"El saber de mis hijos hará mi grandeza"

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Chapter 1

Existence theory for optimal policies

In this chapter, we will give the important results for the proof of the theorem of existence of optimal control pairs. In addition to presenting some problems related to these optimal pairs [1].

In what follows, we denote $\mathbb{R}_+ = [0, \infty)$. For any initial pair $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, consider the following ordinary differential equation:

$$\begin{cases} \dot{X}(s) = f(s, u(s), X(s)\ s \in [t, \infty), \\ X(t) = x, \end{cases} \tag{1.1}$$

Where $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is a given map. In the above, $X(\cdot)$ is called the state trajectory, taking values in $\mathbb{R}^n$ and $u(\cdot)$ is called the control, taking values in some metric space U. We call (1.1) a control system.

For any $0 \leq t < T < \infty$, we define the following:

$$\mathcal{U}[t, T] = \{u : [t, T] \to U | u(\cdot) \text{ is measurable}\},$$

and

$$\mathcal{U}[t, \infty) = \{u : [t, \infty) \to U | u(\cdot) \text{ is measurable}\}.$$

Any $u(\cdot) \in \mathcal{U}[t,T]$, respectively $u(\cdot) \in \mathcal{U}[t,\infty)$, is called a feasible control on $[t,T]$, respectively $[t,\infty)$.

Under proper conditions, for any initial pair (t,x), and feasible control $u(\cdot)$, (1.1) admits a unique solution $X(\cdot) = X(\cdot;t,x,u(\cdot))$ deined on $[t,\infty)$. Clearly different choices of $U(\cdot)$ will result in different state trajectories $X(\cdot)$. We reer to $(u(\cdot),X(\cdot))$ as a state-control pair of the control system (1.1).

Next, we recall that $2^{\mathbb{R}^n}$ is the set of all subsets of $\mathbb{R}^n$. Any map $M : \mathbb{R}_+ \to 2^{\mathbb{R}^n}$ is called a moving target in $\mathbb{R}^n$ i for any $t \in \mathbb{R}_+$, $M(t)$ is a measurable set in $\mathbb{R}^n$. WE allow $M(t)$ to be empty for some or all t, which will give us some fleaxibility below. In most situations, for any $t \in \mathbb{R}_+$, $M(t)$ is assumed to be closed or open.

Problem (C). Let $M(\cdot)$ be a moving target set in $\mathbb{R}^n$. For given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, ind a control $u(\cdot) \in \mathcal{U}[t,\infty)$ such that for some $\tau \geq t$,

$$X(\tau;t,x,u(\cdot)) \in M(\tau).$$

The above is called a controllability problem for the system (1.1) with the moving target set $M(\cdot)$. For a moving set $M(\cdot)$ in $\mathbb{R}^n$, and $T \in (0\infty)$, we define

$$\mathcal{U}_x^{M(\cdot)}[t,T] = \{u(\cdot) \in \mathcal{U}[t,\infty) | X(\tau;t,x,u(\cdot)) \in M(\tau), \text{ for some } \tau \in [t,T]\},\ (t,x) \in [0,T] \times \mathbb{R}^n,$$

and

$$\mathcal{U}_x^{M(\cdot)}[t,\infty) = \bigcup_{T \geq t} \mathcal{U}_x^{M(\cdot)}[t,T], \text{ for every } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Let $M \subseteq \mathbb{R}^n$ be non-empty and closed, $T \in (0,\infty)$, and let

$$M(t) = \emptyset I_{\mathbb{R}_+ \setminus T}(t) + M I_{\{T\}}(t),\ t \in \mathbb{R}_+.$$

Then, for any $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$\mathcal{U}_x^{M(\cdot)}[t,T] = \{u(\cdot) \in \mathcal{U}[t,T] | X(T;t,x,u(\cdot)) \in M\} \equiv \tilde{\mathcal{U}}_x^M[t,T].$$

And define the cost functional

$$J(t,x;u(\cdot)) = \int_t^T g(s,u(s),X(s))ds + h(T,X(T)) \equiv J^T(t,x,u(\cdot)).$$

For some maps $g(\cdot)$ and $h(\cdot)$. Now we introduce the following problem.

Problem $(OC)^T$. For given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ with $\tilde{\mathcal{U}}_x^M[t,T] \neq \emptyset$, find a $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]$ such that

$$J^T(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]} J^T(t,x;u(\cdot)) \equiv V(t,x).$$

The above is called an optimal control problem, with a fixed terminal time and a terminal state constraint.

1.1 Background

In this section, we present some results which will be used in the following sections. These results are presented mostly in [1].

Let us introduce some spaces. For any $0 \leq t < T < \infty$ and $1 \leq p < \infty$, define

$$C([t,T];\mathbb{R}^n) = \{\varphi : [t,T] \to \mathbb{R}^n | \varphi(\cdot) \text{ is continuous}\},$$

$$L^\infty(t,T;\mathbb{R}^n) = \{\varphi : [t,T] \to \mathbb{R}^n | \varphi(\cdot) \text{ measurable, } \operatorname{esssup}_{s\in[t,T]} |\varphi(s)| < \infty\},$$

$$L^p(t,T;\mathbb{R}^n) = \{\varphi : [t,T] \to \mathbb{R}^n | \varphi(\cdot) \text{ measurable}, \int_t^T |\varphi(s)|^p ds < \infty\},$$

which are Banach space under the following norms, respectively,

$$||\varphi(\cdot)||_{C([t,T];\mathbb{R}^n)} = \sup_{s\in[t,T]} |\varphi(s)|, \text{ for every } \varphi(\cdot) \in C([t,T];\mathbb{R}^n),$$

$$||\varphi(\cdot)||_{L^\infty(t,T;\mathbb{R}^n)} = \text{esssup}_{s\in[t,T]}|\varphi(s)|, \text{ for every } \varphi(\cdot) \in L^\infty(t,T;\mathbb{R}^n),$$

where $\text{esssup} f := \inf\{M|\mu(\{x : f(x) > M\}) = 0\}$,

$$||\varphi(\cdot)||_{L^p(t,T;\mathbb{R}^n)} = \left(\int_t^T |\varphi(s)|^p ds\right)^{\frac{1}{p}}, \text{ for every } \varphi(\cdot) \in L^p(t,T;\mathbb{R}^n).$$

We now present some standard results.

Theorem 1. *[Banach fixed point theorem] Let $\mathbb{X}$ be a Banach space, and $S : \mathbb{X} \to \mathbb{X}$ be a map satisfying*

$$||S(x) - S(y)|| \leq \alpha||x - y||, \textit{ for every } x, y \in \mathbb{X}, \tag{1.2}$$

with $\alpha \in (0,1)$. There exists a unique $\bar{x} \in \mathbb{X}$ such that $S(\bar{x}) = \bar{x}$.

Proof. Let see that the map S is continuous. Given $\epsilon > 0$, and $||x - y|| < \delta$ with $\delta = \frac{\epsilon}{\alpha}$, we have by 1.2

$$||S(x) - S(y)|| \leq \alpha||x - y|| < \epsilon.$$

For every $x, y \in \mathbb{X}$. Now pick any $x_0 \in \mathbb{X}$, and define the sequence $x_k = S^k(x_0)$, $k \geq 1$. Then for any $k, l \geq 1$,

$$\begin{aligned} ||x_{k+l} - x_k|| &\leq ||\sum_{i=k+1}^{k+l} (x_i - x_{i-1})|| = ||\sum_{i=k+1}^{k+l} (S^i(x_0) - S^{i-1}(x_0))|| \\ &\leq \sum_{i=k+1}^{k+l} ||(S^i(x_0) - S^{i-1}(x_0))|| \leq \sum_{i=k+1}^{k+l} \alpha^k ||x_1 - x_0||. \end{aligned}$$

Thus, $\{x_k\}_{k>0}$ is a Cauchy sequence. Consequently, there exists a unique $\bar{x} \in \mathbb{X}$ such that

$$\lim_{k\to\infty} ||x_k - \bar{x}|| = 0.$$

Then by continuity of S, we obtain

$$\bar{x} = \lim_{k\to\infty} = \lim_{k\to\infty} S(x_{k-1}) = S(\bar{x}).$$

This means that $\bar{x}$ is a fixed point of S. Finally, if $\bar{x}$ and $\tilde{x}$ are two fixed point. Then

$$||\bar{x} - \tilde{x}|| = ||S(\bar{x}) - S(\tilde{x})|| \leq \alpha ||\bar{x} - \tilde{x}||.$$

Hence, $\bar{x} = \tilde{x}$, proving the uniqueness. ■

Theorem 2. *[Arzela-Ascoli] Let $\mathcal{Z} \subseteq C([t,T]; \mathbb{R}^n)$ be an infinite set which is uniformly bounded and equicontinuous, i.e.*

$$\sup_{\varphi(\cdot)\in\mathcal{Z}} ||\varphi(\cdot)||_{C([t,T];\mathbb{R}^n)} < \infty,$$

and for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\varphi(t) - \varphi(s)| < \epsilon, \textit{ for every } |t - s| < \delta,\ \varphi(\cdot) \in \mathcal{Z}.$$

Then there exists a sequence $\varphi_k(\cdot) \in \mathcal{Z}$ such that

$$\lim_{k\to\infty} |\varphi_k(\cdot) - \bar{\varphi}(\cdot)||_{C([t,T];\mathbb{R}^n)} = 0,$$

for some $\bar{\varphi}(\cdot) \in C([t,T];\mathbb{R}^n)$.

