Modeling optimal phytosanitary policies in crops of economic importance in the state of Sonora.

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Contents

Tomato Leaf Curl Virus

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Others Controls

Cultural Control

- physical barriers,
- planting dates,
- replanting of infected plants,
- host plant resistance.

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- flupyradifurone,
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R. E. Shun-xiang, W. A. Zhen-zhong, Q. I. Bao-li, X. I. Yuan.
The pest status of bemisia tabaci in china and non-chemical control strategies*.
Insect Science, 8(3):279–288, 2001.

H. A. Smith and M. C. Giurcanu.
New Insecticides for Management of Tomato Yellow Leaf Curl, a Virus Vectored by the Silverleaf Whitefly, Bemisia tabaci.
Journal of Insect Science, 14(1):4–7, jan 2014.

Objetive

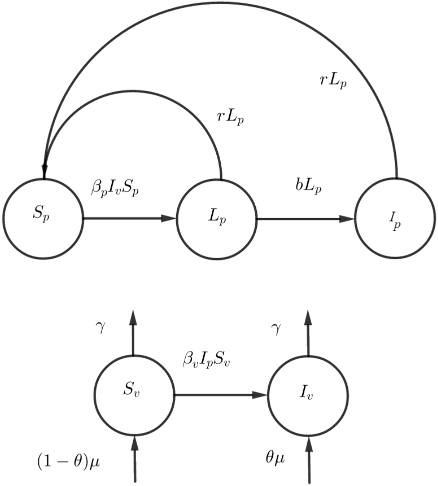
Model **optimal phytosanitary policies** for diseases in farm crops using ODE, PDE, SDE.

J. Holt, J. Colvin, and V. Muniyappa.
Identifying control strategies for tomato leaf curl virus disease using an epidemiological model.
Journal of Applied Ecology, 36(5):625–633, oct 1999.

Plant model without control

Hypothesis:

- Plants become latent by infected vectors,
- replanting latent and infected plants,
- latent plants become infectious plants,
- vectors become infected by infected plants,
- vectors die or depart per day,
- immigration from alternative hosts.



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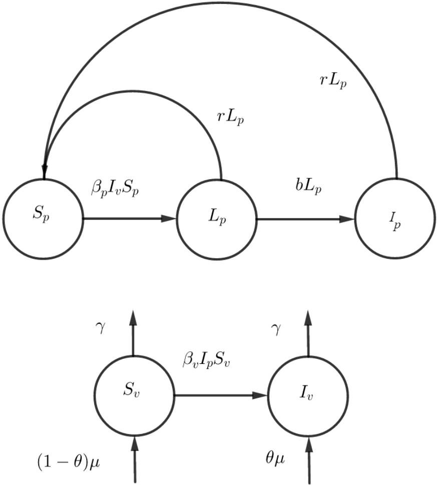

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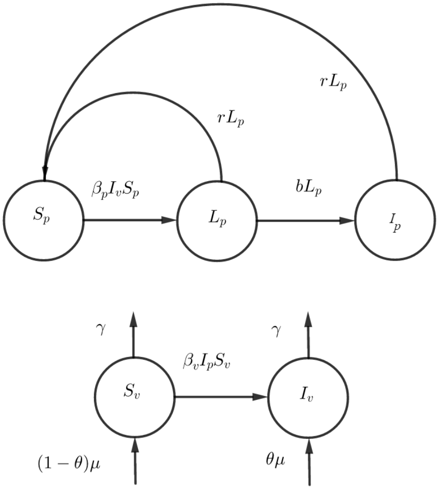

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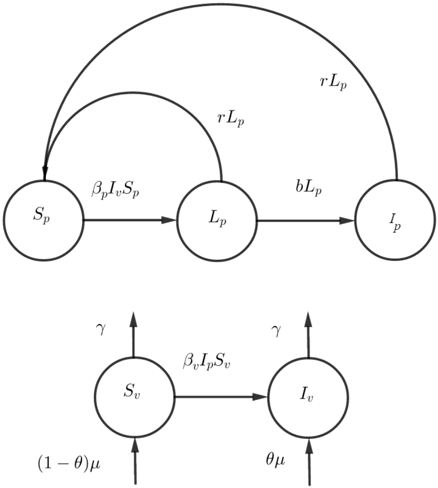

