1 Notes

- $V(X)$ – vertices of structure X. Will be written as X when obvious.
- $a - b$, when a and b are nodes – a and b are neighbors.
- $a \cdots b$, when a and b are nodes – a and b are not neighbors.
- $a - X$, when a is a node and X is a set of nodes – a has a neighbor in X.
- $a \cdots X$, when a is a node and X is a set of nodes – a has a nonneighbor in X.
- $a \blacktriangleleft X$, when a is a node and X is a set of nodes – a is complete to X.
- $a \ntriangleleft X$, when a is a node and X is a set of nodes – a is anticomplete to X.
- $X \blacksquare Y$, when X and Y are set of nodes – X is complete to Y.
- $X \not\square Y$, when X and Y are set of nodes – X is anticomplete to Y.
- $[n]$ – $\{1, \ldots, n\}$.
- $L(BS(K_4))$– a line-graph of a biparite subdivision of K_4.
- $a \leftarrow b$ – let a be equal b.
- $a :\in X$ – let a be equal to any element of X
- $a \veebar b$ – a xor b

2 Algorithms

Color-Good-Partition($G, (K_1, K_2, K_3, L, R), c_1, c_2$)

Input: G – square-free, Berge graph
    (K_1, K_2, K_3, L, R) – good partition
    c_1, c_2 – colorings of $G \setminus R$ and $G \setminus L$ (possibly NULL)

Output: $\omega(G)$-coloring of G

$G_1 \leftarrow G \setminus R$
$G_2 \leftarrow G \setminus L$
if $c_1, c_2 =$ NULL **then**
    $c_1 \leftarrow$ Color-Graph(G_1)
    $c_2 \leftarrow$ Color-Graph(G_2)
foreach $u \in K_1 \cup K_2$ **do**
    relabel c_2, so that $c_1(u) = c_2(u)$
$B \leftarrow \{u \in K_3 : c_1(u) \neq c_2(u)\}$
if $B = \emptyset$ **then return** $c_1 \cup c_2$
foreach $h \in [2]$, distinct colors i, j **do**
    $G_h^{i,j} \leftarrow$ subgraph induced on G_h by $\{v \in G_h : c_h(v) \in \{i, j\}\}$
foreach $u \in K_3$ **do**
    $C_h^{i,j}(u) \leftarrow$ component of $G_h^{i,j}$ containing u
ASSERT: $C_h^{c_1(u),c_2(u)}(u) \cap K_2 = \emptyset$
if $\exists u \in B, h \in [2] : C_h^{c_1(u),c_2(u)}(u) \cap K_1 = \emptyset$ **then**
    $c_1' \leftarrow c_1$ with colors i and j swapped in $C_1^{i,j}(u)$
    ASSERT: c_1' and c_2 agree on $K_1 \cup K_2$
    ASSERT: $\forall u \in K_3 \setminus B : c_1'(u) = c_1(u)$
    ASSERT: $c_1'(u) = j = c_2(u)$
    return Color-Good-Partition($G, K_1, K_2, K_3, L, R, c_1', c_2$)
else
    $w \leftarrow$ vertex of B with nost neighbors in K_1
    ASSERT: $\forall u \in B : N(u) \cap K_1 \subset N(w) \cap K_1$
    relabel c_1, c_2, so that $c_1(w) = 1, c_2(w) = 2$
    $P \leftarrow$ chordless path $w - p_1 - \ldots - p_k - a$ in $C_1^{1,2}(w)$ so that
        $k \geq 1$, $p_1 \in K_3 \cup L$, $p_2 \ldots p_k \in L$, $a \in K$, $c_1(a) \in [2]$
    $Q \leftarrow$ chordless path $w - q_1 - \ldots - q_l - a$ in $C_2^{1,2}(w)$ so that
        $l \geq 1$, $q_1 \in K_3 \cup R$, $q_2 \ldots q_l \in R$, $a \in K$, $c_2(a) \in [2]$
    $i \leftarrow c_1(a)$
    $j \leftarrow 3 - i$
    ASSERT: exactly one of the colors 1 and 2 appears in K_1 (as in Lemma 2.2.(3))
    ASSERT: $|P|$ and $|Q|$ have different parities
    ASSERT: $p_1 \in K_3 \vee p_2 \in K_3$ (as in Lemma 2.2.(4))
    ASSERT: $\nexists y \in K_3 : c_1(y) = 2 \wedge c_2(y) = 1$ (as in Lemma 2.2.(5))
    if $p_1 \in K_3$ **then**
        ASSERT: $c_2(p_1) \notin [2]$
        relabel c_2, so that $c_2(p_1) = 3$

// **else** // $\nexists u \in B, h \in [2] : C_h^{c_1(u), c_2(u)}(u) \cap K_1 = \emptyset$

