Noname manuscript No.
(will be inserted by the editor)

Threshold behavior of a stochstic vector plant model.

The Tomato Yellow Curl Virus

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Received: December 4, 2020/ Accepted: date

Keywords Keywords

Abstract BACKGROUND
PROBLEM SETUP
FINDINGS
IMPLICATIONS

1 Introduction

Review the structure

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Par.	Unit	Description
β_p	vector/day	infection rate of susceptible plants by infected vectors
r_1, r_2	day^{-1}	replanting rate of latent and infected plants, respectively.
b	day^{-1}	latency rate, plant latent becomes infectious
γ	day^{-1}	vector death rate
μ	plant/day	vector migration rates from alternative plants to crop
θ		proportion vector migration rate
β_v	plant/day	infection rate of susceptible vectors by an infected plant

Table 1 Parameters description and values of deterministic dynamics in ODE (*).

2 Deterministic base dynamics

$$
\begin{aligned}
\dot{S_p} &= -\beta_p S_p \frac{I_v}{N_v} + \tilde{r}_1 L_p + \tilde{r}_2 I_p \\
\dot{L_p} &= \beta_p S_p \frac{I_v}{N_v} - bL_p - \tilde{r}_1 L_p \\
\dot{I_p} &= bL_p - \tilde{r}_2 I_p \\
\dot{S_v} &= -\beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} S_v + (1-\theta)\mu \\
\dot{I_v} &= \beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} I_v + \theta\mu
\end{aligned}
\tag{1}
$$

Make a table for description of all parameters

Redact this conservation law to the entire system (1). Write a introductory paragraph to Thm 1

Theorem 1 *With the notation of ODE (1), let*

$$
\begin{aligned}
N_v(t) &:= S_v(t) + I_v(t) \\
N_v^{\infty} &:= \frac{\mu}{\gamma}.
\end{aligned}
$$

Then for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^{\top} \in (0,\infty) \times (0, N_v^{\infty})$, *she plant and vector total populations respectively satisfies*

$$
\begin{aligned}
&\frac{dN_p}{dt} = \frac{d}{dt}(S_p + L_p + I_p) = 0, \\
&\lim_{t\to\infty} N_v(t) = N_v^{\infty}.
\end{aligned}
$$

2.1 Deterministic fixed points

Fix notation to distinguish between free disease and endemic

Here we compute the determinsitic fixed points of system (1) and show that its unicity. Thus by definition of we solve

$$
\begin{aligned}
-\beta_p S_p^* \frac{I_v^*}{N_v} + r(N_p - S_p^*) &= 0\\
\beta_p S_p^* \frac{I_v^*}{N_v} - bL_p^* - rL_p^* &= 0\\
bL_p^* - rI_p^* &= 0\\
-\beta_v S_v^* \frac{I_p^*}{N_p} - \gamma S_v^* + (1-\theta)\mu &= 0\\
\beta_v S_v^* \frac{I_p^*}{N_p} - \gamma I_v^* + \theta\mu &= 0.
\end{aligned}
\tag{2}
$$

to determine our fixed points. There is two fixed points—free disease equilibrium and the endemic equilibrium. We characterize the fist the relation $L_p^* = I_p^* = I_v^* = 0$, wich implies that

$$r(N_p - S_p^*) = 0,$$

and therefore, we obtain $S_p^* = N_p$.F or the vector population we have by Theorem (1) that $S_v^* + I_v^* \to \frac{\mu}{\gamma}$ as $\to \infty$, then $S_v^* \to \frac{\mu}{\gamma}$ when we have $I_v^* = 0$. The free disease equilibrium point is $(N_p, 0, 0, \frac{\mu}{\gamma}, 0)^\top$. For the case of endemic equilibrium point, we need suppose that $L_p^*, I_p^*, I_v^* \neq 0$ and solve each right hand side of system (1) in terms of other variable. From $\dot{S_p}$, we can obtain

$$S_p^* = \frac{rN_pN_v}{rN_v + I_v^*\beta_p},$$

and similar for the other equations we obtain

$$
\begin{aligned}
L_p^* &= \frac{\beta_p S_p^* I_v^*}{N_v\,(b+r)},\\
I_p^* &= \frac{bL_p^*}{r},\\
S_v^* &= \frac{(1-\theta)\,\mu\, N_p}{\gamma N_p + I_p^*\beta_v},
\end{aligned}
$$

Expresing the above coordinate in terms of I_v, we obtain rewrite as align

$$
\begin{aligned}
S_p^* &= \frac{rN_pN_v}{rN_v + I_v^*\beta_p},\\
L_p^* &= \frac{\beta_p r N_p I_v^*}{(b+r)\,(rN_v + I_v^*\beta_p)},\\
I_p^* &= \frac{b\beta_p N_p I_v^*}{(b+r)\,(rN_v + I_v^*\beta_p)},\\
S_v^* &= \frac{(1-\theta)\,\mu\,(b+r)(rN_v + \beta_p I_v^*)}{\gamma(b+r)(rN_v + \beta_p I_v^*) + b\beta_p\beta_v I_v^*},
\end{aligned}
$$

We only need substituting the above expression into the differential equation of I_v and solve the following quadratic equation

$$\begin{aligned}-N_p[b\gamma^2 r I_v^* N_v + b\gamma^2 (I_v^*)^2\beta_p - b\gamma\mu r\theta N_v - b\gamma\mu\theta I_v^*\beta_p + \\ b\gamma(I_v^*)^2\beta_p\beta_v + b\mu\theta I_v^*\beta_p^2 - b\mu\theta I_v^*\beta_p\beta_v + \gamma^2 r^2 I_v^* N_v + \\ \gamma^2 r (I_v^*)^2\beta_p - \gamma\mu r^2\theta N_v - \gamma\mu r\theta I_v^* + \beta_p - b\mu I_v^*\beta_p^2] = 0\,. \end{aligned} \tag{3}$$

In sake of clearnes we define

$$\begin{aligned} a_1 &:= b\gamma^2\beta_p + b\gamma\beta_p\beta_v + \gamma^2 r\beta_p, \\ a_2 &:= -b\gamma\mu\theta\beta_p + b\mu\theta\beta_p^2 - b\mu\theta\beta_p\beta_v + \gamma^2 r^2 N_v - \gamma\mu r\theta\beta_p - b\mu\beta_p^2 + \gamma^2 r N_v, \\ a_3 &:= -b\gamma\mu r\theta N_v - \gamma\mu r^2\theta N_v, \end{aligned}$$

and rewrite the above eqution in this new notation as

$$\underbrace{()}_{:=a_1} I_v^{*2} + \underbrace{()}_{:=a_2} I_v + \underbrace{()} \,. \tag{4}$$

Fill according to each term

We need a positive solution, then according to discriminant, we obtain

$$\begin{aligned} \Delta &= a_2^2 - 4a_1a_3 \\ &= (-b\gamma\mu\theta\beta_p + b\mu\theta\beta_p^2 - b\mu\theta\beta_p\beta_v + \gamma^2 r^2 N_v - \gamma\mu r\theta\beta_p - b\mu\beta_p^2 + \gamma^2 r N_v)^2 \\ &+ 4(b\gamma^2\beta_p + b\gamma\beta_p\beta_v + \gamma^2 r\beta_p)(b\gamma\mu r\theta N_v + \gamma\mu r^2\theta N_v), \end{aligned}$$

which ever is positive, then we have two different real solution, since we require the positive, we deduce that

$$I_v^* = \frac{-a_2 + \sqrt{a_2^2 - 4a_1a_3}}{2a_1}.$$

3 Stochastic extension

Why we want to normalize?

write here the parameters

Following ideas from [referencia], we quantify uncertinity in replanting rate of plants, and died rate of vector, r_1, r_2 and γ, to this end, we perturb parameters $r_1 \ldots$ whit a Winner process to obtain a stochastic differential equation(SDE). Here, the perturbation describe stochastic environmental noise on each population. In symbols $dB(t) = B(t+dt) - B(t)$ denotes the increment of a standard Wiener process, thus we perturb potentially replanting r_1, r_2, and vector death γ in the infinitiesimal time interval $[t, t+dt)$ by

$$\begin{aligned} r_1 dt &\rightsquigarrow r_1 dt + \sigma_L dB_p(t), \\ r_2 dt &\rightsquigarrow r_2 dt + \sigma_I dB_p(t), \\ \gamma dt &\rightsquigarrow \gamma dt + \sigma_v dB_v(t). \end{aligned} \tag{5}$$