Proof. Let define $D := \{t_k\}_{k\geq 1}$ be a dense set of $[t,T]$. For any $k \geq 1$, the set $\{\varphi(t_1)|\varphi(\cdot) \in \mathcal{Z}\}$ is bounded. Thus, there exists aa sequence denoted by $\{\varphi_{\sigma_1(i)}(t_1)\}$ converging some point in $\mathbb{R}^n$, denoted by $\bar{\varphi}(t_1)$. Next, the set $\{\varphi_{\sigma_1(i)}(t_2)\}$ is bounded.

Thus, we may let $\{\varphi_{\sigma_2(i)}(t_2)\}$ be a subsequence of $\{\varphi_{\sigma_1(i)}(t_2)\}$, which is convergent to some point in $\mathbb{R}^n$ denoted by $\bar{\varphi}(t_2)$. Continue this process, we obtain a function $\bar{\varphi} : D \to \mathbb{R}$. by letting

$$\bar{\varphi}(\cdot) = \varphi_{\sigma_i(i)}(\cdot),\ i \geq 1.$$

We have

$$\lim_{i\to\infty} \bar{\varphi}_i(s) = \bar{\varphi}(s) \,\text{for every}\, s \in D.$$

By equi-continuity of the sequence $\{\varphi_k(\cdot)\}$, we see that for any $\epsilon > 0$ there exits a $\delta = \delta(\epsilon) > 0$, independent of $i \geq 1$ such that

$$|\bar{\varphi}_i(s_1) - \bar{\varphi}_i(s_2)| < \epsilon \,\text{for every}\, s_1, s_2 \in D, |s_1 - s_2| < \delta. \tag{1.3}$$

Then letting $i \to \infty$, we obtain

$$|\bar{\varphi}(s_1) - \bar{\varphi}(s_2)| \leq \epsilon \text{ for every}\, s_1, s_2 \in D, |s_1 - s_2| < \delta.$$

This means that $\bar{\varphi} : D \to \mathbb{R}^n$ is uniformly continuous on D. Consequently, we may extend $\bar{\varphi}(\cdot)$ on $\bar{D} = [t,T]$ which is still continuous. Finally, for any $\epsilon > 0$, let $\delta > 0$ be such that (1.3) holds and by compactness of $[t,T]$, let $S_\delta = \{s_j, 1 \le j \le M\} \subseteq D$ with $M > 1$ depending on $\epsilon > 0$ such that

$$[t,T] \subseteq \bigcup_{j=1}^{M} (s_j - \delta, s_j + \delta).$$

Next, we may let $i_0 > 1$ such that

$$|\bar{\varphi}_i(s_j) - \bar{\varphi}(s_j)| < \epsilon,\ i \ge i_0,\ 1 \le j \le M.$$

Then for any $s \in [t,T]$, there is an $s_j \in S_\delta$ such that $|s - s_j| < \delta$. Consequently

$$|\bar{\varphi}_i(s) - \bar{\varphi}(s)| \le |\bar{\varphi}_i(s) - \bar{\varphi}_i(s_j)| + |\bar{\varphi}_i(s_j) - \bar{\varphi}(s_j)| + |\bar{\varphi}(s_j) - \bar{\varphi}(s)| \le 3\epsilon.$$

This show that $\varphi_i(\cdot)$ converges to $\bar{\varphi}(\cdot)$ uniformly in $s \in [t,T]$. ■

Theorem 3. *[Banach-Saks] Let $\varphi_k(\cdot) \in L^2(a,b;\mathbb{R}^n)$ be a sequence which is weakly convergent to $\bar{\varphi}(\cdot) \in L^2(a,b;\mathbb{R}^n)$, i.e.,*

$$\lim_{k\to\infty} \int_a^b \langle \varphi_k(s) - \bar{\varphi}(s), \eta(s)\rangle ds, \textit{ for every } \eta \in L^2(a,b;\mathbb{R}^n).$$

Then there is a subsequence $\{\varphi_{k_j}(\cdot)\}$ such that

$$\lim_{k\to\infty} ||\frac{1}{N}\sum_{j=1}^{N} \varphi_{k_j}(\cdot) - \bar{\varphi}(\cdot)||_{L^2(a,b;\mathbb{R}^n)} = 0.$$

Proof. Whitout loss of generality, first we consider that $\bar{\varphi}(\cdot) = 0$. Let $k_1 = 1$. By the weak convergence of $\varphi_k(\cdot)$, we may find $k_1 < k_2 < \cdots < k_N$ such that

$$|\int_a^b \langle \varphi_{k_i}(s), \varphi_{k_j}(s)\rangle ds| < \frac{1}{N}\ 1 \le i < j \le N$$

observe

$$\begin{aligned}
||\frac{1}{N}\sum_{i=1}^{N}\varphi_{k_i}(\cdot)||^2_{L^2(a,b;\mathbb{R}^2)} &= \frac{1}{N^2}\int_a^b|\sum_{i=1}^{N}\varphi_{k_i}(s)|^2ds = \frac{1}{N^2}\int\sum_{i,j=1}^{N}\langle\varphi_{k_i}(s),\varphi_{k_j}(s)\rangle ds\\
&= \frac{1}{N^2}\sum_{i=1}^{N}||\varphi_{k_i}(\cdot)||^2_{L^2(a,b;\mathbb{R}^n)} + \frac{2}{N^2}\sum_{1\leq i<j\leq N}\int_a^b\langle\varphi_{k_i}(s),\varphi_{k_j}(s)\rangle ds\\
&\leq \frac{1}{N}\sup_{i\geq 1}||\varphi_{k_i}(\cdot)||^2_{L^2(a,b;\mathbb{R}^n)} + \frac{2}{N^3}\frac{N(N-1)}{2}\\
&\leq \frac{1}{N}\sup_{i\geq 1}||\varphi_{k_i}(\cdot)||^2_{L^2(a,b;\mathbb{R}^n)} + \frac{1}{N}\to 0,
\end{aligned}$$

when $N\to 0$. Now, consider that $\bar{\varphi}(\cdot)\neq 0$, thus $\varphi_k(\cdot)-\bar{\varphi}(\cdot)=0$, and we can apply the previous steps. ∎

Lemma 4. *[Filippov] Let U be a complete separable metric space whose metric is denoted by $d(\cdot,\cdot)$. Let $g:[0,T]\times U\to\mathbb{R}^n$ be a map which is measurable in $t\in[0,T]$ and*

$$|g(t,u)-g(t,v)|\leq\omega(d(u,v)),\ \textit{for every}\ u,v\in U,\ t\in[0,t],$$

for some continuous and increasing function $\omega:\mathbb{R}_+\to\mathbb{R}_+$ with $\omega(0)=0$, called a modulus of continuity. Moreover,

$$0\in g(t,U)\ \ a.e.\ t\in[0,t].$$

Then there exists a measurable map $u:[0,T]\to U$, such that

$$g(t,u(t))=0\ a.e.\ t\in[0,T]. \tag{1.4}$$

Proof. Define $\bar{d}(u,v)=\frac{d(u,v)}{1+d(u,v)}<1$ for every $u,v\in U$. Then with this new metric $\bar{d}$, U still complet and separable. Hence without loss of generality, we assume that the original metric $d(\cdot,\cdot)$ already satisfies $d(u,v)<1$ for every $u,v\in U$.