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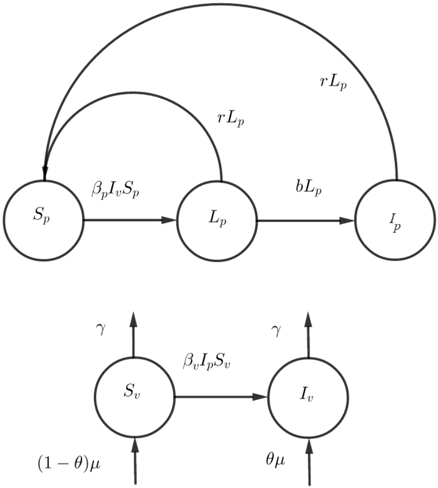

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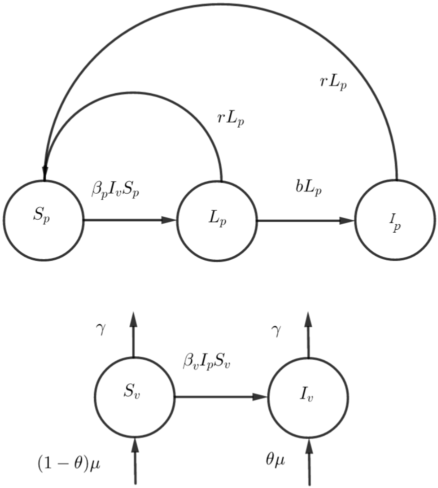

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\frac{dS_p}{dt} &= -\beta_p S_p I_v + r(L_p + I_p),\\
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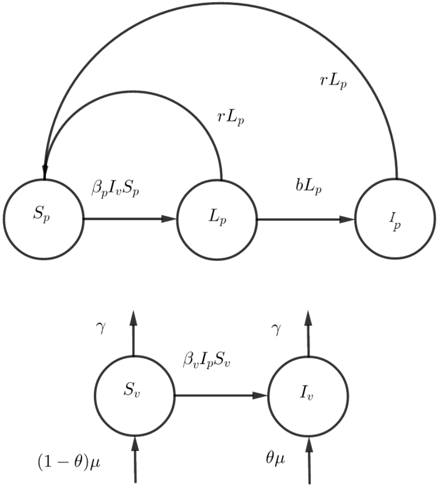

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Par.	Unit	value
β_p	vector^{-1}day^{-1}	0.1
r	day^{-1}	0.01
b	day^{-1}	0.075
γ	day^{-1}	0.06
μ	plant^{-1}day^{-1}	0.3
θ	proportion	0.2
β_v	plant^{-1}day^{-1}	0.003

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + r(L_p + I_p),$$
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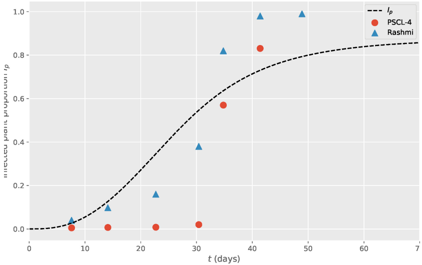

Basic reproductive number:

$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2 (r+b)\gamma}}.$$

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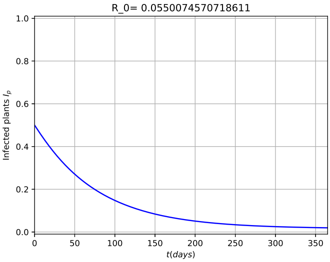
If $R_0 < 1$,

$$\lim_{t\to\infty}(S_p, L_p, I_p, S_v, I_v) = (N_p, 0, 0, \frac{\mu}{\gamma}, 0).$$



