Modeling of optimal phytosanitary policies in crops of economic importance in the state of Sonora.

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Contents

1. Motivation
2. Objetive
3. Epidemical Model
   - Model without control
   - Controlled Model
4. Existence Theory
5. Optimal Control Characterization
6. Simulations
7. Perspective

Tomato Leaf Curl Virus

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Others Controls

Cultural Control

- physical barriers,
- planting dates,
- replanting of infected plants,
- host plant resistance.

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R. E. Shun-xiang, W. A. Zhen-zhong, Q. I. Bao-li, X. I. Yuan.
The pest status of bemisia tabaci in china and non-chemical control strategies*.
Insect Science, 8(3):279–288, 2001.

H. A. Smith and M. C. Giurcanu.
New Insecticides for Management of Tomato Yellow Leaf Curl, a Virus Vectored by the Silverleaf Whitefly, Bemisia tabaci.
Journal of Insect Science, 14(1):4–7, jan 2014.

Objetive

Model optimal phytosanitary policies for diseases in farm crops using ODE, PDE, SDE.

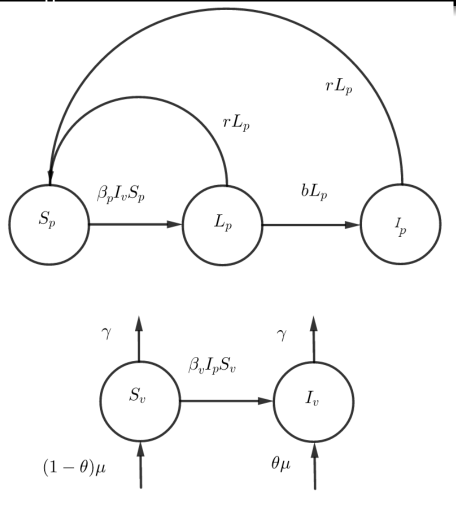
J. Holt, J. Colvin, and V. Muniyappa.
Identifying control strategies for tomato leaf curl virus disease using an epidemiological model.
Journal of Applied Ecology, 36(5):625–633, oct 1999.

Plant Model without control

Tomato Leaf Curl Virus Disease Using an Epidemiological Model

Hypothesis:

- Remove from latent and infected plants,
- plants become latent by infected vectors,
- latent plants become infectious plants,
- vectors become infected by infected plants,
- vectors die or depart per day,
- immigration from alternative hosts.



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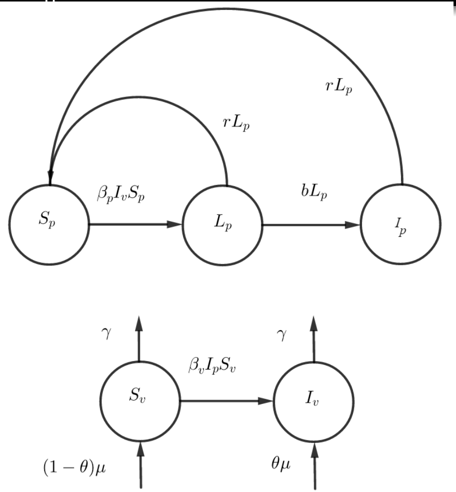

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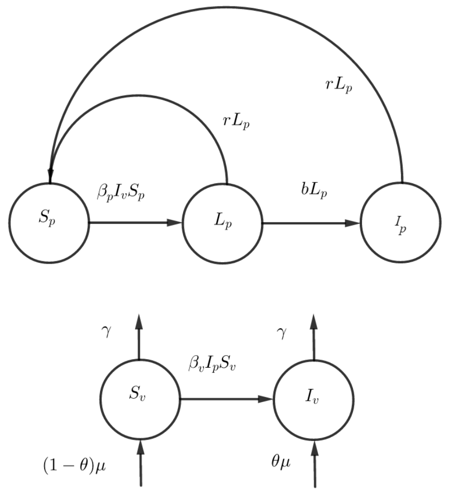

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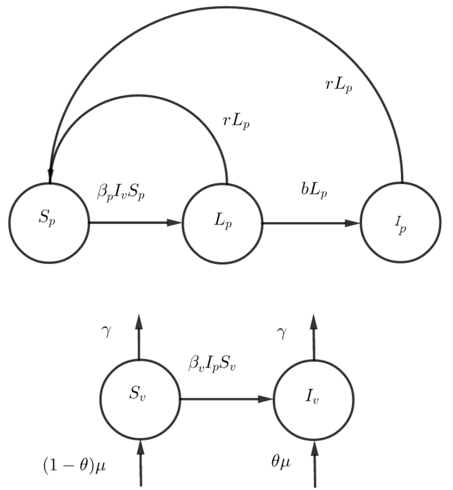

