

# Modeling optimal phytosanitary policies in crops of economic importance in the state of Sonora.

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# Tomato Leaf Curl Virus



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# Others Controls

## Cultural Control

- physical barriers,
- planting dates,
- replanting of infected plants,
- host plant resistance.

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- Pymetrozine,
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R. E. Shun-xiang, W. A. Zhen-zhong, Q. I. Bao-li, X. I. Yuan.

The pest status of *bemisia tabaci* in china and non-chemical control strategies\*.

*Insect Science*, 8(3):279–288, 2001.



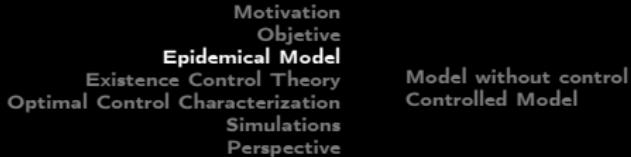
H. A. Smith and M. C. Giurcanu.

New Insecticides for Management of Tomato Yellow Leaf Curl, a Virus Vectored by the Silverleaf Whitefly, *Bemisia tabaci*.

*Journal of Insect Science*, 14(1):4–7, jan 2014.

## Objetive

Model **optimal phytosanitary policies** for diseases in farm crops using ODE, PDE, SDE.



J. Holt, J. Colvin, and V. Muniyappa.

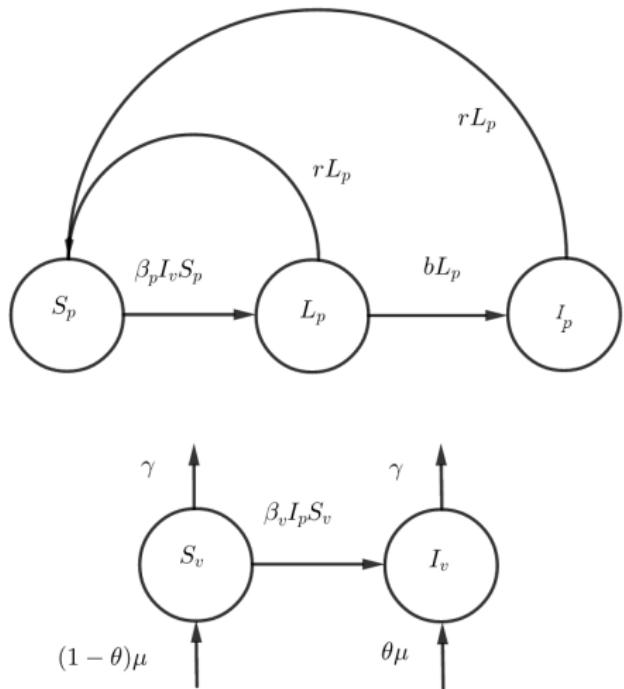
Identifying control strategies for tomato leaf curl virus disease using an epidemiological model.

*Journal of Applied Ecology*, 36(5):625–633, oct 1999.

# Plant model without control

## Hypothesis:

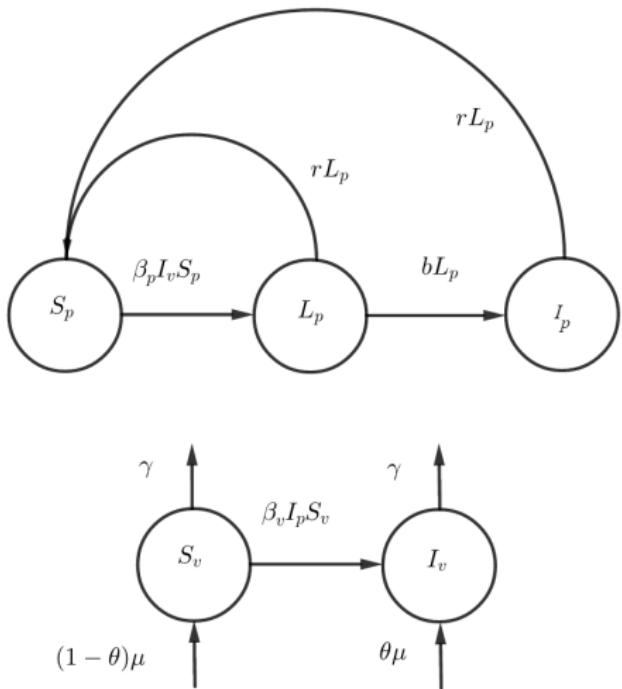
- Plants become latent by infected vectors,
- replanting latent and infected plants,
- latent plants become infectious plants,
- vectors become infected by infected plants,
- vectors die or depart per day,
- immigration from alternative hosts.



