

# Modeling optimal phytosanitary policies in crops of economic importance in the state of Sonora.

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# Tomato Leaf Curl Virus



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# Others Controls

## Cultural Control

- physical barriers,
- planting dates,
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- host plant resistance.

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R. E. Shun-xiang, W. A. Zhen-zhong, Q. I. Bao-li, X. I. Yuan.

The pest status of *bemisia tabaci* in china and non-chemical control strategies\*.

*Insect Science*, 8(3):279–288, 2001.



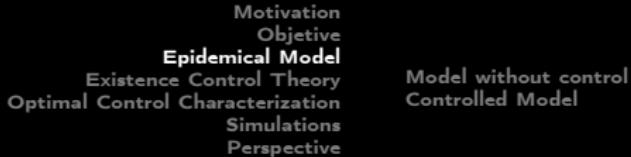
H. A. Smith and M. C. Giurcanu.

New Insecticides for Management of Tomato Yellow Leaf Curl, a Virus Vectored by the Silverleaf Whitefly, *Bemisia tabaci*.

*Journal of Insect Science*, 14(1):4–7, jan 2014.

## Objetive

Model optimal phytosanitary policies for diseases in farm crops using ODE, PDE, SDE.



J. Holt, J. Colvin, and V. Muniyappa.

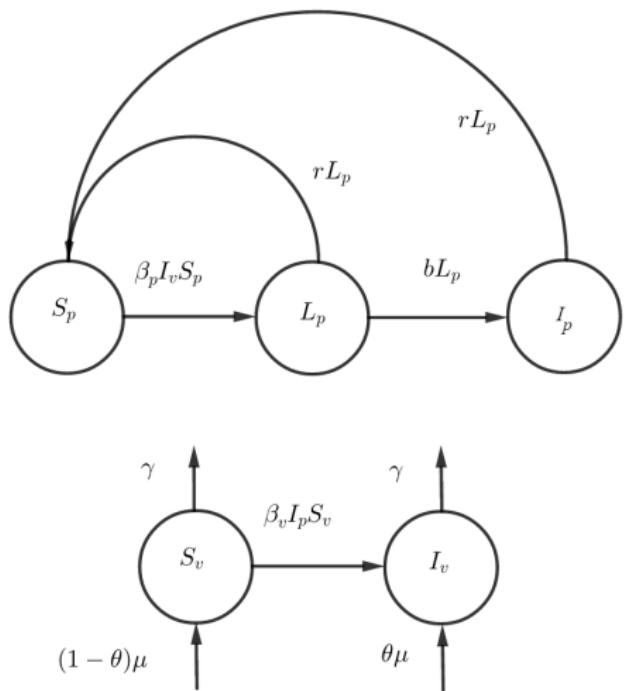
Identifying control strategies for tomato leaf curl virus disease using an epidemiological model.

*Journal of Applied Ecology*, 36(5):625–633, oct 1999.

# Plant Model without control

## Hypothesis:

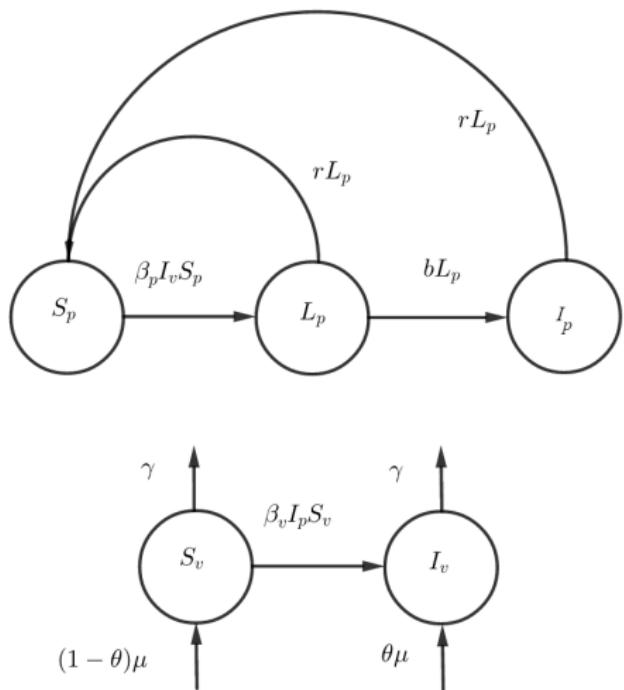
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- replanting latent and infected plants,
- latent plants become infectious plants,
- vectors become infected by infected plants,
- vectors die or depart per day,
- immigration from alternative hosts.