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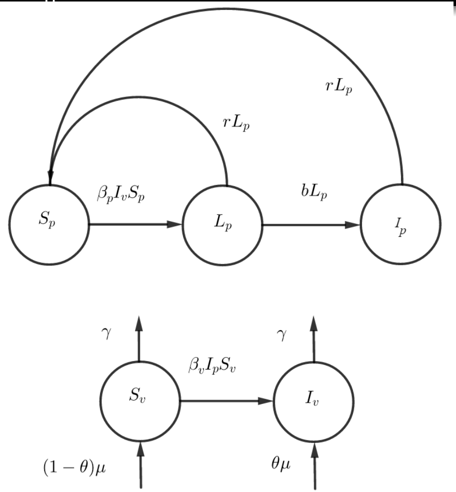

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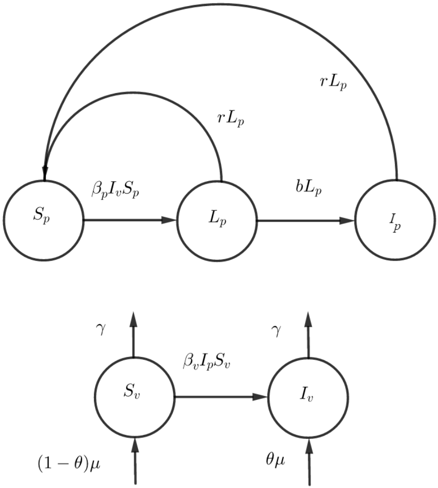

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + r(L_p + I_p),$$
$$\frac{dL_p}{dt} = \beta_p S_p I_v - bL_p - rL_p,$$
$$\frac{dI_p}{dt} = bL_p - rI_p,$$
$$\frac{dS_v}{dt} = -\beta_v S_v I_p - \gamma S_v + (1-\theta)\mu,$$
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$$S_p(0) = S_{p_0}, L_p(0) = L_{p_0}, I_p(0) = I_{p_0},$$
$$S_v(0) = S_{v_0}, I_v(0) = I_{v_0}.$$

Par.	Unit	Descrip.
β_p	vector^{-1}day^{-1}	plants became latently infected rate
r	day^{-1}	replanting rate
b	day^{-1}	latent class to the infectious class rate
γ	day^{-1}	vector die or depart rate
μ	plant^{-1}day^{-1}	immigration rate
θ	proportion	proportion of infected vector arrival to crop
β_v	plant^{-1}day^{-1}	vector infectious acquisition rate

$$
\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + r(L_p + I_p),\\
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\end{aligned}
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$$S_p(0) = S_{p_0}, L_p(0) = L_{p_0}, I_p(0) = I_{p_0},$$

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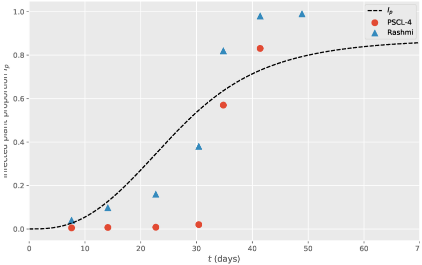

Basic reproductive number:

$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2(r+b)\gamma}}.$$

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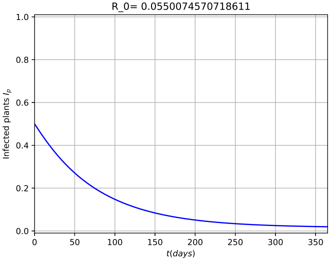
If $R_0 < 1$,

$$\lim_{t\to\infty} (S_p, L_p, I_p, S_v, I_v) = (N_p, 0, 0, \frac{\mu}{\gamma}, 0).$$



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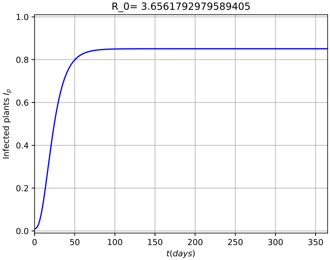
$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2 (r+b)\gamma}}.$$

If $R_0 > 1$,

$$\lim_{t\to\infty} (S_p, L_p, I_p, S_v, I_v) = (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*).$$



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\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + (r + u_1)L_p + (r + u_2)I_p,\\
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Controls:

- u_1: replanting latent plant,
- u_2: replanting infected plants,
- u_3: fumigation,

$$u_i^{min} < u_i < u_i^{max}.$$

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$$u_i^{min} < u_i < u_i^{max}.$$