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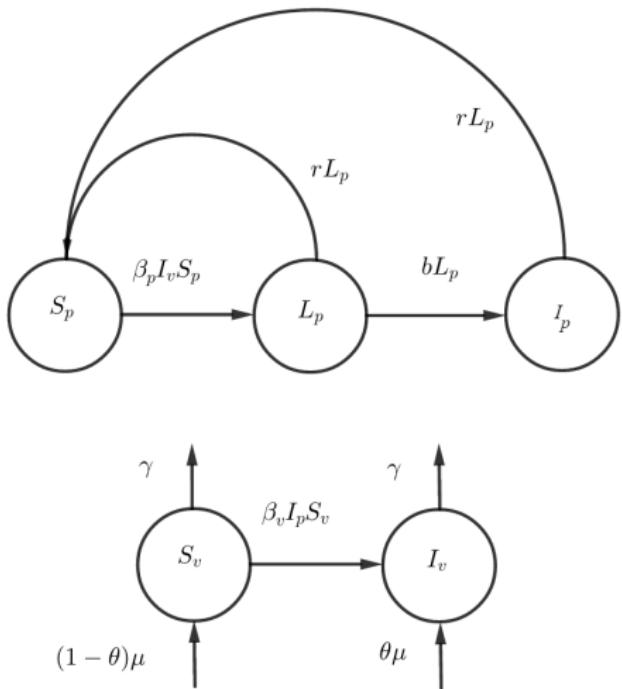
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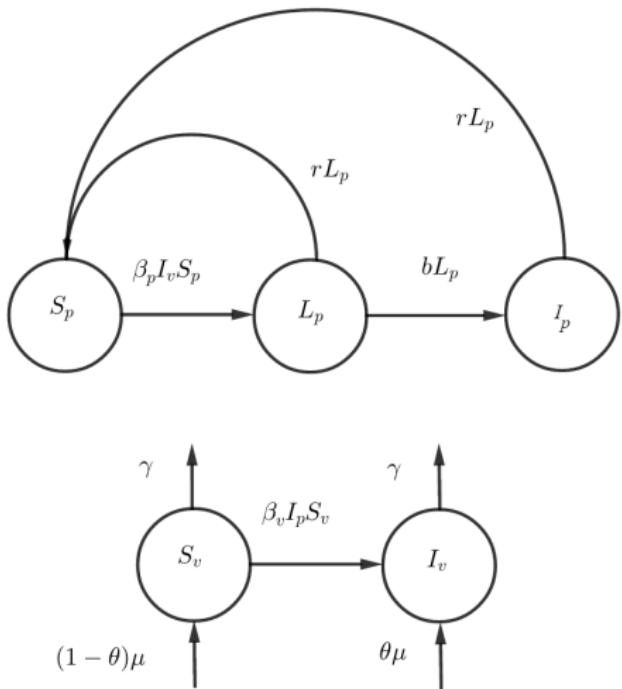
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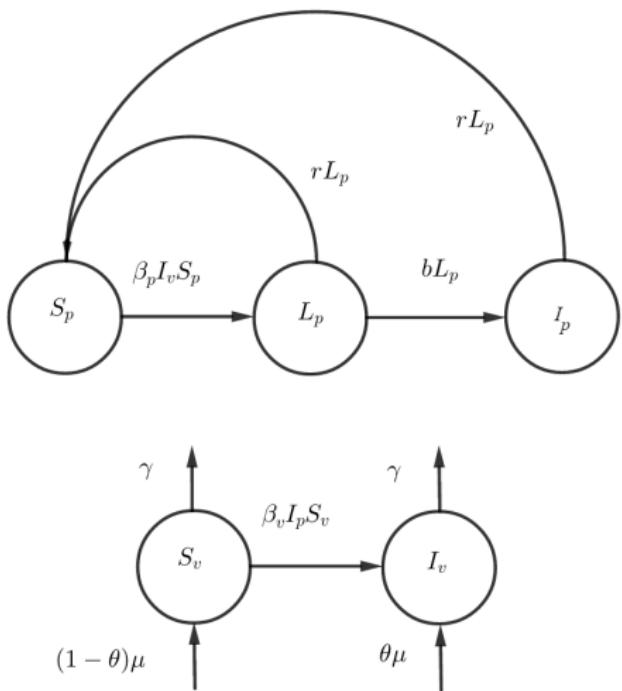
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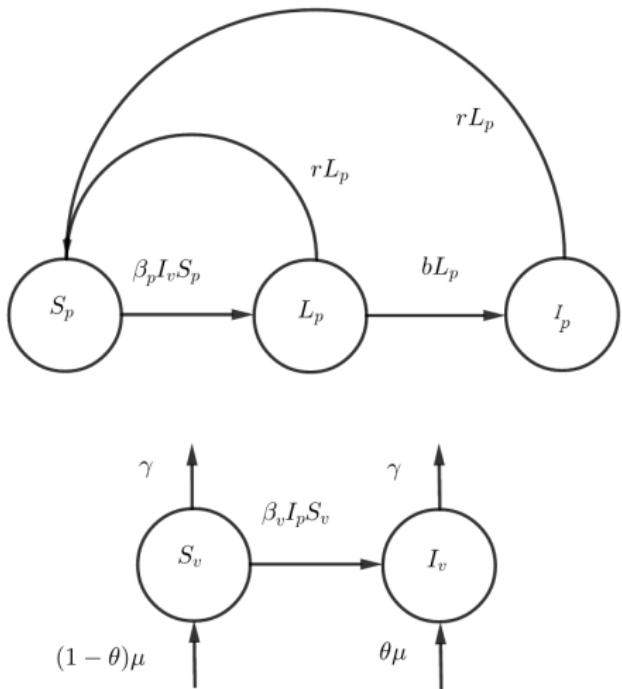
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$$\frac{dS_p}{dt} = -\beta_p S_p I_v + \textcolor{blue}{r}(L_p + I_p),$$