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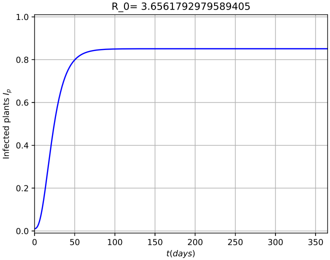

Controlled model

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Controls:

- u_1: replanting latent plant,
- u_2: replanting infected plants,
- u_3: fumigation,

$$u_i^{min} < u_i < u_i^{max}.$$

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$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (r + u_1)L_p + (r + u_2)I_p,$$
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Cost Functional

$$\int_0^T (A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2) dt,$$

OC

$$\min_{\bar{u}(\cdot)\in\mathcal{U}_{x_0}[t_0,T]} J(u_1,u_2,u_3)$$

s.t.

$$\begin{aligned}
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& u_i(t) \in [0, u_i^{max}]
\end{aligned}$$

Existence Theory

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) \; s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{T} g(s, u(s), x(s)) ds + h(x(T)).$$

Problem (OC)

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ with $\mathcal{U}_{x_0}[t_0, T] \neq \emptyset$, find a $\bar{u}(\cdot) \in \mathcal{U}_{x_0}[t_0, T]$ s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

Hypothesis:

(C-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies a lipchitz condition in x, $|f(t,u,0)| \leq L$, $\forall\,(t,u) \in \mathbb{R}_+ \times U$.

(C-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

$$|g(s,u,x_1) - g(s,u,x_2)| + |h(x_1) - h(x_2)| \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|)$$

$\forall\,(s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

(C-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\ x \in \mathbb{R}^n$.

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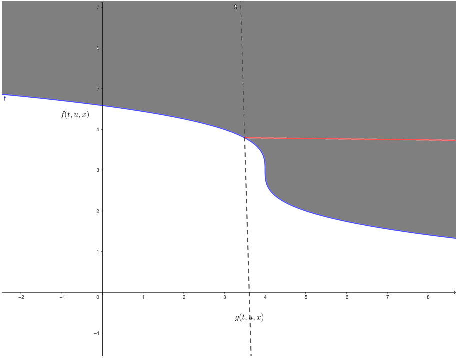
Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, increasing, $\omega(r,0) = 0$ $\forall r \geq 0$.



$\bar{co}(\mathbf{E})$: closed convex hull of $\mathbf{E}$,

$$\mathbf{E}(t,x) = \{(z^0, z) \in \mathbb{R} \times \mathbb{R}^n | \; z^0 \geq g(t,u,x), \; z = f(t,u,x), \; u \in U\}.$$

Hypothesis:

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Cesari property

$$\bigcap_{\delta > 0} \bar{co}\mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t,x)$$

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(C-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies a lipchitz condition in x, $|f(t,u,0)| \leq L$, $\forall (t,u) \in \mathbb{R}_+ \times U$.

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(C-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\ x \in \mathbb{R}^n$.

Existence Theorem

Let **(C-1)**-**(C-3)** hold. Then problem (OC) admits at least one optimal pair.

Optimal Control Characterization

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) \ s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{T} g(s, u(s), x(s)) ds + h(x(T)).$$

Hamiltonian:

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

Additional hypothesis:

(**C**-4)

$$x \mapsto (f(t,u,x), g(t,u,x), h(x))$$

is differentiable,

$$(u,x) \mapsto (f(t,u,x), f_x(t,u,x), g(t,u,x), g_x(t,u,x), h_x(x))$$

is continuous.