Cost Functional

$$\int_0^T (A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2)dt,$$

OC

$$\min_{\bar{u}(\cdot)\in\tilde{\mathcal{U}}_{x_0}[t_0,T]} J(u_1, u_2, u_3)$$

s.t.

$$\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + (r+u_1)L_p + (r+u_2)I_p,\\
\frac{dL_p}{dt} &= \beta_p S_p I_v - bL_p - (r+u_1)L_p,\\
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& S_p(0) = S_{p_0}, L_p(0) = L_{p_0},\\
& I_p(0) = I_{p_0}, S_v(0) = S_{v_0}, I_v(0) = I_{v_0}.\\
& u_i(t) \in [0, u_i^{max}]
\end{aligned}$$

Existence Theory

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) \; s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{T} g(s, u(s), x(s)) ds + h(x(T)).$$

Problem (OC)

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ with $\tilde{\mathcal{U}}_{x_0}[t_0, T] \neq \emptyset$, find a $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]$ s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

Hypothesis:

(**C**-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies a lipchitz condition in x, $|f(t,u,0)| \le L$, $\forall\,(t,u) \in \mathbb{R}_+ \times U$.

(**C**-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

$$|g(s,u,x_1) - g(s,u,x_2)| + |h(x_1) - h(x_2)| \le \omega(|x_1| \vee |x_2|, |x_1 - x_2|)$$

$\forall\,(s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

(**C**-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\, x \in \mathbb{R}^n$.

Hypothesis:

(C-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies a lipchitz condition in x, $|f(t,u,0)| \leq L$, $\forall\,(t,u) \in \mathbb{R}_+ \times U$.

(C-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

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(C-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\; x \in \mathbb{R}^n$.

Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, increasing, $\omega(r,0) = 0$ $\forall r \geq 0$.

Hypothesis:

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(**C**-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\ x \in \mathbb{R}^n$.

$\bar{co}(\mathbf{E})$: closed convex hull of $\mathbf{E}$,

$$\mathbf{E}(t,x) = \{(z^0, z) \in \mathbb{R} \times \mathbb{R}^n |\ z^0 \geq g(t,u,x),\ z = f(t,u,x),\ u \in U\}.$$

Cesari property

$$\bigcap_{\delta} \bar{co}\mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t,x)$$

Hypothesis:

(**C**-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies a lipchitz condition in x, $|f(t, u, 0)| \le L$, $\forall\,(t, u) \in \mathbb{R}_+ \times U$.

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$$|g(s, u, x_1) - g(s, u, x_2)| + |h(x_1) - h(x_2)| \le \omega(|x_1| \vee |x_2|, |x_1 - x_2|)$$

$\forall\,(s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

(**C**-3) For a.a. $t \in [0, T]$, Cesari property holds $\forall\; x \in \mathbb{R}^n$.

Existence Theorem

Let (**C**-1)-(**C**-3) hold. Then problem (OC) admits at least one optimal pair.

Optimal Control Characterization

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) \; s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{T} g(s, u(s), x(s))ds + h(x(T)).$$

Hamiltonian:

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

Optimallity Constraints:

$$\frac{\partial H}{\partial u_i}(t, \bar{x}(t), \bar{u}) = 0.$$

Additional hypothesis:

(**C**-4)

$$x \mapsto (f(t,u,x), g(t,u,x), h(x))$$

is differentiable,

$$(u,x) \mapsto (f(t,u,x), f_x(t,u,x), g(t,u,x), g_x(t,u,x), h_x(x))$$

is continuous.

Pontryagin's Maximum Principle

If $\bar{u}(t)$ and $\bar{x}(t)$ are optimal for the problem (OC), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ s.t.

$$H(t, \bar{x}(t), u(t), \lambda(t)) \leq H(t, \bar{x}(t), \bar{u}(t), \lambda(t))$$

$\forall\ u$ at t,

$$\begin{aligned} \lambda'(t) &= -\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial x}, \\ \lambda(T) &= 0. \end{aligned}$$

