# 1 Notes

- $V(X)$ – vertices of structure $X$. Will be written as $X$ when obvious.
- $a - b$, when $a$ and $b$ are nodes – $a$ and $b$ are neighbors.
- $a \cdots b$, when $a$ and $b$ are nodes – $a$ and $b$ are not neighbors.
- $a - X$, when $a$ is a node and $X$ is a set of nodes – $a$ has a neighbor in $X$.
- $a \cdots X$, when $a$ is a node and $X$ is a set of nodes – $a$ has a nonneighbor in $X$.
- $a \blacktriangleleft X$, when $a$ is a node and $X$ is a set of nodes – $a$ is complete to $X$.
- $a \not\triangleleft X$, when $a$ is a node and $X$ is a set of nodes – $a$ is anticomplete to $X$.
- $X \blacksquare Y$, when $X$ and $Y$ are set of nodes – $X$ is complete to $Y$.
- $X \not\square Y$, when $X$ and $Y$ are set of nodes – $X$ is anticomplete to $Y$.
- $[n]$ – $\{1, \ldots, n\}$.
- $L(BS(K_4))$– a line-graph of a biparite subdivision of $K_4$.
- $a \leftarrow b$ – let a be equal b.
- $a :\in X$ – let a be equal to any element of $X$
- $a \veebar b$ – $a$ xor $b$

# 2 Algorithms

Color-Good-Partition($G, (K_1, K_2, K_3, L, R), c_1, c_2$)

**Input:** $G$ – square-free, Berge graph
$(K_1, K_2, K_3, L, R)$ – good partition
$c_1$, $c_2$ – colorings of $G \setminus R$ and $G \setminus L$ (possibly NULL)

**Output:** $\omega(G)$-coloring of $G$

$G_1 \leftarrow G \setminus R$
$G_2 \leftarrow G \setminus L$
**if** $c_1, c_2 =$ NULL **then**
  $c_1 \leftarrow$ Color-Graph($G_1$)
  $c_2 \leftarrow$ Color-Graph($G_2$)
**foreach** $u \in K_1 \cup K_2$ **do**
  relabel $c_2$, so that $c_1(u) = c_2(u)$
$B \leftarrow \{u \in K_3 : c_1(u) \neq c_2(u)\}$
**if** $B = \emptyset$ **then return** $c_1 \cup c_2$
**foreach** $h \in [2]$, distinct colors $i, j$ **do**
  $G_h^{i,j} \leftarrow$ subgraph induced on $G_h$ by $\{v \in G_h : c_h(v) \in \{i, j\}\}$
**foreach** $u \in K_3$ **do**
  $C_h^{i,j}(u) \leftarrow$ component of $G_h^{i,j}$ containing $u$
ASSERT: $C_h^{c_1(u),c_2(u)}(u) \cap K_2 = \emptyset$
**if** $\exists u \in B, h \in [2] : C_h^{c_1(u),c_2(u)}(u) \cap K_1 = \emptyset$ **then**
  $c_1' \leftarrow c_1$ with colors $i$ and $j$ swapped in $C_1^{i,j}(u)$
  ASSERT: $c_1'$ and $c_2$ agree on $K_1 \cup K_2$
  ASSERT: $\forall u \in K_3 \setminus B : c_1'(u) = c_1(u)$
  ASSERT: $c_1'(u) = j = c_2(u)$
  **return** Color-Good-Partition($G, K_1, K_2, K_3, L, R, c_1', c_2$)
**else**
  $w \leftarrow$ vertex of $B$ with nost neighbors in $K_1$
  ASSERT: $\forall u \in B : N(u) \cap K_1 \subset N(w) \cap K_1$
  relabel $c_1, c_2$, so that $c_1(w) = 1, c_2(w) = 2$
  $P \leftarrow$ chordless path $w - p_1 - \ldots - p_k - a$ in $C_1^{1,2}(w)$ so that
    $k \geq 1$, $p_1 \in K_3 \cup L$, $p_2 \ldots p_k \in L$, $a \in K$, $c_1(a) \in [2]$
  $Q \leftarrow$ chordless path $w - q_1 - \ldots - q_l - a$ in $C_2^{1,2}(w)$ so that
    $l \geq 1$, $q_1 \in K_3 \cup R$, $q_2 \ldots q_l \in R$, $a \in K$, $c_2(a) \in [2]$
  $i \leftarrow c_1(a)$
  $j \leftarrow 3 - i$
  ASSERT: exactly one of the colors 1 and 2 appears in $K_1$ (as in Lemma 2.2.(3))
  ASSERT: $|P|$ and $|Q|$ have different parities
  ASSERT: $p_1 \in K_3 \vee p_2 \in K_3$ (as in Lemma 2.2.(4))
  ASSERT: $\nexists y \in K_3 : c_1(y) = 2 \wedge c_2(y) = 1$ (as in Lemma 2.2.(5))
  **if** $p_1 \in K_3$ **then**
    ASSERT: $c_2(p_1) \notin [2]$
    relabel $c_2$, so that $c_2(p_1) = 3$