Note that right hand side of (5) is a random perturbations of parameters r_1, r_2, γ, with mean $\mathbb{E}[r_1 dt + \sigma_L dB_p(t)]$ and variance $\mathrm{Var}[r_1 dt + \sigma_L dB_p(t)] = \sigma_L^2 dt$,

Note that here we will use the latex proba package, plase use the same commands in the remain of the manuscript

$\mathbb{E}(\tilde{r}_2 dt) = r_2 dt$ and $Var(\tilde{r}_2 dt) = \sigma_I^2 dt$ and $\mathbb{E}(\tilde{\gamma} dt) = \gamma dt$ and $Var(\tilde{\gamma} dt) = \sigma_v^2 dt$. Thus, we establish an stochastic extencion from deterministic tomato model (1) by the Itô SDE

$$
\begin{aligned}
dS_p &= \left(-\beta_p S_p \frac{I_v}{N_v} + r_1 L_p + r_2 I_p\right) dt + (\sigma_L L_p + \sigma_I I_p) dB_p(t) \\
dL_p &= \left(\beta_p S_p \frac{I_v}{N_v} - b L_p - r_1 L_p\right) dt - \sigma_L L_p dB_p(t) \\
dI_p &= (b L_p - r_2 I_p) dt - \sigma_I I_p dB_p(t) \\
dS_v &= \left(-\beta_v S_v \frac{I_p}{N_p} - \gamma S_v + (1-\theta)\mu\right) dt - \sigma_v S_v dB_v(t) \\
dI_v &= \left(\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta\mu\right) dt - \sigma_v I_v dB_v(t).
\end{aligned}
\tag{6}
$$

4 Existence of a unique positive solution

Thereom *.* of [Mao Book] assures ths existence of unique solution of (6) in a compact interval. Since we study asymptotic behaviour, we have to assure the existence of unique-globally-positive invariant solution of SDE (*). To this end, let $\mathbb{R}^n_+$ the first octant of $\mathbb{R}^n$ and consider

$$
\mathbf{E} := \left\{ (S_p, L_p, I_p, S_v, I_v)^\top \in \mathbb{R}^5_+ : \quad 0 \leq S_p + L_p + I_p \leq N_p, \quad S_v + I_v \leq \frac{\mu}{\gamma} \right\},
$$

the following result prove that this set is positive invariant.

Theorem 2 *For any initial values $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^\top \in \mathbf{E}$, exists unique a.s. invariant global positive solution to SDE (6) in $\mathbf{E}$, that is,*

$$
\mathbb{P}\left[(L_p(t), I_p(t), S_v(t), I_v(t)) \in \mathbf{E}, \quad \forall t \geq 0\right] = 1.
$$

Proof Since the right hand side of system (6) are quadratic, linear and constans terms, this imply that they are locally Lipschitz. We know by [ref Mao], that for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^\top \in \mathbf{E}$ there is a unique maximal local solution $(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top$ at $t \in [0, \tau_e)$, where τ_e is the explosion time. Let $k_0 > 0$ be sufficiently large, and define the stopping time

$$
\begin{aligned}
\tau_k = \inf \Bigg\{ t \in [0, \tau_e) : L_p(t) \notin \left(\frac{1}{k_0}, N_p - \frac{1}{k_0}\right) \bigcup I_p(t) \notin \left(\frac{1}{k_0}, N_p - \frac{1}{k_0}\right) \\
\bigcup I_v(t) \notin \left(\frac{1}{k_0}, N_v - \frac{1}{k_0}\right) \Bigg\},
\end{aligned}
\tag{7}
$$

We know that $\tau_k \nearrow \tau_\infty$. In other words, $\tau_\infty = \infty$ a.s. implies Give an argument

$$
(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top \in \mathbf{E} \tag{8}
$$

a.s. for all $t \geq 0$. Thus, we show that $\tau_\infty = \infty$ a.s. To this end, we proced by contradiction. Suppose that the above statement is false for a given time t, then there is a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ such that some component from L_p, I_p, I_v, or L_p, get-outs from its corresponding interval

$$\left(\frac{1}{k_0}, N_\bullet - \frac{1}{k_0}\right),$$

that is, $\mathbb{P}[\tau_\infty \leq T] > \varepsilon$. Hence, there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}[\tau_k \leq T] > \varepsilon, \quad \forall k \geq k_1. \tag{9}$$

Define a function $V_p : (0, N_p) \to \mathbb{R}_+$ by

$$V_p(x) := \frac{1}{x} + \frac{1}{N_p - x}.$$

Write auxiliar results in a fucking appendix

According to the inifinitesimal operation $\mathscr{L}$ see **??** APPENDIX By diffusion operator, we have, for any $t \in [0,T]$ and $k \geq k_1$

$$\begin{aligned}\mathscr{L}[V_p(L_p)] = &\left[-\frac{1}{L_p^2} + \frac{1}{(N_p - L_p)^2}\right]\left[\beta_p S_p \frac{I_v}{N_v} - (b + r_1)L_p\right] \\ &+ \frac{1}{2}\left[\frac{2}{L_p^3} + \frac{2}{(N_p - L_p)^3}\right]\sigma_p^2 \frac{L_p^2 S_p^2}{N_p^2}.\end{aligned}$$

Expanding each term, we have

$$\begin{aligned}\mathscr{L}[V_p(L_p)] = &-\beta_p \frac{S_p I_v}{L_p^2 N_v} + \beta_p \frac{S_p I_v}{(N_p - L_p)^2 N_v} + \frac{(b + r_1)}{L_p} - \frac{(b + r_1)L_p}{(N_p - L_p)^2} \\ &+ \left[\frac{1}{L_p^3} + \frac{1}{(N_p - L_p)^3}\right]\sigma_p^2 \frac{L_p^2 S_p^2}{N_p^2}.\end{aligned}$$

Droping negative terms, we bound the above relation by

$$\mathscr{L}[V_p(L_p)] \leq \beta_p \frac{S_p}{(N_p - L_p)^2} + \frac{(b + r_1)}{L_p} + \left[\frac{1}{L_p^3} + \frac{1}{(N_p - L_p)^3}\right]\sigma_p^2 \frac{L_p^2 S_p^2}{N_p^2}.$$

Rewview this steap

Moreover we see that $S_p \leq N_p - L_p = S_p + I_p$, thus

$$\mathscr{L}[V_p(L_p)] \leq q \frac{\beta_p}{N_p - L_p} + \frac{(b + r_1)}{L_p} + \sigma_p^2 \left[\frac{1}{L_p} + \frac{L_p^2}{N_p^2 (N_p - L_p)}\right].$$

explain why

And this implies that

$$\mathscr{L}[V_p(L_p)] \leq \frac{b+r_1}{L_p} + \frac{\beta_p}{N_p - L_p} + \sigma_p^2 \left[\frac{1}{L_p} + \frac{1}{N_p - L_p}\right].$$

Now define $C := (b+r_1) \vee \beta_p + \sigma_p^2$, we obtain the following inequality

$$\mathscr{L}[V(L_p)] \leq CV_p(L_p). \tag{10}$$

By Itô's formula and applying expectation, we have, for any $t \in [0,T]$ and $k \geq k_1$

$$\mathbb{E}V(L_p(t \wedge \tau_k)) = V(L_p(0)) + \mathbb{E}\int_0^{t\wedge\tau_k} \mathscr{L}[V(L_p(s))]ds.$$

By equation (10) and Fubini's Theorem, we have

$$\mathbb{E}V(L_p(t \wedge \tau_k)) \leq V(L_p(0)) + C\int_0^t \mathbb{E}V(L_p(s \wedge \tau_k))ds.$$