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

Example

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$
$$\frac{dL_p}{dt} = \beta_p S_p I_v - bL_p - (r + u_1) L_p,$$
$$\frac{dI_p}{dt} = bL_p - (r + u_2) I_p,$$
$$\frac{dS_v}{dt} = -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta)\mu,$$
$$\frac{dI_v}{dt} = \beta_v S_v I_p - (\gamma + u_3) I_v + \theta\mu,$$

$$\begin{aligned}
H = {} & A_1 I_v + A_2 L_p + A_3 I_v \\
& + \sum_{i=1}^{3} c_i u_i^2 \\
& + \lambda_1(-\beta_p S_p I_v + (r + u_1) L_p \\
& + (r + u_2) I_p) \\
& + \lambda_2(\beta_p S_p I_v - bL_p \\
& - (r + u_1) L_p) \\
& + \lambda_3(bL_p - (r + u_2) I_p) \\
& + \lambda_4(-\beta_v S_v I_p - (\gamma + u_3) S_v \\
& + (1 - \theta)\mu) \\
& + \lambda_5(\beta_v S_v I_p - (\gamma + u_3) I_v \\
& + \theta\mu).
\end{aligned}$$

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \beta_p(\lambda_1 - \lambda_2), \\
\frac{d\lambda_2}{dt} &= -A_2 + (r + u_1)(\lambda_2 - \lambda_1) + b(\lambda_2 - \lambda_3), \\
\frac{d\lambda_3}{dt} &- A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5), \\
\frac{d\lambda_4}{dt} &= \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4, \\
\frac{d\lambda_5}{dt} &= -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,
\end{aligned}
$$

$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$
\begin{aligned}
\bar{u}_1 &= \min\left(\max\left(0, \frac{L_p(\lambda_2 - \lambda_1)}{2c_1}\right), u_1^{max}\right) \\
\bar{u}_2 &= \min\left(\max\left(0, \frac{I_p(\lambda_3 - \lambda_1)}{2c_2}\right), u_2^{max}\right) \\
\bar{u}_3 &= \min\left(\max\left(0, \frac{S_v\lambda_4 + I_v\lambda_5}{2c_3}\right), u_i^{max}\right)
\end{aligned}
$$

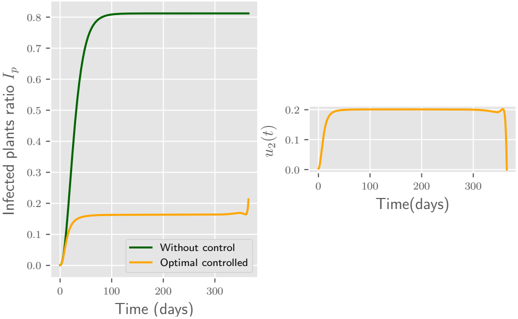
The most popular

Algorithm 2 Forward Backward Sweep

Input: $t_0, t_f, x_0, h, \text{tol}, \lambda_f$
Output: x^*, u^*, λ

procedure FORWARD_BACKWARD_SWEEP($g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{max}$)
  while $\epsilon > \text{tol}$ **do**
    $u_{\text{old}} \leftarrow u$
    $x_{\text{old}} \leftarrow x$
    $x \leftarrow$ RUNGE_KUTTA_FORWARD(g, u, x_0, h)
    $\lambda_{\text{old}} \leftarrow \lambda$
    $\lambda \leftarrow$ RUNGE_KUTTA_BACKWARD($\lambda_{\text{function}}, x, \lambda_f, h$)
    $u_1 \leftarrow$ OPTIMALITY_CONDITION(u, x, λ)
    $u \leftarrow \alpha u_1 + (1-\alpha)u_{old}, \qquad \alpha \in [0,1]$ ▷ convex combination
    $\epsilon_u \leftarrow \dfrac{||u - u_{\text{old}}||}{||u||}$
    $\epsilon_x \leftarrow \dfrac{||x - x_{\text{old}}||}{||x||}$ ▷ relative error
    $\epsilon_\lambda \leftarrow \dfrac{||\lambda - \lambda_{\text{old}}||}{||\lambda||}$
    $\epsilon \leftarrow \max\{\epsilon_u, \epsilon_x, \epsilon_\lambda\}$
  end while
  return x^*, u^*, λ ▷ Optimal pair
end procedure

