2}\right][bL_p - r_2 I_p] + \frac{1}{2}\left[\frac{2}{I_p^3} + \frac{1}{(N - I_p)^3}\right]\sigma_I I_p^2.$$

And we can see that

$$LV(I_p) \leq \frac{bN_p}{N_p - I_p} + \frac{r_2}{I_p} + \sigma_I^2\left[\frac{1}{I_p} + \frac{1}{N_p - I_p}\right] \leq C_2 V(I_p),$$

where $C_2 = (bN_p) \vee (r_2) + \sigma_I^2$. Sustituting this into (9), we get

$$\begin{aligned}\mathbb{E}V(I_p(t \wedge \tau_k)) &\leq V(I_{p_0}) + \mathbb{E}\int_0^{t\wedge\tau_k} C_2 V(I_p(s))ds \\ &\leq V(I_{p_0}) + C_2 \int_0^t \mathbb{E}V(I_p(s \wedge \tau_k))ds.\end{aligned}$$

The Gronwall inequality yields that

$$\mathbb{E}V(I_p(T \wedge \tau_k)) \leq V(I_{p_0})e^{C_2 T} \tag{11}$$

And the last argument for I_v, we obtain the following

$$LV(I_v) = \left[-\frac{1}{I_v^2} + \frac{1}{(N - I_v)^2}\right]\left[\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta\mu\right] + \frac{1}{2}\left[\frac{2}{I_p^3} + \frac{1}{(N - I_p)^3}\right]\sigma_I I_v^2.$$

And we can see that

$$LV(I_v) \leq \frac{\beta_v N_v + \theta\mu}{N_p - I_v} + \frac{\gamma}{I_v} + \sigma_v^2\left[\frac{1}{I_v} + \frac{1}{N_p - I_v}\right] \leq C_3 V(I_v),$$

where $C_3 := (\beta_v N_v + \theta\mu) \vee (\gamma) + \sigma_v^2$. Sustituting this into (9), we get

$$\mathbb{E}V(I_v(t\wedge\tau_k)) \le V(I_{v_0}) + \mathbb{E}\int_0^{t\wedge\tau_k} C_3 V(I_v(s))ds$$
$$\le V(I_{v_0}) + C_3\int_0^t \mathbb{E}V(I_v(s\wedge\tau_k))ds.$$

The Gronwall inequality yields that

$$\mathbb{E}V(I_v(T\wedge\tau_k)) \le V(I_{v_0})e^{C_3 T} \tag{12}$$

Set $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$ and, by (8), $\mathbb{P}(\Omega_k) \ge \epsilon$. Note that, for every $\omega \in \Omega_k$, $L_p(\tau_k,\omega), I_p(\tau_k,\omega), I_v(\tau_k,\omega)$ equals either $\frac{1}{k}$ or $N - \frac{1}{k}$, and hence

$$V(L_p(\tau_k,\omega)) \ge k,$$
$$V(I_p(\tau_k,\omega)) \ge k,$$
$$V(I_v(\tau_k,\omega)) \ge k.$$

It follows from (10),(11),(12), that

$$V(L_{p_0})e^{C_1 T} \ge \mathbb{E}\left[I_{\{\Omega_k\}}(\omega)V(L_p(\tau_k,\omega))\right] \ge k\mathbb{P}(\Omega_k) \ge \epsilon k,$$
$$V(I_{p_0})e^{C_2 T} \ge \mathbb{E}\left[I_{\{\Omega_k\}}(\omega)V(I_p(\tau_k,\omega))\right] \ge k\mathbb{P}(\Omega_k) \ge \epsilon k,$$
$$V(I_{v_0})e^{C_3 T} \ge \mathbb{E}\left[I_{\{\Omega_k\}}(\omega)V(I_v(\tau_k,\omega))\right] \ge k\mathbb{P}(\Omega_k) \ge \epsilon k.$$

Letting $k \to \infty$ leads to the contradiction

$$\infty > V(L_{p_0})e^{C_1 T} = \infty,$$
$$\infty > V(I_{p_0})e^{C_2 T} = \infty,$$
$$\infty > V(L_{v_0})e^{C_3 T} = \infty$$

so we must therefore have $\tau_\infty = \infty$ a.s., whence the proof is complete. ■