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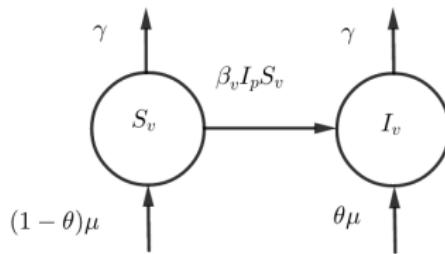
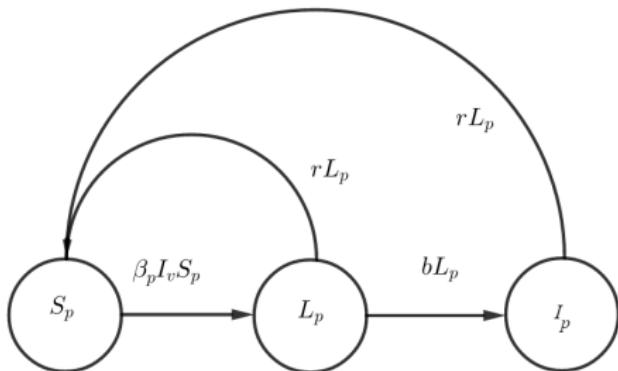
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Par.	Unit	value
$\beta_p$	vector $^{-1}$ day $^{-1}$	0.1
$r$	day $^{-1}$	0.01
$b$	day $^{-1}$	0.075
$\gamma$	day $^{-1}$	0.06
$\mu$	plant $^{-1}$ day $^{-1}$	0.3
$\theta$	proportion	0.2
$\beta_v$	plant $^{-1}$ day $^{-1}$	0.003

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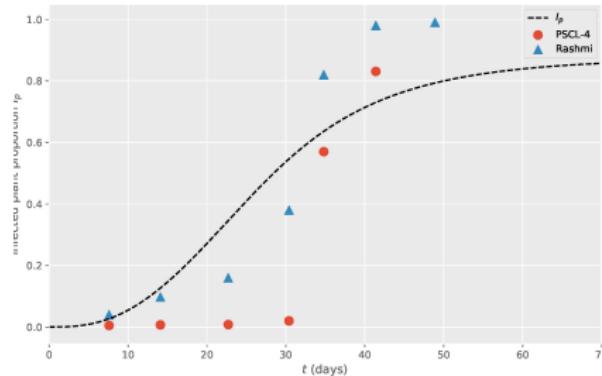
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**Basic reproductive number:**

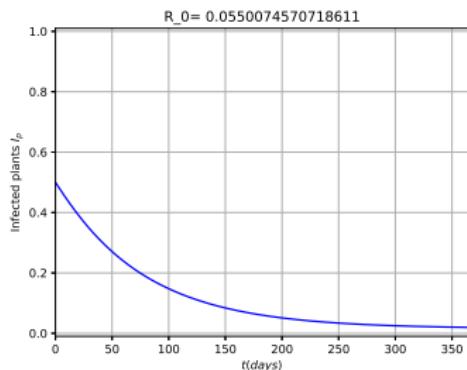
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If  $R_0 < 1$ ,

$$\lim_{t \rightarrow \infty} (S_p, L_p, I_p, S_v, I_v) = (N_p, 0, 0, \frac{\mu}{\gamma}, 0).$$

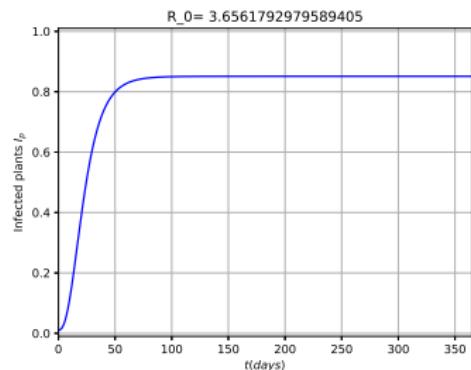


## Basic reproductive number:

$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2(r+b)\gamma}}.$$

If  $R_0 > 1$ ,

$$\lim_{t \rightarrow \infty} (S_p, L_p, I_p, S_v, I_v) = (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*).$$



## Controlled model

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## Controls:

- $u_1$ : replanting latent plant,
- $u_2$ : replanting infected plants,
- $u_3$ : fumigation,

$$u_i^{\min} < u_i < u_i^{\max}.$$

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$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

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## Cost Functional

$$\int_0^T (A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2) dt,$$

$$\min_{\bar{u}(\cdot) \in \mathcal{U}_{x_0}[t_0, T]} J(u_1, u_2, u_3)$$

s.t.