Next, we define

$$\Gamma(t) := \{u \in U | g(t,u) = 0\},\ t \in [0,t].$$

We have that $\Gamma(t) \neq \emptyset$, because $0 \in g(t,U)$ *a.e.* $t \in [0,t]$. Let $U_0 := \{v_k | k \geq 1\}$ be a countable dense subset of U. We claim that for any $u \in U$ and $0 \leq c < 1$,

$$\{t \in [0,T] |\, d(u,\Gamma(t)) \leq c\} = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{t \in [0,T] |\, d(u,v_j) \leq c + \frac{1}{i}, |g(t,v_j)| \leq \frac{1}{i}\}, \tag{1.5}$$

where

$$d(u,\Gamma(t)) := \inf_{v \in \Gamma(t)} d(u,v).$$

To show (1.5), we note that $t \in [0,T]$, with $d(u,\Gamma(t)) \leq c$ if and only if there exists a sequence $u_k \in \Gamma(t)$, i.e., $g(t,u_k) = 0$, such that

$$d(u,u_k) \leq c + \frac{1}{k}.$$

Since $\bar{U}_0 = U$, there exists a sequence $v_{j_k} \in U_0$ such that

$$d(u_k, v_{j_k}) < \frac{1}{k}.$$

Hence, by triangle inequality

$$d(u, v_{j_k}) \leq d(u,u_k) + d(u_k, v_{j_k}) \leq c + \frac{2}{k}.$$

Next, by the uniform continuity of $u \mapsto g(t,u)$, we have

$$|g(t,v_{j_k})| \leq |g(t,v_{j_k}) - g(t,u_k)| \leq \omega(d(v_{j_k},u_k)) \leq \omega\left(\frac{1}{k}\right).$$

Hence, one has

$$\begin{cases} \lim_{k\to\infty} d(u, v_{j_k}) \le c, \\ \lim_{k\to\infty} g(t, v_{j_k}) = 0. \end{cases}$$

Thus, $d(u, \Gamma(t)) \le c$ if and only if for any $i \ge 1$, there exists a $j \ge i$, such that

$$\begin{cases} \bar{d}(u, v_j) \le c + \frac{1}{i}, \\ |g(t, v_j)| \le \frac{1}{i}. \end{cases}$$

This prove (1.5). Since the right-hand side of (1.5) is measurable, so is the let-hand side. On the other hand,

$$\begin{cases} \{t \in [0,T] |\, d(u, \Gamma(t)) \le c\} = [0,T], \text{ for every}\, c \ge 1, \\ \{t \in [0,T] |\, d(u, \Gamma(t)) \le c\} = \emptyset, \text{ for every}\, c < 0. \end{cases}$$

Hence, the function $t \mapsto d(u, \Gamma(t))$ is measurable. Now, we define

$$u_0(t) := v_1, \text{ for every}\, t \in [0,T].$$

Clearly, $u_0(t)$ is measurable and

$$d(u_0(t), \Gamma(t)) < 1\, \text{for every}\, t \in [0,T].$$

Suppose that we have defined $u_{k-1}(\cdot)$ such that

$$\begin{cases} d(u_{k-1}(t), \Gamma(t)) \le 2^{1-k}, \\ d(u_{k-1}(t), u_{k-2}(t)) \le 2^{2-k}, \end{cases} \quad t \in [0,T]. \tag{1.6}$$

We define the sets

$$\begin{cases} C_i^k := \{t \in [0,T] | \, d(v_i, \Gamma(t)) < 2^{-k}\}, \\ D_i^k := \{t \in [0,T] | \, d(v_i, u_{k-1}(t)) < 2^{1-k}\}. \end{cases}$$

Since $t \mapsto d(v_i, \Gamma(t))$ is measurable, C_i^k is measurable. Likewise, D_i^k is also measurable. Set

$$A_i^k = C_i^k \cap D_i^k, \; k, i \geq 1.$$

Then A_i^k is measurable as well. We claim that

$$[0,T] = \bigcup_{i=1}^{\infty} A_i^k \text{ for every } k \geq 1. \tag{1.7}$$

In fact, for any $t \in [0,T]$, by (1.6), there exists a $u \in \Gamma(t)$ such that

$$d(u_{k-1}(t), u) < 2^{1-k}.$$

By the density of U_0 in U, there exists an $i \geq 1$ such that

$$d(v_i, \Gamma(t)) \leq \begin{cases} d(v_i, u) < 2^{-k}, \\ d(v_i, u_{k-1}(t)) < 2^{1-k}, \end{cases}$$

which means $t \in A_i^k$, proving (1.7). Now we define $u_k(\cdot) : [0,T] \to U_0 \subseteq U$ as follows:

$$u_k(t) = v_i, \text{ for every } t \in A_i^k \backslash \bigcup_{j=1}^{\infty} A_j^k.$$

By $t \in C_i^k$, we have

$$d(u_k(t), \Gamma(t)) < 2^{-k},$$

and by $t \in D_i^k$, we have

$$d(u_k(t), u_{k-1}(t)) < 2^{1-k}.$$

This completes the construction of the sequence $\{u_k(\cdot)\}$ inductively. Clearly, (1.6) holds for every $k \geq 1$. This als implies that for each $t \in [0, T]$, $\{u_k(t)\}$ is Cauchy in U. By completeness of Um we obtain

$$\lim_{k\to\infty} u_k(t) = u(t),\ t \in [0, t].$$

Of course, $u(\cdot)$ is measurable, and moreover, by the closeness of $\Gamma(t)$, we have

$$u(t) \in \Gamma(t), \text{ for every } t \in [0, T].$$

This means (1.4) holds. ∎

Proposition 1. *[Gronwall's Inequality] Let $\theta : [a, b] \to \mathbb{R}_+$ be continuous and satisfy*

$$\theta(s) \leq \alpha(s) + \int_a^s \beta(r)\theta(r)dr,\ \ s \in [a, b],$$

for some $\alpha(\cdot), \beta(\cdot) \in L^1(a, b; \mathbb{R}_+)$. Then

$$\theta(s) \leq \alpha(s) + \int_a^s \alpha(\tau)\beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau,\ \ s \in [a, b]. \tag{1.8}$$

In particular, if $\alpha(\cdot) = \alpha$ is a constant, then

$$\theta(s) \leq \alpha e^{\int_a^s \beta(r)dr},\ \ s[a, b]. \tag{1.9}$$

Proof. Let $\varphi(s) = \int_a^s \beta(r)\theta(r)dr.$, by the fundamental theorem of calculus, we have

$$\dot{\varphi}(s) = \beta(s)\theta(s) \leq \beta(s)[\alpha(s) + \varphi(s)].$$

This leads to

$$[\varphi(s)e^{-\int_a^s \beta(r)dr}]' \leq \alpha(s)\beta(s)e^{-\int_a^s \beta(r)dr}.$$

Consequently,

$$\varphi(s)e^{-\int_a^s \beta(r)dr} \leq \int_a^s \alpha(\tau)\beta(\tau)e^{-\int_a^\tau \beta(r)dr}d\tau.$$

$$\varphi(s) \leq \int_a^s \alpha(\tau)\beta(\tau)e^{-\int_a^\tau \beta(r)dr}e^{\int_a^s \beta(r)dr}d\tau.$$

rewriting

$$\varphi(s) \leq \int_a^s \alpha(\tau)\beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau.$$

Hence,

$$\theta(s) \leq \alpha(s) + \int_a^s \alpha(\tau)\beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau.$$

(1.8) holds. Now, consider α constant, then

$$\theta(s) \leq \alpha + \int_a^s \alpha\beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau.$$

By integration rules, if $u = \alpha, dv = \beta(\tau)e^{\int_\tau^s \beta(r)dr}d\tau$. In the other hand,

$$\frac{d}{d\tau}e^{\int_\tau^s \beta(r)dr} = \frac{d}{d\tau}e^{-\int_s^\tau \beta(r)dr} = -\beta(\tau)e^{-\int_s^\tau \beta(r)dr} = -\beta(\tau)e^{\int_\tau^s \beta(r)dr}$$

Then

$$\theta(s) \leq \alpha - \alpha\int_a^s \frac{d}{d\tau}e^{\int_\tau^s \beta(r)dr}d\tau = \alpha - \alpha[e^0 - e^{\int_a^s \beta(r)dr}].$$

Therefore,

$$\theta(s) \leq \alpha e^{\int_a^s \beta(r)dr},$$

(1.9) holds. ■

1.2 Control Systems

In this section, we presente the assumptions for the prove of existence of a optimal pair.

We assume that U is a non-empty closed subset in $\mathbb{R}^n$ (it could be generally a separable complete metric space). For any initial pair $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, we rewrite the control system here:

$$\begin{cases} \dot{X}(s) = f(s, u(s), X(s)), \; s \in [t, \infty) \\ X(t) = x. \end{cases} \tag{1.10}$$

Let us begin with the following assuption:

(C1) The map $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable and there exists a constan $L > 0$ such that

$$\begin{cases} |f(t, u, x_1) - f(t, u, x_2)| \leq L|x_1 - x_2|, \; (t, u) \in \mathbb{R}_+ \times U, \; x_1, x_2 \in \mathbb{R}^n, \\ |f(t, u, 0)| \leq L, \text{ for every } (t, u) \in \mathbb{R}_+ \times U. \end{cases}$$

Note that this conditions imply

$$|f(t, u, x)| \leq L(1 + |x|), \; (t, u, x) \in \mathbb{R}_+ \times U \times \mathbb{R}^n.$$

This condition is also usually called the lipschitz condition of the function f. A key feature of the above is that the bond of $|f(t, u, x)|$, depending on $|x|$, is uniform in u.

Proposition 2. *Let (C1) hold. Then, for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$, and $u(\cdot)\in\mathcal{U}[t,\infty)$, there exits a unique solution $X(\cdot)\equiv X(\cdot;t,x,u(\cdot))$ to the state equation (1.10). Moreover, the following estimate hold:*

$$\begin{cases} |X(s;t,x,u(\cdot))|\le e^{L(s-t)}(1+|x|)-1, \\ |X(s;t,x,u(\cdot))-x|\le [e^{L(s-t)}-1](1+|x|),\ s\in(t,\infty],\ u(\cdot)\in\mathcal{U}[t,\infty). \end{cases} \tag{1.11}$$

Further, for any $t\in\mathbb{R}_+$, $x_1,x_2\in\mathbb{R}^n$, and $u(\cdot)\in\mathcal{U}[t,\infty)$,

$$|X(s;t,x_1,u(\cdot))-X(s;t,x_2,u(\cdot))|\le e^{L(s-t)}|x_1-x_2|,\ \textit{for every}\, s\in[t,\infty). \tag{1.12}$$

Proof. It suffices to prove our conclusion on any interval $[t,T]$ with $0\le t<T<\infty$, Since $[t,\infty)=\bigcup_{T\ge t}[t,T]$.