3 Extinction of the disease

Theorem 3 *[Extinction by noise] If* $\frac{\beta_p^2}{2\sigma_L^2} + \frac{r_2^2}{2\sigma_I^2} + 2\beta_p - r_1 < 0$, $\frac{\beta_v^2}{2\sigma_v^2} + \beta_v - \gamma + \theta\mu < 0$*, then the disease will exponientially extinguish with probability one. that is, for any initial condition* $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^T \in \mathbb{R}_+^5$

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(L_p + I_p) < 0 \ \textit{and} \ \limsup_{t\infty\to\infty} \frac{1}{t}\ln(I_v) < 0 \ \textit{a.s.}$$

Proof. The main idea is apply the Itô formula to a conveniently function and deduce conditions. Let $V(S_p, L_p, I_p) = \ln(L_p + I_p)$, then the Itô formula gives

$$\begin{aligned} d\ln(L_p + I_p) &= \left(\frac{1}{L_p + I_p}\right)\left(\frac{\beta_p}{N_v^\infty} S_p I_v - (b + r_1)L_p - \frac{1}{2}\sigma_L^2 \frac{L_p^2}{(L_p + I_p)^2}\right) dt \\ &- \sigma_L \frac{L_p}{L_p + I_p} dB_p(t) \\ &\leq \left(\frac{1}{L_p + I_p}\right)\left(\beta_p S_p - (b + r_1) - \frac{1}{2}\sigma_L^2 \frac{L_p^2}{(L_p + I_p)^2}\right) dt \\ &- \sigma_L \frac{L_p}{L_p + I_p} dB_p(t). \end{aligned}$$

Let $x := \frac{L_p}{L_p+I_p}$, then

$$\begin{aligned} d\ln(L_p + I_p) &\leq \left(\beta_p \frac{S_p}{L_p + I_p} - (b + r_1) - \frac{1}{2}\sigma_L^2 x^2\right) dt - \sigma_L x dB_p(t) \\ &\leq \left(\beta_p \frac{N_p}{L_p + I_p} - (b + r_1) - \frac{1}{2}\sigma_L^2 x^2\right) dt - \sigma_L x dB_p(t) \\ &\leq \left(\beta_p x + 2\beta_p - (b + r_1) - \frac{1}{2}\sigma_L^2 x^2\right) dt - \sigma_L x dB_p(t) \\ &= \left(-\frac{1}{2}\sigma_L^2 x^2 + \beta_p x + 2\beta_p - (b + r_1)\right) dt - \sigma_L x dB_p(t). \end{aligned}$$

Hence,

$$\begin{aligned} \ln(L_p + I_p) &\leq -\frac{\sigma_L^2}{2}\int_0^t \left(\left(x - \frac{\beta_p}{\sigma_L^2}\right)^2 + \frac{\beta_p^2}{2\sigma_L^2} + 2\beta_p - (b + r_1)\right) du \\ &- \int_0^t \sigma_L x dB_p(u) + \ln(L_p(0) + I_p(0)), \end{aligned}$$

which implies,

$$\begin{aligned} \frac{1}{t}\ln(L_p + I_p) &\leq -\frac{\sigma_L^2}{2t}\int_0^t \left(x - \frac{\beta_p}{\sigma_L^2}\right)^2 du + \frac{\beta_p^2}{2\sigma_L^2} - (b + r_1) + 2\beta_p \\ &- \frac{1}{t}\int_0^t \sigma_L x dB_p(u) + \frac{1}{t}\ln(S_p(0) + L_p(0) + I_p(0)), \end{aligned} \tag{13}$$

let $M_t := \frac{1}{t}\int_0^t \sigma_L x dB_p(t) + \frac{1}{t}\ln(L_p(0) + I_p(0))$. Since the integral in the term M_t is a martingale, the strong law of large numbers for martingales Mao, implies that

$$\lim_{t\to\infty} M_t = 0 \text{ a.s.}$$

Thus, from relation (13) we obtain

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(L_p + I_p) < \frac{\beta_p^2}{2\sigma_L^2} + 2\beta_p - (b + r_1) \tag{14}$$

A similar argument also shows that

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(L_p + I_p) < \frac{r_2^2}{2\sigma_I^2} + b \tag{15}$$

Through the equations (14) and (15), we obtain

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(L_p + I_p) < \frac{\beta_p^2}{2\sigma_L^2} + \frac{r_2^2}{2\sigma_I^2} + 2\beta_p - r_1$$

and

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(I_v) < \frac{\beta_v^2}{2\sigma_v^2} + \beta_v - \gamma + \theta\mu$$

■

Our analysis needs the following function and conditions.

