"El saber de mis hijos hará mi grandeza"

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Chapter 1

Existence theory for optimal policies

In this chapter, we will give the important results for the proof of the theorem of existence of optimal control pairs. In addition to presenting some problems related to these optimal pairs [1].

In what follows, we denote $\mathbb{R}_+ = [0, \infty)$. For any initial pair $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, consider the following ordinary differential equation:

$$\begin{cases} \dot{X}(s) = f(s, u(s), X(s) \; s \in [t, \infty), \\ X(t) = x, \end{cases} \tag{1.1}$$

Where $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is a given map. In the above, $X(\cdot)$ is called the state trajectory, taking values in $\mathbb{R}^n$ and $u(\cdot)$ is called the control, taking values in some metric space U. We call (1.1) a control system.

For any $0 \leq t < T < \infty$, we define the following:

$$\mathcal{U}[t, T] = \{u : [t, T] \to U | u(\cdot) \text{ is measurable}\},$$

and

$$\mathcal{U}[t, \infty) = \{u : [t, \infty) \to U | u(\cdot) \text{ is measurable}\}.$$

Any $u(\cdot) \in \mathcal{U}[t,T]$, respectively $u(\cdot) \in \mathcal{U}[t,\infty)$, is called a feasible control on $[t,T]$, respectively $[t,\infty)$.

Under proper conditions, for any initial pair (t,x), and feasible control $u(\cdot)$, (1.1) admits a unique solution $X(\cdot) = X(\cdot;t,x,u(\cdot))$ deined on $[t,\infty)$. Clearly different choices of $U(\cdot)$ will result in different state trajectories $X(\cdot)$. We reer to $(u(\cdot),X(\cdot))$ as a state-control pair of the control system (1.1).

Next, we recall that $2^{\mathbb{R}^n}$ is the set of all subsets of $\mathbb{R}^n$. Any map $M : \mathbb{R}_+ \to 2^{\mathbb{R}^n}$ is called a moving target in $\mathbb{R}^n$ i for any $t \in \mathbb{R}_+$, $M(t)$ is a measurable set in $\mathbb{R}^n$. WE allow $M(t)$ to be empty for some or all t, which will give us some fleaxibility below. In most situations, for any $t \in \mathbb{R}_+$, $M(t)$ is assumed to be closed or open.

Problem (C). Let $M(\cdot)$ be a moving target set in $\mathbb{R}^n$. For given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, ind a control $u(\cdot) \in \mathcal{U}[t,\infty)$ such that for some $\tau \geq t$,

$$X(\tau;t,x,u(\cdot)) \in M(\tau).$$

The above is called a controllability problem for the system (1.1) with the moving target set $M(\cdot)$. For a moving set $M(\cdot)$ in $\mathbb{R}^n$, and $T \in (0\infty)$, we define

$$\mathcal{U}_x^{M(\cdot)}[t,T] = \{u(\cdot) \in \mathcal{U}[t,\infty) | X(\tau;t,x,u(\cdot)) \in M(\tau), \text{ for some } \tau \in [t,T]\},\ (t,x) \in [0,T] \times \mathbb{R}^n,$$

and

$$\mathcal{U}_x^{M(\cdot)}[t,\infty) = \bigcup_{T \geq t} \mathcal{U}_x^{M(\cdot)}[t,T],\ \text{for every}\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Let $M \subseteq \mathbb{R}^n$ be non-empty and closed, $T \in (0,\infty)$, and let

$$M(t) = \emptyset I_{\mathbb{R}_+ \setminus T}(t) + M I_{\{T\}}(t),\ \ t \in \mathbb{R}_+.$$

Then, for any $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$\mathcal{U}_x^{M(\cdot)}[t,T] = \{u(\cdot) \in \mathcal{U}[t,T] | X(T;t,x,u(\cdot)) \in M\} \equiv \tilde{\mathcal{U}}_x^M[t,T].$$

And define the cost functional

$$J(t,x;u(\cdot)) = \int_t^T g(s,u(s),X(s))ds + h(T,X(T)) \equiv J^T(t,x,u(\cdot)).$$

For some maps $g(\cdot)$ and $h(\cdot)$. Now we introduce the following problem.

Problem $(OC)^T$. For given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ with $\tilde{\mathcal{U}}_x^M[t,T] \neq \emptyset$, find a $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]$ such that

$$J^T(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\tilde{\mathcal{U}}_x^M[t,T]} J^T(t,x;u(\cdot)) \equiv V(t,x).$$

The above is called an optimal control problem, with a fixed terminal time and a terminal state constraint.

1.1 Background

In this section, we present some results which will be used in the following sections. These results are presented mostly in [1].