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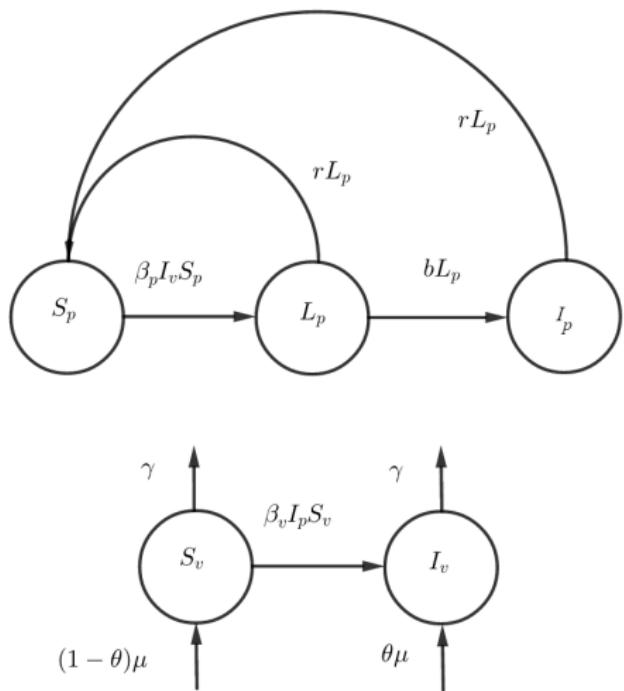
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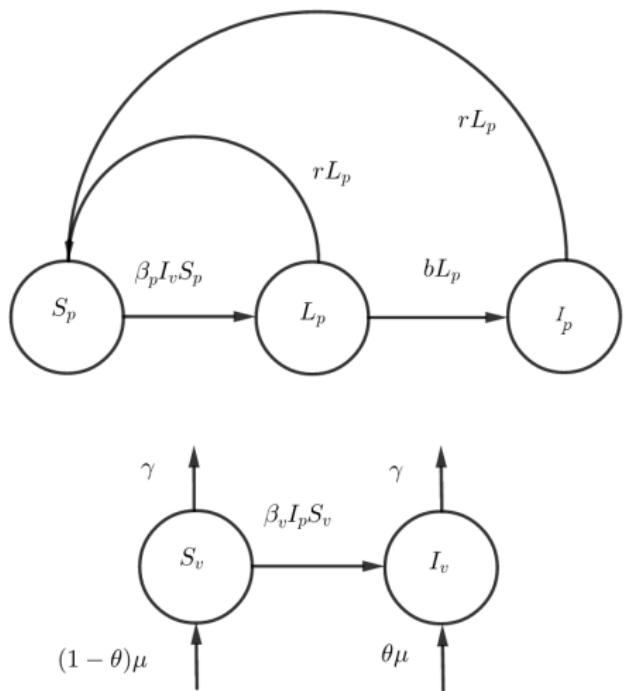
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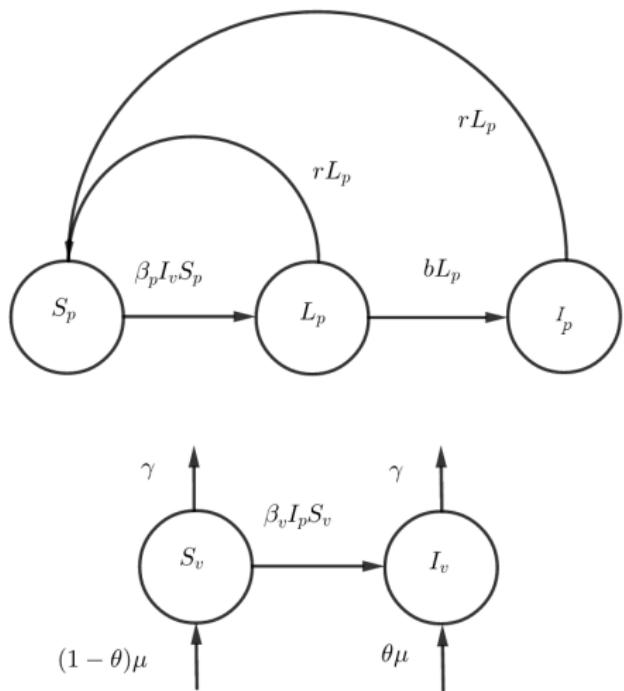
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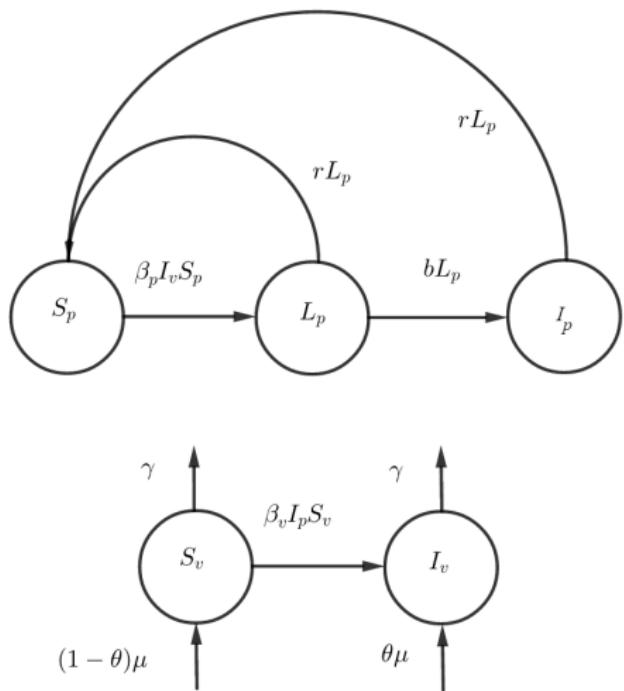
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$$S_v(0) = S_{v_0}, I_v(0) = I_{v_0}.$$