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$$u_i(t) \in [0, u_i^{\max}]$$

# Existence Theory

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^T g(s, u(s), x(s)) ds + h(x(T)).$$

## **Problem** (*OC*)

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$  with  $\mathcal{U}_{x_0}[t_0, T] \neq \emptyset$ , find a  $\bar{u}(\cdot) \in \mathcal{U}_{x_0}[t_0, T]$  s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

## Hypothesis:

(C-1)  $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is measurable, satisfies a lipchitz condition in  $x$ ,  
 $|f(t, u, 0)| \leq L, \forall (t, u) \in \mathbb{R}_+ \times U.$

(C-2)  $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  
 $h : \mathbb{R}^n \rightarrow \mathbb{R}$  are measurable,  
and

$$\begin{aligned} & |g(s, u, x_1) - g(s, u, x_2)| + \\ & |h(x_1) - h(x_2)| \\ & \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$

$$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$

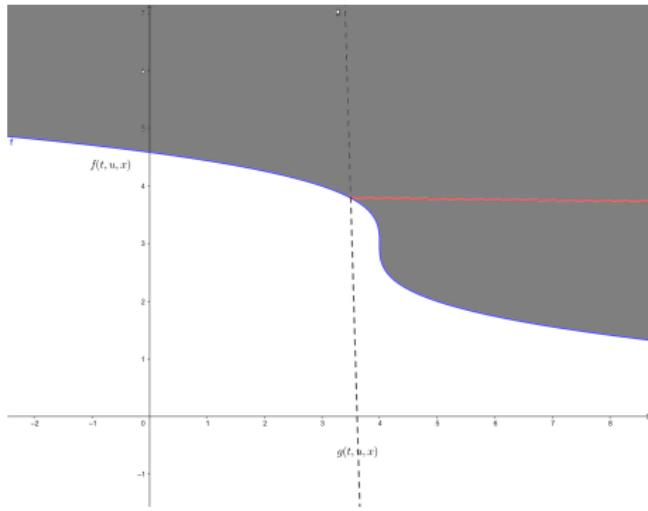
(C-3) For a.a.  $t \in [0, T]$ , Cesari property holds  $\forall x \in \mathbb{R}^n$ .

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- (C-3) For a.a.  $t \in [0, T]$ , Cesari property holds  $\forall x \in \mathbb{R}^n$ .

## Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  
increasing,  $\omega(r, 0) = 0$   
 $\forall r \geq 0$ .



$\bar{co}(\mathbf{E})$ : closed convex hull  
of  $\mathbf{E}$ ,

$$\begin{aligned}\mathbf{E}(t, x) &= \{(z^0, z) \in \mathbb{R} \times \mathbb{R}^n \mid \\ z^0 &\geq g(t, u, x), \\ z &= f(t, u, x), u \in U\}.\end{aligned}$$

## Hypothesis:

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- (C-2)  $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  are measurable, and
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## Cesari property

$$\bigcap_{\delta > 0} \bar{co}\mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t, x)$$

## Hypothesis:

- (C-1)  $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is measurable, satisfies a lipchitz condition in  $x$ ,  $|f(t, u, 0)| \leq L, \forall (t, u) \in \mathbb{R}_+ \times U$ .
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- $$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$
- (C-3) For a.a.  $t \in [0, T]$ , Cesari property holds  $\forall x \in \mathbb{R}^n$ .

## Existence Theorem

Let (C-1)-(C-3) hold. Then problem (*OC*) admits at least one optimal pair.

# Optimal Control Characterization

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^T g(s, u(s), x(s)) ds + h(x(T)).$$

**Hamiltonian:**

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

## Additional hypothesis:

(C-4)

$$x \mapsto (f(t, u, x), g(t, u, x), h(x))$$

is differentiable,

$$(u, x) \mapsto (f(t, u, x), f_x(t, u, x), g(t, u, x), g_x(t, u, x), h_x(x))$$

is continuous.

# Optimal control characterization

## Pontryagin's Maximum Principle

Let **(C-1)-(C-4)** hold. If  $\bar{u}(t)$  and  $\bar{x}(t)$  are optimal for the problem  $(OC)$ , then there exists a piecewise differentiable adjoint variable  $\lambda(t)$  s.t.

$$H(t, \bar{x}(t), \bar{u}(t), \lambda(t)) = \min_{u(\cdot) \in \mathcal{U}_{x_0}[t_0, T]} H(t, \bar{x}(t), u(t), \lambda(t))$$

$\forall u$  at  $t$ ,

$$\begin{aligned}\lambda'(t) &= -\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial x}, \\ \lambda(T) &= 0.\end{aligned}$$

---

$$\frac{\partial H}{\partial u_i}(t, \bar{x}, \bar{u}) = 0.$$

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

## Example

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (\textcolor{red}{r} + \textcolor{blue}{u}_1) L_p + (\textcolor{red}{r} + \textcolor{blue}{u}_2) I_p,$$

$$\frac{dL_p}{dt} = \beta_p S_p I_v - b L_p - (\textcolor{red}{r} + \textcolor{blue}{u}_1) L_p,$$

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$$\frac{dS_v}{dt} = -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta)\mu,$$

$$\frac{dI_v}{dt} = \beta_v S_v I_p - (\gamma + u_3) I_v + \theta\mu,$$

$$\begin{aligned} H = & A_1 I_v + A_2 L_p + A_3 I_p \\ & + \sum_{i=1}^3 c_i u_i^2 \\ & + \lambda_1 (-\beta_p S_p I_v + (r + u_1) L_p \\ & + (r + u_2) I_p) \\ & + \lambda_2 (\beta_p S_p I_v - b L_p \\ & - (r + u_1) L_p) \\ & + \lambda_3 (b L_p - (r + u_2) I_p) \\ & + \lambda_4 (-\beta_v S_v I_p - (\gamma + u_3) S_v \\ & + (1 - \theta)\mu) \\ & + \lambda_5 (\beta_v S_v I_p - (\gamma + u_3) I_v \\ & + \theta\mu). \end{aligned}$$

$$\frac{d\lambda_1}{dt} = \beta_p(\lambda_1 - \lambda_2),$$

$$\frac{d\lambda_2}{dt} = -A_2 + (r + u_1)(\lambda_2 - \lambda_1) + b(\lambda_2 - \lambda_3),$$

$$\frac{d\lambda_3}{dt} = A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5),$$

$$\frac{d\lambda_4}{dt} = \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4,$$

$$\frac{d\lambda_5}{dt} = -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,$$