Let us introduce some spaces. For any $0 \leq t < T < \infty$ and $1 \leq p < \infty$, define

$$C([t,T];\mathbb{R}^n) = \{\varphi : [t,T] \to \mathbb{R}^n | \varphi(\cdot) \text{ is continuous}\},$$

$$L^\infty(t,T;\mathbb{R}^n) = \{\varphi : [t,T] \to \mathbb{R}^n | \varphi(\cdot) \text{ measurable, } \operatorname{esssup}_{s\in[t,T]}|\varphi(s)| < \infty\},$$

$$L^p(t,T;\mathbb{R}^n) = \{\varphi : [t,T] \to \mathbb{R}^n | \varphi(\cdot) \, \text{measurable}, \int_t^T |\varphi(s)|^p ds < \infty\},$$

which are Banach space under the following norms, respectively,

$$||\varphi(\cdot)||_{C([t,T];\mathbb{R}^n)} = \sup_{s\in[t,T]} |\varphi(s)|, \text{ for every}\, \varphi(\cdot) \in C([t,T];\mathbb{R}^n),$$

$$||\varphi(\cdot)||_{L^\infty(t,T;\mathbb{R}^n)} = \text{esssup}_{s\in[t,T]}|\varphi(s)|, \text{ for every}\, \varphi(\cdot) \in L^\infty(t,T;\mathbb{R}^n),$$

where $\text{esssup} f := \inf\{M|\mu(\{x : f(x) > M\}) = 0\}$,

$$||\varphi(\cdot)||_{L^p(t,T;\mathbb{R}^n)} = \left(\int_t^T |\varphi(s)|^p ds\right)^{\frac{1}{p}}, \text{ for every}\, \varphi(\cdot) \in L^p(t,T;\mathbb{R}^n).$$

We now present some standard results.

Theorem 1. *[Banach fixed point theorem] Let $\mathbb{X}$ be a Banach space, and $S : \mathbb{X} \to \mathbb{X}$ be a map satisfying*

$$||S(x) - S(y)|| \leq \alpha||x - y||, \textit{ for every}\, x, y \in \mathbb{X}, \tag{1.2}$$

with $\alpha \in (0,1)$. There exists a unique $\bar{x} \in \mathbb{X}$ such that $S(\bar{x}) = \bar{x}$.

Proof. Let see that the map S is continuous. Given $\epsilon > 0$, and $||x - y|| < \delta$ with $\delta = \frac{\epsilon}{\alpha}$, we have by 1.2

$$||S(x) - S(y)|| \leq \alpha||x - y|| < \epsilon.$$

For every $x, y \in \mathbb{X}$. Now pick any $x_0 \in \mathbb{X}$, and define the sequence $x_k = S^k(x_0)$, $k \geq 1$. Then for any $k, l \geq 1$,

$$\begin{aligned} ||x_{k+l} - x_k|| &\leq ||\sum_{i=k+1}^{k+l} (x_i - x_{i-1})|| = ||\sum_{i=k+1}^{k+l} (S^i(x_0) - S^{i-1}(x_0))|| \\ &\leq \sum_{i=k+1}^{k+l} ||(S^i(x_0) - S^{i-1}(x_0))|| \leq \sum_{i=k+1}^{k+l} \alpha^k ||x_1 - x_0||. \end{aligned}$$

Thus, $\{x_k\}_{k>0}$ is a Cauchy sequence. Consequently, there exists a unique $\bar{x} \in \mathbb{X}$ such that

$$\lim_{k \to \infty} ||x_k - \bar{x}|| = 0.$$

Then by continuity of S, we obtain

$$\bar{x} = \lim_{k \to \infty} = \lim_{k \to \infty} S(x_{k-1}) = S(\bar{x}).$$

This means that $\bar{x}$ is a fixed point of S. Finally, if $\bar{x}$ and $\tilde{x}$ are two fixed point. Then

$$||\bar{x} - \tilde{x}|| = ||S(\bar{x}) - S(\tilde{x})|| \leq \alpha ||\bar{x} - \tilde{x}||.$$

Hence, $\bar{x} = \tilde{x}$, proving the uniqueness. ■

Theorem 2. *[Arzela-Ascoli] Let $\mathcal{Z} \subseteq C([t, T]; \mathbb{R}^n)$ be an infinite set which is uniformly bounded and equicontinuous, i.e.*

$$\sup_{\varphi(\cdot) \in \mathcal{Z}} ||\varphi(\cdot)||_{C([t,T];\mathbb{R}^n)} < \infty,$$

and for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\varphi(t) - \varphi(s)| < \epsilon, \text{ for every } |t - s| < \delta,\ \varphi(\cdot) \in \mathcal{Z}.$$

Then there exists a sequence $\varphi_k(\cdot) \in \mathcal{Z}$ such that

$$\lim_{k\to\infty} |\varphi_k(\cdot) - \bar{\varphi}(\cdot)||_{C([t,T];\mathbb{R}^n)} = 0,$$

for some $\bar{\varphi}(\cdot) \in C([t,T];\mathbb{R}^n)$.

Proof. Let define $D := \{t_k\}_{k\geq 1}$ be a dense set of $[t,T]$. For any $k \geq 1$, the set $\{\varphi(t_1)|\varphi(\cdot) \in \mathcal{Z}\}$ is bounded. Thus, there exists aa sequence denoted by $\{\varphi_{\sigma_1(i)}(t_1)\}$ converging some point in $\mathbb{R}^n$, denoted by $\bar{\varphi}(t_1)$. Next, the set $\{\varphi_{\sigma_1(i)}(t_2)\}$ is bounded.