Applying the Gronwall inequality yields that

$$\mathbb{E}V(L_p(t \wedge \tau_k)) \leq V(L_p(0))e^{CT}. \tag{11}$$

Set $\Omega_k = \{\omega : \tau_k \leq T\}$ for $k \geq k_1$, note that by relation in Equation 9, $\mathbb{P}(\Omega_k) > \varepsilon$. For every $\omega \in \Omega_k$, we have $L_p(t,\omega) \in \left(\frac{1}{k_0}, N_p - \frac{1}{k_0}\right)^{\complement}$, and hence

$$\begin{aligned} V_p(L_p(t,\omega)) &= \frac{1}{L_p} + \frac{1}{N_p - L_p} \\ &\geq k + \frac{1}{N_p - \frac{1}{k}} \\ &\geq k. \end{aligned}$$

It follows from equation (11), that

$$V_p(L_p(0))e^{CT} \geq \mathbb{E}\left[\mathbf{1}_{\{\Omega_k\}}(\omega)V_p(L_p(\tau_k,\omega))\right] \geq k\mathbb{P}(\Omega_k) \geq \varepsilon k.$$

Thus, letting $k \to \infty$ leads to the contradiction

$$\infty > V_p(L_p(0))e^{CT} \geq \infty.$$

Therfore we have $\tau_\infty = \infty$ a.s., and the proof is complete. $\square$

5 Disiease extinction

In this section we will study when the disease can be extinguished, for this we will give the necessary conditions so that this phenomenon can occur through two different cases. The first case will be when due to the intensity of the noise.The theorem presented below shows that under conditions on the parameters we can make the disease tend to become extinct.

Theorem 3 *[Extinction by noise] If*

$$\frac{\beta_p^2}{2\sigma_L^2} + \frac{r_2^2}{2\sigma_I^2} + 2\beta_p - r_1 < 0, \qquad \frac{\beta_v^2}{2\sigma_v^2} + \beta_v - \gamma + \theta\mu < 0,$$

then the disease will exponentially extinguish with probability one. That is, for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^\top \in \mathbb{R}_+^5$

$$\limsup_{t\infty\to\infty} \frac{1}{t}\ln(L_p + I_p) < 0 \text{ and } \limsup_{t\infty\to\infty} \frac{1}{t}\ln(I_v) < 0 \text{ a.s.}$$

Proof The main idea is apply the Itô formula to a conveniently function and deduce conditions. Let $V(S_p, L_p, I_p) = \ln(L_p + I_p)$, then the Itô formula gives

$$\begin{aligned}
d\ln(L_p + I_p) = &\left(\frac{1}{L_p + I_p}\right)\left(\frac{\beta_p}{N_v^\infty} S_p I_v - (b + r_1)L_p - \frac{1}{2}\sigma_L^2 \frac{L_p^2}{(L_p + I_p)^2}\right) dt \\
&- \sigma_L \frac{L_p}{L_p + I_p} dB_p(t) \\
\leq &\left(\frac{1}{L_p + I_p}\right)\left(\beta_p S_p - (b + r_1) - \frac{1}{2}\sigma_L^2 \frac{L_p^2}{(L_p + I_p)^2}\right) dt \\
&- \sigma_L \frac{L_p}{L_p + I_p} dB_p(t).
\end{aligned}$$

Let $x := \frac{L_p}{L_p + I_p}$, then

$$\begin{aligned}
d\ln(L_p + I_p) &\leq \left(\beta_p \frac{S_p}{L_p + I_p} - (b + r_1) - \frac{1}{2}\sigma_L^2 x^2\right) dt - \sigma_L x dB_p(t) \\
&\leq \left(\beta_p \frac{N_p}{L_p + I_p} - (b + r_1) - \frac{1}{2}\sigma_L^2 x^2\right) dt - \sigma_L x dB_p(t) \\
&\leq \left(\beta_p x + 2\beta_p - (b + r_1) - \frac{1}{2}\sigma_L^2 x^2\right) dt - \sigma_L x dB_p(t) \\
&= \left(-\frac{1}{2}\sigma_L^2 x^2 + \beta_p x + 2\beta_p - (b + r_1)\right) dt - \sigma_L x dB_p(t).
\end{aligned}$$

Hence,

$$\ln(L_p+I_p) \le -\frac{\sigma_L^2}{2}\int_0^t \left(\left(x-\frac{\beta_p}{\sigma_L^2}\right)^2+\frac{\beta_p^2}{2\sigma_L^2}+2\beta_p-(b+r_1)\right)du$$
$$-\int_0^t \sigma_L x dB_p(u)+\ln(L_p(0)+I_p(0)),$$

which implies,

$$\begin{aligned}\frac{1}{t}\ln(L_p+I_p) \le &-\frac{\sigma_L^2}{2t}\int_0^t \left(x-\frac{\beta_p}{\sigma_L^2}\right)^2 du+\frac{\beta_p^2}{2\sigma_L^2}-(b+r_1)+2\beta_p\\ &-\frac{1}{t}\int_0^t \sigma_L x dB_p(u)+\frac{1}{t}\ln(S_p(0)+L_p(0)+I_p(0)),\end{aligned} \tag{12}$$

let

$$M_t := \frac{1}{t}\int_0^t \sigma_L x dB_p(t)+\frac{1}{t}\ln(L_p(0)+I_p(0)).$$

Since the integral in the term M_t is a martingale, the strong law of large numbers for martingales [**?**], implies that

$$\lim_{t\to\infty} M_t = 0 \text{ a.s.}$$

Thus, from relation (12) we obtain

$$\limsup_{t\infty\to\infty}\frac{1}{t}\ln(L_p+I_p) < \frac{\beta_p^2}{2\sigma_L^2}+2\beta_p-(b+r_1) \tag{13}$$

A similar argument also shows that

$$\limsup_{t\infty\to\infty}\frac{1}{t}\ln(L_p+I_p) < \frac{r_2^2}{2\sigma_I^2}+b \tag{14}$$

Through the equations (13) and (14), we obtain

$$\limsup_{t\infty\to\infty}\frac{1}{t}\ln(L_p+I_p) < \frac{\beta_p^2}{2\sigma_L^2}+\frac{r_2^2}{2\sigma_I^2}+2\beta_p-r_1$$

and

$$\limsup_{t\infty\to\infty}\frac{1}{t}\ln(I_v) < \frac{\beta_v^2}{2\sigma_v^2}+\beta_v-\gamma+\theta\mu$$

Remark 1 Theorem 3 shows that, under certain conditions on the parameters can cause disease exponentially towards zero whenever the noise intensity is large enough.

The next case of extinction of the disease is through the basic reproductive number. For the deterministic case, defining the basic reproductive number is done using the next generation matrix [Van der drish], but in the stochastic case it is not possible to give such a definition.

To define the stochastic reproductive number we will use the techniques used in [Agwar], in which, by means of algebraic procedures, this parameter can be defined. As our deterministic base structure this paramenters summarizes the behavior of extinction and persistence according to a threshold.

Our analysis needs the following function and conditions.

(H-1) According to SDE (6), replatin rates satisfies $r = r_1 + r_2$.
(H-2) The replanting noise intesities are equal $\sigma_L = \sigma_I = \sigma_p$.

Given a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, define an operator $LV : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\mathscr{L}[V(x,t)] = V_t(x,t) + V_x(x,t) f(x,t) + \frac{1}{2}\operatorname{trace}(g^T(x,t) V_{xx}(x,t) g(x,t)) \tag{15}$$

which is called the diffusion operator of the Itô process associated with the $C^{2,1}$ function V. With this diffusion operator, the Itô formula can be written as

$$dV(x(t),t) = \mathscr{L}V(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t) \qquad a.s. \tag{16}$$

We define the repoductive nomber of our stochastic model in SDE (6) by

$$\mathscr{R}_0^s = \frac{\beta_p \beta_v}{\gamma r} \tag{17}$$

rename labels

Theorem 4 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of SDE (6) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p) \times (0, N_p) \times (0, N_p)$ $\times (0, N_v)$. If $0 \leq \mathscr{R}_0^s < 1$, then the following conditions holds*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathscr{R}_0^s - 1]I_p - rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma} I_v I_p \right] dr \leq \frac{1}{2}\sigma^2 N_p, \, a.s.,$$

namely, the infected individual tends to zero exponentially a.s, i.e the disease will die out with probability one.