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= \beta_p(\lambda_1 - \lambda_2), \\
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\frac{d\lambda_4}{dt} &= \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4, \\
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\end{aligned}$$

$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$\frac{d\lambda_1}{dt} = \beta_p(\lambda_1 - \lambda_2),$$

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$$\frac{d\lambda_4}{dt} = \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4,$$

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$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$\bar{u}_1 = \min \left( \max \left( 0, \frac{\bar{L}_p(\lambda_2 - \lambda_1)}{2c_1} \right), u_1^{\max} \right)$$

$$\bar{u}_2 = \min \left( \max \left( 0, \frac{\bar{I}_p(\lambda_3 - \lambda_1)}{2c_2} \right), u_2^{\max} \right)$$

$$\bar{u}_3 = \min \left( \max \left( 0, \frac{\bar{S}_v \lambda_4 + \bar{I}_v \lambda_5}{2c_3} \right), u_i^{\max} \right)$$

# Indirect method

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**Algorithm 2** Forward Backward Sweep

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**Input:**  $t_0, t_f, x_0, h, \text{tol}, \lambda_f$   
**Output:**  $x^*, u^*, \lambda$

**procedure** FORWARD\\_BACKWARD\\_SWEEP( $g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{\max}$ )

**while**  $\epsilon > \text{tol}$  **do**

$u_{\text{old}} \leftarrow u$

$x_{\text{old}} \leftarrow x$

$x \leftarrow \text{RUNGE\_KUTTA\_FORWARD}(g, u, x_0, h)$

$\lambda_{\text{old}} \leftarrow \lambda$

$\lambda \leftarrow \text{RUNGE\_KUTTA\_BACKWARD}(\lambda_{\text{function}}, x, \lambda_f, h)$

$u_1 \leftarrow \text{OPTIMALITY\_CONDITION}(u, x, \lambda)$

$u \leftarrow \alpha u_1 + (1 - \alpha) u_{\text{old}}, \quad \alpha \in [0, 1]$  ▷ convex combination

$\epsilon_u \leftarrow \frac{\|u - u_{\text{old}}\|}{\|u\|}$

$\epsilon_x \leftarrow \frac{\|x - x_{\text{old}}\|}{\|x\|}$  ▷ relative error

$\epsilon_\lambda \leftarrow \frac{\|\lambda - \lambda_{\text{old}}\|}{\|\lambda\|}$

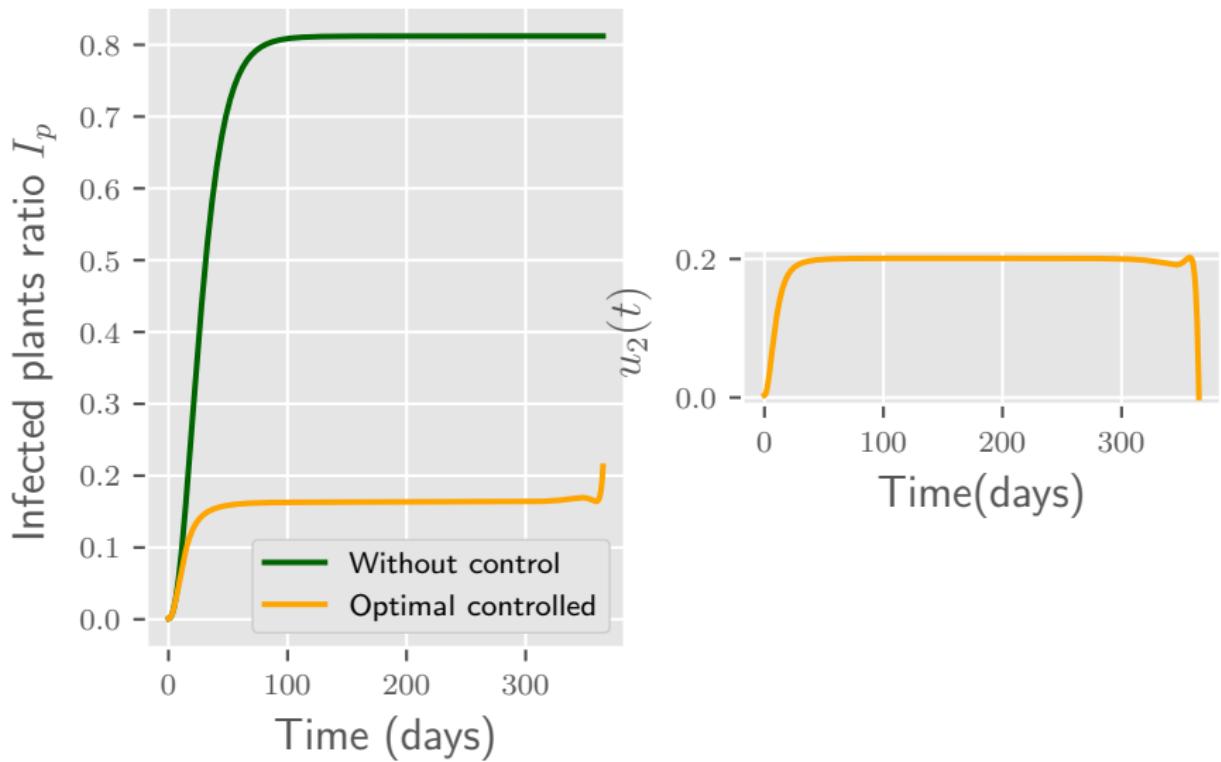
$\epsilon \leftarrow \max \{\epsilon_u, \epsilon_x, \epsilon_\lambda\}$