Thus, we may let $\{\varphi_{\sigma_2(i)}(t_2)\}$ be a subsequence of $\{\varphi_{\sigma_1(i)}(t_2)\}$, which is convergent to some point in $\mathbb{R}^n$ denoted by $\bar{\varphi}(t_2)$. Continue this process, we obtain a function $\bar{\varphi} : D \to \mathbb{R}$. by letting

$$\bar{\varphi}(\cdot) = \varphi_{\sigma_i(i)}(\cdot),\ i \geq 1.$$

We have

$$\lim_{i\to\infty} \bar{\varphi}_i(s) = \bar{\varphi}(s) \text{ for every } s \in D.$$

By equi-continuity of the sequence $\{\varphi_k(\cdot)\}$, we see that for any $\epsilon > 0$ there exits a $\delta = \delta(\epsilon) > 0$, independent of $i \geq 1$ such that

$$|\bar{\varphi}_i(s_1) - \bar{\varphi}_i(s_2)| < \epsilon \text{ for every } s_1, s_2 \in D, |s_1 - s_2| < \delta. \tag{1.3}$$

Then letting $i \to \infty$, we obtain

$$|\bar{\varphi}(s_1) - \bar{\varphi}(s_2)| \leq \epsilon \text{ for every } s_1, s_2 \in D, |s_1 - s_2| < \delta.$$

This means that $\bar{\varphi} : D \to \mathbb{R}^n$ is uniformly continuous on D. Consequently, we may extend $\bar{\varphi}(\cdot)$ on $\bar{D} = [t,T]$ which is still continuous. Finally, for any $\epsilon > 0$, let $\delta > 0$ be such that (1.3) holds and by compactness of $[t,T]$, let $S_\delta = \{s_j, 1 \leq j \leq M\} \subseteq D$ with $M > 1$ depending on $\epsilon > 0$ such that

$$[t,T] \subseteq \bigcup_{j=1}^{M} (s_j - \delta, s_j + \delta).$$

Next, we may let $i_0 > 1$ such that

$$|\bar{\varphi}_i(s_j) - \bar{\varphi}(s_j)| < \epsilon,\ i \geq i_0,\ 1 \leq j \leq M.$$

Then for any $s \in [t,T]$, there is an $s_j \in S_\delta$ such that $|s - s_j| < \delta$. Consequently

$$|\bar{\varphi}_i(s) - \bar{\varphi}(s)| \leq |\bar{\varphi}_i(s) - \bar{\varphi}_i(s_j)| + |\bar{\varphi}_i(s_j) - \bar{\varphi}(s_j)| + |\bar{\varphi}(s_j) - \bar{\varphi}(s)| \leq 3\epsilon.$$

This show that $\varphi_i(\cdot)$ converges to $\bar{\varphi}(\cdot)$ uniformly in $s \in [t,T]$. ■

Theorem 3. *[Banach-Saks] Let $\varphi_k(\cdot) \in L^2(a,b;\mathbb{R}^n)$ be a sequence which is weakly convergent to $\bar{\varphi}(\cdot) \in L^2(a,b;\mathbb{R}^n)$, i.e.,*

$$\lim_{k\to\infty} \int_a^b \langle \varphi_k(s) - \bar{\varphi}(s), \eta(s)\rangle ds, \text{ for every } \eta \in L^2(a,b;\mathbb{R}^n).$$

Then there is a subsequence $\{\varphi_{k_j}(\cdot)\}$ such that

$$\lim_{k\to\infty} \Big|\Big|\frac{1}{N}\sum_{j=1}^{N} \varphi_{k_j}(\cdot) - \bar{\varphi}(\cdot)\Big|\Big|_{L^2(a,b;\mathbb{R}^n)} = 0.$$

Proof. Whitout loss of generality, first we consider that $\bar{\varphi}(\cdot) = 0$. Let $k_1 = 1$. By the weak convergence of $\varphi_k(\cdot)$, we may find $k_1 < k_2 < \cdots < k_N$ such that

$$\Big|\int_a^b \langle \varphi_{k_i}(s), \varphi_{k_j}(s)\rangle ds\Big| < \frac{1}{N}\ 1 \leq i < j \leq N$$

observe

$$\begin{aligned}
||\frac{1}{N}\sum_{i=1}^{N}\varphi_{k_i}(\cdot)||^2_{L^2(a,b;\mathbb{R}^2)} &= \frac{1}{N^2}\int_a^b|\sum_{i=1}^{N}\varphi_{k_i}(s)|^2 ds = \frac{1}{N^2}\int\sum_{i,j=1}^{N}\langle\varphi_{k_i}(s),\varphi_{k_j}(s)\rangle ds \\
&= \frac{1}{N^2}\sum_{i=1}^{N}||\varphi_{k_i}(\cdot)||^2_{L^2(a,b;\mathbb{R}^n)} + \frac{2}{N^2}\sum_{1\le i<j\le N}\int_a^b\langle\varphi_{k_i}(s),\varphi_{k_j}(s)\rangle ds \\
&\le \frac{1}{N}\sup_{i\ge 1}||\varphi_{k_i}(\cdot)||^2_{L^2(a,b;\mathbb{R}^n)} + \frac{2}{N^3}\frac{N(N-1)}{2} \\
&\le \frac{1}{N}\sup_{i\ge 1}||\varphi_{k_i}(\cdot)||^2_{L^2(a,b;\mathbb{R}^n)} + \frac{1}{N} \to 0,
\end{aligned}$$

when $N \to 0$. Now, consider that $\bar{\varphi}(\cdot) \neq 0$, thus $\varphi_k(\cdot) - \bar{\varphi}(\cdot) = 0$, and we can apply the previous steps. ■