For any $X(\cdot)\in C([t,T];\mathbb{R}^n)$, we define

$$[\mathcal{S}X(\cdot)](s):=x+\int_t^s f(r,u(r),X(r))dr,\ s\in[t,T].$$

Then for any $X_1(\cdot),X_2(\cdot)\in C([t,T];\mathbb{R}^n)$ and by condition (C1), we have

$$\begin{aligned} ||[\mathcal{S}X_1(\cdot)]-[\mathcal{S}X_2(\cdot)]||_{C([t,T];\mathbb{R}^n)} &= ||\int_t^s f(r,u(r),X_1(r))dr-\int_t^s f(r,u(r),X_2(r))dr|| \\ &\le L||\int_t^s (X_1(r)-X_2(r))dr|| \\ &\le L||X_1(\cdot)-X_2(\cdot)||_{C([t,T];\mathbb{R}^n)}|t-s|. \end{aligned}$$

Define $\delta=|t-s|$, then

$$||[\mathcal{S}X_1(\cdot)]-[\mathcal{S}X_2(\cdot)]||_{C([t,T];\mathbb{R}^n)}\le\delta L||X_1(\cdot)-X_2(\cdot)||_{C([t,T];\mathbb{R}^n)}.$$

Therefore, by chosing $\delta < \frac{1}{L}$, we see that $\mathcal{S} : C([t,T];\mathbb{R}^n) \to C([t,T];\mathbb{R}^n)$ is contractive. Hence, by Contraction Mapping Theorem (1), the control system (1.10) admits a unique solution on $[t, t+\delta]$. Repeating the argument, we can obtain that (1.10) admits a unique solution on $[t, T]$.

Next, for the unique solution $X(\cdot)$ of (1.10), we have

$$|X(s)| \leq |x| + L\int_t^s (1 + |X(r)|)dr,\ s \in [t,T].$$

If we denote the right-hand side of the above by $\theta(s)$, then

$$\dot{\theta}(s) = L + L|X(s)| \leq L + L\theta(s),$$

which leads to

$$\dot{\theta}(s) \leq e^{L(s-t)}|x| + L\int_t^s e^{L(s-r)dr} = e^{L(s-t)}|x| + e^{L(s-t)} - 1,$$

thus, $|X(s)| \leq e^{L(s-t)}(1+|x|) - 1$. This gives the first estimate in (1.11). Next, we will apply the first estimate

$$\begin{aligned} |X(s) - x| &= |\int_t^s f(r, u(r), X(r))dr| \leq \int_t^S |f(r, u(r), X(r))|dr \\ &\leq L\int_t^s (1 + |X(r)|)dr \leq L\int_t^s e^{L(r-t)}(1+|x|)dr \\ &= (1+|x|)[e^{L(s-t)} - 1]. \end{aligned}$$

This prove the second estimate in (1.11). finally, for $x_1, x_2 \in \mathbb{R}^n$, let us denote $X_i(\cdot) = X(\cdot; t, x_i, u(\cdot))$. Then

$$|X_1(s) - X_2(s)| \leq L\int_t^s |X_1(r) - X_2(r)|dr.$$

By Gronwall's Inequality (1), we obtain (1.12). ∎

We can see that estimates (1.11) are uniform in $u(\cdot) \in \mathcal{U}[t,T]$. This will paly an interesting role later.

1.3 Existence Theory

Let $T \in (0,\infty)$ be fixed. Consider the control system

$$\begin{cases} \dot{X}(s) = f(s,u(s),X(s) \; s \in [t,T], \\ X(t) = x, \end{cases}$$

with termimal state constraint

$$X(T;t,x,u(\cdot)) \in M,$$

where $M \subseteq \mathbb{R}^n$ is fixed, and with cost functional

$$J^T(t,x;u(\cdot)) = \int_t^T g(s,u(s),X(s))ds + h(X(T)).$$

We recall

$$\tilde{\mathcal{U}}_x^M[t,T] = \{u(\cdot) \in \mathcal{U}[t,T] | X(T;t,x,u(\cdot)) \in M\},$$

and recall the following optimal control problem:

Problem $(OC)^T$. For given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ with $\tilde{\mathcal{U}}_x^M[t,T] \neq \emptyset$, find a $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]$ such that

$$J^T(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]} J^T(t,x;u(\cdot)) \equiv V(t,x). \tag{1.13}$$

Any $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]$ satisfying (1.13) is called an optimal control, the corresponding $\bar{X}(\cdot) \equiv X(\cdot;t,x,\bar{u}(\cdot))$ is called an optimal state trajectory, and $(\bar{u}(\cdot),\bar{X}(\cdot))$ is called optimal pair, we call $V(\cdot,\cdot)$ the value functional of problem $(OC)^T$.

We introduce the following assumption for the cost functional:

(C2) The maps $g:\mathbb{R}_+\times U\times\mathbb{R}^n\to\mathbb{R}$ and $h.\mathbb{R}^n\to\mathbb{R}$ are measurable and there exists a continuous function $\omega:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, called a local modulus of continuity, which is increasing in each argument, and $\omega(r,0)=0$ for every $r\geq 0$, such that

$$|g(s,u,x_1)-g(s,u,x_2)|+|h(x_1)-h(x_2)|\leq\omega(|x_1|\vee|x_2|,|x_1-x_2|) \text{ for every } (s,u)\in\mathbb{R}_+\times U, x_1,x_2\in\mathbb{R}^n,$$

where $|x_1|\vee|x_2|=\max\{|x_1|,|x_2|\}$, and

$$\sup_{(s,u)\in\mathbb{R}_+\times U}|g(s,u,0)|\equiv g_0<\infty.$$

For any $(t,x)\in[0,T]\times\mathbb{R}^n$, let us introduce the following set

$$\mathbb{E}(t,x)=\{(z^0,z)\in\mathbb{R}\times\mathbb{R}_+|z^0\geq g(t,u,x), z=f(t,u,x),\ u\in U\}.$$

The following assumption gives some compability between the control system and the cost functional.

(C3) For almost all $t\in[0,T]$, the following Cesari property holds at any $x\in\mathbb{R}^n$,

$$\bigcap_{\delta>0}\bar{co}\mathbb{E}(t,B_\delta(x))=\mathbb{E}(t,x),$$

where, we recall that $B_\delta(x)$ is the open ball centered at x with radius $\delta>0$, and $\bar{co}(E)$ stands for the closed convex hull of the set E (the smallest closed convex set containing E).

It is cleary that if $\mathbb{E}(t,x)$ has Cesari property at x, the $\mathbb{E}(t,x)$ is convex and closed.

Theorem 5. *Let (C1)-(C3) hold. Let $M \subseteq \mathbb{R}^n$ be a non-empty closed set. Let $(t,x) \in [0,T] \times \mathbb{R}^n$ be given and $\tilde{\mathcal{U}}_x^M[t,T] \neq \emptyset$. Then problem $(OC)^T$ admits at least one optimal pair.*

Proof. Let $u_k(\cdot) \in \tilde{\mathcal{U}}_x^M[t,T]$ be a minimizing sequence. By proposition (2)

$$|X_k(s)| \leq e^{L(s-t)}(1+|x|) - 1,\ s \in [t,T]\ k \geq 1. \tag{1.14}$$

and for any $t \leq \tau < s \leq T$,

$$\begin{aligned} |X_k(s) - X_k(\tau)| &= |X(s;t,x,u_k(\cdot)) - X(\tau;t,x,u(\cdot))| \\ &\leq [e^{L(s-\tau)} - 1][1 + X(\tau;t,x,u_k(\cdot))] \\ &\leq [e^{L(s-\tau)} - 1]e^{L(\tau-t)}(1+|x|). \end{aligned}$$

Hence, the sequence $\{X_k(\cdot)\}$ is uniformly bounded and equicontinuous. Therefore, By Arzela-Ascoli Theorem (2), we may assume that the sequence is convergent to some $\bar{X}(\cdot)$ in $C([t,T];\mathbb{R}^n)$. On the other hand,

$$|f(d,u_k(s),X_k(s)| \leq L(1+|X_k(s)) \leq Le^{L(s-t)}(1+|x|).$$

Also, by (1.14) and (C2), we have

$$\begin{aligned} |g(s,u_k(s),X_k(s))| &\leq |g(s,u_k(s),0)| + |g(s,u_k(s),X_k(s)) - g(s,u_k(s),0)| \\ &\leq g_0 + \omega(|X_k(s)|,|X_k(s)|) \leq g_0 + \omega(e^{LT}(1+|x|),e^{LT}(1+|x|)) \\ &\leq K,\ s \in [t,T],\ k \geq 1. \end{aligned}$$