Pontryagin's Maximum Principle

Let (**C**-1)-(**C**-4) hold. If $\bar{u}(t)$ and $\bar{x}(t)$ are optimal for the problem (OC), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ s.t.

$$H(t, \bar{x}(t), u(t), \lambda(t)) \leq H(t, \bar{x}(t), \bar{u}(t), \lambda(t))$$

$\forall$ u at t,

$$\begin{aligned} \lambda'(t) &= -\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial x}, \\ \lambda(T) &= 0. \end{aligned}$$

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

Optimallity Constraints:

$$\frac{\partial H}{\partial u_i}(t, \bar{x}, \bar{u}) = 0.$$

Example

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\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + (r + u_1)L_p + (r + u_2)I_p, \\
\frac{dL_p}{dt} &= \beta_p S_p I_v - bL_p - (r + u_1)L_p, \\
\frac{dI_p}{dt} &= bL_p - (r + u_2)I_p, \\
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$$\begin{aligned} H = {} & A_1 I_v + A_2 L_p + A_3 I_v \\ & + \sum_{i=1}^{3} c_i u_i^2 \\ & + \lambda_1(-\beta_p S_p I_v + (r + u_1)L_p \\ & + (r + u_2)I_p) \\ & + \lambda_2(\beta_p S_p I_v - bL_p \\ & - (r + u_1)L_p) \\ & + \lambda_3(bL_p - (r + u_2)I_p) \\ & + \lambda_4(-\beta_v S_v I_p - (\gamma + u_3)S_v \\ & + (1 - \theta)\mu) \\ & + \lambda_5(\beta_v S_v I_p - (\gamma + u_3)I_v \\ & + \theta\mu). \end{aligned}$$

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \beta_p(\lambda_1 - \lambda_2), \\
\frac{d\lambda_2}{dt} &= -A_2 + (r + u_1)(\lambda_2 - \lambda_1) + b(\lambda_2 - \lambda_3), \\
\frac{d\lambda_3}{dt} &- A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5), \\
\frac{d\lambda_4}{dt} &= \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4, \\
\frac{d\lambda_5}{dt} &= -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,
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\frac{d\lambda_3}{dt} &- A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5),\\
\frac{d\lambda_4}{dt} &= \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4,\\
\frac{d\lambda_5}{dt} &= -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,
\end{aligned}
$$

$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \beta_p(\lambda_1 - \lambda_2),\\
\frac{d\lambda_2}{dt} &= -A_2 + (r + u_1)(\lambda_2 - \lambda_1) + b(\lambda_2 - \lambda_3),\\
\frac{d\lambda_3}{dt} &- A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5),\\
\frac{d\lambda_4}{dt} &= \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4,\\
\frac{d\lambda_5}{dt} &= -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,
\end{aligned}
$$

$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$
\begin{aligned}
\bar{u}_1 &= \min\left(\max\left(0, \frac{\bar{L}_p(\lambda_2 - \lambda_1)}{2c_1}\right), u_1^{max}\right)\\
\bar{u}_2 &= \min\left(\max\left(0, \frac{\bar{I}_p(\lambda_3 - \lambda_1)}{2c_2}\right), u_2^{max}\right)\\
\bar{u}_3 &= \min\left(\max\left(0, \frac{\bar{S}_v\lambda_4 + \bar{I}_v\lambda_5}{2c_3}\right), u_i^{max}\right)
\end{aligned}
$$