**end while**

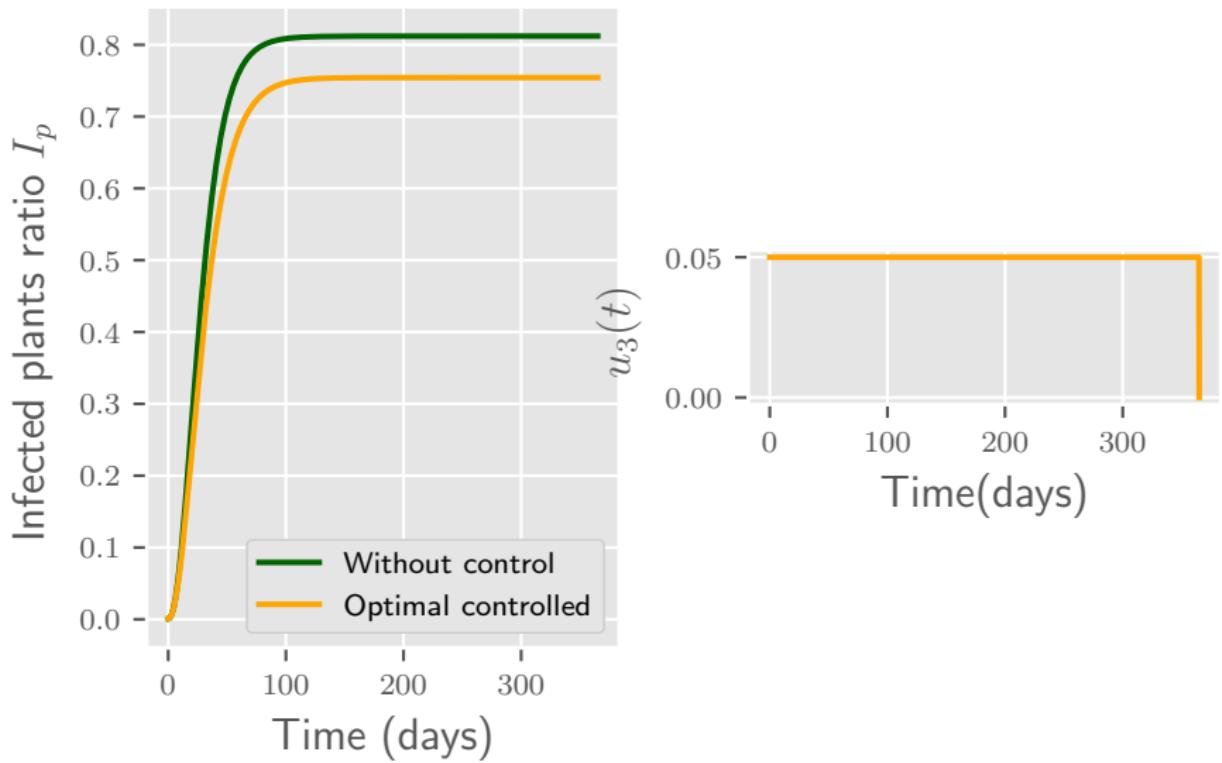
**return**  $x^*, u^*, \lambda$  ▷ Optimal pair