Par.	Unit	value
$\beta_p$	vector $^{-1}$ day $^{-1}$	0.1
$r$	day $^{-1}$	0.01
$b$	day $^{-1}$	0.075
$\gamma$	day $^{-1}$	0.06
$\mu$	plant $^{-1}$ day $^{-1}$	0.3
$\theta$	proportion	0.2
$\beta_v$	plant $^{-1}$ day $^{-1}$	0.003

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + \textcolor{red}{r}(L_p + I_p),$$

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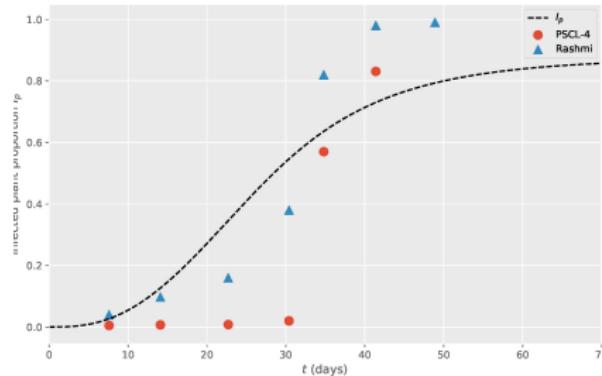
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**Basic reproductive number:**

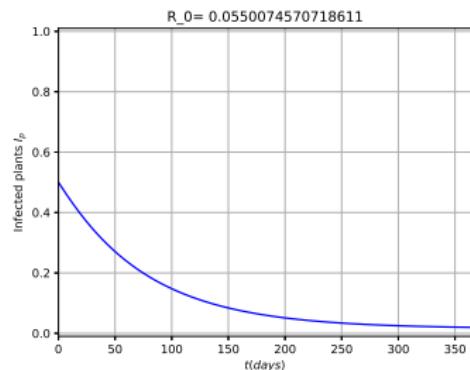
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If  $R_0 < 1$ ,

$$\lim_{t \rightarrow \infty} (S_p, L_p, I_p, S_v, I_v) = (N_p, 0, 0, \frac{\mu}{\gamma}, 0).$$