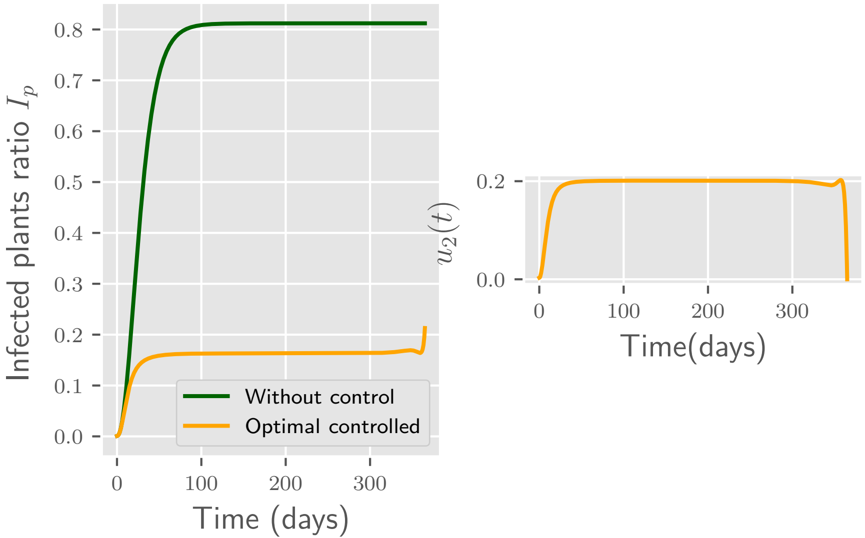
Indirect method

Algorithm 2 Forward Backward Sweep

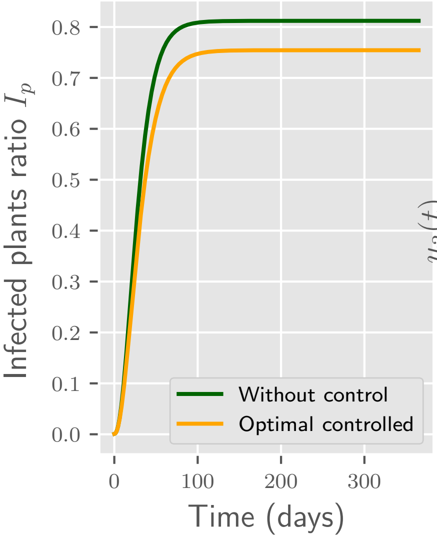
Input: $t_0, t_f, x_0, h, \mathrm{tol}, \lambda_f$
Output: x^*, u^*, λ

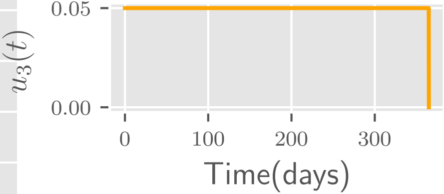
procedure FORWARD_BACKWARD_SWEEP($g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{max}$)
  while $\epsilon > \mathrm{tol}$ **do**
    $u_{\text{old}} \leftarrow u$
    $x_{\text{old}} \leftarrow x$
    $x \leftarrow$ RUNGE_KUTTA_FORWARD(g, u, x_0, h)
    $\lambda_{\text{old}} \leftarrow \lambda$
    $\lambda \leftarrow$ RUNGE_KUTTA_BACKWARD($\lambda_{\text{function}}, x, \lambda_f, h$)
    $u_1 \leftarrow$ OPTIMALITY_CONDITION(u, x, λ)
    $u \leftarrow \alpha u_1 + (1-\alpha) u_{old}, \qquad \alpha \in [0,1]$ ▷ convex combination
    $\epsilon_u \leftarrow \dfrac{||u - u_{\text{old}}||}{||u||}$
    $\epsilon_x \leftarrow \dfrac{||x - x_{\text{old}}||}{||x||}$ ▷ relative error
    $\epsilon_\lambda \leftarrow \dfrac{||\lambda - \lambda_{\text{old}}||}{||\lambda||}$
    $\epsilon \leftarrow \max\{\epsilon_u, \epsilon_x, \epsilon_\lambda\}$
  end while
  return x^*, u^*, λ ▷ Optimal pair
end procedure