Lemma 4. *[Filippov] Let U be a complete separable metric space whose metric is denoted by $d(\cdot,\cdot)$. Let $g : [0,T] \times U \to \mathbb{R}^n$ be a map which is measurable in $t \in [0,T]$ and*

$$|g(t,u) - g(t,v)| \le \omega(d(u,v)), \text{ for every } u, v \in U,\ t \in [0,t],$$

for some continuous and increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$, called a modulus of continuity. Moreover,

$$0 \in g(t,U) \ \text{ a.e. } t \in [0,t].$$

Then there exists a measurable map $u : [0,T] \to U$, such that

$$g(t,u(t)) = 0 \text{ a.e. } t \in [0,T]. \tag{1.4}$$

Proof. Define $\bar{d}(u,v) = \frac{d(u,v)}{1+d(u,v)} < 1$ for every $u, v \in U$. Then with this new metric $\bar{d}$, U still complet and separable. Hence without loss of generality, we assume that the original metric $d(\cdot,\cdot)$ already satisfies $d(u,v) < 1$ for every $u, v \in U$.

Next, we define

$$\Gamma(t) := \{u \in U | g(t,u) = 0\},\ t \in [0,t].$$

We have that $\Gamma(t) \neq \emptyset$, because $0 \in g(t,U)$ *a.e.* $t \in [0,t]$. Let $U_0 := \{v_k | k \geq 1\}$ be a countable dense subset of U. We claim that for any $u \in U$ and $0 \leq c < 1$,

$$\{t \in [0,T] |\, d(u,\Gamma(t)) \leq c\} = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{t \in [0,T] |\, d(u,v_j) \leq c + \frac{1}{i}, |g(t,v_j)| \leq \frac{1}{i}\}, \qquad (1.5)$$

where

$$d(u,\Gamma(t)) := \inf_{v \in \Gamma(t)} d(u,v).$$

To show (1.5), we note that $t \in [0,T]$, with $d(u,\Gamma(t)) \leq c$ if and only if there exists a sequence $u_k \in \Gamma(t)$, i.e., $g(t,u_k) = 0$, such that

$$d(u,u_k) \leq c + \frac{1}{k}.$$

Since $\bar{U}_0 = U$, there exists a sequence $v_{j_k} \in U_0$ such that

$$d(u_k, v_{j_k}) < \frac{1}{k}.$$

Hence, by triangle inequality

$$d(u, v_{j_k}) \leq d(u,u_k) + d(u_k, v_{j_k}) \leq c + \frac{2}{k}.$$

Next, by the uniform continuity of $u \mapsto g(t,u)$, we have

$$|g(t,v_{j_k})| \leq |g(t,v_{j_k}) - g(t,u_k)| \leq \omega(d(v_{j_k},u_k)) \leq \omega\left(\frac{1}{k}\right).$$

Hence, one has

$$\begin{cases} \lim_{k\to\infty} d(u, v_{j_k}) \le c, \\ \lim_{k\to\infty} g(t, v_{j_k}) = 0. \end{cases}$$

Thus, $d(u, \Gamma(t)) \le c$ if and only if for any $i \ge 1$, there exists a $j \ge i$, such that

$$\begin{cases} \bar{d}(u, v_j) \le c + \frac{1}{i}, \\ |g(t, v_j)| \le \frac{1}{i}. \end{cases}$$

This prove (1.5). Since the right-hand side of (1.5) is measurable, so is the let-hand side. On the other hand,

$$\begin{cases} \{t \in [0, T] |\, d(u, \Gamma(t)) \le c\} = [0, T], \text{ for every}\, c \ge 1, \\ \{t \in [0, T] |\, d(u, \Gamma(t)) \le c\} = \emptyset, \text{ for every}\, c < 0. \end{cases}$$

Hence, the function $t \mapsto d(u, \Gamma(t))$ is measurable. Now, we define

$$u_0(t) := v_1, \text{ for every}\, t \in [0, T].$$

Clearly, $u_0(t)$ is measurable and

$$d(u_0(t), \Gamma(t)) < 1 \, \text{for every}\, t \in [0, T].$$

Suppose that we have defined $u_{k-1}(\cdot)$ such that

$$\begin{cases} d(u_{k-1}(t), \Gamma(t)) \le 2^{1-k}, \\ d(u_{k-1}(t), u_{k-2}(t)) \le 2^{2-k}, \end{cases} \quad t \in [0, T]. \tag{1.6}$$

We define the sets

$$\begin{cases} C_i^k := \{t \in [0,T] \,|\, d(v_i, \Gamma(t)) < 2^{-k}\}, \\ D_i^k := \{t \in [0,T] \,|\, d(v_i, u_{k-1}(t)) < 2^{1-k}\}. \end{cases}$$

