Stochastic Controlled Tomato leaf curving disease

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Abstract

1 Tomato Model

En esta sección, vamos a definir el modelo básico que trabajaremos, consideraremos que las plantas se dividen en tres tipos: plantas susceptibles, latentes e infectadas. Las moscas blancas, las cuales llamaremos vectores, se dividen en susceptibles e infectadas.

Las plantas susceptibles pasan a ser plantas latentes cuando un vector infectado se alimenta de ella a una tasa de β_p, continuando el proceso cuando las plantas latentes se convierten en plantas infectadas a una tasa de b, en cada uno de estos casos consideraremos que estaremos revisando los cultivos para el cual removeremos plantas latentes e infectadas si se detecta que dicha planta esta infectada a una tasa de r_1 y r_2 respectivamente.

Plants become latent by infected vectors, replanting latent and infected plants, latent plants become infectious plants, vectors become infected by infected plants, vectors die or depart per day, immigration from alternative hosts.

$$\dot{S_p} = -\beta_p S_p \frac{I_v}{N_v} + \tilde{r_1} L_p + \tilde{r_2} I_p \tag{1}$$

$$\dot{L_p} = \beta_p S_p \frac{I_v}{N_v} - b L_p - \tilde{r_1} L_p \tag{2}$$

$$\dot{I_p} = b L_p - \tilde{r_2} I_p \tag{3}$$

$$\dot{S_v} = -\beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} S_v + (1-\theta)\mu \tag{4}$$

$$\dot{I_v} = \beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} I_v + \theta\mu \tag{5}$$

donde β_p: tasa de infección de las plantas susceptibles mediante un vector infectado. r_1: tasa de replanteo de plantas latentes. r_2: tasa de replanteo de plantas infecciosas. b: tasa de latencia (planta latente se convierte en infecciosa). β_v: tasa de infección de los vectores susceptibles mediante una planta infectada.

γ: tasa de muerte o alejamiento de los vectores, μ: migración de los vectores de plantas hospederas alternas, θ: proporción de migración de los vectores.

Theorem 1 *With the same notation of SDE (1), let*

$$N_v(t) := S_v(t) + I_v(t),\ N_v^0 := S_v(0) + I_v(0),\ N_v^\infty := \frac{\mu}{\gamma}.$$

Then for any initial condition N_v^0 in $(0, N_v^\infty]$, the whole vector population satisfies

$$N_v(t) = N_v^\infty + (N_v^0 - N_v^\infty)e^{-\gamma t},\ t \geq 0.$$

We first going to adimensionality the system (1), with the following variable change:

$$x = \frac{S_p}{N_p}, y = \frac{L_p}{N_p}, z = \frac{I_p}{N_p}, v = \frac{I_p}{N_v}, w = \frac{I_v}{N_v}$$

Then, the system (1) becomes

$$\dot{x} = -\beta_p xw + \tilde{r_1}y + \tilde{r_2}z \tag{6}$$
$$\dot{y} = \beta_p xw - (b + \tilde{r_1})y \tag{7}$$
$$\dot{z} = by - \tilde{r_2}z \tag{8}$$
$$\dot{v} = -\beta_v vz + (1 - \theta - v)\frac{\mu}{N_v} \tag{9}$$
$$\dot{w} = \beta_v vz + (\theta - w)\frac{\mu}{N_v} \tag{10}$$

Folowwing [referencia], we are interested in a model where the replanting rate of plants, and died rate of vector, r_1, r_2 and γ are now a random variables. This could be doubt to some stochastic environmental factor acts simultaneously on each plant in the crop. More precisely each replanting, died rate, makes

$$\tilde{r}_1 dt = r_1 dt + \sigma_L dB(t), \tag{11}$$

$$\tilde{r}_2 dt = r_2 dt + \sigma_I dB(t), \tag{12}$$

$$\tilde{\gamma} dt = \gamma dt + \sigma_v dB(t), \tag{13}$$

potentially replanting, and vector death in $[t, t + dt)$. Here $dB(t) = B(t + dt) - B(t)$is the increment of a standard Wiener process or Brownian motion. Note that $\tilde{r}_1$,$\tilde{r}_2$,$\tilde{\gamma}$ are just a random perturbations of r_1,r_2,γ, with $\mathbb{E}(\tilde{r}_1 dt) = r_1 dt$ and $Var(\tilde{r}_1 dt) = \sigma_L^2 dt$, $\mathbb{E}(\tilde{r}_2 dt) = r_2 dt$ and $Var(\tilde{r}_2 dt) = \sigma_I^2 dt$ and $\mathbb{E}(\tilde{\gamma} dt) = \gamma dt$ and $Var(\tilde{\gamma} dt) = \sigma_v^2 dt$.