// **else** // $\nexists u \in B, h \in [2] : C_h^{c_1(u),c_2(u)}(u) \cap K_1 = \emptyset$

// **if** $p_1 \in K_3$ **then**

ASSERT: color 3 does not appear in $K_2$

ASSERT: color 3 does not appear in $K_1$

ASSERT: $C_2^{j,3}(p_1) \cap K_1 = \emptyset$

$c_2' \leftarrow c_2$ with colors $j$ and 3 swapped in $C_2^{j,3}(p_1)$

ASSERT: $j = 2$

**return** COLOR-GOOD-PARTITION($G$, $K_1$, $K_2$, $K_3$, $L$, $R$, $c_1$, $c_2'$)

**else**

relabel $c_1$, so that $c_1(q_1) = 3$

**if** 3 *does not appear in* $K_1$ **then**

ASSERT: $C_1^{j,3}(q_1) \cap K_1 = \emptyset$

ASSERT: $j = 1$

$c_1' \leftarrow c_1$ with colors $j$ and 3 swapped in $C_1^{j,3}(q_1)$

**return** COLOR-GOOD-PARTITION($G$, $K_1$, $K_2$, $K_3$, $L$, $R$, $c_1'$, $c_2$)

**else**

ASSERT: $q_1 \ntriangleleft \{a, a_3\}$

ASSERT: $C_1^{i,3}(q_1) \cap K_1 = \emptyset$

ASSERT: $i = 1$

$c_1' \leftarrow c_1$ with colors $i$ and 3 swapped in $C_1^{i,3}(q_1)$

**return** COLOR-GOOD-PARTITION($G$, $K_1$, $K_2$, $K_3$, $L$, $R$, $c_1'$, $c_2$)

Grow-Hyperprism($G, G, M, F$) // Lemma 3.3

**Input:** $G$ – square-free, Berge graph
$H = (A_1, \dots B_3)$ – a hyperprism in $G$
$M$ – the set of major neighbors of $H$ in $G$
$F$ – a minimal component of $G \setminus (H \cup M)$ with a set of attachments in $H$ not local.

**Output:** $H'$ – a larger hyperprism, or
$L$ – a $L(BS(K_4))$

$X \leftarrow$ set of attachments of $F$ in $H$

**if** $\exists i : X \cap C_i \neq \emptyset$ **then**
  relabel strips of $H$, so that $X \cap C_1 \neq \emptyset$
  $x_1 :\in X \cap C_1$
  ASSERT: $X \cap S_2 \neq \emptyset$
  $x_2 :\in X \cap S_2$
  $R_1 \leftarrow$ 1-rung of $H$, so that $x_1 \in V(R_1)$
  $R_2 \leftarrow$ 2-rung of $H$, so that $x_2 \in V(R_2)$
  $R_3 \leftarrow$ a 3-rung of $H$
  $\forall i \in [3] : a_i, b_i \leftarrow$ends of $R_i$, so that $a_i \in A_i, b_i \in B_i$
  $K \leftarrow$ a prism $(R_1, R_2, R_3)$
  ASSERT: no vertex in $F$ is major w.r.t. $K$ (as in SPGT 10.5)
  $f_1 - \dots - f_n \leftarrow$ a minimal path in $F$, so that
    $f_1 \blacktriangleleft \{a_2, a_3\}$,
    $f_n - R \setminus \{a_1\}$
    there are no other edges between $\{f_1, \dots f_n\}$ and $V(K) \setminus \{a_1\}$
  ASSERT: $F = \{f_1, \dots, f_n\}$
  ASSERT: $f_1 \blacktriangleleft A_3$
  $A'_1 \leftarrow A_1 \cup \{f_1\}$
  $C'_1 \leftarrow C_1 \cup \{f_2, \dots, f_n\}$
  **return** $H' \leftarrow (A'_1, A_2, \dots, B_3, C'_1, C_2, C_3)$
**else**
  relabel strips of $H$, so that there is $\{x_1 :\in A_1, x_2 :\in A_2\} \subset X$ that is not local
  find a path $x - f_1 - \dots - f_n - x_2$
  ASSERT: $F = \{f_1, \dots f_n\}$
  **if** $n$ is even and $H$ is even, or $n$ is odd and $H$ is odd **then**
    ASSERT: $f_1 - a_3 \veebar f_n - b_3$
    **if** $f_1 - a_3$ **then**
      $H' \leftarrow$ mirrored $H$ – every $A_i$ and $B_i$ are swapped
      check if $M$ and $F$ are OK **return** Grow-Hyperprism($G, H', M, F$)
    **else**
      **if** $f_n \blacktriangleleft B_2 \cup B_3$ **then**
        $B'_1 \leftarrow B_1 \cup \{f_n\}$
        $C'_1 \leftarrow C_1 \cup \{f_1, \dots, f_{n-1}\}$