  // **if** $p_1 \in K_3$ **then**

    ASSERT: color 3 does not appear in K_2

    ASSERT: color 3 does not appear in K_1

    ASSERT: $C_2^{j,3}(p_1) \cap K_1 = \emptyset$

    $c_2' \leftarrow c_2$ with colors j and 3 swapped in $C_2^{j,3}(p_1)$

    ASSERT: $j = 2$

    return `Color-Good-Partition`$(G, K_1, K_2, K_3, L, R, c_1, c_2')$

  else

    relabel c_1, so that $c_1(q_1) = 3$

    if 3 *does not appear in* K_1 **then**

      ASSERT: $C_1^{j,3}(q_1) \cap K_1 = \emptyset$

      ASSERT: $j = 1$

      $c_1' \leftarrow c_1$ with colors j and 3 swapped in $C_1^{j,3}(q_1)$

      return `Color-Good-Partition`$(G, K_1, K_2, K_3, L, R, c_1', c_2)$

    else

      ASSERT: $q_1 \not\triangleleft \{a, a_3\}$

      ASSERT: $C_1^{i,3}(q_1) \cap K_1 = \emptyset$

      ASSERT: $i = 1$

      $c_1' \leftarrow c_1$ with colors i and 3 swapped in $C_1^{i,3}(q_1)$

      return `Color-Good-Partition`$(G, K_1, K_2, K_3, L, R, c_1', c_2)$

`GROW-HYPERPRISM`(G, G, M, F) `// Lemma 3.3`

Input: G – square-free, Berge graph
$H = (A_1, \dots B_3)$ – a hyperprism in G
M – the set of major neighbors of H in G
F – a minimal component of $G \setminus (H \cup M)$ with a set of attachments in H not local.

Output: H' – a larger hyperprism, or
L – a $L(BS(K_4))$

$X \leftarrow$ set of attachments of F in H
if $\exists i : X \cap C_i \neq \emptyset$ **then**
  relabel strips of H, so that $X \cap C_1 \neq \emptyset$
  $x_1 :\in X \cap C_1$
  ASSERT: $X \cap S_2 \neq \emptyset$
  $x_2 :\in X \cap S_2$
  $R_1 \leftarrow$ 1-rung of H, so that $x_1 \in V(R_1)$
  $R_2 \leftarrow$ 2-rung of H, so that $x_2 \in V(R_2)$
  $R_3 \leftarrow$ a 3-rung of H
  $\forall i \in [3] : a_i, b_i \leftarrow$ends of R_i, so that $a_i \in A_i, b_i \in B_i$
  $K \leftarrow$ a prism (R_1, R_2, R_3)
  ASSERT: no vertex in F is major w.r.t. K (as in SPGT 10.5)
  $f_1 - \ldots - f_n \leftarrow$ a minimal path in F, so that
    $f_1 \blacktriangleleft \{a_2, a_3\}$,
    $f_n - R \setminus \{a_1\}$
    there are no other edges between $\{f_1, \ldots f_n\}$ and $V(K) \setminus \{a_1\}$
  ASSERT: $F = \{f_1, \ldots, f_n\}$
  ASSERT: $f_1 \blacktriangleleft A_3$
  $A_1' \leftarrow A_1 \cup \{f_1\}$
  $C_1' \leftarrow C_1 \cup \{f_2, \ldots, f_n\}$
  return $H' \leftarrow (A_1', A_2, \ldots, B_3, C_1', C_2, C_3)$
else
  relabel strips of H, so that there is $\{x_1 :\in A_1, x_2 :\in A_2\} \subset X$ that is not local
  find a path $x - f_1 - \ldots - f_n - x_2$
  ASSERT: $F = \{f_1, \ldots f_n\}$
  if n is even and H is even, or n is odd and H is odd **then**
    ASSERT: $f_1 - a_3 \veebar f_n - b_3$
    if $f_1 - a_3$ **then**
      $H' \leftarrow$ mirrored H – every A_i and B_i are swapped
      # TODO: check if M and F are OK
      return `GROW-HYPERPRISM`(G, H', M, F)
    else
      if $f_n \blacktriangleleft B_2 \cup B_3$ **then**
        $B_1' \leftarrow B_1 \cup \{f_n\}$
        $C_1' \leftarrow C_1 \cup \{f_1, \ldots, f_{n-1}\}$