Thus, the stochastic tomato model is given by the following system of coupled Ito's SDE's

$$dS_p = \left(-\beta_p S_p \frac{I_v}{N_v} + r_1 L_p + r_2 I_p\right) dt + (\sigma_L L_p + \sigma_I I_p) dB(t) \tag{14}$$
$$dL_p = \left(\beta_p S_p \frac{I_v}{N_v} - b L_p - r_1 L_p\right) dt - \sigma_L L_p dB(t) \tag{15}$$
$$dI_p = (b L_p - r_2 I_p)\, dt - \sigma_I I_p dB(t) \tag{16}$$
$$dS_v = \left(-\beta_v S_v \frac{I_p}{N_p} - \gamma S_v + (1-\theta)\mu\right) dt - \sigma_v S_v dB(t) \tag{17}$$
$$dI_v = \left(\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta\mu\right) dt - \sigma_v I_v dB(t) \tag{18}$$

and the corresponding change of variable, the system (14) can be reewriting as

$$dx(t) = (-\beta_p x w + r_1 y + r_2 z)dt + (\sigma_L y + \sigma_I z)dB(t) \tag{19}$$
$$dy(t) = (\beta_p x w - (b + r_1)y)dt - \sigma_L y dB(t) \tag{20}$$
$$dz(t) = (by - r_2 z)dt - \sigma_I z dB(t) \tag{21}$$
$$dv(t) = \left(-\beta_v v z + (1 - \theta - v)\frac{\mu}{N_v}\right) dt \tag{22}$$
$$dw(t) = \left(\beta_v v z + (\theta - w)\frac{\mu}{N_v}\right) dt \tag{23}$$

2 Existence of unique positive solution

Theorem 2 *For any initial values $(L_p(0), I_p(0), S_v(0), I_v(0)) \in (0, N_p)\times(0, N_p)$ $\times(0, N_v)\times(0, N_v)$, the SDE (14) has a unique global positive solution $(L_p(t), I_p(t)$ $, S_v(t), I_v(t)) \in (0, N_p) \times (0, N_p) \times (0, N_v) \times (0, N_v)$ for all $t \geq 0$ with probability one, namely,*

$$\mathbb{P}[(L_p(t), I_p(t), S_v(t), I_v(t)) \in (0, N_p) \times (0, N_p) \times (0, N_v) \times (0, N_v)\ \forall t \geq 0] = 1.$$

Proof.
■

3 Extinction of the disease

Theorem 3 *[Extinction by noise] If $\min\{\sigma, \sigma_v\} > max\{,\}$, then the disease will exponientially extinfuish with probability one. that is, for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^T \in \mathbb{R}^5_+$*

$$\limsup_{t\infty\to\infty} \frac{1}{t} \ln(S_p + L_p + I_p) < 0 \ \ and \ \ \limsup_{t\infty\to\infty} \frac{1}{t} \ln(S_v + I_v) < 0 \ \ a.s.$$

Proof. The main idea is apply the Itô formula to a conveniently function and deduce conditions. Let $V(S_p, L_p, I_p) = \ln(S_p + L_p + I_p)$, then theItô formula gives

$$
\begin{aligned}
d\ln(S_p+L_p+I_p) = {} & \left(\frac{1}{S_p+L_p+I_p}\right)\left(\frac{\beta_p}{N_v^{\infty}}S_pI_v-(b+r_1)L_p-\frac{1}{2}\sigma_L^2\frac{L_p^2}{(S_p+L_p+I_p)^2}\right)dt \\
& -\sigma_L\frac{L_p}{S_p+L_p+I_p}dB_t \\
\leq {} & \left(\frac{1}{S_p+L_p+I_p}\right)\left(\beta_pS_p-(b+r_1)-\frac{1}{2}\sigma_L^2\frac{L_p^2}{(S_p+L_p+I_p)^2}\right)dt \\
& -\sigma_L\frac{L_p}{S_p+L_p+I_p}dB_t.
\end{aligned}
$$