Proof The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$V(S_p, L_p, I_p, I_v) = \left(S_p - S_p^0 - S_p^0 \ln\frac{S_p}{S_p^0}\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v^\infty} I_v,$$

Let f, g respectively be the dirft and difussion of SDE (11). Applyng the diffusion operator $\mathscr{L}$ we have

$$
\begin{aligned}
V_x f &= \left(1-\frac{S_p^0}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty}S_pI_v + rN_p - rS_p\right) + \frac{\beta_p}{N_v^\infty}S_pI_v - (b+r)L_p \\
&+ bL_p - rI_p + \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p}I_p - \frac{\beta_v}{N_v^\infty}I_vI_p - \gamma I_v\right) \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty}S_pI_v + \frac{\beta_p}{N_v^\infty}I_vS_p^0 + \frac{\beta_p}{N_v^\infty}S_pI_v - r(L_p+I_p) \\
&+ \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p}I_p - \frac{\beta_v}{N_v^\infty}I_vI_p - \gamma I_v\right) \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty}I_vS_p^0 - r(L_p+I_p) + \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v N_v}{N_p}I_p \\
&- \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_v^\infty}I_vI_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\gamma I_v
\end{aligned}
$$

Then,

$$
\begin{aligned}
V_x f &= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\gamma - \frac{\beta_p N_p}{N_v^\infty}\right]I_v + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v\frac{N_v^\infty}{N_p} - r\right]I_p \\
&- rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p}I_vI \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v\frac{N_v^\infty}{N_p} - r\right]I_p - rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p}I_vI_p \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\frac{\beta_p\beta_v}{\gamma r} - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p.
\end{aligned}
$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathscr{R}_0^s$ we get

$$
V_x f = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p.
$$

Moreover,

$$
\begin{aligned}
\frac{1}{2}\operatorname{trace}(g^T V_{xx} g) &= \frac{1}{2}\sigma^2 N_p\left(\frac{N_p - S_p}{S_p}\right)^2 \\
&\leq \frac{1}{2}\sigma^2 N_p.
\end{aligned}
$$

The stochastic terms are not necessary, because they do a martingale process and therefore, when we use integral and expectation they vanishing.

Combining the above terms, we obtain

$$\begin{aligned} dV(X) &= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p\left(\frac{N_p - S_p}{S_p}\right)^2 \\ &\le -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p. \end{aligned}$$

Define $\mathscr{L}V(X)$ as

$$\mathscr{L}V(X) = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p.$$

Using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$\begin{aligned} 0 &\le \mathbb{E}V(t) - \mathbb{E}V(0) \le \mathbb{E}\int_0^t LV(X(s))ds \\ &\le -\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - r\left[\mathscr{R}_0^s - 1\right]I_p + rL_p + \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p \right] ds + \frac{1}{2}\sigma^2 N_p \end{aligned}$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[-rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p \right] ds \le \frac{1}{2}\sigma^2 N_p.$$

Remark 2 Theorem 4 shows that, if the basic stochastic reproductive number $\mathscr{R}_0^s$ is less than one, we have the solutions $X(t) = (S_p(t), L_p(t), (t)I_p(t), S_v(t), I_v(t))^\top$ tend to the equilibrium point $(N_p, 0, 0, N_v^\infty, 0)^\top$, when $t \to \infty$.

Rewrite hypothesis according to this version. Stress the condition for noise intensities.

Theorem 5 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of SDE (6) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0,N_p)\times(0,N_p)\times(0,N_p)$ $\times(0,N_v)$. If $0 \le \mathscr{R}_0^s < 1$, then the following conditions holds*

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathscr{R}_0^s - 1]I_p - rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma}I_vI_p \right] dr \le \frac{1}{2}\sigma^2 N_p, \ , a.s.,$$

namely, the infected individual tends to zero exponentially a.s, i.e the disease will die out with probability one.

Proof The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$
\begin{aligned}
V(S_p, L_p, I_p, S_v, I_v) = &\left(S_p - N_p - N_p \ln \frac{S_p}{N_p}\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v^\infty} I_v \\
&+ \left(S_v - N_v - N_v \ln \frac{S_v}{N_v}\right).
\end{aligned}
$$

Let f, g respectively be the dirft and difussion of SDE (11). Applyng the inifinitesimal opreator $\mathscr{L}$ we have

$$
\begin{aligned}
V_x f = &\left(1 - \frac{N_p}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + r N_p - r S_p\right) + \frac{\beta_p}{N_v^\infty} S_p I_v - (b + r) L_p \\
&+ b L_p - r I_p + \left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1 - \theta)\mu\right) \\
&+ \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta \mu\right).
\end{aligned}
\tag{18}
$$

Expanded the first term and factoring the term S_p, we obtain

$$
\begin{aligned}
&\left(1 - \frac{N_p}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + r N_p - r S_p\right) \\
&\qquad = \left(1 - \frac{N_p}{S_p}\right)\left(-r S_p \left(1 - \frac{N_p}{S_p}\right) - \frac{\beta_p}{N_v^\infty} S_p I_v\right) \\
&\qquad = -r S_p \left(1 - \frac{N_p}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty} S_p I_v + \frac{\beta_p}{N_v^\infty} N_p I_v.
\end{aligned}
\tag{19}
$$

For the second term, since $(1-\theta)\mu \leq \gamma N_v$ we can bounded by the following rewrite

$$
\begin{aligned}
&\left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1 - \theta)\mu\right) \\
&\qquad \leq \left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + \gamma N_v\right) \\
&\qquad \leq \left(1 - \frac{N_v}{S_v}\right)\left(-\gamma S_v \left(1 - \frac{N_v}{S_v}\right) - \frac{\beta_v}{N_p} S_v I_p\right) \\
&\qquad \leq -\gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 - \frac{\beta_v}{N_p} S_v I_p + \frac{\beta_v}{N_p} N_v I_p .
\end{aligned}
\tag{20}
$$

Same way from above calculation, and since $\theta\mu \leq \theta\gamma N_v$, we obtain

$$
\begin{aligned}
\frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta \mu\right) &\leq \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta \gamma N_v\right) \\
&\leq \frac{\beta_p \beta_v S_v I_p}{\gamma N_v} - \frac{\beta_p N_p}{N_v^\infty} I_v + \beta_p \theta N_p.
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
V_x f \leq & -rS_p\left(1-\frac{N_p}{S_p}\right)^2+\left[\frac{\beta_p}{N_v^\infty}N_p-\frac{\beta_p N_p}{N_v^\infty}\right]I_v-r(L_p+I_p)\\
& -\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-\frac{\beta_v}{N_p}S_v I_p+\frac{\beta_v}{N_p}N_v I_p\\
& +\frac{\beta_p\beta_v S_v I_p}{\gamma N_v}+\beta_p\theta N_p\ .
\end{aligned}
$$