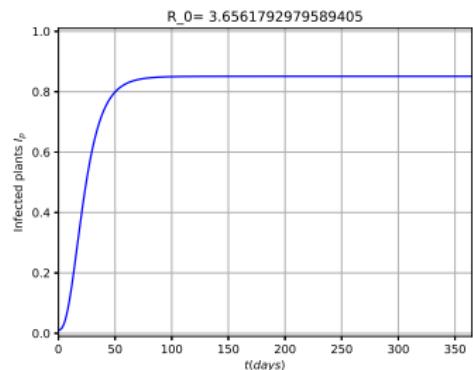


## Basic reproductive number:

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If  $R_0 > 1$ ,

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## Controls:

- $u_1$ : replanting latent plant,
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## Cost Functional

$$\int_0^T (A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2) dt,$$

$$\min_{\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]} J(u_1, u_2, u_3)$$

s.t.

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

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$$u_i(t) \in [0, u_i^{\max}]$$

# Existence Theory

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^T g(s, u(s), x(s)) ds + h(x(T)).$$

## Problem ( $OC$ )

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$  with  $\tilde{\mathcal{U}}_{x_0}[t_0, T] \neq \emptyset$ , find a  $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]$  s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

## Hypothesis:

(C-1)  $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is measurable, satisfies a lipchitz condition in  $x$ ,  
 $|f(t, u, 0)| \leq L, \forall (t, u) \in \mathbb{R}_+ \times U$ .

(C-2)  $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  
 $h : \mathbb{R}^n \rightarrow \mathbb{R}$  are measurable,  
and

$$\begin{aligned} & |g(s, u, x_1) - g(s, u, x_2)| + \\ & |h(x_1) - h(x_2)| \\ & \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$

$$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$

(C-3) For a.a.  $t \in [0, T]$ , Cesari property holds  $\forall x \in \mathbb{R}^n$ .

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## Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  
increasing,  $\omega(r, 0) = 0$   
 $\forall r \geq 0$ .

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- (C-3) For a.a.  $t \in [0, T]$ , Cesari property holds  $\forall x \in \mathbb{R}^n$ .

$\bar{co}(\mathbf{E})$ : closed convex hull of  $\mathbf{E}$ ,

$$\begin{aligned} \mathbf{E}(t, x) = & \{(z^0, z) \in \mathbb{R} \times \mathbb{R}^n | \\ & z^0 \geq g(t, u, x), \\ & z = f(t, u, x), u \in U\}. \end{aligned}$$

## Cesari property

$$\bigcap_{\delta > 0} \bar{co}\mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t, x)$$

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- $$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$
- (C-3) For a.a.  $t \in [0, T]$ , Cesari property holds  $\forall x \in \mathbb{R}^n$ .

## Existence Theorem

Let (C-1)-(C-3) hold. Then problem (*OC*) admits at least one optimal pair.

# Optimal Control Characterization

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^T g(s, u(s), x(s)) ds + h(x(T)).$$

**Hamiltonian:**

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

**Optimallity  
Constraints:**

$$\frac{\partial H}{\partial u_i}(t, \bar{x}(t), \bar{u}) = 0.$$

## Additional hypothesis:

(C-4)

$$x \mapsto (f(t, u, x), g(t, u, x), h(x))$$

is differentiable,

$$(u, x) \mapsto (f(t, u, x), f_x(t, u, x), g(t, u, x), g_x(t, u, x), h_x(x))$$

is continuous.

## Pontryagin's Maximum Principle

Let **(C-1)-(C-4)** hold. If  $\bar{u}(t)$  and  $\bar{x}(t)$  are optimal for the problem (*OC*), then there exists a piecewise differentiable adjoint variable  $\lambda(t)$  s.t.

$$H(t, \bar{x}(t), u(t), \lambda(t)) \leq H(t, \bar{x}(t), \bar{u}(t), \lambda(t))$$

$\forall u$  at  $t$ ,

$$\lambda'(t) = -\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial x},$$

$$\lambda(T) = 0.$$

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

## Example

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

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$$\frac{dI_v}{dt} = \beta_v S_v I_p - (\gamma + u_3) I_v + \theta\mu,$$