Hence, by exctrcting a subsequence if necessary, we may assume that

$$\begin{cases} g(\cdot, u_k(\cdot), X_k(\cdot)) \to \bar{g}(\cdot), \text{ weakly in } L^2([t,T];\mathbb{R}^n), \\ f(\cdot, u_k(\cdot), X_k(\cdot)) \to \bar{f}(\cdot), \text{ weakly in } L^2([t,T];\mathbb{R}^n), \end{cases}$$

for some $\bar{g}(\cdot)$ and $\bar{f}(\cdot)$. Then by Banach-Saks Theorem (3), we have

$$\begin{cases} \tilde{g}_k(\cdot) := \frac{1}{k}\sum_{i=1}^k g(\cdot, u_i(\cdot), X_i(\cdot)) \to \bar{g}(\cdot), \text{ strongly in } L^2([t,T];\mathbb{R}^n), \\ \tilde{f}_k(\cdot) := \frac{1}{k}\sum_{i=1}^k f(\cdot, u_i(\cdot), X_i(\cdot)) \to \bar{f}(\cdot), \text{ strongly in } L^2([t,T];\mathbb{R}^n). \end{cases} \tag{1.15}$$

On the other hand, by (C1) and the convergence of $X_k(\cdot) \to \bar{X}(\cdot)$ in $C([t,T];\mathbb{R}^n)$, we have

$$\begin{aligned} |\tilde{f}_k(s) - \frac{1}{k}\sum_{i=1}^k f(s, u_i(s), \bar{X}(s)| &\le \frac{1}{k}\sum_{i=1}^k |f(s, u_i(s), X_i(s)) - f(s, u_i(s), \bar{X}(s))| \\ &\le \frac{1}{k}\sum_{i=1}^k |X_i(s) - \bar{X}(s)| \to 0 \ k \to \infty, \end{aligned}$$

uniformly in $s \in [t,T]$. Similarly, by (C2),

$$\begin{aligned} |\tilde{g}_k(s) - \frac{1}{k}\sum_{i=1}^k g(s, u_i(s), \bar{X}(s)| &\le \frac{1}{k}\sum_{i=1}^k |g(s, u_i(s), X_i(s)) - g(s, u_i(s), \bar{X}(s))| \\ &\le \frac{1}{k}\sum_{i=1}^k \omega(|X_i(s)| \vee |\bar{X}(s)|, |X_i(s) - \bar{X}(s)|) \to 0 \ k \to \infty, \end{aligned}$$

uniformly in $s \in [t,T]$. Next, by the definition of $\mathbb{E}(t,x)$, we have

$$\begin{pmatrix} g(s, u_i(s), X_i(s)) \\ f(s, u_i(s), X_i(s)) \end{pmatrix} \in \mathbb{E}(s, X_i(s)), \ i \ge 1, \ s \in [t,T].$$

Hence, for any $\delta > 0$, there exits a $K_\delta > 0$ such that

$$\begin{pmatrix} \tilde{g}_k(s) \\ \tilde{f}_k(s) \end{pmatrix} \in \bar{co}\mathbb{E}(s, B_\delta(\bar{X}(s))),\ K \geq K_\delta,\ s \in [t, T]. \tag{1.16}$$

Combinig (1.15) and (1.16), using (C3), we obtain

$$\begin{pmatrix} \bar{g}(s) \\ \bar{f}(s) \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} \tilde{g}_k(s) \\ \tilde{f}_k(s) \end{pmatrix} \in \bigcap_{\delta>0} \bar{co}\mathbb{E}(s, B_\delta(\bar{X}(s))) = \mathbb{E}(s, \bar{X}(s)).$$

Then by Filippov's Lemma (4), there exits a $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ such that

$$\bar{g}(s) \geq g(s, \bar{u}(s), \bar{X}(s)),\ \bar{f}(s) = f(s, \bar{u}(s), \bar{X}(s)),\ s \in [t, T].$$

This means $\bar{X}(\cdot) = X(\cdot; t, x, \bar{u})$. On the other hand, since

$$\bar{X}_k(T) \equiv X(T; t, x, \bar{u}_k(\cdot)) \in M,\ k \geq 1$$

one has

$$\bar{X}(T) \equiv X(T; t, x, \bar{u}(\cdot)) \in M,$$

which means that $\bar{u}(\cdot) \in \tilde{\mathcal{U}}^M_x[t, T]$. finally, by Fatou's Lemma

$$
\begin{aligned}
J^T(t,x;\bar{u}(\cdot)) &\leq \int_t^T \bar{g}(s)ds + h(\bar{X}(T)) \\
&\leq \liminf_{k\to\infty}[\int_t^T \tilde{g}_k(s)ds + h(X_k(T))] \\
&= \lim_{k\to\infty}\frac{1}{k}J^T(t,x;u_i(\cdot)) \\
\lim_{k\to\infty} J^T(t,x;u_k(\cdot)) & \\
&= \inf_{u(\cdot)\in\tilde{\mathcal{U}}_x^M[t,T]} J^T(t,x;u(\cdot)).
\end{aligned}
$$

This means that $(\bar{u}(\cdot),\bar{X}(\cdot))$ is an optimal pair.

■

1.4 Aplication

In this section, we see a optimal Control applied to vaccination and Treatment for SIR model [refer].

Optimal control theory is applied to suggest the most effective mitigation strategy to minimize the number of individuals who become infected in the course of an infection while efficiently balancing vaccination and treatment applied to the models with various cost scenarios. Optimal control is a mathematical technique derived from the calculus of variations.

Accidental and intentional introduction of infectious diseases to previously naive geographic regions has brought more focus and attention to the development of response plans to such scenarios.

All levels of government and public health officials are searching for answers to identify the best strategies for intervention prior to what may be an inevitable event.

The first step is to represent the epidemiology of the disease divided in subpopulations, called compartments, that represent the various stage of the disease progression.

For example, Pontryagin's maximum principle allows the calculation of the optimal control for an ordinary differential equation model system with a given constraint.

The question we study in this case is whether this underlying epidemic structure significantly impacts the predicted optimal control strategy for administering vaccination and / or treatment.

We present the SIR model analysis in detail including, the proof of uniqueness and existence of the optimal control system. In this model, we consider a logistic growth as defined by the following state equation:

$$\dot{S} = \mu N - \beta\frac{SI}{N} - \nu S - \mu\frac{NS}{K}, \tag{1.17}$$

$$\dot{I} = \beta\frac{SI}{N} - (\gamma + \tau + \delta)I - \mu\frac{IN}{K}, \tag{1.18}$$

$$\dot{R} = (\gamma + \tau)I + \nu S - \mu\frac{NR}{K}, \tag{1.19}$$

subject to the boundary conditions

$$S(0) = S_0,\ I(0) = I_0,\ R(0) = R_0. \tag{1.20}$$

The variables, S, I, R represent the susceptible, infections and removed classes, respectively. We assume a logistic growth of the total population $N = S + I + R$ with carrying capacity K, and assume the new births enter in the susceptible class.

β : approximates the average number of contacts with infectious individuals needed to make one person ill in each unit of time. μ : approximates the average number of individuals that birth/death in each unit of time.

γ : approximates the average number of infectious individuals removed from infection class to removed class in each unit of time.

δ : approximates the average number of individuals that dies by the disease in each unit of time.

We see that $\beta\frac{SI}{N}$ is the number of susceptibles becomes infectious in each unit of time, $\mu\frac{SN}{K}$ is the number of susceptibles that dies in each unit of time, similarly in the other terms.

$\nu(t)$: control function measures the rates at which susceptibles are vaccinated in each time period.

$\tau(t)$: control function measures the rates at which infectious individuals are treated in each time period.

Note that control ν moves individuals from the S class to the R class and the control τ moves individuals from the I class to the R class.

Given initial population sizes S_0, I_0, R_0, we seek the best mitigation strategy for the outbreak modeled in equation (**??**)-(**??**) by optimally defining bounded, Lebesgue integrable control functions $\nu(t)$ and $\tau(t)$.

Our goal is to minimize the number of people who become infected, and thus also the number of people who die due to the infection, while also minimizing the effort of vaccinating and treating the population. Thus we seek minimizing the following objective functional

$$J(\nu,\tau) = \int_0^T [B_1 I(t) + B_2[\frac{R(t)}{K}]^m \nu^2(t) + B_3\tau^2(t)]dt, \tag{1.21}$$

where $m \geq 1$.

The constants B_1, B_2, B_3 have a dual role. On one hand, they are needed to balance the units in the integrand and ν and τ are treatment rates and will be necessarily lies between 0 and 1.

The remaining term in our objective functional seeks to increase the expense of vaccination when most of the population has either been vaccinated or has immunity from a prior infection.