Let $x := \frac{L_p}{S_p+L_p+I_p}$, then

$$
\begin{aligned}
d\ln(S_p+L_p+I_p) &\leq \left(\beta_p\frac{S_p}{S_p+L_p+I_p}-(b+r_1)-\frac{1}{2}\sigma_L^2x^2\right)dt-\sigma_LxdB_t \\
&\leq \left(\beta_p\frac{N_p}{S_p+L_p+I_p}-(b+r_1)-\frac{1}{2}\sigma_L^2x^2\right)dt-\sigma_LxdB_t \\
&\leq \left(\beta_px+2\beta_p-(b+r_1)-\frac{1}{2}\sigma_L^2x^2\right)dt-\sigma_LxdB_t \\
&= \left(-\frac{1}{2}\sigma_L^2x^2+\beta_px+2\beta_p-(b+r_1)\right)dt-\sigma_LxdB_t.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\ln(S_p+L_p+I_p) \leq {} & -\frac{\sigma_L^2}{2}\int_0^t\left(\left(x-\frac{\beta_p}{\sigma_L^2}\right)^2+\frac{\beta_p^2}{2\sigma_L^2}+2\beta_p-(b+r_1)\right)du \\
& -\int_0^t\sigma_LxdB_u+\ln(S_p(0)+L_p(0)+I_p(0)),
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\frac{1}{t}\ln(S_p+L_p+I_p) \leq {} & -\frac{\sigma_L^2}{2t}\int_0^t\left(x-\frac{\beta_p}{\sigma_L^2}\right)^2du+\frac{\beta_p^2}{2\sigma_L^2}-(b+r_1)+2\beta_p \\
& -\frac{1}{t}\int_0^t\sigma_LxdB_u+\frac{1}{t}\ln(S_p(0)+L_p(0)+I_p(0)), \qquad (24)
\end{aligned}
$$

let $M_t := \frac{1}{t}\int_0^t \sigma_L x dB_u + \frac{1}{t}\ln(S_p(0)+L_p(0)+I_p(0))$. Since the integral in the term M_t is a martingale, the strong law of large numbers for martingales Mao, implies that

$$\lim_{t\to\infty} M_t = 0 \text{ a.s.}$$

Thus, from relation (24) we obtain

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(S_p+L_p+I_p) < \frac{\beta_p^2}{2\sigma_L^2} + 2\beta_p - (b+r_1) \tag{25}$$

A similar argument also shows that

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(S_p+L_p+I_p) < \frac{r_2^2}{2\sigma_I^2} + b \tag{26}$$

Through the ecuaciones (25) and (26), we obtain

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(S_p+L_p+I_p) < \frac{\beta_p^2}{2\sigma_L^2} + 2\beta_p - (b+r_1) + \frac{r_2^2}{2\sigma_I^2} + b$$

and

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(S_v+I_v) < \frac{\beta_v^2}{2\sigma_v^2} + \beta_v - \gamma + \theta\mu$$

■

Our analysis needs the following function and conditions.

$$V(S_p,L_p,I_p,S_v,I_v) = \left(S_p - S_p^0 - S_p^0\ln\left(\frac{S_p}{S_p^0}\right)\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v} I_v \tag{27}$$

We consider in the system (14) that $r = r_1 = r_2$ and $\sigma = \sigma_L = \sigma_I$. These conditions provided us with the necessary factors for the construction of the following stochastic basic reproductive number. (posiblemente umbral),

$$\mathcal{R}_0^s = \frac{\beta_p\beta_v}{\gamma r} \tag{28}$$

Theorem 4 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (14) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0,N_p)\times(0,N_p)\times(0,N_p)$ $\times(0,N_v)$. If $\mathcal{R}_0^s < 1$, then the following conditions holds*

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathcal{R}_0^s - 1]I_p - rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma} I_v I_p \right] dr \le \frac{1}{2}\sigma^2 N_p, \ a.s.,$$

namely, the infected individual tends to zero exponentially a.s, i.e the disease will die out with probability one.

Proof. The main idea is to propose a Lyapunov function $V : \mathbb{R}^4 \to \mathbb{R}_+$ defined as

$$V(S_p, L_p, I_p, I_v) = \left(S_p - S_p^0 - S_p^0 \ln \frac{S_p}{S_p^0}\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v^\infty} I_v,$$

and to verify the hypotheses of Khasminskii Theorem. To apply Itô formula we proceed as

$$\begin{aligned}
V_x f &= \left(1 - \frac{S_p^0}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + rN_p - rS_p\right) + \frac{\beta_p}{N_v^\infty} S_p I_v - (b+r)L_p \\
&+ bL_p - rI_p + \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p} I_p - \frac{\beta_v}{N_v^\infty} I_v I_p - \gamma I_v\right) \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty} S_p I_v + \frac{\beta_p}{N_v^\infty} I_v S_p^0 + \frac{\beta_p}{N_v^\infty} S_p I_v - r(L_p + I_p) \\
&+ \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p} I_p - \frac{\beta_v}{N_v^\infty} I_v I_p - \gamma I_v\right) \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty} I_v S_p^0 - r(L_p + I_p) + \frac{\beta_p N_p}{\gamma N_v^\infty} \frac{\beta_v N_v}{N_p} I_p \\
&- \frac{\beta_p N_p}{\gamma N_v^\infty} \frac{\beta_v}{N_v^\infty} I_v I_p - \frac{\beta_p N_p}{\gamma N_v^\infty} \gamma I_v
\end{aligned}$$