Dynamic control by infected replanting

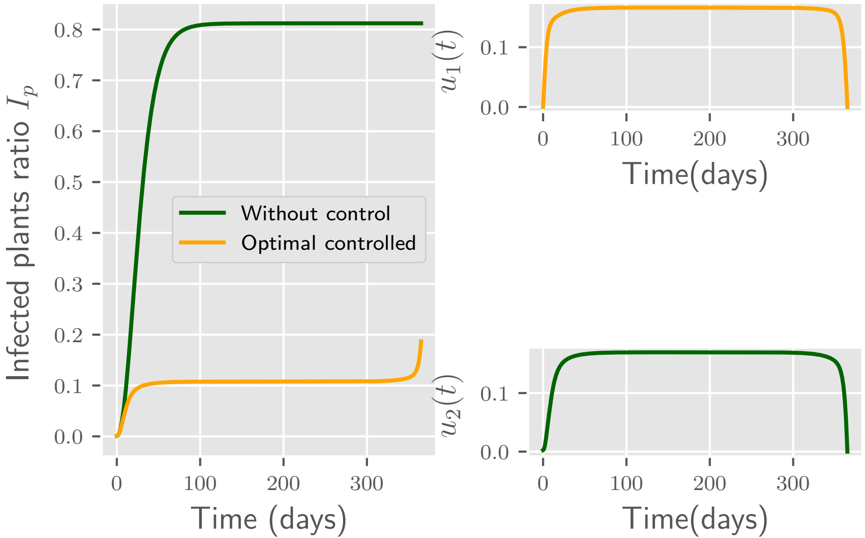


Dynamic control by fumigation

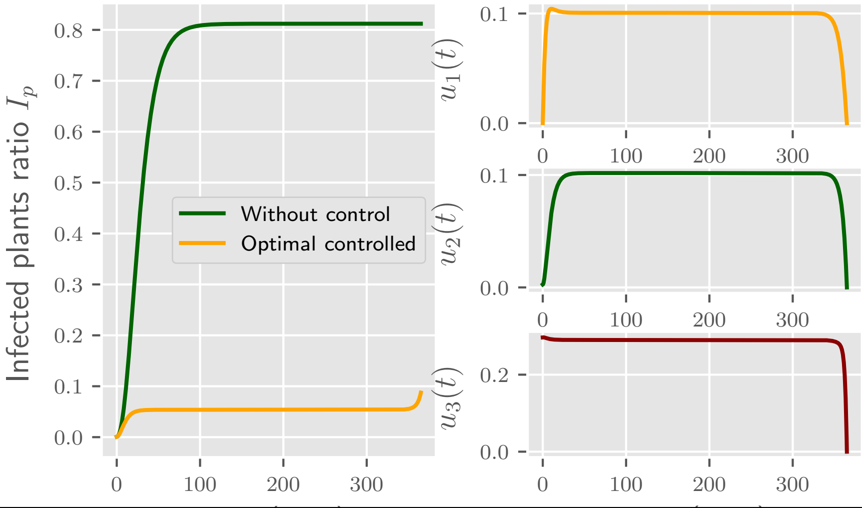




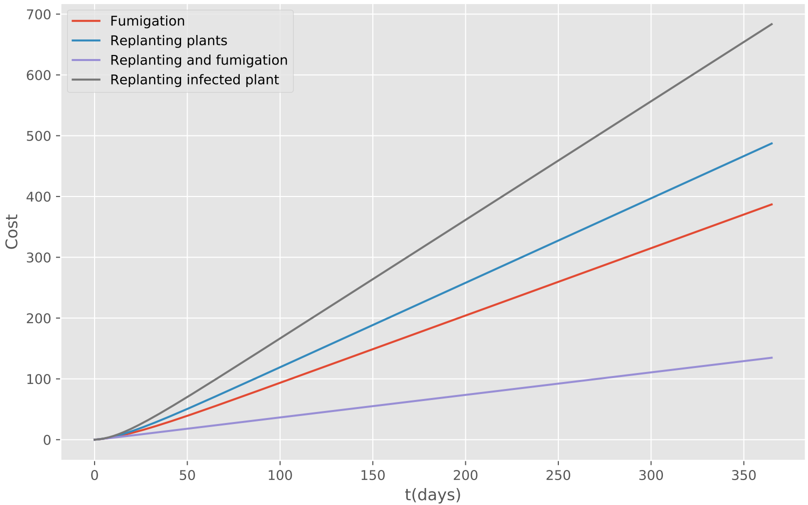
Dynamic control by latent and infected replanting