(H-1) According to SDE (6), replatin rates satisfies $r_1 = r_2 = r$.

(H-2) The replanting noise intesities are equal $\sigma_L = \sigma_I = \sigma$.

Given a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, define an operator $LV : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace(g^T(x,t)V_{xx}(x,t)g(x,t)) \tag{16}$$

which is called the diffusion operator of the Itô process associated with the $C^{2,1}$ function V. With this diffusion operator, the Itô formula can be written as

$$dV(x(t),t) = LV(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t) \ a.s. \tag{17}$$

We define the repoductive nomber of our stochastic model in SDE (6) by

$$\mathcal{R}_0^s = \frac{\beta_p\beta_v}{\gamma r} \tag{18}$$

As our deterministic base structure this paramenters summarizes the behavior of extinction and persistence according with a threshold.

Theorem 4 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of SDE (6) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p) \times (0, N_p) \times (0, N_p)$ $\times (0, N_v)$. If $0 \leq \mathcal{R}_0^s < 1$, then the following conditions holds*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathcal{R}_0^s - 1]I_p - rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma}I_vI_p \right] dr \leq \frac{1}{2}\sigma^2 N_p, \ a.s.,$$

namely, the infected individual tends to zero exponentially a.s, i.e the disease will die out with probability one.

Proof. The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$V(S_p, L_p, I_p, I_v) = \left(S_p - S_p^0 - S_p^0 \ln\frac{S_p}{S_p^0}\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v^\infty} I_v,$$

Let f, g respectively be the dirft and difussion of SDE (10). Applyng the inifinitesimal opreator $\mathcal{L}$ we have

$$\begin{aligned}
V_x f &= \left(1 - \frac{S_p^0}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty}S_pI_v + rN_p - rS_p\right) + \frac{\beta_p}{N_v^\infty}S_pI_v - (b+r)L_p \\
&+ bL_p - rI_p + \frac{\beta_pN_p}{\gamma N_v^\infty}\left(\frac{\beta_vN_v}{N_p}I_p - \frac{\beta_v}{N_v^\infty}I_vI_p - \gamma I_v\right) \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty}S_pI_v + \frac{\beta_p}{N_v^\infty}I_vS_p^0 + \frac{\beta_p}{N_v^\infty}S_pI_v - r(L_p + I_p) \\
&+ \frac{\beta_pN_p}{\gamma N_v^\infty}\left(\frac{\beta_vN_v}{N_p}I_p - \frac{\beta_v}{N_v^\infty}I_vI_p - \gamma I_v\right) \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty}I_vS_p^0 - r(L_p + I_p) + \frac{\beta_pN_p}{\gamma N_v^\infty}\frac{\beta_vN_v}{N_p}I_p \\
&- \frac{\beta_pN_p}{\gamma N_v^\infty}\frac{\beta_v}{N_v^\infty}I_vI_p - \frac{\beta_pN_p}{\gamma N_v^\infty}\gamma I_v
\end{aligned}$$

Then,

$$\begin{aligned}
V_x f &= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\gamma - \frac{\beta_p N_p}{N_v^\infty}\right] I_v + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p \\
&- rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p - rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\frac{\beta_p\beta_v}{\gamma r} - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p.
\end{aligned}$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathcal{R}_0^s$ we get

$$V_x f = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p.$$

Moreover,

$$\begin{aligned}
\frac{1}{2} trace(g^T V_{xx} g) &= \frac{1}{2}\sigma^2 N_p \left(\frac{N_p - S_p}{S_p}\right)^2 \\
&\leq \frac{1}{2}\sigma^2 N_p.
\end{aligned}$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising.