Since $t \mapsto d(v_i, \Gamma(t))$ is measurable, C_i^k is measurable. Likewise, D_i^k is also measurable. Set

$$A_i^k = C_i^k \cap D_i^k, \; k, i \geq 1.$$

Then A_i^k is measurable as well. We claim that

$$[0,T] = \bigcup_{i=1}^{\infty} A_i^k \text{ for every } k \geq 1. \tag{1.7}$$

In fact, for any $t \in [0,T]$, by (1.6), there exists a $u \in \Gamma(t)$ such that

$$d(u_{k-1}(t), u) < 2^{1-k}.$$

By the density of U_0 in U, there exists an $i \geq 1$ such that

$$d(v_i, \Gamma(t)) \leq \begin{cases} d(v_i, u) < 2^{-k}, \\ d(v_i, u_{k-1}(t)) < 2^{1-k}, \end{cases}$$

which means $t \in A_i^k$, proving (1.7). Now we define $u_k(\cdot) : [0,T] \to U_0 \subseteq U$ as follows:

$$u_k(t) = v_i, \text{ for every } t \in A_i^k \backslash \bigcup_{j=1}^{\infty} A_j^k.$$

By $t \in C_i^k$, we have

$$d(u_k(t), \Gamma(t)) < 2^{-k},$$

and by $t \in D_i^k$, we have

$$d(u_k(t), u_{k-1}(t)) < 2^{1-k}.$$

This completes the construction of the sequence $\{u_k(\cdot)\}$ inductively. Clearly, (1.6) holds for every $k \geq 1$. This als implies that for each $t \in [0, T]$, $\{u_k(t)\}$ is Cauchy in U. By completeness of Um we obtain

$$\lim_{k \to \infty} u_k(t) = u(t),\ t \in [0, t].$$

Of course, $u(\cdot)$ is measurable, and moreover, by the closeness of $\Gamma(t)$, we have

$$u(t) \in \Gamma(t), \text{ for every } t \in [0, T].$$

This means (1.4) holds. ■

Proposition 1. *[Gronwall's Inequality] Let $\theta : [a, b] \to \mathbb{R}_+$ be continuous and satisfy*

$$\theta(s) \leq \alpha(s) + \int_a^s \beta(r)\theta(r)dr,\ \ s \in [a, b],$$

for some $\alpha(\cdot), \beta(\cdot) \in L^1(a, b; \mathbb{R}_+)$. Then

$$\theta(s) \leq \alpha(s) + \int_a^s \alpha(\tau)\beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau,\ \ s \in [a, b]. \tag{1.8}$$

In particular, if $\alpha(\cdot) = \alpha$ is a constant, then

$$\theta(s) \leq \alpha e^{\int_a^s \beta(r)dr},\ \ s[a, b]. \tag{1.9}$$

Proof. Let $\varphi(s) = \int_a^s \beta(r)\theta(r)dr.$, by the fundamental theorem of calculus, we have

$$\dot{\varphi}(s) = \beta(s)\theta(s) \leq \beta(s)[\alpha(s) + \varphi(s)].$$

This leads to

$$[\varphi(s)e^{-\int_a^s \beta(r)dr}]' \leq \alpha(s)\beta(s)e^{-\int_a^s \beta(r)dr}.$$

Consequently,

$$\varphi(s)e^{-\int_a^s \beta(r)dr} \leq \int_a^s \alpha(\tau)\beta(\tau)e^{-\int_a^\tau \beta(r)dr}d\tau.$$

$$\varphi(s) \leq \int_a^s \alpha(\tau)\beta(\tau)e^{-\int_a^\tau \beta(r)dr}e^{\int_a^s \beta(r)dr}d\tau.$$

rewriting

$$\varphi(s) \leq \int_a^s \alpha(\tau)\beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau.$$

Hence,

$$\theta(s) \leq \alpha(s) + \int_a^s \alpha(\tau)\beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau.$$

(1.8) holds. Now, consider α constant, then

$$\theta(s) \leq \alpha + \int_a^s \alpha\beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau.$$

By integration rules, if $u = \alpha, dv = \beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau$. In the other hand,

$$\frac{d}{d\tau}e^{\int_\tau^s \beta(r)dr} = \frac{d}{d\tau}e^{-\int_s^\tau \beta(r)dr} = -\beta(\tau)e^{-\int_s^\tau \beta(r)dr} = -\beta(\tau)e^{\int_\tau^s \beta(r)dr}$$

Then

$$\theta(s) \leq \alpha - \alpha\int_a^s \frac{d}{d\tau}e^{\int_\tau^s \beta(r)dr}d\tau = \alpha - \alpha[e^0 - e^{\int_a^s \beta(r)dr}].$$

Therefore,

$$\theta(s) \leq \alpha e^{\int_a^s \beta(r)dr},$$

(1.9) holds. ■

1.2 Control Systems

In this section, we presente the assumptions for the prove of existence of a optimal pair.