// **else** // $\forall_{i \in [3]} X \cap C_i = \emptyset$
  // **if** $n$ is even and $H$ is even, or $n$ is odd and $H$ is odd **then**
    // **else** // $f_n - b_3$
      // **if** $f_n \blacktriangleleft B_2 \cup B_3$ **then**
        **return** $H' \leftarrow \begin{pmatrix} A_1 & C_1' & B_1' \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{pmatrix}$
      **else**
        $\forall_{i \in [3]} : A_i' \leftarrow$ neighbors of $f_1$ in $A_i$
        $\forall_{i \in [3]} : A_i'' \leftarrow A_i \setminus A_i'$
        $\forall_{i \in [3]} : B_i'' \leftarrow$ neighbors of $f_n$ in $B_i$
        $\forall_{i \in [3]} : B_i' \leftarrow B_i \setminus B_i''$
        ASSERT: Every $i$-rung is between $A_i'$ and $B_i'$ or $A_i''$ and $B_i''$
        $\forall_{i \in [3]} : C_i' \leftarrow$ union of interiors of $i$-rings between $A_i'$ and $B_i'$
        $\forall_{i \in [3]} : C_i'' \leftarrow$ union of interiors of $i$-rings between $A_i''$ and $B_i''$
        ASSERT: $C_i = C_i' \cup C_i''$, $C_i' \cap C_i'' = \emptyset$
        ASSERT: $A_i' \cup C_i' \not\square C_i'' \cup B_i''$, $A_i'' \cup C_i'' \not\square C_i \cup B_i$
        ASSERT: $A_i' \blacksquare A_i''$, $B_i' \blacksquare B_i''$
        ASSERT: $A_1', A_2'', A_3', A_3'' \neq \emptyset$
        $H' \leftarrow \begin{pmatrix} A_1' & C_1' & B_1' \\ A_2' \cup A_3' & C_2' \cup C_3' & B_2' \cup B_3' \\ \bigcup_i A_i'' \cup \{f_1\} & \bigcup_i C_i'' \cup \{f_2, \ldots, f_n\} & \bigcup_i B_i'' \end{pmatrix}$
        **return** $H'$
  **else**
    $a_1 \leftarrow$ neighbor of $f_1$ in $A_1$
    $R_1 \leftarrow$ 1-rung with end $a_1$
    $b_1 \leftarrow$ the other end of $R_1$
    $b_2 \leftarrow$ neighbor of $f_2$ in $B_2$
    $R_2 \leftarrow$ 2-rung with end $b_2$
    $a_2 \leftarrow$ the other end of $R_2$
    ASSERT: $b_1 \in X$, $a_2 \in X$
    ASSERT: $(b_1 - f_1 \wedge a_2 - f_n) \veebar (b_1 - f_n \wedge a_2 - f_1)$
    **if** $f_1 - b_1$ **then**
      ASSERT: $H$ is odd
      $R_3 \leftarrow$ any 3-rung with ends $a_3, b_3$, such that $\{a_3, b_3\} \not\square \{f_1, f_n\}$
      **return** $V(R_1) \cup V(R_2) \cup V(R_3) \cup \{f_1, \ldots, f_n\}$ - a $L(BS(K_4))$
      Is it valid input for part of ALG I?