Incorporation all terms calculate above, we obtain

$$\begin{aligned}
dV(X) &= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p \left(\frac{N_p - S_p}{S_p}\right)^2 \\
&\leq -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p.
\end{aligned}$$

Define $LV(X)$ as

$$LV(X) = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p.$$

Using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$
\begin{aligned}
0 &\leq \mathbb{E}V(t) - \mathbb{E}V(0) \leq \mathbb{E}\int_0^t LV(X(s))ds \\
&\leq -\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - r\left[\mathcal{R}_0^s - 1\right]I_p + rL_p + \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p\right]ds + \frac{1}{2}\sigma^2 N_p
\end{aligned}
$$

Therefore,

$$
\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[-rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p\right]ds \leq \frac{1}{2}\sigma^2 N_p.
$$

■

4 Persistence

Theorem 5 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (6) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p)\times(0, N_p)\times(0, N_p)\times(0, N_v)$. If $\mathcal{R}_0^s > 1$, then the system (6) is globally asymptotically stable at endemic equilibrium point if*

$$
\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[\frac{rS_p^*}{S_pS_p^*}(S_p^* - S_p)^2 + \frac{\beta_p}{N_v}S_p^*I_v^*A_1 + \frac{\beta_v}{N_p}\frac{I_p}{I_v}(I_v - I_v^*)^2 + \gamma I_v^* A_2\right]dr \leq A_3.
$$

namely, the disease will persist with probability one.

Proof. Let us define the following Lyapunov function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$

$$
\begin{aligned}
V(S_p, L_P, I_p, I_v) &= (S_p + L_p + I_p + I_v) - (S_p^* + L_p^* + I_p^* + I_v^*) \\
&\quad - \left(S_p^*\ln\frac{S_p}{S_p^*} + L_p^*\ln\frac{L_p}{L_p^*} + I_p^*\ln\frac{I_p}{I_p^*} + I_v^*\ln\frac{I_v}{I_v^*}\right).
\end{aligned}
$$

Computing the Itô formula terms as:

$$
\begin{aligned}
V_xf &= \left(1-\frac{S_p^*}{S_p}\right)\left(rN_p - \beta_pS_p\frac{I_v}{N_v^\infty} - rS_p\right) + \left(1-\frac{L_p^*}{L_p}\right)\left(\beta_pS_p\frac{I_v}{N_v^\infty} - (r+b)L_p\right) \\
&\quad + \left(1-\frac{I_p^*}{I_p}\right)(bL_p - rI_p) + \left(1-\frac{I_v^*}{I_v}\right)\left(\beta_vN_v\frac{I_p}{N_p} - \beta_v\frac{I_p}{N_p}I_v - \gamma I_v\right).
\end{aligned}
$$

The system (6) satisfy the following relations at equilibrium point

$$
\begin{aligned}
rN_p &= \beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* \\
(r+b) &= \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} \\
r &= b\frac{L_p^*}{I_p^*} \\
\beta_v \frac{N_v}{N_p} &= \frac{\beta_v}{N_p} I_v^* + \gamma \frac{I_v^*}{I_p^*}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
V_x f = & \left(1 - \frac{S_p^*}{S_p}\right)\left(\beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) \\
& + \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} L_p\right) + \left(1 - \frac{I_p^*}{I_p}\right)\left(bL_p - b\frac{L_p^*}{I_p^*} I_p\right) \\
& + \left(1 - \frac{I_v^*}{I_v}\right)\left(\frac{\beta_v}{N_p} I_v^* I_p + \gamma \frac{I_v^*}{I_p^*} I_p - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right) \\
& = rS_p^*\left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p}{S_p^*}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p I_v}{S_p^* I_v^*}\right) \\
& + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p^*}{L_p}\right)\left(\frac{S_p I_v}{S_p^* I_v^*} - \frac{L_p}{L_p^*}\right) + bL_p^*\left(1 - \frac{I_p^*}{I_p}\right)\left(\frac{L_p}{L_p^*} - \frac{I_p}{I_p^*}\right) \\
& + \left(1 - \frac{I_v^*}{I_v}\right)\left(-\frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right) + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v}{I_v^*}\right)\right) \\
& = rS_p^*\left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right) - \frac{S_p^*}{S_p}\right) \\
& + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(\frac{S_p I_v}{S_p^* I_v^*}\left(1 - \frac{L_p^*}{L_p}\right) - \frac{L_p}{L_p^*}\left(1 - \frac{L_p^*}{L_p}\right)\right) + bL_p^*\left(1 + \frac{L_p}{L_p^*} - \frac{I_p}{I_p^*} - \frac{I_p^* L_p}{I_p L_p^*}\right) \\
& - \frac{\beta_v}{N_v} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v^* I_p}{I_v I_p^*} - \frac{I_v}{I_v^*} + 1\right) \\
& = rS_p^*\left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p} - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right)\right) \\
& + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p}{L_p^*} - \frac{S_p I_v}{S_p^* I_v^*}\left(\frac{L_p^*}{L_p} - 1\right)\right) + bL_p^*\left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right) \\
& - \frac{\beta_v}{N_v^\infty} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
V_x f &= rS_p^*\left(2-\frac{S_p}{S_p^*}-\frac{S_p^*}{S_p}\right)+\frac{\beta_p}{N_v^\infty}S_p^*I_v^*\left(2-\frac{S_p^*}{S_p}-\frac{L_p}{L_p^*}-\frac{I_v}{I_v^*}\left(\frac{S_pL_p^*}{S_p^*L_p}-1\right)\right)\\
&+bL_p^*\left(1-\frac{I_p}{I_p^*}+\frac{L_p}{L_p^*}\left(1-\frac{I_p^*}{I_p}\right)\right)-\frac{\beta_v}{N_p}I_vI_p\left(1-\frac{I_v^*}{I_v}\right)^2\\
&+\gamma I_v^*\left(1-\frac{I_v}{I_v^*}-\frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v}-1\right)\right).
\end{aligned}
$$