Dynamic control by replanting and fumigation



Cost Comparation



Stochastic optimal control theory

$(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$,
$W(t)$: m-dimensional Brownian motion.

$$dx(t) = f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t)$$
$$x(0) = x_0,$$

$f : [0, T] \times U \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0, T] \times U \times \mathbb{R}^n \to \mathbb{R}^{n+m}$,

U : separable metric space.

$$\mathcal{U}[0, T] := \{u : [0, T] \times \Omega \to U | u(\cdot) \text{ is } \{\mathscr{F}_t\}_{t\geq 0}\text{-adapted}\}$$

Weak formulation of optimal control

A 6-tuple $\pi = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}, W(\cdot), u(\cdot))$, $u(\cdot)$ is a w-admissible control, $(u(\cdot), x(\cdot))$ is a w-admissible pair, if

- $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions,
- $W(t)$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$,
- $u(\cdot)$ is an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process on $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in U,
- $x(\cdot)$ is unique solution,
- some prescribed state constraints are satisfied,
- $g(\cdot, u(\cdot), x(\cdot)) \in L^1_{\mathscr{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathscr{F}_T}(\Omega; \mathbb{R})$

Weak formulation of optimal control

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- $x(\cdot)$ is unique solution,
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- $g(\cdot, u(\cdot), x(\cdot)) \in L^1_{\mathscr{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathscr{F}_T}(\Omega; \mathbb{R})$

$$J(u(\cdot)) = \mathbb{E}\left\{\int_0^T g(t, u(t), x(t))dt + h(x(T))\right\}$$

(WS)

$$J(\bar{\pi}) = \inf_{\pi \in \mathcal{U}_{ad}^{w}[0,T]} J(\pi) \qquad (*)$$

s.t.

$$\begin{aligned} dx(t) &= f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t) \\ x(0) &= x_0, \end{aligned}$$

---

problem (WS) is finite, if r.h.s. of (*) is finite.

Hypothesis:

(**SE**-1) (U, d) is a compact metric space and $T > 0$,

(**SE**-2) f, σ, g, and h are all continous, and $\exists L > 0$ s.t. $\psi(t, u, x) = \{f(t, u, x), \sigma(t, u, x), g(t, u, x), h(x)\}$,

$$\begin{aligned} &|\psi(t, u, x) - \psi(t, u, \hat{x})| \leq L|x - \hat{x}|, \\ &\qquad\qquad \forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u \in U, \\ &\qquad |\psi(t, u, 0)| \leq L \forall (t, u) \in [0, T] \times U. \end{aligned}$$

(**SE**-3) $\forall (t, x) \in [0, T] \times \mathbb{R}^n$, the set

$$\begin{aligned} (f, \sigma\sigma^T, g)(t, x, U) := \{&(f_i(t, u, x), (\sigma\sigma^T)^{ij}(t, u, x), g(t, u, x)) | \\ &u \in U, i = 1, \ldots, n, j = 1, \ldots, m\} \end{aligned}$$

is convex in $\mathbb{R}^{m+nm+1}$,

(**SE**-4) $S(t) \equiv \mathbb{R}^n$.

Hypothesis:

(SE-1) (U, d) is a compact metric space and $T > 0$,

(SE-2) f, σ, g, and h are all continous, and $\exists L > 0$ s.t.
$\psi(t, u, x) = \{f(t, u, x), \sigma(t, u, x), g(t, u, x), h(x)\}$,

$$|\psi(t, u, x) - \psi(t, u, \hat{x})| \leq L|x - \hat{x}|, \quad \forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u \in U,$$
$$|\psi(t, u, 0)| \leq L \forall (t, u) \in [0, T] \times U.$$

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is convex in $\mathbb{R}^{m+nm+1}$,

(SE-4) $S(t) \equiv \mathbb{R}^n$.

Existence Theorem (weak formulation)

Under (**SE**1)-(**SE**4), if (WS) is finite, then it admits an optimal control.

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Source code

https://github.com/AdrianSalcedo
/Tomato_control