Moreover, since $S_v+I_v\leq N_v$, we can obtain the following relation

$$
\begin{aligned}
V_x f \leq & -rS_p\left(1-\frac{N_p}{S_p}\right)^2-r(L_p+I_p)\\
& -\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2+\frac{\beta_v}{N_p}I_v I_p\\
& +\frac{\beta_p\beta_v I_p}{\gamma}-\frac{\beta_p\beta_v I_v I_p}{\gamma N_v}+\beta_p\theta N_p\ .
\end{aligned}
$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathscr{R}_0^s$ we get

$$
\begin{aligned}
V_x f = & -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2-\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-rL_p-r\left[1-\mathscr{R}_0^s\right]I_p\\
& -\left[\frac{\beta_p\beta_v}{\gamma N_v^\infty}-\frac{\beta_v}{N_p}\right]I_v I_p-\frac{\beta_v}{N_p}S_v I_p+\beta_p\theta N_p\ .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{1}{2}\operatorname{trace}(g^T V_{xx} g) &= \frac{1}{2}\frac{(\sigma_p(L_p+I_p))^2}{N_p}+\frac{1}{2}\sigma_v^2 N_v\\
&\leq \frac{1}{2}\sigma_p^2 N_p+\frac{1}{2}\sigma_v^2 N_v\ .
\end{aligned}
$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising. Incorporation all terms calculate above, we obtain

$$
\begin{aligned}
\mathscr{L}V(X) \leq & -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2-\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-rL_p-r\left[1-\mathscr{R}_0^s\right]I_p\\
& -\left[\frac{\beta_p\beta_v}{\gamma N_v^\infty}-\frac{\beta_v}{N_p}\right]I_v I_p-\frac{\beta_v}{N_p}S_v I_p+\beta_p\theta N_p+\frac{1}{2}\sigma_p^2 N_p+\frac{1}{2}\sigma_v^2 N_v\ .
\end{aligned}
$$

Define $\sigma_{p,v} := \beta_p \theta N_p + \frac{1}{2}\sigma_p^2 N_p + \frac{1}{2}\sigma_v^2 N_v$, then

$$\mathscr{L}V(X) \le -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 - rL_p - r\left[1-\mathscr{R}_0^s\right] I_p \\ - \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} - \frac{\beta_v}{N_p}\right] I_v I_p - \frac{\beta_v}{N_p} S_v I_p + \sigma_{p,v} \ .$$

Since $V(x) \ge 0$, using the integral form of Itô's formula and taking expectation yields

$$0 \le \mathbb{E}V(t) - \mathbb{E}V(0) \le \mathbb{E}\int_0^t \mathscr{L}V(X(s))ds \\ \le -\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathscr{R}_0^s\right] I_p \right. \\ \left. + \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p - \sigma_{p,v}\right] ds \ .$$

Therefore,

$$\frac{1}{t}\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathscr{R}_0^s\right] I_p \right. \\ \left. + \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p\right] ds \le \sigma_{p,v} \ .$$

This implies that,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathscr{R}_0^s\right] I_p \right. \\ \left. + \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p\right] ds \le \sigma_{p,v}$$

Taking θ, σ_p, and σ_v such that $0 < \sigma_{p,v} < 1$, we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathscr{R}_0^s\right] I_p \right. \\ \left. + \left(\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right) I_v I_p + \frac{\beta_v}{N_p} S_v I_p\right] ds \le \log\sigma_{p,v} < 0 \ .$$

Therefore,

$$\lim_{t\to\infty}\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathscr{R}_0^s\right] I_p + \right. \\ \left. \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p\right] ds \le \lim_{t\to\infty} e^{\sigma_{p,v}t} = 0 \ .$$

Thus, letting $t \to \infty$ we obtain

rewrite

$$\lim_{t\to\infty}(S_p, L_p, I_p, S_v, I_v)_t^\top = (N_p, 0, 0, N_v, 0)$$

exponentially a.s.

6 Persistence

In the case of deterministic models, one of the problems taken into account is to determine under what conditions the endemic equilibrium point is attractor or asymptotically stable. In the case of stochastic models, said endemic equilibrium point is not an equilibrium point. To determinate the persistence in the stochastic cases, we use the following definition.

So how do we determine if the disease is going to persist? In this section we will give the conditions under which the difference between the solution of the system (6) and $(S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$ is small if the noise is weak, reflecting that the disease is prevalent.

Definition 1 Let $x = (S_p, L_p, I_p, S_v, I_v)^\top$ be the solution of system (6). We said that this solution process is persistent in mean if

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(r)dr > 0, \qquad a.s. \tag{22}$$

For establish the persistent of the endemic equilibrium point of the system (6), we need consider the opposite conditions of Theorem *Our analysis require the following hypothesis.

Who is theorem 2. Change label to a explicit meanin not fucking numbers

(A) According to Theorem we need consider

$$\frac{\beta_p^2 + r^2}{2\sigma_p^2} + 2\beta_p - r > 0,$$

(B) and

$$\frac{\beta_v^2}{2\sigma_v^2} + \beta_v - \gamma + \theta\mu > 0$$

.

The following Theorem gives a upper bounds for the system (6).

Disucssuion about relation between R_0^D R_0^S

Theorem 6 *Let $R_0^d > 1$ and conditions (A)-(B) holds. Consider the endemic deterministic fixed point $(S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$. Then*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\int_0^t (r(1-2\rho_1)\left((S_p - S_p^*)^2 + (L_p - L_p^*)^2 + (I_p - I_p^*)^2\right) \\
&+ \gamma(1-2\rho_2)(S_v - S_v)^2 - \gamma\left(1 - \frac{1}{4\rho_2}\right)(I_v - I_v)^2)ds \\
&\leq K_2\alpha_1 + K_1\alpha_2 + \frac{1}{2}\left(\sigma_p^2(L_p^* K_1 + I_p^* K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) \qquad a.s.
\end{aligned} \tag{23}$$

where $K_1 = \frac{N_p^2}{L_p^*}$, $K_2 = \frac{N_p^2}{I_p^*}$, $\rho_1 \in (0, \frac{1}{2})$ and $\rho_2 \in (\frac{1}{4}, \frac{1}{2})$.

Proof By hypothesis $(S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$ is the endemic equilibrium of system (1), we have

$$\begin{aligned} rN_p &= rS_p^* + \frac{\beta_p}{N_v} S_p^* I_v^*, \qquad \frac{\beta_p}{N_v} S_p^* I_v^* = (b+r)L_p^*, \\ bL_p^* &= rI_p^*, \qquad (1-\theta)\mu = \frac{\beta_v}{N_p} S_v^* I_p^* + \gamma S_v^*, \\ \theta\mu &= \gamma I_v^* - \frac{\beta_v}{N_p} S_v^* I_p^*. \end{aligned} \tag{24}$$

Let consider the following Lyapunov function

$$\begin{aligned} V(S_p, L_p, I_p, S_v, I_v) = K_1 \left(L_p - L_p^* - L_p^* \log\left(\frac{L_p}{L_p^*}\right)\right) + K_2 \left(I_p - I_p^* - I_p^* \log\left(\frac{I_p}{I_p^*}\right)\right) \\ + \frac{1}{2}\left((S_p - S_p^*) + (L_p - L_p^*) + (I_p - I_p^*)\right)^2 + \frac{1}{2}\left((S_v - S_v^*) + (I_v - I_v^*)\right)^2 \end{aligned}$$

We can rename the Lyapunov function as the follows

$$V(S_p, L_p, I_p, S_v, I_v) = K_1 V_1 + K_2 V_2 + V_3 + V_4, \tag{25}$$

and we work with each V_i. For V_1, we have

$$\begin{aligned} \mathscr{L}V_1 &= \left(1 - \frac{L_P^*}{L_p}\right)\left(\frac{\beta_p}{N_v} S_p I_v - (b+r)L_p\right) + \frac{1}{2}\frac{\sigma_p^2 S_p^2 L_p^*}{N_p^2} \\ &= \left(1 - \frac{L_P^*}{L_p}\right)\left(\frac{\beta_p}{N_v} S_p I_v - \frac{\beta_p}{N_v} S_p^* I_v^* \frac{L_p}{L_p^*}\right) + \frac{1}{2}\frac{\sigma_p^2 S_p^2 L_p^*}{N_p^2} \\ &= \frac{\beta_p}{N_v}\left(1 - \frac{L_P^*}{L_p}\right)\left(S_p I_v - \frac{S_p^* I_v^* L_p}{L_p^*}\right) + \frac{1}{2}\frac{\sigma_p^2 S_p^2 L_p^*}{N_p^2} \\ &= \frac{\beta_p}{L_p N_v}\left(L_p - L_p^*\right)\left(S_p I_v - S_p^* I_v^* \frac{L_p}{L_p^*}\right) + \frac{1}{2}\frac{\sigma_p^2 S_p^2 L_p^*}{N_p^2}. \end{aligned}$$