$$\begin{aligned} H = & A_1 I_v + A_2 L_p + A_3 I_p \\ & + \sum_{i=1}^3 c_i u_i^2 \\ & + \lambda_1 (-\beta_p S_p I_v + (r + u_1) L_p \\ & + (r + u_2) I_p) \\ & + \lambda_2 (\beta_p S_p I_v - b L_p \\ & - (r + u_1) L_p) \\ & + \lambda_3 (b L_p - (r + u_2) I_p) \\ & + \lambda_4 (-\beta_v S_v I_p - (\gamma + u_3) S_v \\ & + (1 - \theta)\mu) \\ & + \lambda_5 (\beta_v S_v I_p - (\gamma + u_3) I_v \\ & + \theta\mu). \end{aligned}$$

$$\frac{d\lambda_1}{dt} = \beta_p(\lambda_1 - \lambda_2),$$

$$\frac{d\lambda_2}{dt} = -A_2 + (r + u_1)(\lambda_2 - \lambda_1) + b(\lambda_2 - \lambda_3),$$

$$\frac{d\lambda_3}{dt} = A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5),$$

$$\frac{d\lambda_4}{dt} = \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4,$$

$$\frac{d\lambda_5}{dt} = -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,$$

$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$\bar{u}_1 = \min \left( \max \left( 0, \frac{L_p(\lambda_2 - \lambda_1)}{2c_1} \right), u_1^{\max} \right)$$

$$\bar{u}_2 = \min \left( \max \left( 0, \frac{I_p(\lambda_3 - \lambda_1)}{2c_2} \right), u_2^{\max} \right)$$

$$\bar{u}_3 = \min \left( \max \left( 0, \frac{S_v \lambda_4 + I_v \lambda_5}{2c_3} \right), u_i^{\max} \right)$$

# The most popular

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**Algorithm 2** Forward Backward Sweep

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**Input:**  $t_0, t_f, x_0, h, \text{tol}, \lambda_f$   
**Output:**  $x^*, u^*, \lambda$

**procedure** FORWARD\\_BACKWARD\\_SWEEP( $g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{\max}$ )

**while**  $\epsilon > \text{tol}$  **do**

$u_{\text{old}} \leftarrow u$

$x_{\text{old}} \leftarrow x$

$x \leftarrow \text{RUNGE\_KUTTA\_FORWARD}(g, u, x_0, h)$

$\lambda_{\text{old}} \leftarrow \lambda$

$\lambda \leftarrow \text{RUNGE\_KUTTA\_BACKWARD}(\lambda_{\text{function}}, x, \lambda_f, h)$

$u_1 \leftarrow \text{OPTIMALITY\_CONDITION}(u, x, \lambda)$

$u \leftarrow \alpha u_1 + (1 - \alpha) u_{\text{old}}, \quad \alpha \in [0, 1]$  ▷ convex combination

$\epsilon_u \leftarrow \frac{\|u - u_{\text{old}}\|}{\|u\|}$

$\epsilon_x \leftarrow \frac{\|x - x_{\text{old}}\|}{\|x\|}$  ▷ relative error

$\epsilon_\lambda \leftarrow \frac{\|\lambda - \lambda_{\text{old}}\|}{\|\lambda\|}$

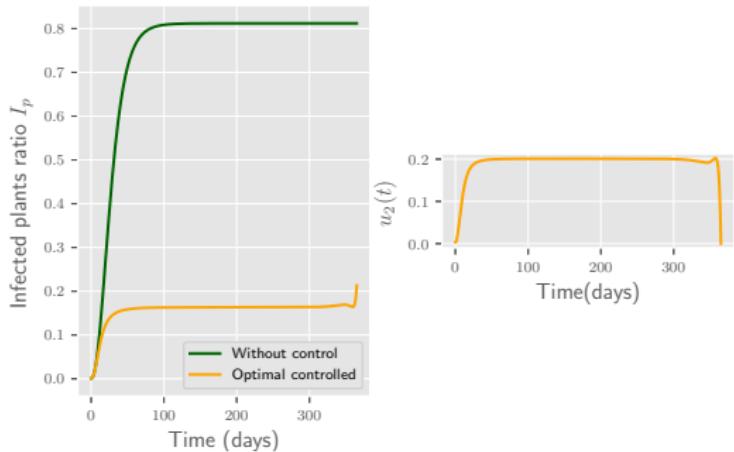
$\epsilon \leftarrow \max \{\epsilon_u, \epsilon_x, \epsilon_\lambda\}$