We assume there are practical limitations on the maximum rate at which individuals who may be vaccinated or treated in a given time period and we define positive constants ν_{max} and τ_{max} accordingly. We define the set of admissible controls to be

$$\Omega = \{(\nu, \tau) \in L^1(0,T) | (\nu(t), \tau(t)) \in [0, \nu_{max}] \times [0, \tau_{max}], \text{for every}\, t \in [0,T]\}. \tag{1.22}$$

We seek an optimal control pair (ν^*, τ^*) such that

$$J(\nu^*, \tau^*) = \min_{\Omega} J(\nu, \tau). \tag{1.23}$$

Now, we need deduce the sufficient conditions for the existence of a solution to the optimal control problem.

A high death rate from disease could theoretically cause the population to vanish, and the existence theorem must assume that the external death rate δ is smaller that the population intrinsic growth, or turnover, rate μ.

Theorem 6. *There exists an optimal control pair $\nu^*(t), \tau^*(t)$, and corresponding solution S^*, I^*, R^* to the state initial value problem (1.17)-(1.20) that minimizes $J(\nu, \tau)$ over Ω.*

Proof. We referer to the conditions in Theorem III.4.1 and its Corollary in Flaming and Rishel []. The requirements there on the set of admissible controls Ω and on the set of end conditions are clearly met here.

The following nontrivial requirements from Fleming and Rishel's theorem are listed and later verified below:

The set of all solutions to system (1)-(4) with corresponding control functions in Ω is nonempty. The state system can be written as a linear function of the control variables with coefficients dependent on time and the state variables. The integrand L in equation (1.22) is convex on Ω and additionally satisfies $L(t, S, I, R, \nu, \tau) \geq c_1 |(\nu, \tau)|^{\beta} - c_2$, where

$c_1 > 0$ and $\beta > 1$. In order to establish condition a), we refer to Theorem 3.1, by Picard-Lindelof, (existence theorem of solution for EDO). If the solutions to the state equations are a priori bounded and if the state equations are continuous and Lipschitz in the state variables, then there is a unique solution corresponding to every admissible control pair in Ω.

Thus, we begin establishing bounds on $N = S + I + R$, and, by extension, S, I, R. Note that N satisfies the modified logistic equation

$$\dot{N} = \mu N(1 - \frac{N}{K}) - \delta I, \tag{1.24}$$

and assume that $N(0) = N_0$. We can see that, if $N > K$, then N is decreasing. Thus, N is bounded about by N_0. If $N < K$, then N is increasing. Thus, again N is bounded by K. This imply that N is bounded by $\max(N_0, K)$ and $\min(N_0, K)$. Note that by extension, we can see that S, I, R are upper bounded by the same bound for N.

With bounds established above, its follows that the state system is continuous and bounded. it is equally direct to show the boundedness of the partial derivates with respect to the state variables in the state system, which establishes that the system is Lipschitz with respect to the state variables. This completes the proof that condition a) holds.

Condition b) is verified by observing the linear dependence of the state equations on controls ν and τ.

Finally, to verify condition c) we note that since the integrand L of the objective functional is quadratic in the controls. L is convex in the controls. To prove that bound on the L we note that by the definition of Ω we have $B_2\nu^2 \leq B_2$, and $B_2\nu^2 - B_2 \leq 0$. Thus $L = B_1 I(t) + B_2[\frac{R}{K}]^m \nu^2(t) + B_3\tau^2(t) \geq B_3\tau^2(t) \geq B_2\nu^2 + B_3\tau^2 - B_2 \geq \min(B_2, B_3)(\nu^2 + \tau^2) - B_2$. ■

Now, we apply Pontryagin's Maximun Principle and convert the optimization problem to the problem of finding the point-wise minimun relative to ν and τ of the Hamiltonian.

Theorem 7. *Given the optimal control pair (ν^*, τ^*) and corresponding solutions to the state system (1.17)-(1.20) S^*, I^*, R^* that minimize the objetive functional (1.21) there exist adjoint variables λ_1, λ_2 and λ_3 satisying*

$$\begin{aligned}
\dot{\lambda_1} &= \frac{\mu}{K}[\lambda_1(S+N-K)+\lambda_2 I+\lambda_3 R]+\beta\frac{I(N-S)}{N^2}(\lambda_1-\lambda_2)+\nu(\lambda_1-\lambda_3), && (1.25)\\
\dot{\lambda_2} &= -B_1+\frac{\mu}{K}[\lambda_1(S-K)+\lambda_2(N+I)+\lambda_3 R]+\beta\frac{S(N-I)}{N^2}(\lambda_1-\lambda_2)\\
&+ \delta\lambda_2+(\gamma+\tau)(\lambda_2-\lambda_3), && (1.26)\\
\dot{\lambda_3} &= mB_2\nu^2\frac{R^{m-1}}{K^m}+\frac{\mu}{K}[(S-K)\lambda_1+\lambda_2 I+(N+R)\lambda_3]+\beta\frac{SI}{N^2}(\lambda_2-\lambda_1), && (1.27)
\end{aligned}$$

with transversality conditions

$$\lambda_1(T)=\lambda_2(T)=\lambda_3(T)=0. \tag{1.28}$$

Fuethermore, as long as the optimal removed class R^* is nonzero, we may characterize the optimal pair by the continuous functions

$$\begin{aligned}
\nu^* &= \min\left(\max\left(0,\frac{S^*(\lambda_1-\lambda_3)}{2B_2[\frac{R^*}{K}]^m}\right),\nu_{max}\right),\\
\tau^* &= \min\left(\max\left(0,\frac{I^*(\lambda_2-\lambda_3)}{2B_3}\right),\tau_{max}\right). && (1.29)
\end{aligned}$$

Proof. The result follows from a direct application o a version of Pontryagin's Maximum Principle for bounded controls [Morton]. We form the Hamiltonian H:

$$H=g(t,u(t),X(t))+\lambda_1\dot{S}+\lambda_2\dot{I}+\lambda_3\dot{R}.$$

Thus,

$$
\begin{aligned}
H &= B_1 I + B_2[\frac{R}{K}]^m \nu^2 + B_3\tau^2 + \lambda_1[\nu N - \mu\frac{N}{K}S - \beta\frac{SI}{N} - \nu S] \\
&+ \lambda_2[\beta\frac{SI}{N} - (\delta+\gamma+\tau)I - \mu\frac{NI}{K}] + \lambda_3[(\gamma+\tau)I + \nu S - \mu\frac{NR}{K}]. \qquad (1.30)
\end{aligned}
$$

Again, by the Pontryagin's Maximum Principle, The adjoint equations are given by the equations $\dot{\lambda_1} = -\frac{\partial H}{\partial S}$, $\dot{\lambda_2} = -\frac{\partial H}{\partial I}$, $\dot{\lambda_3} = -\frac{\partial H}{\partial R}$, and must satisfy transversality conditions $\lambda_i(T) = 0$ for $i = 1, 2, 3$.

Finally, the optimal conditions dictate that $\frac{\partial H}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0$, then

$$\frac{\partial H}{\partial \nu} = 2B_2\left[\frac{R}{K}\right]^m \nu - \lambda_1 S + \lambda_3 S\,\Big|_{(S^*,I^*,R^*)} = 0,$$

and we get

$$\nu^* = \frac{S^*(\lambda_1 - \lambda_3)}{2B_2\left[\frac{R}{K}\right]^m}.$$

$$\frac{\partial H}{\partial \tau} = 2B_3\tau - \lambda_2 I + \lambda_3 I\,\Big|_{(S^*,I^*,R^*)} = 0,$$

and we get

$$\tau^* = \frac{I^*(\lambda_2 - \lambda_3)}{2B_3}.$$

This values are not necessay positive, then we need max $\left(0, \frac{S^*(\lambda_1-\lambda_3)}{2B_2\left[\frac{R}{K}\right]^m}\right)$, max $\left(0, \frac{I^*(\lambda_2-\lambda_3)}{2B_3}\right)$, and by boundness we get (1.29). Observe that by the characterization of the controls given in (1.29) and the nonzero assumption for R, it follows that the controls are continuous in time. ■

The optimality system is defined as is the compilation of the state equations (1.17)-(1.19), the initial conditions (1.20), the adjoint equations (1.25)-(1.27), and the transversality conditions (1.28), with the optimality equations (1.29) sustituted into the state and adjoint equations.

In the previous sections we present results or which we guarantee the existence of an optimal pair, but the uniqueness off this pair can not be guaranteed. Therefore, we present, for this system, the following result in which we guarantee the uniqueness of the optimal pair for a small time interval.