// **else** // $\forall_{i \in [3]} X \cap C_i = \emptyset$

// **else** // $n$ is odd and $H$ is even, or $n$ is even and $H$ is odd

**else** // $f_1 - a_2$

$\forall_{i \in [3]} : A'_i \leftarrow A_i \cap X,\ A''_i \leftarrow A_i \setminus X$

$\forall_{i \in [3]} : B'_i \leftarrow B_i \cap X,\ B''_i \leftarrow B_i \setminus X$

$\forall_{i \in [3]} : C'_i \leftarrow$ union of $i$-rungs between $A'_i$ and $B'_i$

$\forall_{i \in [3]} : C''_i \leftarrow$ union of $i$-rungs between $A''_i$ and $B''_i$

ASSERT: $C_i = C'_i \cup C''_i,\ C'_i \cap C''_i = \emptyset$

**if** $f_1$ *is complete to at least two of* $A_i$ **then**

relabel strips of $H$, so that $f_1$ is complete to $A_1$ and $A_2$

ASSERT: $f_n$ is complete to $B_1$ and $B_2$

ASSERT: $n > 1$ (as in SPGT 10.5 OK for odd $H$?)

**return** $\begin{pmatrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 \cup \{f_1\} & C_3 \cup \{f_2, \dots, f_{n-1}\} & B_3 \cup \{f_n\} \end{pmatrix}$

**else**

ASSERT: $A'_i \blacksquare A''_i$ OK odd H?

ASSERT: $B'_i \blacksquare B''_i$ OK odd H?

**return**

$$\begin{pmatrix} A'_1 & C'_1 & B'_1 \\ A'_2 \cup A'_3 & C_2 \cup C'_3 & B'_2 \cup C'_3 \\ \bigcup_i A''_i \cup \{f_1\} & \bigcup_i C''_i \cup \{f_2, \dots, f_{n-1}\} & \bigcup_i B''_i \cup \{f_n\} \end{pmatrix}$$

Good-Partition-From-Even-Hyperprism($G, H, M$)

**Input:** $G$ – square-free, Berge graph containing no $L(BS(K_4))$
$H = (A_1, \dots, B_3)$ – maximal even hyperprism in $G$
$M$ – set of major neighbors of $H$

**Output:** A good partition of $G$

$Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$
relabel strips of $H$, so that $M \cup A_1$ and $M \cup B_1$ are cliques
$F_1 \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_1 \cup B_1 \cup C_1$ $\}$
ASSERT: $M$ is a clique
ASSERT: $M \cup A_i$ is a clique for at least two values of $i$
ASSERT: $M \cup B_j$ is a clique for at least two values of $j$
$K_1 \leftarrow A_1$, $K_2 \leftarrow M$, $K_3 \leftarrow B_1$
$R \leftarrow C_1 \cup F_1 \cup Z$
$L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$
**return** $(K_1, K_2, K_3, L, R)$

Good-Partition-From-Odd-Hyperprism($G, H, M$)

**Input:** $G$ – square-free, Berge graph containing no $L(BS(K_4))$
$H = (A_1, \ldots, B_3)$ – maximal odd hyperprism in $G$
$M$ – set of major neighbors of $H$

**Output:** A good partition of $G$

$Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$
relabel strips of $H$, so that $A_1 \not\square B_1$ and $A_2 \not\square B_2$
ASSERT: $C_1 \neq \emptyset, C_2 \neq \emptyset$
$\forall_{i \in [3]} F_i \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_i \cup B_i \cup C_i$ $\}$
$F_B \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z \cup F_1 \cup F_2 \cup F_3\}$ that attaches to $B_1 \cup B_2 \cup B_3$ $\}$
$F_i, F_A, F_B$ are from algIV, make sure it is correct
ASSERT: At least two of $A_i$ and at least two of $B_i$ are cliques
ASSERT: $M$ is complete to at least two of $A_i$ and at least two of $B_i$
ASSERT: $M$ is a clique
ASSERT: For at least two $i$ : $A_i \cup M$ is a clique
ASSERT: For at least two $j$ : $A_j \cup M$ is a clique
choose $h$, so that $M \cup A_h$ and $M \cup B_h$ are cliques
**if** $h = 1 \vee h = 2$ **then** `// make sure` $h = 2$ `is ok`
    $K_1 \leftarrow A_1$, $K_2 \leftarrow M$, $K_3 \leftarrow B_1$
    $R \leftarrow C_1 \cup F_1 \cup Z$
    $L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$
    **return** $(K_1, K_2, K_3, L, R)$
**else**
    relabel $H$ so that $M \cup A_1$ and $M \cup B_2$ are cliques
    $K_1 \leftarrow B_2 \cup B_3$, $K_2 \leftarrow M$, $K_3 \leftarrow A_1 \cup A_3$
    $L \leftarrow B_1 \cup C_1 \cup F_1 \cup F_B$
    $R \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup L\}$
    **return** $(K_1, K_2, K_3, L, R)$