**end while**

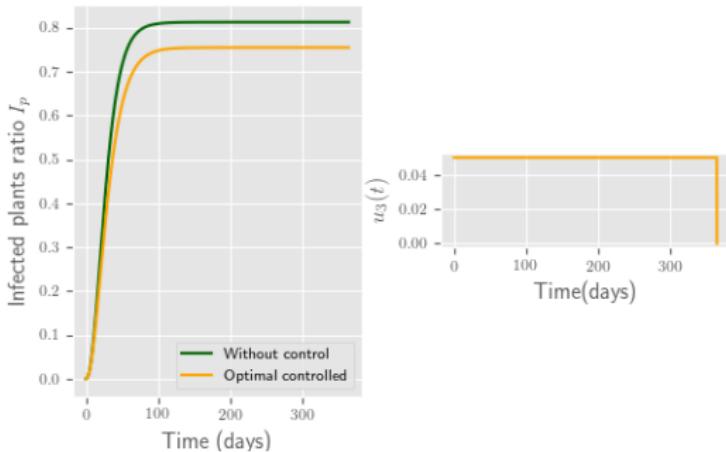
**return**  $x^*, u^*, \lambda$  ▷ Optimal pair

**end procedure**

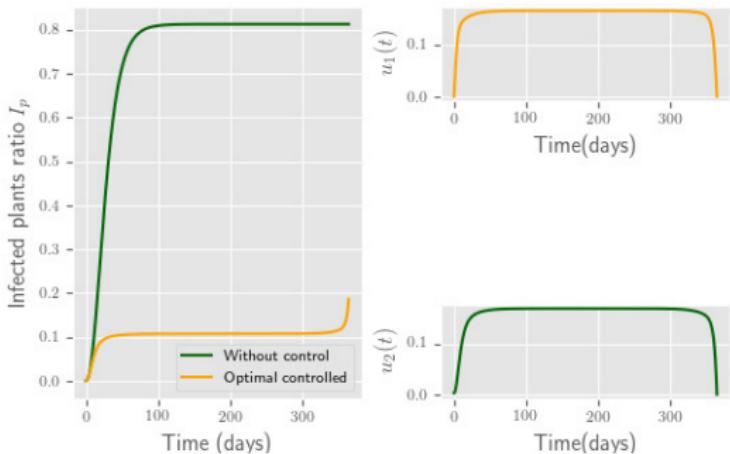
# Case with one controls



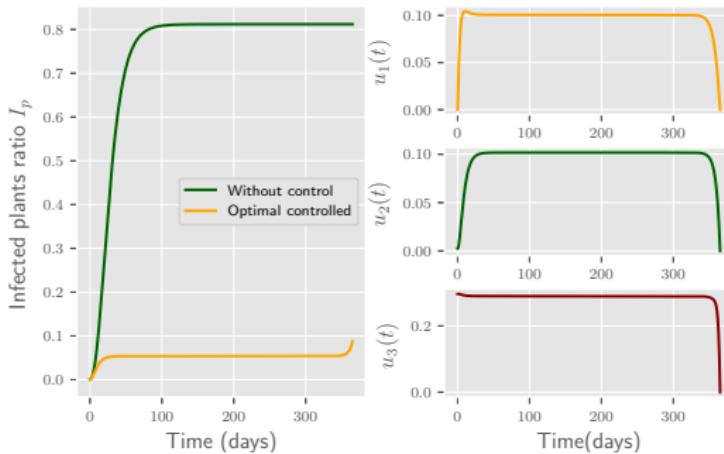
## Case with one controls



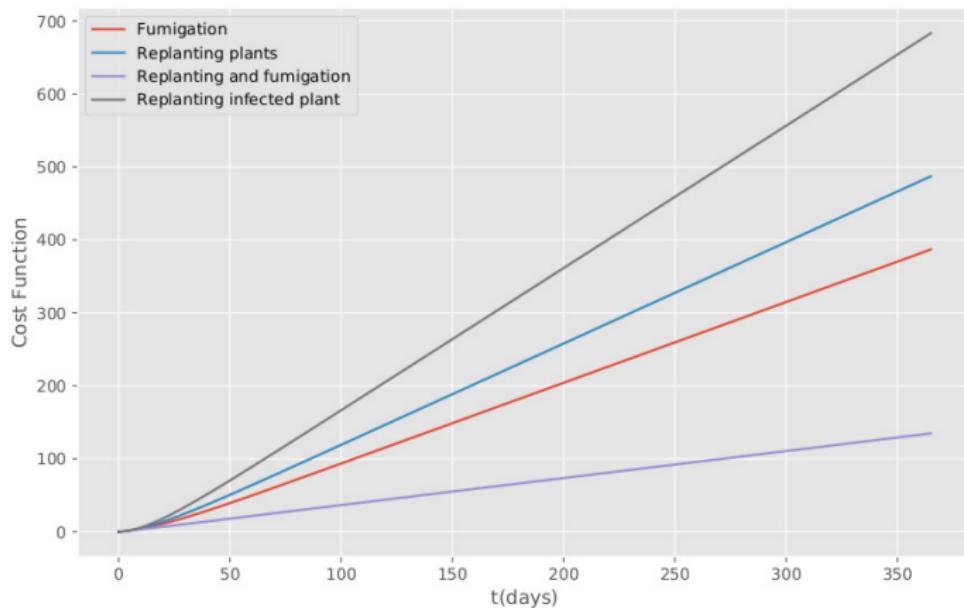
# Case with two controls



## Case with three controls



# Cost Function Comparation



# Stochastic control theory

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ ,  
 $W(t)$  :  $m$ -dimensional Brownian motion.

$$dx(t) = f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t)$$

$$x(0) = x_0,$$

$$f : [0, T] \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma : [0, T] \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^{n+m},$$

$U$  : separable metric space.

$$\mathcal{U}[0, T] := \{u : [0, T] \times \Omega \rightarrow U | u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted}\}$$

## Definition (weak formulation)

A 6-tuple  $\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W(\cdot), u(\cdot))$ ,  $u(\cdot)$  is a w-admissible control,  $(u(\cdot), x(\cdot))$  is a w-admissible pair, if

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  is a filtered probability space satisfying the usual conditions,
- $W(t)$  is an  $m$ -dimensional standard Brownian motion defined on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ ,
- $u(\cdot)$  is an  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process on  $(\Omega, \mathcal{F}, \mathbb{P})$  taking values in  $U$ ,
- $x(\cdot)$  is unique solution,
- some prescribed state constraints are satisfied,