Theorem 8. *For T sufficiently small the optimality system is unique.*

Proof. We must first consider bounds on the adjoint system. Note that bounds for the state system were established in Theorem 6. To see that the adjoint system is bounded, we rearrange equations (1.25)-(1.27):

$$
\begin{aligned}
\dot{\lambda_1} &= \lambda_1\left[\frac{\mu}{K}(S+N-K)+\beta\frac{I(N-S)}{N^2}+\nu\right]+\lambda_2\left[\frac{\mu}{K}I-\beta\frac{I(N-S)}{N^2}\right]+\lambda_3\left[\frac{\mu}{K}R-\nu\right],\\
\dot{\lambda_2} &= -B_1+\lambda_1\left[\frac{\mu}{K}(S-K)+\beta\frac{S(N-I)}{N^2}\right]+\lambda_2\left[\frac{\mu}{K}(N+I)-\beta\frac{S(N-I)}{N^2}+\delta+\gamma+\tau\right]\\
&+ \lambda_3\left[\frac{\mu}{K}R-\gamma-\tau\right],\\
\dot{\lambda_3} &= mB_2\nu^2\frac{R^{m-1}}{K^m}+\lambda_1\left[\frac{\mu}{K}(S-K)-\beta\frac{SI}{N^2}\right]+\lambda_2\left[\frac{\mu}{K}I+\beta\frac{SI}{N^2}\right]+\lambda_3\left[\frac{\mu}{K}(N+R)\right].
\end{aligned}
$$

We recall that $N=S+I+R$, that S,I,R are nonnegative, and that $N>0$. Thus, $0\le\frac{I(N-S)}{N^2},\frac{SI}{N^2},\frac{S(N-I)}{N^2}\le 1$. We recall the bounds $0\le\nu\le\nu_{max}$ and $0\le\tau\le\tau_{max}$, and with thid substitution we see the following bounds for each equation:

$$
\begin{aligned}
\dot{\lambda_1} &= \lambda_1\left[\frac{\mu}{K}(S+N-K)+\beta\frac{I(N-S)}{N^2}+\nu\right]+\lambda_2\left[\frac{\mu}{K}I-\beta\frac{I(N-S)}{N^2}\right]+\lambda_3\left[\frac{\mu}{K}R-\nu\right]\\
&\le \lambda_1\left[\frac{\mu}{K}(S+N-K)+\beta+\nu_{max}\right]+\lambda_2\left[\frac{\mu}{K}I-\beta\right]+\lambda_3\left[\frac{\mu}{K}R-\nu_{max}\right].
\end{aligned}
$$

$$\begin{aligned}
\dot{\lambda_2} &= -B_1 + \lambda_1\left[\frac{\mu}{K}(S-K) + \beta\frac{S(N-I)}{N^2}\right] + \lambda_2\left[\frac{\mu}{K}(N+I) - \beta\frac{S(N-I)}{N^2} + \delta + \gamma + \tau\right] \\
&+ \lambda_3\left[\frac{\mu}{K}R - \gamma - \tau\right] \leq -B_1 + \lambda_1\left[\frac{\mu}{K}(S-K) + \beta\right] + \lambda_2\left[\frac{\mu}{K}(N+I) - \beta + \delta + \gamma + \tau_{max}\right] \\
&+ \lambda_3\left[\frac{\mu}{K}R - \gamma - \tau_{max}\right].
\end{aligned}$$

$$\begin{aligned}
\dot{\lambda_3} &= mB_2\nu^2\frac{R^{m-1}}{K^m} + \lambda_1\left[\frac{\mu}{K}(S-K) - \beta\frac{SI}{N^2}\right] + \lambda_2\left[\frac{\mu}{K}I + \beta\frac{SI}{N^2}\right] + \lambda_3\left[\frac{\mu}{K}(N+R)\right] \\
&\leq mB_2\nu_{max}^2 + \lambda_1\left[\frac{\mu}{K}(S-K) - \beta\right] + \lambda_2\left[\frac{\mu}{K}I + \beta\right] + \lambda_3\left[\frac{\mu}{K}(N+R)\right]
\end{aligned}$$

Hence, the adjoint system is bounded by linear systems with bounded coefficients. Thus, the sub- and super-solutions are uniformly bounded, establishing bounds for the adjoint system in finite time.

Now, suppose that there are two solutionsto the optimality system:

$$(S(t), I(t), R(t), \lambda_1(t), \lambda_2(t), \lambda_3(t)) \text{ and } (\bar{S}(t), \bar{I}(t), \bar{R}(t), \bar{\lambda}_1(t), \bar{\lambda}_2(t), \bar{\lambda}_3(t)).$$

To show that the two solutions are equivalent, it is convenient to make a change of variables. We define $s, i, r, \phi_1, \phi_2, \phi_3, \bar{s}, \bar{i}, \bar{r}, \bar{\phi}_1, \bar{\phi}_2$ and $\bar{\phi}_3$ so that

$$S(t) = e^{\alpha t}s(t),\ I(t) = e^{\alpha t}i(t)\ R(t) = e^{\alpha t}r(t),$$

$$\lambda_1(t) = e^{-\alpha t}\phi_1(t), \lambda_2(t) = e^{-\alpha t}\phi_2(t),\ \lambda_3(t) = e^{-\alpha t}\phi_3(t),$$

where α is a constant to be chosen later. The barred variables are transformed similarly. Note that the bounds for the state and adjoint variables can be extended to bounds for the new variables. With the new variables the optimality conditions become

$$\nu = \min\left(\max\left(0, \frac{e^{2\alpha t}s(t)(\phi_1(t) - \phi_3(t))}{2B_2\left[\frac{e^{\alpha t}r(t)}{K}\right]^m}\right), \nu_{max}\right), \tag{1.31}$$

$$\tau = \min\left(\max\left(0, \frac{e^{2\alpha t}i(t)(\phi_2(t) - \phi_3(t))}{2B_3}\right), \tau_{max}\right), \tag{1.32}$$

and likewise the optimality conditions for the barred equations could be defined. For convenience we define $n(t) = s(t) + i(t) + r(t)$ and we note that $N(t) = e^{\alpha t}n(t)$. Note that we omit the dependence on time in the following except in the case that a specific time is intended. In the next part, we consider the difference of the resulting equations for s and $\bar{s}$, i and $\bar{i}$, and so on, and we the simpliy the resulting equations by integration with the use of appropiate integrating factors.

$$\begin{aligned}
\alpha e^{\alpha t}s + e^{\alpha t}\dot{s} &= \mu e^{\alpha t}n - \beta\frac{e^{2\alpha t}si}{e^{\alpha t}n} - \nu e^{\alpha t}s - \mu\frac{e^{2\alpha t}ns}{K}, \\
\alpha s + \dot{s} &= \mu n - \beta\frac{si}{n} - \nu s - \mu e^{\alpha t}\frac{ns}{k}.
\end{aligned}$$

Now, we substract from the above the corresponding barred equation to get:

$$\alpha(s - \bar{s}) + (\dot{s} - \dot{\bar{s}}) = \mu(n - \bar{n}) - \beta\left[\frac{si}{n} - \frac{\bar{s}\bar{i}}{\bar{n}}\right] - (\nu s - \bar{\nu}\bar{s}) - \mu e^{\alpha t}\left[\frac{sn}{K} - \frac{\bar{s}\bar{n}}{K}\right].$$

We multiply by $(s - \bar{s})$ and integrate from 0 (at which the state equation variables are equivalent) to T yielding

$$\begin{aligned}
\frac{1}{2}(s(T) - \bar{s}(T))^2 &+ \alpha\int_0^T (s - \bar{s})^2 dt = \mu\int_0^T (n - \bar{n})(s - \bar{s})dt - \beta\int_0^T\left[\frac{si}{n} - \frac{\bar{s}\bar{i}}{\bar{n}}\right](s - \bar{s})dt \\
&- \int_0^T (\nu s - \bar{\nu}\bar{s})(s - \bar{s})dt - \frac{\mu}{K}\int_0^T e^{\alpha t}\left[sn - \bar{s}\bar{n}\right](s - \bar{s})dt. \qquad (1.33)
\end{aligned}$$

The remaining equations are manipulated similarly, with derivations and dependence on time omitted in the interest of space. The specific characterization of the controls as given in equations (1.31) and (1.32) is represented simply by $\nu, \bar{\nu}, \tau$ and $\bar{\tau}$ in the six manipulated equations:

$$\begin{aligned}\frac{1}{2}(i(T)-\bar{i}(T))^2 \;&+\; \alpha\int_0^T (i-\bar{i})^2 dt = \beta\int_0^T \left[\frac{si}{n}-\frac{\bar{s}\bar{i}}{\bar{n}}\right](i-\bar{i})dt - (\gamma+\delta)\int_0^T (i-\bar{i})^2 dt \\ &-\; \int_0^T (\tau i-\bar{\tau}\bar{i})(i-\bar{i})dt - \frac{\mu}{K}\int_0^T e^{\alpha t}\left[in-\bar{i}\bar{n}\right](i-\bar{i})dt. \qquad (1.34)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(r(T)-\bar{r}(T))^2 \;&+\; \alpha\int_0^T (r-\bar{r})^2 dt = \gamma\int_0^T (i-\bar{i})(r-\bar{r})dt + \int_0^T (\tau i-\bar{\tau}\bar{i})(r-\bar{r})dt \\ &+\; \int_0^T (\nu s-\bar{\nu}\bar{s})(r-\bar{r})dt - \frac{\mu}{K}\int_0^T e^{\alpha t}\left[rn-\bar{r}\bar{n}\right](r-\bar{r})dt. \qquad (1.35)\end{aligned}$$