## 2.1 ALG VI

### 2.1.1 Notes

- $T_{uv} := N_u \cap N_v$
- Strip $S_{uv}$ is *rich* iff $S_{uv} \setminus T_{uv} \neq \emptyset$

Good-Partition-From-J-Strip-System$(G, J, (S, N), M)$

**Input:** $G$ – square-free, Berge graph
$J$ – a maximal 3-connected graph with appearance in $G$
$(S, N)$ – a maximal $J$-strip system
$M$ – a set of major vertices w.r.t. $(S, N)$

**Output:** A good partition of $G$

$S^*_{uv} \leftarrow S_{uv} \cup$ ( components of $G \setminus V(S, N)$ that attach in $S_{uv}$ only )
$T_{uv} \leftarrow N_u \cap N_v$
ASSERT: $T_{uv}$ ■ $N_u \setminus N_{uv}$, $T_{uv}$ ■ $N_v \setminus N_{vu}$
ASSERT: $M \cup T_{uv}$ is a clique
**if** $\exists S_{uv}$ *– a rich strip in* $(S, N)$ **then**
    **if** $\exists S_{uv}$ *– a rich strip in* $(S, N)$*, such that* $M \cup (N_u \setminus N_{uv})$ *and* $M \cup (N_v \setminus N_{vu})$ *are cliques* **then**
        $K_1 \leftarrow N_u \setminus N_{uv}$, $K_2 \leftarrow M \cup T_{uv}$, $K_3 \leftarrow N_v \setminus N_{vu}$
        $L \leftarrow (S^*_{uv} \setminus T_{uv}) \cup$ ( components of $G \setminus V(S, N)$ that attach only to $N_u$ and those that attach only to $N_v$ )
        $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$
        **return** $(K_1, K_2, K_3, L, R)$
    **else**
        $S_{uv} \leftarrow$ a rich strip in $(S, N)$, such that $M \cup (N_u \setminus N_{uv})$ is not a clique and $M \cup (N_v \setminus N_{vu})$ is a clique
        $K_1 \leftarrow N_{uv} \setminus T_{uv}$, $K_2 \leftarrow M \cup T_{uv}$, $K_3 \leftarrow N_v \setminus N_{vu}$
        $R \leftarrow (S^*_{uv} \setminus N_u) \cup$ ( components of $G \setminus V(S, U)$ that attach only to $N_v$ )
        $L \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup R)$
        **return** $(K_1, K_2, K_3, L, R)$
**else**
    ASSERT: $\forall uv \in E(J) : S_{uv} = T_{uv}, S_{uv}$ is a clique
    $S_{uv} \leftarrow$ any strip
    $K_1 \leftarrow N_u \setminus S_{uv}$, $K_2 \leftarrow M$, $K_3 \leftarrow N_v \setminus S_{uv}$
    $L \leftarrow S^*_{uv} \cup$ ( components of $G \setminus V(S, N)$ that attach only to $N_u$ and only to $N_v$ )
    $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$
    **return** $(K_1, K_2, K_3, L, R)$

Good-Partition-From-Special-Strip-System($G, J, (S, N), M$)

**Input:** $G$ – square-free, Berge graph
$(S, N)$ – a special $K_4$ strip system
$M$ – a set of major vertices w.r.t. $(S, N)$