**end procedure**

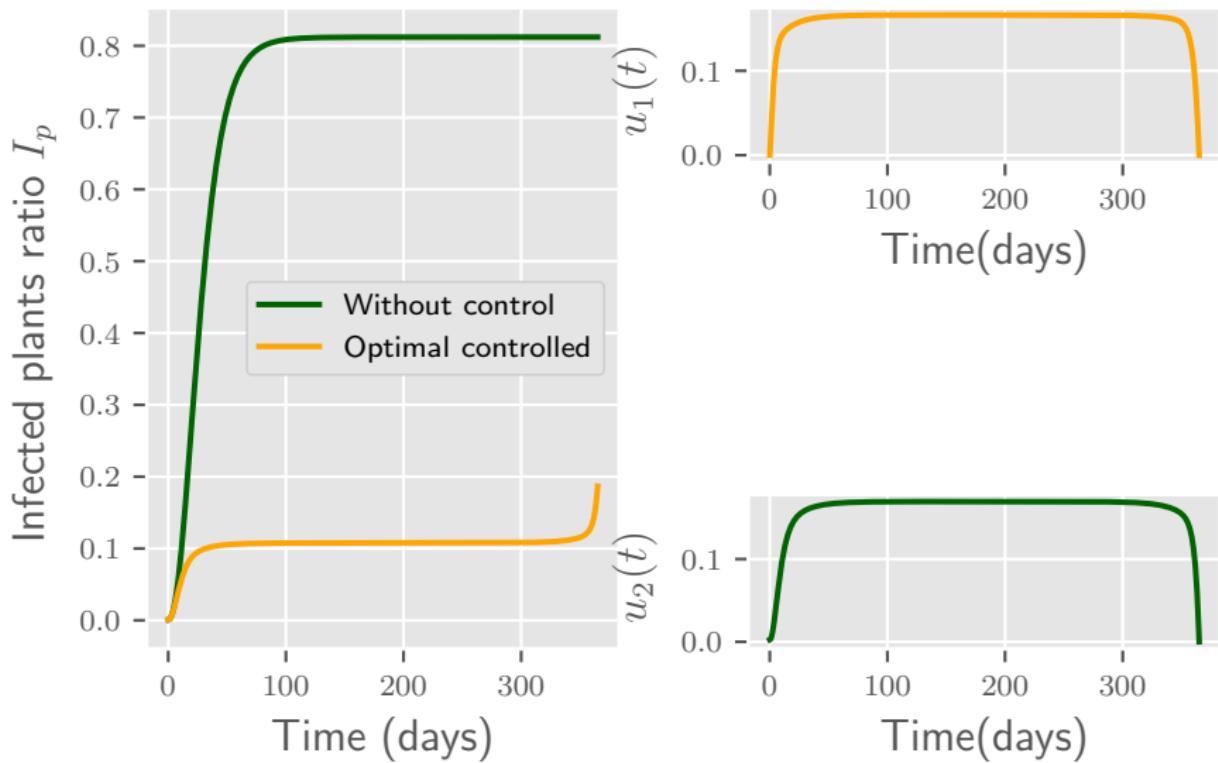
# Dynamic control by infected replanting