We assume that U is a non-empty closed subset in $\mathbb{R}^n$ (it could be generally a separable complete metric space). For any initial pair $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, we rewrite the control system here:

$$\begin{cases} \dot{X}(s) = f(s, u(s), X(s)), \; s \in [t, \infty) \\ X(t) = x. \end{cases} \tag{1.10}$$

Let us begin with the following assuption:

(C1) The map $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable and there exists a constan $L > 0$ such that

$$\begin{cases} |f(t, u, x_1) - f(t, u, x_2)| \leq L|x_1 - x_2|, \; (t, u) \in \mathbb{R}_+ \times U, \; x_1, x_2 \in \mathbb{R}^n, \\ |f(t, u, 0)| \leq L, \text{ for every } (t, u) \in \mathbb{R}_+ \times U. \end{cases}$$

Note that this conditions imply

$$|f(t, u, x)| \leq L(1 + |x|), \; (t, u, x) \in \mathbb{R}_+ \times U \times \mathbb{R}^n.$$

This condition is also usually called the lipschitz condition of the function f. A key feature of the above is that the bond of $|f(t, u, x)|$, depending on $|x|$, is uniform in u.

Proposition 2. *Let (C1) hold. Then, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, and $u(\cdot) \in \mathcal{U}[t,\infty)$, there exits a unique solution $X(\cdot) \equiv X(\cdot;t,x,u(\cdot))$ to the state equation (1.10). Moreover, the following estimate hold:*

$$\begin{cases} |X(s;t,x,u(\cdot))| \leq e^{L(s-t)}(1+|x|) - 1, \\ |X(s;t,x,u(\cdot)) - x| \leq [e^{L(s-t)} - 1](1+|x|),\ s \in (t,\infty],\ u(\cdot) \in \mathcal{U}[t,\infty). \end{cases} \tag{1.11}$$

Further, for any $t \in \mathbb{R}_+$, $x_1, x_2 \in \mathbb{R}^n$, and $u(\cdot) \in \mathcal{U}[t,\infty)$,

$$|X(s;t,x_1,u(\cdot)) - X(s;t,x_2,u(\cdot))| \leq e^{L(s-t)}|x_1 - x_2|, \text{ for every } s \in [t,\infty). \tag{1.12}$$

Proof. It suffices to prove our conclusion on any interval $[t,T]$ with $0 \leq t < T < \infty$, Since $[t,\infty) = \bigcup_{T \geq t}[t,T]$.

For any $X(\cdot) \in C([t,T];\mathbb{R}^n)$, we define

$$[\mathcal{S}X(\cdot)](s) := x + \int_t^s f(r,u(r),X(r))dr,\ s \in [t,T].$$

Then for any $X_1(\cdot), X_2(\cdot) \in C([t,T];\mathbb{R}^n)$ and by condition (C1), we have

$$\begin{aligned} ||[\mathcal{S}X_1(\cdot)] - [\mathcal{S}X_2(\cdot)]||_{C([t,T];\mathbb{R}^n)} &= ||\int_t^s f(r,u(r),X_1(r))dr - \int_t^s f(r,u(r),X_2(r))dr|| \\ &\leq L||\int_t^s (X_1(r) - X_2(r))dr|| \\ &\leq L||X_1(\cdot) - X_2(\cdot)||_{C([t,T];\mathbb{R}^n)}|t-s|. \end{aligned}$$

Define $\delta = |t-s|$, then

$$||[\mathcal{S}X_1(\cdot)] - [\mathcal{S}X_2(\cdot)]||_{C([t,T];\mathbb{R}^n)} \leq \delta L||X_1(\cdot) - X_2(\cdot)||_{C([t,T];\mathbb{R}^n)}.$$

Therefore, by chosing $\delta < \frac{1}{L}$, we see that $\mathcal{S} : C([t,T];\mathbb{R}^n) \to C([t,T];\mathbb{R}^n)$ is contractive. Hence, by Contraction Mapping Theorem (1), the control system (1.10) admits a unique solution on $[t, t+\delta]$. Repeating the argument, we can obtain that (1.10) admits a unique solution on $[t,T]$.

Next, for the unique solution $X(\cdot)$ of (1.10), we have

$$|X(s)| \leq |x| + L\int_t^s (1 + |X(r)|)dr, \; s \in [t,T].$$

If we denote the right-hand side of the above by $\theta(s)$, then

$$\dot{\theta}(s) = L + L|X(s)| \leq L + L\theta(s),$$

which leads to

$$\dot{\theta}(s) \leq e^{L(s-t)}|x| + L\int_t^s e^{L(s-r)dr} = e^{L(s-t)}|x| + e^{L(s-t)} - 1,$$

thus, $|X(s)| \leq e^{L(s-t)}(1+|x|) - 1$. This gives the first estimate in (1.11). Next, we will apply the first estimate

$$\begin{aligned}
|X(s) - x| &= |\int_t^s f(r, u(r), X(r))dr| \leq \int_t^S |f(r, u(r), X(r))|dr \\
&\leq L\int_t^s (1 + |X(r)|)dr \leq L\int_t^s e^{L(r-t)}(1+|x|)dr \\
&= (1+|x|)[e^{L(s-t)} - 1].
\end{aligned}$$

This prove the second estimate in (1.11). finally, for $x_1, x_2 \in \mathbb{R}^n$, let us denote $X_i(\cdot) = X(\cdot; t, x_i, u(\cdot))$. Then

$$|X_1(s) - X_2(s)| \leq L\int_t^s |X_1(r) - X_2(r)|dr.$$

By Gronwall's Inequality (1), we obtain (1.12). ∎

We can see that estimates (1.11) are uniform in $u(\cdot) \in \mathcal{U}[t,T]$. This will paly an interesting role later.