And for adjoint variables we get:

$$\begin{aligned}-\frac{1}{2}(\phi_1(0)-\bar{\phi}_1(0))^2 \;&-\; \alpha\int_0^T (\phi_1-\bar{\phi}_1)^2 dt = \frac{\mu}{K}\int_0^T e^{\alpha t}(\phi_1-\bar{\phi}_1)\{[\phi_1(n+s)+\phi_2 i+\phi_3 r] \\ &-\; [\bar{\phi}_1(\bar{n}+\bar{s})+\bar{\phi}_2\bar{i}+\bar{\phi}_3\bar{r}]\}dt - \mu\int_0^T (\phi_1-\bar{\phi}_1)^2 dt \\ &+\; \beta\int_0^T e^{\alpha t}(\phi_1-\bar{\phi}_1)\left[\frac{i(n-s)}{n^2}(\phi_1-\phi_2)-\frac{\bar{i}(\bar{n}-\bar{s})}{\bar{n}^2}(\bar{\phi}_1-\bar{\phi}_2)\right]dt \\ &+\; \int_0^T (\phi_1-\bar{\phi}_1)[\nu(\phi_1-\phi_3)-\bar{\nu}(\bar{\phi}_1-\bar{\phi}_3))]dt. \qquad (1.36)\end{aligned}$$

$$
\begin{aligned}
-\frac{1}{2}(\phi_2(0)-\bar{\phi}_2(0))^2 &- \alpha\int_0^T(\phi_2-\bar{\phi}_2)^2dt = \frac{\mu}{K}\int_0^T e^{\alpha t}(\phi_1 s-\bar{\phi}_1\bar{s})(\phi_2-\bar{\phi}_2)dt \\
&- \mu\int_0^T(\phi_1-\bar{\phi}_1)(\phi_2-\bar{\phi}_2)dt+\frac{\mu}{K}\int_0^T e^{\alpha t}[\phi_2(n+i)-\bar{\phi}_2(\bar{n}+\bar{i}) \\
&+ (\phi_3 r-\bar{\phi}_3\bar{r})](\phi_2-\bar{\phi}_2)dt+\beta\int_0^T\left[\frac{s(n-i)}{n^2}(\phi_1-\phi_2)-\frac{\bar{s}(\bar{n}-\bar{i})}{\bar{n}^2}(\bar{\phi}_1-\bar{\phi}_2)\right] \\
&+ (\phi_2-\bar{\phi}_2)dt+\delta\int_0^T(\phi_2-\bar{\phi}_2)^2dt+\gamma\int_0^T[(\phi_2-\phi_3)-(\bar{\phi}_2-\bar{\phi}_3)](\phi_2-\bar{\phi}_2)dt \\
&+ \int_0^T[\tau(\phi_2-\phi_3)-\bar{\tau}(\bar{\phi}_2-\bar{\phi}_3)](\phi_2-\bar{\phi}_2)dt. \qquad (1.37)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{2}(\phi_3(0)-\bar{\phi}_3(0))^2 &- \alpha\int_0^T(\phi_3-\bar{\phi}_3)^2dt = \frac{mB_2}{K^m}\int_0^T e^{m\alpha t}[\nu^2 r^{m-1}-\bar{\nu}^2\bar{r}^{m-1}](\phi_3-\bar{\phi}_3)dt \\
&+ \frac{\mu}{K}\int_0^T e^{\alpha t}(\phi_1 s-\bar{\phi}_1\bar{s})(\phi_3-\bar{\phi}_3)dt-\mu\int_0^T(\phi_1-\bar{\phi}_1)(\phi_3-\bar{\phi}_3)dt \\
&+ \beta\int_0^T\left[\frac{si}{n^2}(\phi_2-\phi_1)-\frac{\bar{s}\bar{i}}{\bar{n}^2}(\bar{\phi}_2-\bar{\phi}_1)\right](\phi_3-\bar{\phi}_3)dt+\frac{\mu}{K}\int_0^T e^{\alpha t}\{[\\
&\phi_2 i+\phi_3(n+r)]-[\bar{\phi}_2\bar{i}+\bar{\phi}_3(\bar{n}+\bar{r})]\}(\phi_3-\bar{\phi}_3)dt. \qquad (1.38)
\end{aligned}
$$

We define

$$\Psi(t)=(s(t)-\bar{s}(t))^2+(i(t)-\bar{i}(t))^2+(r(t)-\bar{r}(t))^2$$

and

$$\Phi(t)=(\phi_1(t)-\bar{\phi}_1(t))^2+(\phi_2(t)-\bar{\phi}_2(t))^2+(\phi_3(t)-\bar{\phi}_3(t))^2.$$

Observe that $\Psi(t)\geq 0$ and $\Phi(t)\geq 0$ for every t. We multiply equations (1.36)-(1.38) by -1 and then add the resulting equations to equations (1.33)-(1.35). The left-hand side of the resulting equation become

$$\frac{1}{2}(\Psi(T)+\Phi(0))+\alpha\int_0^T(\Psi(t)+\Phi(t))dt,$$

and we work now to bound the right-hand side. An elementary inequality

$$(x-y)^2 \geq 0 \text{ implies } x^2+y^2 \geq \frac{1}{2}(x^2+y^2) \geq xy$$

can be used repeatedly to simplify right-hand expressions. The first is in the common form

$$(x-\bar{x})(y-\bar{y}) \leq (x-\bar{x})^2+(y-\bar{y})^2. \tag{1.39}$$

Another common expression and needed inequality:

$$\begin{aligned}(xy-\bar{x}\bar{y})(w-\bar{w}) &= (xy-\bar{x}y+\bar{x}y-\bar{x}\bar{y})(w-\bar{w})\\ &= y(x-\bar{x})(w-\bar{w})+\bar{x}(y-\bar{y})(w-\bar{w})\\ &\leq y[(x-\bar{x})^2+(w-\bar{w})^2]+\bar{x}[(y-\bar{y})^2+(w-\bar{w})^2]\\ &= [y(x-\bar{x})^2+\bar{x}(y-\bar{y})^2+(y+\bar{x})(w-\bar{w})^2].\end{aligned}$$

Let define $C=(y+\bar{x})$, where C depends on bounds for $\bar{x}$ and y, to get

$$(xy-\bar{x}\bar{y})(w-\bar{w}) \leq C[(x-\bar{x})^2+(y-\bar{y})^2+(w-\bar{w})^2]. \tag{1.40}$$

Several bounds require in turn a bound on $(n-\bar{n})^2$ which follows directly from the definition of n:

$$(n-\bar{n})^2 \leq (s-\bar{s})^2+(i-\bar{i})^2+(r-\bar{r})^2. \tag{1.41}$$

A bound is needed for the expressions with division by n^2 in equations (1.36)-(1.38). We focus on the árticular expression in equation (1.38), and others would be similar. Note that

$$\frac{si}{n^2} - \frac{\bar{s}\bar{i}}{\bar{n}^2} = \frac{1}{n^2\bar{n}^2}[(i-\bar{i})s\bar{n}^2 + \bar{i}\bar{n}^2(s-\bar{s}) + \bar{i}\bar{s}(\bar{n}-n)(\bar{n}+n)].$$

In the following, for simplicity we write $Q = \frac{si}{n^2}$ with similar definition for $\bar{Q}$, and rely on bounds for the state variables established in Theorem 6.

$$\begin{aligned}
\left[\frac{si}{n^2}(\phi_2-\phi_1) - \frac{\bar{s}\bar{i}}{\bar{n}^2}(\bar{\phi}_1-\bar{\phi}_2)\right] &= \left\{(\phi_2-\phi_1)\frac{1}{n^2\bar{n}^2}[(i-\bar{i})s\bar{n}^2 + \bar{i}\bar{n}^2(s-\bar{s})\right.\\
&+ \bar{i}\bar{s}(\bar{n}-n)(\bar{n}+n)] + Q(\phi_2-\bar{\phi}_2)\\
&+ \left.Q(\bar{\phi}_1-\phi_1)\right\}(\phi_3-\bar{\phi}_3)\\
&\leq C(\Psi+\Phi). \qquad (1.42)
\end{aligned}$$

where C depends on the bounds for n, and by extension s, i and r, and the bounds for ϕ_1 and ϕ_2. Equations (1.33) and (1.34) have terms containing $\frac{si}{n}$ and can be bounded similarly. Using the equality

$$\frac{si}{n} - \frac{\bar{s}\bar{i}}{\bar{n}} = \frac{s}{n}(i-\bar{i}) + \frac{s\bar{i}}{n\bar{n}}(\bar{n}-n) + (s-\bar{s})\frac{\bar{i}}{\bar{n}}$$

the following bound can be shown

$$[(i-\bar{i})-(s-\bar{s})]\left[\frac{si}{n} - \frac{\bar{s}\bar{i}}{\bar{n}}\right] \leq C((i-\bar{i})^2 + (n-\bar{n})^2 + (s-\bar{s})^2), \qquad (1.43)$$

with C depends on bounds for s, i, and r. ■

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