Then,

$$\begin{aligned}
V_x f &= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\gamma - \frac{\beta_p N_p}{N_v^\infty}\right] I_v + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p \\
&- rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty} \frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p - rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty} \frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\frac{\beta_p \beta_v}{\gamma r} - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p.
\end{aligned}$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathcal{R}_0^s$ we get

$$V_x f = -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p.$$

Moreover,

$$\begin{aligned}\frac{1}{2} trace(g^T V_{xx} g) &= \frac{1}{2}\sigma^2 N_p \left(\frac{N_p - S_p}{S_p}\right)^2 \\ &\leq \frac{1}{2}\sigma^2 N_p.\end{aligned}$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising.

Incorporation all terms calculate above, we obtain

$$\begin{aligned}dV(X) &= -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p \left(\frac{N_p - S_p}{S_p}\right)^2 \\ &\leq -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p.\end{aligned}$$

Define $LV(X)$ as

$$LV(X) = -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p.$$

Using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$\begin{aligned}&0 \leq \mathbb{E}V(t) - \mathbb{E}V(0) \leq \mathbb{E}\int_0^t LV(X(s))ds \\ &\leq -\mathbb{E}\int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - r\left[\mathcal{R}_0^s - 1\right] I_p + rL_p + \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p\right] ds + \frac{1}{2}\sigma^2 N_p\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[-rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p\right] ds \leq \frac{1}{2}\sigma^2 N_p.$$

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4 Persistence

Theorem 5 *Let* $(S_p(t), L_p(t), I_p(t), I_v(t))$ *be the solution of (14) with initial values* $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p) \times (0, N_p) \times (0, N_p) \times (0, N_v)$*. If* $\mathcal{R}_0^s > 1$*, then the system (14) is globally asymptotically stable at endemic equilibrium point if*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[\frac{rS_p^*}{S_pS_p^*}(S_p^* - S_p)^2 + \frac{\beta_p}{N_v}S_p^*I_v^*A_1 + \frac{\beta_v}{N_p}\frac{I_p}{I_v}(I_v - I_v^*)^2 + \gamma I_v^* A_2\right] dr \leq A_3.$$

namely, the disease will persist with probability one.

Proof. Let us define the following Lyapunov function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$

$$\begin{aligned} V(S_p, L_P, I_p, I_v) &= (S_p + L_p + I_p + I_v) - (S_p^* + L_p^* + I_p^* + I_v^*) \\ &\quad - \left(S_p^* \ln\frac{S_p}{S_p^*} + L_p^* \ln\frac{L_p}{L_p^*} + I_p^* \ln\frac{I_p}{I_p^*} + I_v^* \ln\frac{I_v}{I_v^*}\right). \end{aligned}$$

Computing the Itô formula terms as:

$$\begin{aligned} V_x f = &\left(1 - \frac{S_p^*}{S_p}\right)\left(rN_p - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) + \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - (r+b)L_p\right) \\ &+ \left(1 - \frac{I_p^*}{I_p}\right)(bL_p - rI_p) + \left(1 - \frac{I_v^*}{I_v}\right)\left(\beta_v N_v \frac{I_p}{N_p} - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right). \end{aligned}$$

The system (14) satisfy the following relations at equilibrium point

$$\begin{aligned} rN_p &= \beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* \\ (r+b) &= \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} \\ r &= b\frac{L_p^*}{I_p^*} \\ \beta_v \frac{N_v}{N_p} &= \frac{\beta_v}{N_p} I_v^* + \gamma \frac{I_v^*}{I_p^*} \end{aligned}$$