1.3 Existence Theory

Let $T \in (0,\infty)$ be fixed. Consider the control system

$$\begin{cases} \dot{X}(s) = f(s, u(s), X(s) \ s \in [t,T], \\ X(t) = x, \end{cases}$$

with termimal state constraint

$$X(T; t, x, u(\cdot)) \in M,$$

where $M \subseteq \mathbb{R}^n$ is fixed, and with cost functional

$$J^T(t, x; u(\cdot)) = \int_t^T g(s, u(s), X(s))ds + h(X(T)).$$

We recall

$$\tilde{\mathcal{U}}_x^M[t,T] = \{u(\cdot) \in \mathcal{U}[t,T] | X(T; t, x, u(\cdot)) \in M\},$$

and recall the following optimal control problem:

Problem $(OC)^T$. For given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ with $\tilde{\mathcal{U}}_x^M[t,T] \neq \emptyset$, find a $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]$ such that

$$J^T(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]} J^T(t, x; u(\cdot)) \equiv V(t,x). \tag{1.13}$$

Any $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]$ satisfying (1.13) is called an optimal control, the corresponding $\bar{X}(\cdot) \equiv X(\cdot;t,x,\bar{u}(\cdot))$ is called an optimal state trajectory, and $(\bar{u}(\cdot),\bar{X}(\cdot))$ is called optimal pair, we call $V(\cdot,\cdot)$ the value functional of problem $(OC)^T$.

We introduce the following assumption for the cost functional:

(C2) The maps $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$ and $h.\mathbb{R}^n \to \mathbb{R}$ are measurable and there exists a continuous function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, called a local modulus of continuity, which is increasing in each argument, and $\omega(r,0) = 0$ for every $r \geq 0$, such that

$$|g(s,u,x_1)-g(s,u,x_2)|+|h(x_1)-h(x_2)| \leq \omega(|x_1|\vee|x_2|,|x_1-x_2|) \text{ for every } (s,u) \in \mathbb{R}_+\times U, x_1,x_2 \in \mathbb{R}^n,$$

where $|x_1| \vee |x_2| = \max\{|x_1|,|x_2|\}$, and

$$\sup_{(s,u)\in\mathbb{R}_+\times U} |g(s,u,0)| \equiv g_0 < \infty.$$

For any $(t,x) \in [0,T] \times \mathbb{R}^n$, let us introduce the following set

$$\mathbb{E}(t,x) = \{(z^0,z) \in \mathbb{R} \times \mathbb{R}_+ | z^0 \geq g(t,u,x), z = f(t,u,x),\ u \in U\}.$$

The following assumption gives some compability between the control system and the cost functional.

(C3) For almost all $t \in [0,T]$, the following Cesari property holds at any $x \in \mathbb{R}^n$,

$$\bigcap_{\delta>0} \bar{co}\mathbb{E}(t,B_\delta(x)) = \mathbb{E}(t,x),$$

where, we recall that $B_\delta(x)$ is the open ball centered at x with radius $\delta > 0$, and $\bar{co}(E)$ stands for the closed convex hull of the set E (the smallest closed convex set containing E).

It is cleary that if $\mathbb{E}(t,x)$ has Cesari property at x, the $\mathbb{E}(t,x)$ is convex and closed.

Theorem 5. *Let (C1)-(C3) hold. Let $M \subseteq \mathbb{R}^n$ be a non-empty closed set. Let $(t,x) \in [0,T] \times \mathbb{R}^n$ be given and $\tilde{\mathcal{U}}_x^M[t,T] \neq \emptyset$. Then problem $(OC)^T$ admits at least one optimal pair.*

Proof. Let $u_k(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]$ be a minimizing sequence. By proposition (2)

$$|X_k(s)| \leq e^{L(s-t)}(1+|x|) - 1, \; s \in [t,T] \; k \geq 1. \tag{1.14}$$

and for any $t \leq \tau < s \leq T$,

$$\begin{aligned} |X_k(s) - X_k(\tau)| &= |X(s;t,x,u_k(\cdot)) - X(\tau;t,x,u(\cdot))| \\ &\leq [e^{L(s-\tau)} - 1][1 + X(\tau;t,x,u_k(\cdot))] \\ &\leq [e^{L(s-\tau)} - 1]e^{L(\tau-t)}(1+|x|). \end{aligned}$$

Hence, the sequence $\{X_k(\cdot)\}$ is uniformly bounded and equicontinuous. Therefore, By Arzela-Ascoli Theorem (2), we may assume that the sequence is convergent to some $\bar{X}(\cdot)$ in $C([t,T];\mathbb{R}^n)$. On the other hand,

$$|f(d,u_k(s),X_k(s)| \leq L(1+|X_k(s)) \leq Le^{L(s-t)}(1+|x|).$$

Also, by (1.14) and (C2), we have

$$\begin{aligned} |g(s,u_k(s),X_k(s))| &\leq |g(s,u_k(s),0)| + |g(s,u_k(s),X_k(s)) - g(s,u_k(s),0)| \\ &\leq g_0 + \omega(|X_k(s)|,|X_k(s)|) \leq g_0 + \omega(e^{LT}(1+|x|),e^{LT}(1+|x|)) \\ &\leq K, \; s \in [t,T], \; k \geq 1. \end{aligned}$$