**Output:** A good partition of $G$

$\forall_{i,j \in [4]} O_{ij} \leftarrow$ set of vertices in $V(G) \setminus V(S, N)$ that are complete to $(N_i \cup N_j) \setminus S_{ij}$ and anticomplete to $V(S, N) \setminus (N_i \cup N_j \cup S_{ij})$
**if** $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ *and* $(N_2 \setminus N_{12}) \cup M \cup O_{12}$ *are cliques* **then**
  $K_1 \leftarrow N_1 \setminus N_{12}$, $K_2 \leftarrow O_{12} \cup M$, $K_3 \leftarrow N_2 \setminus N_{12}$
  $L \leftarrow$ union of those components of $G \setminus (K_1 \cup K_2 \cup K_3)$ that contain vertices of $S_{12}$
  $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$
  **return** $(K_1, K_2, K_3, L, R)$
**else**
  **if** $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ *is a clique* **then**
    relabel 1 and 2 in $J$ so that $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ is not a clique
  ASSERT: $N_{12} \cup M \cup O_{12}$ is a clique
  **if** $N_{21} \cup M \cup O_{12}$ *is a clique* **then**
    $X \leftarrow N_{21}$
  **else**
    $X \leftarrow N_2 \setminus N_{21}$
  $K_1 \leftarrow N_{12}$, $K_2 \leftarrow M \cup O_{12}$, $K_3 \leftarrow X$
  $L \leftarrow$ component of $G \setminus (K_1 \cup K_2 \cup K_3)$ that contains $N_1 \setminus N_{12}$
  $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$
  **return** $(K_1, K_2, K_3, L, R)$

FIND-SPECIAL-K4-STRIP-SYSTEM($G, J, (S, N), m$)

**Input:** $G$ – square-free, Berge graph
$J$ – a 3-connected graph with appearance in $G$
$(S, N)$ – a $J$-strip system
$m \in G \setminus (S, N)$ that is major w.r.t. some choice of rungs of $(S, N)$ but not w.r.t. $(S, N)$

**Output:** $(S', N')$ – A special $K_4$-strip system, or
$(S'', N'')$ – a bigger $J$-strip system

$X \leftarrow N(m)$
$M \leftarrow$ vertices of $G \setminus V(S, N)$ major w.r.t. $(S, N)$
$M^* \leftarrow$ vertices of $G \setminus V(S, N)$ major w.r.t. some choice of rungs
ASSERT: $J = K_4$ (as in SPGT 8.4)
$V(J) \leftarrow [4]$
$\forall_{i \neq j \in [4]}$: choose rungs $R_{ij}, R_{ij}$, forming line graphs $L(H)$ and $L(H')$ so that $X$ saturates $L(H)$ but does not saturate $L(H')$
ASSERT: $R_{ij} \neq R'_{ij} \iff \{i, j\} = [2]$
$r_{ij}, r_{ji} \leftarrow$ ends of each $R_{ij}$
$r'_{ij}, r'_{ji} \leftarrow$ ends of each $R'_{ij}$
$\forall_{i \in [4]} T_i \leftarrow \{r_{ij}, j \in [4] \setminus \{i\}\}$
$\forall_{i \in [4]} T'_i \leftarrow \{r'_{ij}, j \in [4] \setminus \{i\}\}$
ASSERT: $X$ has at least two members in each $T_1, \ldots T_4$
ASSERT: There is $T'_i$ that contains at most one member of $X$
ASSERT: $T_3 = T'_3, T_4 = T'_4$
relabel 1 and 2 in $J$, so that $|X \cap T_1| = 2$ and $|X \cap T'_1| = 1$
ASSERT: $r_{12} \in X, r'_{12} \notin X$
ASSERT: $r_{13} \in X \veebar r_{14} \in X$
relabel 3 and 4 in $J$, so that $r_{13} \in X, r_{14} \notin X$
ASSERT: $R_{34}$ is even and $[X \cap V(L(H'))] \setminus V(R_{34}) = \{r_{31}, r_{32}, r_{41}, r_{42}\}$ (as in 6.3.(3))
ASSERT: $R_{14}$ has odd length, $r_{21} \in X$ (as in 6.3.(4))
ASSERT: $R_{12}$ has length 0, every 12-rung has even length (as in 6.3.(5))
ASSERT: $R_{24}$ has length 0 and $R_{23}$ has odd length (as in 6.3.(6))
ASSERT: Every 34-rung has non-zero even length (as in 6.3.(7))
$\forall_{i \neq j \in [4]} O_{ij} \leftarrow$ set of vertices that are not major w.r.t $L(H')$ and are complete to $(T'_i \cup T'_j) \setminus R'_{ij}$
ASSERT: $r_{12} = r_{21} \in O_{12}$, $m \in O_{34}$
ASSERT: $O_{34} = M^* \setminus M$
$(S', N') \leftarrow$ strip system obtained from $(S, N)$ by replacing $S_{12}$ with $S_{12} \setminus O_{12}$
**if** $\exists_{rungR}$ : *adding* $R$ *to* $S'_{12}$ *produces enlargement of* $(S, N)$ **then**
  **return** $(S'', N'')$ – an enlargement of $(S, N)$
**else**
  **return** $(S', N')$ – a special $K_4$ strip system