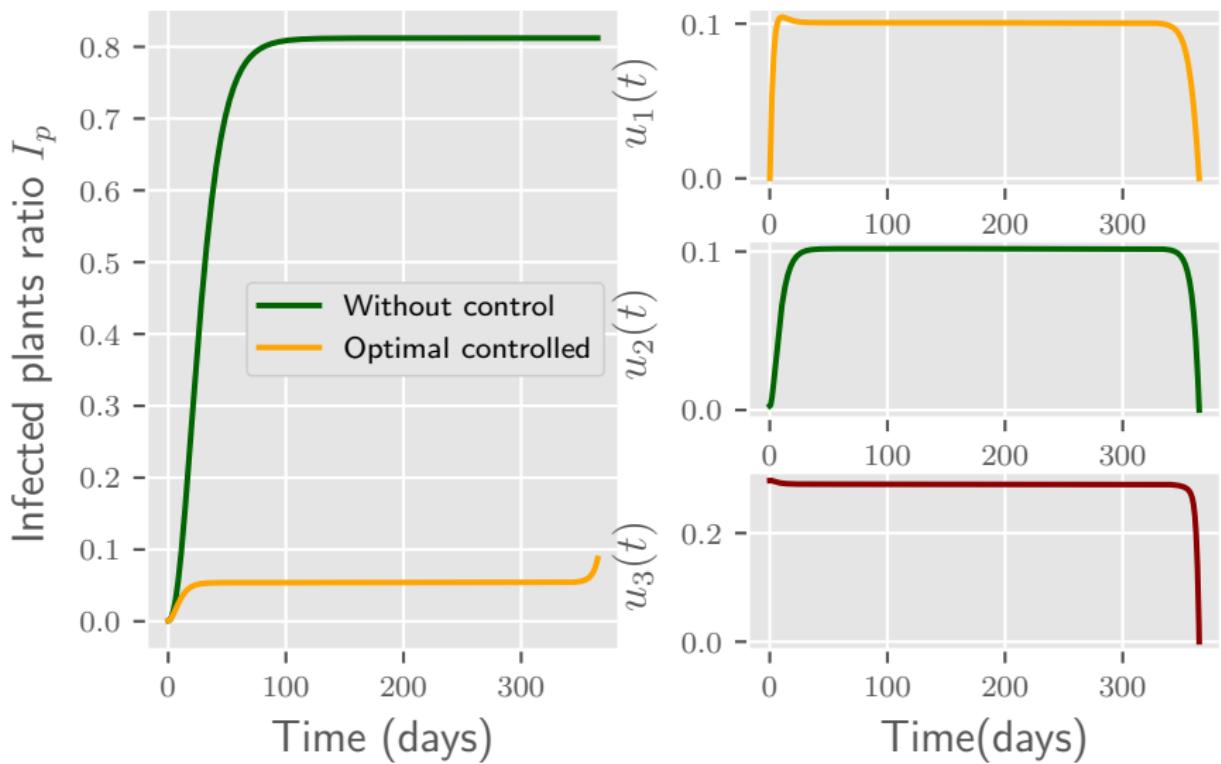
# Dynamic control by fumigation



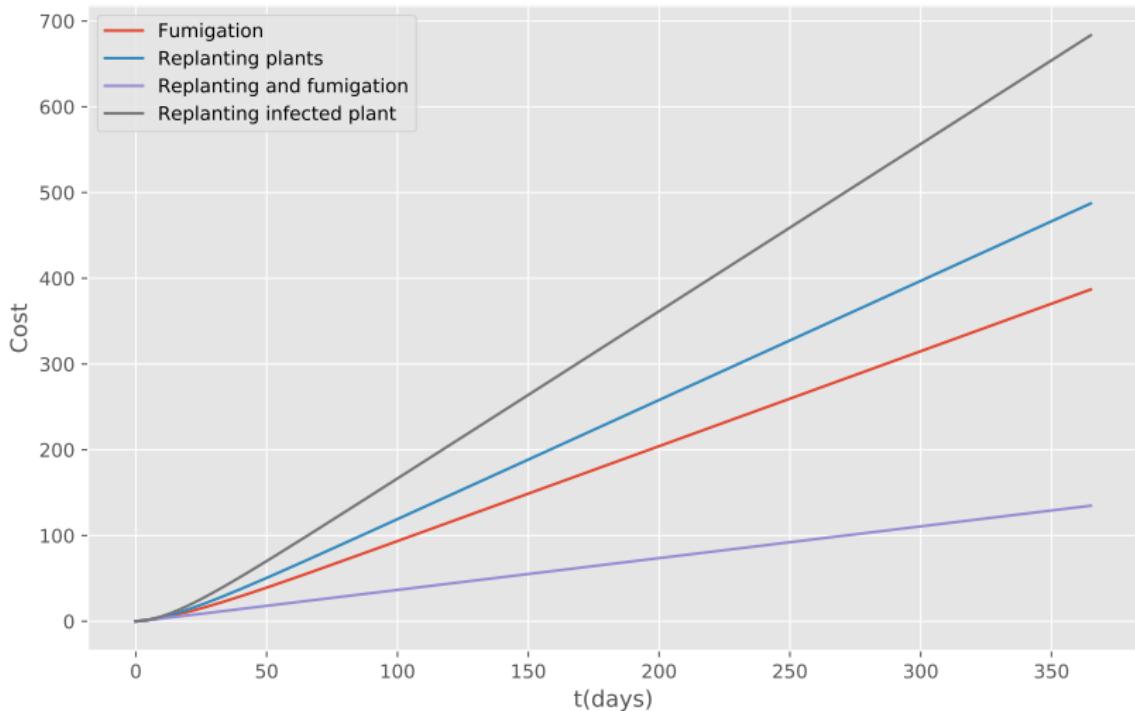
# Dynamic control by latent and infected replanting



# Dynamic control by replanting and fumigation



# Cost Comparation



# Perspectives

# Stochastic optimal control theory

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ ,  
 $W(t)$  :  $m$ -dimensional Brownian motion.

$$dx(t) = f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t)$$

$$x(0) = x_0,$$

$$f : [0, T] \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma : [0, T] \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^{n+m},$$

$U$  : separable metric space.

$$\mathcal{U}[0, T] := \{u : [0, T] \times \Omega \rightarrow U | u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted}\}$$

## Weak formulation of optimal control

A 6-tuple  $\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W(\cdot), u(\cdot))$  is a w-admissible control,  $(u(\cdot), x(\cdot))$  is a w-admissible pair, if

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  is a filtered probability space satisfying the usual conditions,
- $W(t)$  is an  $m$ -dimensional standard Brownian motion defined on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ ,
- $u(\cdot)$  is an  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process on  $(\Omega, \mathcal{F}, \mathbb{P})$  taking values in  $U$ ,
- $x(\cdot)$  is unique solution,
- some prescribed state constraints are satisfied,
- $g(\cdot, u(\cdot), x(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$  and  $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$

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$$J(u(\cdot)) = \mathbb{E} \left\{ \int_0^T g(t, u(t), x(t)) dt + h(x(T)) \right\}$$

(WS)

$$J(\bar{\pi}) = \inf_{\pi \in \mathcal{U}_{ad}^w[0, T]} J(\pi) \quad (*)$$

s.t.

$$\begin{aligned} dx(t) &= f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t) \\ x(0) &= x_0, \end{aligned}$$

---

problem (WS) is finite, if r.h.s. of (\*) is finite.

## Hypothesis:

(SE-1)  $(U, d)$  is a compact metric space and  $T > 0$ ,

(SE-2)  $f, \sigma, g$ , and  $h$  are all continuous, and  $\exists L > 0$  s.t.

$$\psi(t, u, x) = \{f(t, u, x), \sigma(t, u, x), g(t, u, x), h(x)\},$$

$$|\psi(t, u, x) - \psi(t, u, \hat{x})| \leq L|x - \hat{x}|,$$

$$\forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u \in U,$$

$$|\psi(t, u, 0)| \leq L \forall (t, u) \in [0, T] \times U.$$

(SE-3)  $\forall (t, x) \in [0, T] \times \mathbb{R}^n$ , the set

$$(f, \sigma\sigma^T, g)(t, x, U) := \{(f_i(t, u, x), (\sigma\sigma^T)^{ij}(t, u, x), g(t, u, x)) | u \in U, i = 1, \dots, n, j = 1, \dots, m\}$$

is convex in  $\mathbb{R}^{m+nm+1}$ ,

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### Existence Theorem (weak formulation)

Under **(SE1)-(SE4)**, if **(WS)** is finite, then it admits an optimal control.

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# Source code



[https://github.com/AdrianSalcedo  
/Tomato\\_control](https://github.com/AdrianSalcedo/Tomato_control)