Hence, by exctrcting a subsequence if necessary, we may assume that

$$\begin{cases} g(\cdot, u_k(\cdot), X_k(\cdot)) \to \bar{g}(\cdot), \text{ weakly in } L^2([t,T];\mathbb{R}^n), \\ f(\cdot, u_k(\cdot), X_k(\cdot)) \to \bar{f}(\cdot), \text{ weakly in } L^2([t,T];\mathbb{R}^n), \end{cases}$$

for some $\bar{g}(\cdot)$ and $\bar{f}(\cdot)$. Then by Banach-Saks Theorem (3), we have

$$\begin{cases} \tilde{g}_k(\cdot) := \frac{1}{k}\sum_{i=1}^k g(\cdot, u_i(\cdot), X_i(\cdot)) \to \bar{g}(\cdot), \text{ strongly in } L^2([t,T];\mathbb{R}^n), \\ \tilde{f}_k(\cdot) := \frac{1}{k}\sum_{i=1}^k f(\cdot, u_i(\cdot), X_i(\cdot)) \to \bar{f}(\cdot), \text{ strongly in } L^2([t,T];\mathbb{R}^n). \end{cases} \tag{1.15}$$

On the other hand, by (C1) and the convergence of $X_k(\cdot) \to \bar{X}(\cdot)$ in $C([t,T];\mathbb{R}^n)$, we have

$$\begin{aligned} |\tilde{f}_k(s) - \frac{1}{k}\sum_{i=1}^k f(s, u_i(s), \bar{X}(s)| &\leq \frac{1}{k}\sum_{i=1}^k |f(s, u_i(s), X_i(s)) - f(s, u_i(s), \bar{X}(s))| \\ &\leq \frac{1}{k}\sum_{i=1}^k |X_i(s) - \bar{X}(s)| \to 0 \ k \to \infty, \end{aligned}$$

uniformly in $s \in [t,T]$. Similarly, by (C2),

$$\begin{aligned} |\tilde{g}_k(s) - \frac{1}{k}\sum_{i=1}^k g(s, u_i(s), \bar{X}(s)| &\leq \frac{1}{k}\sum_{i=1}^k |g(s, u_i(s), X_i(s)) - g(s, u_i(s), \bar{X}(s))| \\ &\leq \frac{1}{k}\sum_{i=1}^k \omega(|X_i(s)| \vee |\bar{X}(s)|, |X_i(s) - \bar{X}(s)|) \to 0 \ k \to \infty, \end{aligned}$$

uniformly in $s \in [t,T]$. Next, by the definition of $\mathbb{E}(t,x)$, we have

$$\begin{pmatrix} g(s, u_i(s), X_i(s)) \\ f(s, u_i(s), X_i(s)) \end{pmatrix} \in \mathbb{E}(s, X_i(s)),\ i \geq 1,\ s \in [t,T].$$

Hence, for any $\delta > 0$, there exits a $K_\delta > 0$ such that

$$\begin{pmatrix} \tilde{g}_k(s) \\ \tilde{f}_k(s) \end{pmatrix} \in \bar{co}\mathbb{E}(s, B_\delta(\bar{X}(s))),\ K \geq K_\delta,\ s \in [t, T]. \tag{1.16}$$

Combinig (1.15) and (1.16), using (C3), we obtain

$$\begin{pmatrix} \bar{g}(s) \\ \bar{f}(s) \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} \tilde{g}_k(s) \\ \tilde{f}_k(s) \end{pmatrix} \in \bigcap_{\delta>0} \bar{co}\mathbb{E}(s, B_\delta(\bar{X}(s))) = \mathbb{E}(s, \bar{X}(s)).$$

Then by Filippov's Lemma (4), there exits a $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ such that

$$\bar{g}(s) \geq g(s, \bar{u}(s), \bar{X}(s)),\ \bar{f}(s) = f(s, \bar{u}(s), \bar{X}(s)),\ s \in [t, T].$$

This means $\bar{X}(\cdot) = X(\cdot; t, x, \bar{u})$. On the other hand, since

$$\bar{X}_k(T) \equiv X(T; t, x, \bar{u}_k(\cdot)) \in M,\ k \geq 1$$

one has

$$\bar{X}(T) \equiv X(T; t, x, \bar{u}(\cdot)) \in M,$$

which means that $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_x^M[t, T]$. finally, by Fatou's Lemma

$$
\begin{aligned}
J^T(t,x;\bar{u}(\cdot)) &\leq \int_t^T \bar{g}(s)ds + h(\bar{X}(T)) \\
&\leq \liminf_{k\to\infty}[\int_t^T \tilde{g}_k(s)ds + h(X_k(T))] \\
&= \lim_{k\to\infty}\frac{1}{k}J^T(t,x;u_i(\cdot)) \\
\lim_{k\to\infty} J^T(t,x;u_k(\cdot)) & \\
&= \inf_{u(\cdot)\in\tilde{\mathcal{U}}_x^M[t,T]} J^T(t,x;u(\cdot)).
\end{aligned}
$$

This means that $(\bar{u}(\cdot),\bar{X}(\cdot))$ is an optimal pair.

■

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