Now, for V_2 we have

$$\begin{aligned} \mathscr{L}V_2 &= \left(1 - \frac{I_P^*}{I_p}\right)(bL_p - rI_p) + \frac{1}{2}\frac{\sigma_p^2 S_p^2 I_p^*}{N_p^2} \\ &= \frac{1}{I_p}(I_p - I_p^*)\left(\frac{rI_p^*}{L_p^*} - rI_p\right) + \frac{1}{2}\frac{\sigma_p^2 S_p^2 I_p^*}{N_p^2} \\ &= -\frac{r}{I_p}(I_p - I_p^*)\left(I_p - \frac{I_p^*}{L_p^*}\right) + \frac{1}{2}\frac{\sigma_p^2 S_p^2 I_p^*}{N_p^2}. \end{aligned}$$

For V_3, we obtain

$$\begin{aligned}
\mathscr{L}V_3 = &\left((S_p - S_p^*) + (L_p - L_p^*) + (I_p - I_p^*)\right)\left(-\frac{\beta_p}{N_v} S_p I_v + rN_p - rS_p\right. \\
&\left. + \frac{\beta_p}{N_v} S_p I_v - (b+r)L_p + bL_p - rI_p\right) + \sigma_p^2 N_p^2 \\
= &\left((S_p - S_p^*) + (L_p - L_p^*) + (I_p - I_p^*)\right)(rN_p - rS_p - rL_p - rI_p) \\
&+ \sigma_p^2 N_p^2 \\
= &\left((S_p - S_p^*) + (L_p - L_p^*) + (I_p - I_p^*)\right)(rI_p^* + rL_p^* + rS_p^* - rS_p - rL_p - rI_p) + \sigma_p^2 N_p^2 \\
= &\left((S_p - S_p^*) + (L_p - L_p^*) + (I_p - I_p^*)\right)(-r(S_p - S_p^*) - r(L_p - L_p^*) - r(I_p - I_p^*)) + \sigma_p^2 N_p^2 \\
= &-r\left((S_p - S_p^*) + (L_p - L_p^*) + (I_p - I_p^*)\right)^2 + \sigma_p^2 N_p^2.
\end{aligned}$$

For the last function V_4, we have

$$\begin{aligned}
\mathscr{L}V_4 = &((S_v - S_v^*) + (I_v - I_v^*))\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1-\theta)\mu\right. \\
&\left. + \frac{\beta_v}{N_p} S_v I_p - \gamma I_v + \theta\mu\right) + \frac{3}{2}\sigma_v^2 N_v^2 \\
= &((S_v - S_v^*) + (I_v - I_v^*))(-\gamma S_v + \gamma S_v - \gamma I_v + \gamma I_v^*) + \frac{3}{2}\sigma_v^2 N_v^2 \\
= &((S_v - S_v^*) + (I_v - I_v^*))(-\gamma(S_v - S_v^*) - \gamma(I_v - I_v^*)) + \frac{3}{2}\sigma_v^2 N_v^2 \\
= &-\gamma((S_v - S_v^*) + (I_v - I_v^*))^2 + \frac{3}{2}\sigma_v^2 N_v^2.
\end{aligned}$$

Then, we can bound the diffusion operator as follows

$$\begin{aligned}
\mathscr{L}V \leq &-r\left((S_p - S_p^*) + (L_p - L_p^*) + (I_p - I_p^*)\right)^2 - \gamma((S_v - S_v) + (I_v - I_v))^2 \\
&- \frac{K_2 r}{I_p}(I_p - I_p^*)\left(I_p - \frac{I_p^*}{L_p^*}\right) + \frac{\beta_p K_1}{N_v L_p}(L_p - L_p^*)\left(S_p I_v - S_p^* I_v^* \frac{L_p}{L_p^*}\right) \\
&+ \frac{1}{2}\left(\sigma_p^2(L_p^* K_1 + I_p^* K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right)
\end{aligned}$$

We need bound the term, $-\frac{K_2 r}{I_p}(I_p - I_p^*)\left(I_p - \frac{I_p^*}{L_p^*}\right)$, then

$$\begin{aligned}
-\frac{K_2 r}{I_p}(I_p - I_p^*)\left(I_p - \frac{I_p^*}{L_p^*}\right) &= -\frac{K_2 r}{I_p}\left(I_p^2 - \frac{I_p I_p^*}{L_p} - I_p^* I_p + \frac{I_p^{*2}}{L_p^*}\right) \\
&= -K_2 r\left(I_p - \frac{I_p^*}{L_p^*} - I_p^* + \frac{I_p^{*2}}{I_p L_p^*}\right) \\
&\leq K_2 r\left(\frac{I_p^*}{L_p^*} + I_p^*\right).
\end{aligned}$$

Define $\alpha_1 := r\left(\frac{I_p^*}{L_p^*} + I_p^*\right)$, then

$$-\frac{K_2 r}{I_p}(I_p - I_p^*)\left(I_p - \frac{I_p^*}{L_p^*}\right) \leq K_2\alpha_1.$$

Now the term $\frac{\beta_p K_1}{N_v L_p}(L_p - L_p^*)\left(S_p I_v - S_p^* I_v^* \frac{L_p}{L_p^*}\right)$ can be bound as

$$\begin{aligned}
\frac{\beta_p K_1}{N_v L_p}(L_p - L_p^*)\left(S_p I_v - S_p^* I_v^* \frac{L_p}{L_p^*}\right) &= \frac{\beta_p K_1}{N_v L_p}\left(L_p S_p I_v - S_p^* I_v^* \frac{L_p^2}{L_p^*} - L_p^* S_p I_v + S_p^* I_v^* L_p\right) \\
&= \frac{\beta_p K_1}{N_v}\left(S_p I_v - S_p^* I_v^* \frac{L_p}{L_p^*} - \frac{L_p^*}{L_p} S_p I_v + S_p^* I_v^*\right) \\
&\leq \frac{\beta_p K_1}{N_v}\left(S_p I_v - S_p^* I_v^*\right).
\end{aligned}$$

Since $S_p, S_p^* \leq N_p$ and $I_v, I_v^* \leq N_v$, this imply that

$$\frac{\beta_p K_1}{N_v L_p}(L_p - L_p^*)\left(S_p I_v - S_p^* I_v^* \frac{L_p}{L_p^*}\right) \leq 2\frac{\beta_p K_1 N_p}{N_v}.$$

Define $\alpha_2 := 2\frac{\beta_p N_p}{N_v}$, then

$$\frac{\beta_p K_1}{N_v L_p}(L_p - L_p^*)\left(S_p I_v - S_p^* I_v^* \frac{L_p}{L_p^*}\right) \leq K_1\alpha_2.$$

Therefore we can bound the diffusion operator $\mathscr{L}V$ as follows

$$\begin{aligned}
\mathscr{L}V &\leq -r\left((S_p - S_p^*) + (L_p - L_p^*) + (I_p - I_p^*)\right)^2 - \gamma\left((S_v - S_v) + (I_v - I_v)\right)^2 \\
&\quad + K_2\alpha_1 + K_1\alpha_2 + \frac{1}{2}\left(\sigma_p^2(L_p^* K_1 + I_p^* K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) \\
&\leq -3r(S_p - S_p^*)^2 - 3r(L_p - L_p^*)^2 - 3r(I_p - I_p^*)^2 - 2\gamma(S_v - S_v)^2 - 2\gamma(I_v - I_v)^2 \\
&\quad + K_2\alpha_1 + K_1\alpha_2 + \frac{1}{2}\left(\sigma_p^2(L_p^* K_1 + I_p^* K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right).
\end{aligned}$$