Case with one controls



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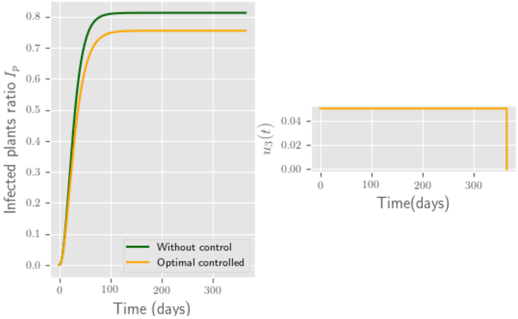
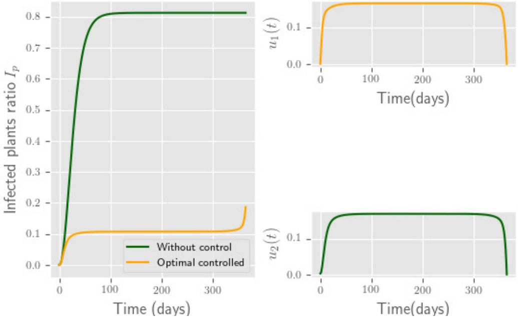
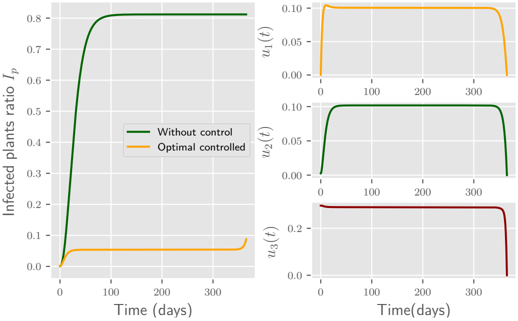


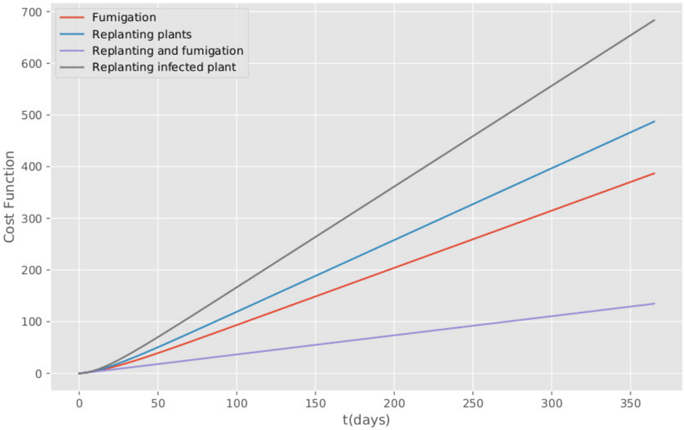
Case with two controls



Case with three controls



Cost Function Comparation



Perspective

$(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$,
$W(t)$: m-dimensional Brownian motion.

$$dx(t) = f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t)$$
$$x(0) = x_0,$$
$$f : [0, T] \times U \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0, T] \times U \times \mathbb{R}^n \to \mathbb{R}^{n+m},$$

U : separable metric space.

$$\mathcal{U}[0, T] := \{u : [0, T] \times \Omega \to U | u(\cdot) \text{ is } \{\mathscr{F}_t\}_{t\geq 0}\text{-adapted}\}$$

Definition

A 6-tuple $\pi = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}, W(\cdot), u(\cdot))$, $u(\cdot)$ is a w-admissible control, $(u(\cdot), x(\cdot))$ is a w-admissible pair, if

- $(\Omega, \mathscr{F}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions,
- $W(t)$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$,
- $u(\cdot)$ is an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process on $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in U,
- $x(\cdot)$ is unique solution,
- some prescribed state constraints are satisfied,
- $g(\cdot, u(\cdot), x(\cdot)) \in L^1_{\mathscr{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathscr{F}_T}(\Omega; \mathbb{R})$

$$J(u(\cdot)) = \mathbb{E}\left\{ \int_0^T g(t, u(t), x(t))dt + h(x(T)) \right\}$$

$(WS)^w$

$$J(\bar{\pi}) = \inf_{\pi \in \mathcal{U}_{ad}^w[0,T]} J(\pi)$$

s.t.

$$\begin{aligned} dx(t) &= f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t) \\ x(0) &= x_0, \end{aligned}$$

Hypothesis:

(**SE**-1) (U, d) is a compact metric space and $T > 0$,

(**SE**-2) f, σ, g, and h are all continous, and $\exists L > 0$ s.t. $\psi(t,u,x) = f(t,u,x), \sigma(t,u,x), g(t,u,x), h(x)$ are lipchitz in x $\forall t \in [0,T]$,$u \in U$, $|\psi(t,u,0)| \leq L$ $\forall (t,u) \in [0,T] \times U$.

(**SE**-3) $\forall (t,x) \in [0,T] \times \mathbb{R}^n$, the set

$$(f, \sigma\sigma^T, g)(t,x,U) := \{(f_i(t,u,x), (\sigma\sigma^T)^{ij}(t,u,x), g(t,u,x)) | \\ u \in U, i = 1, \ldots, n, j = 1, \ldots, m\}$$

is convex in $\mathbb{R}^{m+nm+1}$,

(**SE**-4) $S(t) \equiv \mathbb{R}^n$.

Existence Theorem (weak formulation)

Under (**SE**1)-(**SE**4), if (WS) is finite, then it admits an optimal control.

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