- $g(\cdot, u(\cdot), x(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$  and  $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$

$$J(u(\cdot)) = \mathbb{E} \left\{ \int_0^T g(t, u(t), x(t)) dt + h(x(T)) \right\}$$

(WS)

$$J(\bar{\pi}) = \inf_{\pi \in \mathcal{U}_{ad}^w[0, T]} J(\pi) \quad (*)$$

s.t.

$$\begin{aligned} dx(t) &= f(t, u(t), x(t)) dt + \sigma(t, u(t), x(t)) dW(t) \\ x(0) &= x_0, \end{aligned}$$

problem (WS) is finite, if r.h.s. of (\*) is finite.

## Hypothesis:

(SE-1)  $(U, d)$  is a compact metric space and  $T > 0$ ,

(SE-2)  $f, \sigma, g$ , and  $h$  are all continuous, and  $\exists L > 0$  s.t.

$$\psi(t, u, x) = \{f(t, u, x), \sigma(t, u, x), g(t, u, x), h(x)\},$$

$$|\psi(t, u, x) - \psi(t, u, \hat{x})| \leq L|x - \hat{x}|,$$

$$\forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u \in U,$$

$$|\psi(t, u, 0)| \leq L \forall (t, u) \in [0, T] \times U.$$

(SE-3)  $\forall (t, x) \in [0, T] \times \mathbb{R}^n$ , the set

$$(f, \sigma\sigma^T, g)(t, x, U) := \{(f_i(t, u, x), (\sigma\sigma^T)^{ij}(t, u, x), g(t, u, x)) | u \in U, i = 1, \dots, n, j = 1, \dots, m\}$$

is convex in  $\mathbb{R}^{m+nm+1}$ ,

(SE-4)  $S(t) \equiv \mathbb{R}^n$ .

### Existence Theorem (weak formulation)

Under **(SE1)-(SE4)**, if **(WS)** is finite, then it admits an optimal control.

- Viscosity solutions
  - Optimal Control of PDE.
- 
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# Source code



[https://github.com/AdrianSalcedo  
/Tomato\\_control](https://github.com/AdrianSalcedo/Tomato_control)

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