Now we need compute the term $g^TV_{xx}g$,

$$
g^TV_{xx}g=\begin{bmatrix}\sigma^2\left(\frac{N_p-S_p}{S_p}\right)^2S_p^*+\sigma^2L_p^* & 0\\ 0 & I_p^*\sigma^2+I_v^*\sigma_v^2\end{bmatrix}
$$

therefore,

$$
\begin{aligned}
\frac{1}{2}trace(g^TV_{xx}g) &= \frac{1}{2}\left(\sigma^2\left(\frac{N_p-S_p}{S_p}\right)^2S_p^*+\sigma^2L_p^*+\sigma^2I_p^*+\sigma_v^2I_v^*\right)\\
&\leq \frac{1}{2}\left(\sigma^2S_p^*+\sigma^2L_p^*+\sigma^2I_p^*+\sigma_v^2I_v^*\right)
\end{aligned}
$$

The stochastics terms are not neccesary, because the are a martingale and therefore, when we use integrating and expectation they vanishing, obtaining the following $LV(X)$ operator

$$
\begin{aligned}
LV(X) &= -rS_p^*\frac{\left(S_p^*-S_p\right)^2}{S_pS_p^*}-\frac{\beta_p}{N_v^\infty}S_p^*I_v^*A_1-bL_p^*A_2-\frac{\beta_v}{N_p}I_vI_p\left(1-\frac{I_v^*}{I_v}\right)^2\\
&-\gamma I_v^*A_3+A_4.
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \left(\frac{S_p^*}{S_p}+\frac{L_p}{L_p^*}+\frac{I_v}{I_v^*}\left(\frac{S_pL_p^*}{S_p^*L_p}-1\right)-2\right)>0,\\
A_2 &= \left(\frac{I_p}{I_p^*}-\frac{L_p}{L_p^*}\left(1-\frac{I_p^*}{I_p}\right)-1\right)>0,\\
A_3 &= \left(\frac{I_v}{I_v^*}+\frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v}-1\right)-1\right)>0,\\
A_4 &= \frac{1}{2}\left(\sigma^2S_p^*+\sigma^2L_p^*+\sigma^2I_p^*+\sigma_v^2I_v^*\right)>0.
\end{aligned}
$$

Applying Itô formula, integrating dV from 0 to t and taking expectation gives the following

$$
\begin{aligned}
&0 \le \mathbb{E}V(t) - \mathbb{E}V(0) = \mathbb{E}\int_0^t LV(s)ds \\
&\quad - \mathbb{E}\int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\
&\quad + A_4 t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\
&\qquad \le A_4.
\end{aligned}
$$

■