By the Young's inequality we obtain that,

$$\begin{aligned}
\mathscr{L}V &\leq -r\left(1 - \frac{1}{2\rho_1} - 2\rho_1\right)\left((S_p - S_p^*)^2 + (L_p - L_p^*)^2 + (I_p - I_p^*)^2\right) \\
&\quad - \gamma(1 - 2\rho_2)(S_v - S_v)^2 - \gamma\left(1 - \frac{1}{4\rho_2}\right)(I_v - I_v)^2 \\
&\quad + K_2\alpha_1 + K_1\alpha_2 + \frac{1}{2}\left(\sigma_p^2(L_p^* K_1 + I_p^* K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) \\
&\leq -r(1 - 2\rho_1)\left((S_p - S_p^*)^2 + (L_p - L_p^*)^2 + (I_p - I_p^*)^2\right) \\
&\quad - \gamma(1 - 2\rho_2)(S_v - S_v)^2 - \gamma\left(1 - \frac{1}{4\rho_2}\right)(I_v - I_v)^2 \\
&\quad + K_2\alpha_1 + K_1\alpha_2 + \frac{1}{2}\left(\sigma_p^2(L_p^* K_1 + I_p^* K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right).
\end{aligned}$$

Define $F(t)$ as

$$
\begin{aligned}
F(t) := & -r(1-2\rho_1)\left((S_p - S_p^*)^2 + (L_p - L_p^*)^2 + (I_p - I_p^*)^2\right) \\
& - \gamma(1-2\rho_2)(S_v - S_v)^2 - \gamma\left(1 - \frac{1}{4\rho_2}\right)(I_v - I_v)^2 \\
& + K_2\alpha_1 + K_1\alpha_2 + \frac{1}{2}\left(\sigma_p^2(L_p^* K_1 + I_p^* K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right),
\end{aligned}
$$

therefore

$$
\begin{aligned}
dV \leq F(t)dt + \left(\frac{S_p(\sigma_p L_p + \sigma_p I_p)}{N_p}\right)\left(1 - \frac{N_p}{S_p} - \frac{\sigma_p S_p L_p}{N_p} - \frac{\sigma_p S_p I_p}{N_p}\right) dB_p(t) \\
- \frac{\sigma_v I_v \beta_p N_p dB_v(t)}{\gamma N_v}
\end{aligned}
$$

Integrating both sides from 0 to t yields

$$
\begin{aligned}
V_3(t) - V_3(0) \leq & \int_0^t F(s)ds + \\
& \int_0^t \left(\frac{S_p(\sigma_p L_p + \sigma_p I_p)}{N_p}\left(1 - \frac{N_p}{S_p}\right) - \frac{\sigma_p S_p L_p}{N_p} - \frac{\sigma_p S_p I_p}{N_p}\right) dB_p(s) \\
& - \int_0^t \frac{\sigma_v I_v \beta_p N_p}{\gamma N_v} dB_v(s)
\end{aligned}
$$

Let

$$
\begin{aligned}
M_1(t) &:= \int_0^t \left(\frac{S_p(\sigma_p L_p + \sigma_p I_p)}{N_p}\left(1 - \frac{N_p}{S_p}\right) - \frac{\sigma_p S_p L_p}{N_p}\frac{\sigma_p S_p I_p}{N_p}\right) dB_p(s), \\
M_2(t) &:= \int_0^t \frac{\sigma_v I_v \beta_p N_p dB_v(s)}{\gamma N_v}
\end{aligned}
$$

and compute their quadratic variatiion, then

$$
\begin{aligned}
M_1(t) &:= \int_0^t \left(\frac{S_p(\sigma_p L_p + \sigma_p I_p)}{N_p}\left(1 - \frac{N_p}{S_p}\right) - \frac{\sigma_p S_p L_p}{N_p}\frac{\sigma_p S_p I_p}{N_p}\right) dB_p(s) \\
&\leq \int_0^t \left(\frac{S_p(\sigma_p L_p + \sigma_p I_p)}{N_p}\left(1 - \frac{N_p}{S_p}\right)\right) dB_p(s) \\
&\leq \int_0^t \left(\frac{\sigma_p S_p(L_p + I_p)}{N_p}\left(\frac{S_p - N_p}{S_p}\right)\right) dB_p(s) \\
&\leq \int_0^t \left(-\frac{\sigma_p S_p(L_p + I_p)}{N_p}\left(\frac{L_p + I_p}{S_p}\right)\right) dB_p(s) \\
&\leq \int_0^t 4\sigma_p N_p dB_p(s).
\end{aligned}
$$

Similar for $M_2(t)$, we obtain

$$
M_2(t) \leq \int_0^t \sigma_v \beta_p N_p dB_v(s),
$$

which are local continuous bounded martingale and $M_1(0) = M_2(0) = 0$ with quadratic variation finite. Then by Theorem 1.3.4 of [Mao's Book], we obtain

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0, \qquad \text{a.s., and}$$

$$\lim_{t\to\infty} \frac{M_2(t)}{t} = 0, \qquad \text{a.s.,}$$

by the lim inf and lim sup properties we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t F(s)ds \geq 0 \qquad \text{a.s.}$$
$$-\limsup_{t\to\infty} \frac{1}{t}\int_0^t -F(s)ds \geq 0 \qquad \text{a.s.,}$$

thus

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t -F(s)ds \leq 0 \qquad \text{a.s.}$$

Consequently,

check term like $(S_v - S_v)^2$

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\int_0^t &(r(1-2\rho_1)\left((S_p - S_p^*)^2 + (L_p - L_p^*)^2 + (I_p - I_p^*)^2\right)\\ &+\gamma(1-2\rho_2)(S_v - S_v)^2 - \gamma\left(1 - \frac{1}{4\rho_2}\right)(I_v - I_v)^2)ds\\ &\leq K_2\alpha_1 + K_1\alpha_2 + \frac{1}{2}\left(\sigma_p^2(L_p^* K_1 + I_p^* K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) \qquad \text{a.s.}\end{aligned}$$

Remark 3 The Theorem 6 shows that, under some conditions, the distance between the solution $X(t) = (S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top$ and the fixed point $X^* = (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$ of system (1) has the following form:

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \|X(s) - X^*\|^2 ds \leq C_1 + C_2\|\sigma\|^2, \qquad a.s.,$$

where C_1, C_2 are positive constants. Although the solution ofsystem (6) does not have stability as the deterministic system, we obtain oscillations around deterministic fixed point [*] provided $C_1 + C_2\|\sigma\|^2$ is sufficiently small. In this context, we consider the disease to persist.

(C) According to Theorem 6, we need consider

$$\begin{aligned}&\min\left\{r(1-2\rho_1)(S_p^*)^2, r(1-2\rho_1)(L_p^*)^2, r(1-2\rho_1)(I_p^*)^2,\right.\\ &\left.\gamma(1-2\rho_2)(S_v^*)^2, \gamma\left(1 - \frac{1}{4\rho_2}\right)(I_v^*)^2\right\}\\ &\geq K_1\alpha_1 + K_2\alpha_2 + \frac{1}{2}\left(\sigma_p^2\left(K_1 L_p^* + K_2 I_p^* + 2N_p^2\right) + 3\sigma_v^2 N_v^2\right).\end{aligned}$$

Write the remain expresions as the first

Theorem 7 *Let $R_0^d > 1$ and conditions (A)-(C) holds. Then*

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t S_p dr \geq \frac{S_p^*}{2} - \frac{K_1\alpha_1 + K_2\alpha_2}{r(1-2\rho_1)} + \\ \frac{\sigma_p^2\left(K_1L_p^* + K_2I_p^* + 2N_p^2\right) + 3\sigma_v^2N_v^2}{2r(1-2\rho_1)}\end{aligned} \tag{26}$$

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t L_p dr \geq \frac{L_p^*}{2} - \frac{1}{r(1-2\rho_1)}\left(K_1\alpha_1 + K_2\alpha_2\right.\\ &\left.+\frac{1}{2}\left(\sigma_p^2\left(K_1L_p^* + K_2I_p^* + 2N_p^2\right) + 3\sigma_v^2N_v^2\right)\right) \qquad a.s.\end{aligned} \tag{27}$$