Moreover,

$$
\begin{aligned}
V_x f &= \left(1 - \frac{S_p^*}{S_p}\right)\left(\beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) \\
&+ \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} L_p\right) + \left(1 - \frac{I_p^*}{I_p}\right)\left(bL_p - b\frac{L_p^*}{I_p^*} I_p\right) \\
&+ \left(1 - \frac{I_v^*}{I_v}\right)\left(\frac{\beta_v}{N_p} I_v^* I_p + \gamma \frac{I_v^*}{I_p^*} I_p - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right) \\
&= rS_p^* \left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p}{S_p^*}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p I_v}{S_p^* I_v^*}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p^*}{L_p}\right)\left(\frac{S_p I_v}{S_p^* I_v^*} - \frac{L_p}{L_p^*}\right) + bL_p^* \left(1 - \frac{I_p^*}{I_p}\right)\left(\frac{L_p}{L_p^*} - \frac{I_p}{I_p^*}\right) \\
&+ \left(1 - \frac{I_v^*}{I_v}\right)\left(-\frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right) + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v}{I_v^*}\right)\right) \\
&= rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right) - \frac{S_p^*}{S_p}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(\frac{S_p I_v}{S_p^* I_v^*}\left(1 - \frac{L_p^*}{L_p}\right) - \frac{L_p}{L_p^*}\left(1 - \frac{L_p^*}{L_p}\right)\right) + bL_p^* \left(1 + \frac{L_p}{L_p^*} - \frac{I_p}{I_p^*} - \frac{I_p^* L_p}{I_p L_p^*}\right) \\
&- \frac{\beta_v}{N_v} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v^* I_p}{I_v I_p^*} - \frac{I_v}{I_v^*} + 1\right) \\
&= rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p} - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right)\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p}{L_p^*} - \frac{S_p I_v}{S_p^* I_v^*}\left(\frac{L_p^*}{L_p} - 1\right)\right) + bL_p^* \left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right) \\
&- \frac{\beta_v}{N_v^\infty} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
V_x f = {} & rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(2 - \frac{S_p^*}{S_p} - \frac{L_p}{L_p^*} - \frac{I_v}{I_v^*}\left(\frac{S_p L_p^*}{S_p^* L_p} - 1\right)\right) \\
& + bL_p^* \left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right) - \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 \\
& + \gamma I_v^* \left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).
\end{aligned}
$$

Now we need compute the term $g^T V_{xx} g$,

$$g^T V_{xx} g = \begin{bmatrix} \sigma^2 \left(\frac{N_p - S_p}{S_p}\right)^2 S_p^* + \sigma^2 L_p^* & 0 \\ 0 & I_p^* \sigma^2 + I_v^* \sigma_v^2 \end{bmatrix}$$

therefore,

$$\begin{aligned} \frac{1}{2} trace(g^T V_{xx} g) &= \frac{1}{2} \left(\sigma^2 \left(\frac{N_p - S_p}{S_p} \right)^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^* \right) \\ &\leq \frac{1}{2} \left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^* \right) \end{aligned}$$

The stochastics terms are not neccesary, because the are a martingale and therefore, when we use integrating and expectation they vanishing, obtaining the following $LV(X)$ operator

$$\begin{aligned} LV(X) = &-rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} - \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 - b L_p^* A_2 - \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 \\ &- \gamma I_v^* A_3 + A_4. \end{aligned}$$

where

$$\begin{aligned} A_1 &= \left(\frac{S_p^*}{S_p} + \frac{L_p}{L_p^*} + \frac{I_v}{I_v^*} \left(\frac{S_p L_p^*}{S_p^* L_p} - 1 \right) - 2 \right) > 0, \\ A_2 &= \left(\frac{I_p}{I_p^*} - \frac{L_p}{L_p^*} \left(1 - \frac{I_p^*}{I_p} \right) - 1 \right) > 0, \\ A_3 &= \left(\frac{I_v}{I_v^*} + \frac{I_p}{I_p^*} \left(\frac{I_v^*}{I_v} - 1 \right) - 1 \right) > 0, \\ A_4 &= \frac{1}{2} \left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^* \right) > 0. \end{aligned}$$

Applying Itô formula, integrating dV from 0 to t and taking expectation gives the following

$$\begin{aligned} &0 \leq \mathbb{E}V(t) - \mathbb{E}V(0) = \mathbb{E} \int_0^t LV(s) ds \\ &- \mathbb{E} \int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + b L_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\ &+ A_4 t. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left(rS_p^*\frac{\left(S_p^*-S_p\right)^2}{S_pS_p^*} + \frac{\beta_p}{N_v^\infty}S_p^*I_v^*A_1 + bL_p^*A_2 + \frac{\beta_v}{N_p}I_vI_p\left(1-\frac{I_v^*}{I_v}\right)^2 + \gamma I_v^*A_3 \right) ds \\
&\quad \le A_4.
\end{aligned}
$$

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