Growing-J-Strip$(G, J, (S, N))$

**Input:** $G$ – square-free, Berge graph

$J$ – a 3-connected graph with appearance in $G$

$(S, N)$ – a $J$-strip system

make sure def of $J$ is correct

**Output:** $J'$ and a maximal $J'$-strip system, or a special strip system

$M \leftarrow$ vertices of $G \setminus V(S, N)$ that are major on some choice of Rungs of $(S, N)$

$M$ like ALGI **if** $\exists m : m$ *is not major on some choice of rungs of* $(S, N)$ **then**

  $OUT \leftarrow$ Find-Special-K4-Strip-System$(G, J, (S, N), m)$

  **if** $OUT$ *is a special strip system* **then**

    **return** $OUT$

  **else**

    **return** Growing-J-Strip$(G, OUT)$

**else if** $\exists F$ *: F is a component of* $G \setminus (V(S, N) \cup M)$*, such that no member of F is major w.r.t.* $(S, N)$ *and set of attachments of F on* $H$(?) *is not local* **then**

  ASSERT: (as in 6.2, or actually SPGT 8.5)

  $F \leftarrow$ minimal (component?) with this property

  **if** $\exists v \in V(J) : X \subset \bigcup(S_{uv} : uv \in E(J))$ **then**

    what is X? $x :\in X \cap S_{uv} \setminus N_v$, for some $uv \in E(J)$

    brackets? $x' :\in X \cap S_{u'v}$, for some $u'v \in E(J), u' \neq u$

    ASSERT: $\{x, x'\}$ is not local w.r.t. $(S, N)$

    $L(H) \leftarrow \forall_{i,j \in E(J)}$ choose $ij$-rung $R_{ij}$, so that $x \in V(R_{uv}), x' \in V(R_{u'v})$

    ASSERT: $\{x, x'\}$ is not local w.r.t. $L(H)$

    $D \leftarrow$ a branch of $H$ with ends $d, u$: $\delta_H(d) \setminus E(D) = (X \cap E(H)) \setminus E(D)$

    How can we get $H$ from $L(H)$ $P \leftarrow$ a path with ends $p_1, p_2$, so that:

      $p_1 \blacktriangleleft N_v \setminus N_{vu}$ and no other vertex of $P$ has neighbors in $N_v \setminus N_{uv}$

      $p_2 - x$ and no other vertex of $P$ has neighbors in $S_{uv} \setminus N_v$

    $(S', N') \leftarrow$ add $p_1$ to $N_v$ and $F$ to $S_{uv}$

    **return** Growing-J-Strip $(G, J, (S', N'))$

  **else**

    $K \leftarrow \{uv \in E(J) : X \cap S_{uv} \neq \emptyset\}$

    ASSERT: There are two disjoint edges in $K$ (as in SPGT 8.5.(3))

    $F$ is a vertex set of a path $\leftarrow f_1 - \ldots - f_n$

    ASSERT: Every choice of rungs is broad

// **if** $\exists F \ldots$ **then**

// **else** // $\nexists v \in V(J) : X \subset \bigcup(S_{uv} : uv \in E(J))$

ASSERT: every choice of rungs has the same traversal. (Hard to assert)

$ij \leftarrow$ the traversal edge

$A_1 \leftarrow N_i \setminus S_{ij}$, $A_2 \leftarrow N_j \setminus S_{ij}$

ASSERT: $X \cap (V(S, N) \setminus S_{ij}) = A_1 \cup A_2$

**if** $n = 1$ **then**

$(S', N') \leftarrow$ add $f_1$ to $N_i, N_j, S_{ij}$

**return** Growing-J-Strip $(G, J, (S', N'))$

**else**

$x_1 :\in A_1, x_2 :\in A_2$, so that $x_1$ and $x_2$ are in disjoint strips

ASSERT: $x_1 - f_1 \veebar x_1 - f_n$

**if** $x_1 - f_n$ **then**

relabel $f_1 - \ldots - f_n$ front to back

$(S', N') \leftarrow$ add $f_1$ to $N_i$, $f_n$ to $N_j$ and $F$ to $S_{ij}$

**return** Growing-J-Strip $(G, J, (S', N'))$

**else**

**return** $J, (S, N)$ – a maximal $J$-strip