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_p dr \geq \frac{I_p^*}{2} - \frac{1}{r(1-2\rho_1)}\left(K_1\alpha_1 + K_2\alpha_2\right.\\ &\left.+\frac{1}{2}\left(\sigma_p^2\left(K_1L_p^* + K_2I_p^* + 2N_p^2\right) + 3\sigma_v^2N_v^2\right)\right) \qquad a.s.\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t S_v dr \geq \frac{S_v^*}{2} - \frac{1}{\gamma(1-2\rho_2)}\left(K_1\alpha_1 + K_2\alpha_2\right.\\ &\left.+\frac{1}{2}\left(\sigma_p^2\left(K_1L_p^* + K_2I_p^* + 2N_p^2\right) + 3\sigma_v^2N_v^2\right)\right) \qquad a.s.\end{aligned} \tag{29}$$

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_v dr \geq \frac{I_v^*}{2} - \frac{1}{\gamma(1-\frac{1}{4\rho_2})}\left(K_1\alpha_1 + K_2\alpha_2\right.\\ &\left.+\frac{1}{2}\left(\sigma_p^2\left(K_1L_p^* + K_2I_p^* + 2N_p^2\right) + 3\sigma_v^2N_v^2\right)\right) \qquad a.s.\end{aligned} \tag{30}$$

Proof By the hypothesis of Theorem 7, we have that inequality (23) is satisfied. Then, we have the follows bounds

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\int_0^t (r(1-2\rho_1)\left(S_p - S_p^*\right)^2)dr &\leq K_2\alpha_1 + K_1\alpha_2\\ &+\frac{1}{2}\left(\sigma_p^2(L_p^*K_1 + I_p^*K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right)\end{aligned}$$

Besides,

$$(S_p^*)^2 + (S_p - S_p^*)^2 \geq 2(S_p^*)(S_p^* - S_p).$$

this implies,

$$S_p \geq \frac{S_p^*}{2} - \frac{(S_p - S_p^*)^2}{2S_p^*}.$$

Therefore,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t S_p dr \geq \frac{S_p^*}{2} - \limsup_{t\infty\to\infty}\int_0^t \frac{(S_p - S_p^*)^2}{2S_p^*}dr$$
$$\geq \frac{S_p^*}{2} - \frac{1}{r(1-2\rho_1)}\left(K_2\alpha_1 + K_1\alpha_2 + \right.$$
$$\left.\frac{1}{2}\left(\sigma_p^2(L_p^*K_1 + I_p^*K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) > 0 \qquad \text{a.s.}$$

Similarly, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t L_p dr \geq \frac{L_p^*}{2} - \frac{1}{r(1-2\rho_1)}\left(K_2\alpha_1 + K_1\alpha_2\right.$$
$$\left.+\frac{1}{2}\left(\sigma_p^2(L_p^*K_1 + I_p^*K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) > 0 \qquad \text{a.s.}$$

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_p dr \geq \frac{I_p^*}{2} - \frac{1}{r(1-2\rho_1)}\left(K_2\alpha_1 + K_1\alpha_2\right)$$
$$+\frac{1}{2}\left(\sigma_p^2(L_p^*K_1 + I_p^*K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) > 0$$

and for the vector population, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t S_v dr \geq \frac{S_v^*}{2} - \limsup_{t\to\infty}\int_0^t \frac{(S_v - S_v^*)^2}{2(S_v^*)}dr$$
$$\geq \frac{S_v^*}{2} - \frac{1}{\gamma(1-2\rho_2)}\left(K_2\alpha_1 + K_1\alpha_2 + \frac{1}{2}\left(\sigma_p^2(L_p^*K_1 + I_p^*K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) > 0\right.$$

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_v dr \geq \frac{I_v^*}{2} - \frac{1}{\gamma(1-\frac{1}{4\rho_2})}\left(K_2\alpha_1 + K_1\alpha_2\right.$$
$$\left.+\frac{1}{2}\left(\sigma_p^2(L_\pi^*K_1 + I_p^*K_2 + 2N_p^2) + 3N_v^2\sigma_v^2\right) > 0$$

Remark 4 Theorem 7 shows that, under the assumptions (A)-(C) and the bonud of Theorem 6, the mean of the populations is almost certainly positive. And therefore by the definition 1 we have that the system (6) is persistent in mean.

Theorem 8 *Let $R_0^d > 1$ and conditions (A)-(C) holds. Then the system (6) is persistent in mean.*

Proof The prove of this Theorem is applied the Theorem 6 and 7 and use the definition 1 . ∎

improve redaction

7 Numerical Results

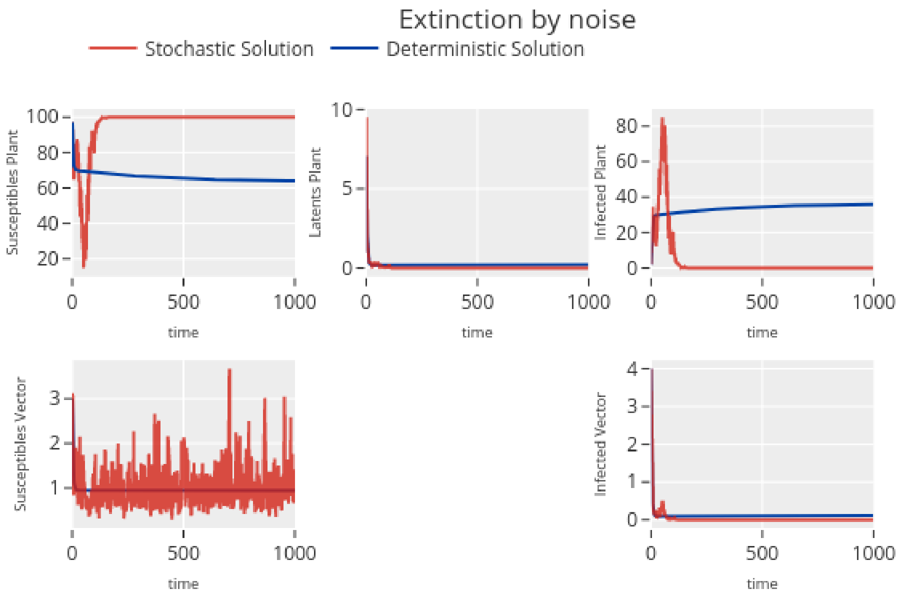

Fig. 1 https://plotly.com/ AdrianSalcedo/2/

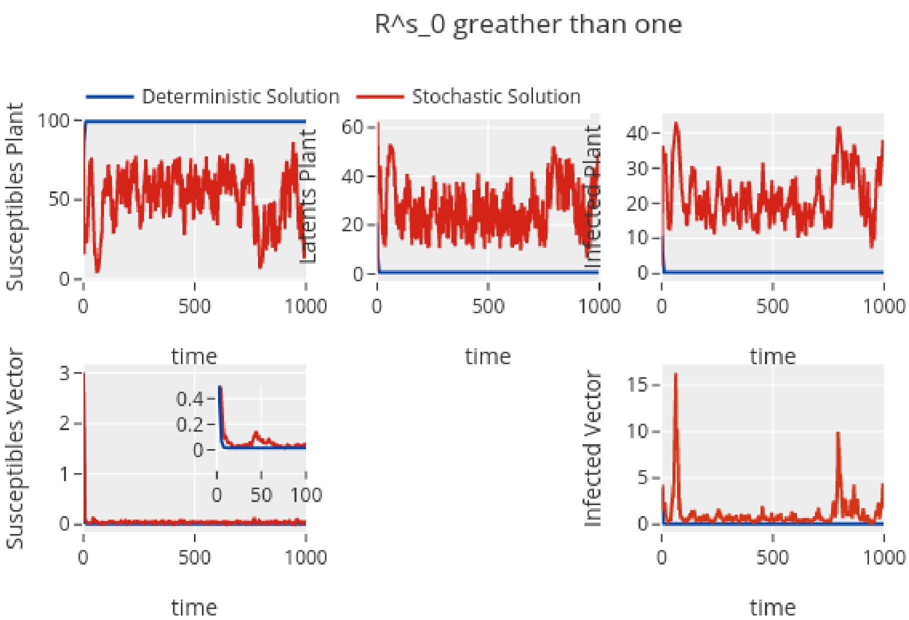

Fig. 2 Caption two

8 Conclusion

A Background

add a tex source with the appendix files