// **else** // $\forall_{i \in [3]} X \cap C_i = \emptyset$

  // **if** n is even and H is even, or n is odd and H is odd **then**

    // **else** // $f_n - b_3$

      // **if** $f_n \blacktriangleleft B_2 \cup B_3$ **then**

        return $H' \leftarrow \begin{pmatrix} A_1 & C_1' & B_1' \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{pmatrix}$

      else

        $\forall_{i \in [3]} : A_i' \leftarrow$ neighbors of f_1 in A_i

        $\forall_{i \in [3]} : A_i'' \leftarrow A_i \setminus A_i'$

        $\forall_{i \in [3]} : B_i'' \leftarrow$ neighbors of f_n in B_i

        $\forall_{i \in [3]} : B_i' \leftarrow B_i \setminus B_i''$

        ASSERT: Every i-rung is between A_i' and B_i' or A_i'' and B_i''

        $\forall_{i \in [3]} : C_i' \leftarrow$ union of interiors of i-rings between A_i' and B_i'

        $\forall_{i \in [3]} : C_i'' \leftarrow$ union of interiors of i-rings between A_i'' and B_i''

        ASSERT: $C_i = C_i' \cup C_i''$, $C_i' \cap C_i'' = \emptyset$

        ASSERT: $A_i' \cup C_i' \not\square C_i'' \cup B_i''$, $A_i'' \cup C_i'' \not\square C_i \cup B_i$

        ASSERT: $A_i' \blacksquare A_i''$, $B_i' \blacksquare B_i''$

        ASSERT: $A_1', A_2'', A_3', A_3'' \neq \emptyset$

        $H' \leftarrow \begin{pmatrix} A_1' & C_1' & B_1' \\ A_2' \cup A_3' & C_2' \cup C_3' & B_2' \cup B_3' \\ \bigcup_i A_i'' \cup \{f_1\} & \bigcup_i C_i'' \cup \{f_2, \dots, f_n\} & \bigcup_i B_i'' \end{pmatrix}$

        return H'

  else

    $a_1 \leftarrow$ neighbor of f_1 in A_1

    $R_1 \leftarrow$ 1-rung with end a_1

    $b_1 \leftarrow$ the other end of R_1

    $b_2 \leftarrow$ neighbor of f_2 in B_2

    $R_2 \leftarrow$ 2-rung with end b_2

    $a_2 \leftarrow$ the other end of R_2

    ASSERT: $b_1 \in X$, $a_2 \in X$

    ASSERT: $(b_1 - f_1 \wedge a_2 - f_n) \veebar (b_1 - f_n \wedge a_2 - f_1)$

    if $f_1 - b_1$ **then**

      ASSERT: H is odd

      $R_3 \leftarrow$ any 3-rung with ends a_3, b_3, such that $\{a_3, b_3\} \not\square \{f_1, f_n\}$

      return $V(R_1) \cup V(R_2) \cup V(R_3) \cup \{f_1, \dots, f_n\}$ - a $L(BS(K_4))$

      # TODO: Is it valid input for part of ALG I?

// **else** // $\forall_{i\in[3]} X \cap C_i = \emptyset$

// **else** // n is odd and H is even, or n is even and H is odd

else // $f_1 - a_2$

$\forall_{i\in[3]} : A'_i \leftarrow A_i \cap X,\ A''_i \leftarrow A_i \setminus X$

$\forall_{i\in[3]} : B'_i \leftarrow B_i \cap X,\ B''_i \leftarrow B_i \setminus X$

$\forall_{i\in[3]} : C'_i \leftarrow$ union of i-rungs between A'_i and B'_i

$\forall_{i\in[3]} : C''_i \leftarrow$ union of i-rungs between A''_i and B''_i

ASSERT: $C_i = C'_i \cup C''_i,\ C'_i \cap C''_i = \emptyset$

if f_1 *is complete to at least two of* A_i **then**

relabel strips of H, so that f_1 is complete to A_1 and A_2

ASSERT: f_n is complete to B_1 and B_2

ASSERT: $n > 1$ (as in SPGT 10.5 OK for odd H?)

return $\begin{pmatrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 \cup \{f_1\} & C_3 \cup \{f_2, \ldots, f_{n-1}\} & B_3 \cup \{f_n\} \end{pmatrix}$

else

ASSERT: $A'_i \blacksquare A''_i$ # TODO: OK odd H?

ASSERT: $B'_i \blacksquare B''_i$ # TODO: OK odd H?

return

$$\begin{pmatrix} A'_1 & C'_1 & B'_1 \\ A'_2 \cup A'_3 & C_2 \cup C'_3 & B'_2 \cup C'_3 \\ \bigcup_i A''_i \cup \{f_1\} & \bigcup_i C''_i \cup \{f_2, \ldots, f_{n-1}\} & \bigcup_i B''_i \cup \{f_n\} \end{pmatrix}$$

Good-Partition-From-Even-Hyperprism(G, H, M)

Input: G – square-free, Berge graph containing no $L(BS(K_4))$
$H = (A_1, \ldots, B_3)$ – maximal even hyperprism in G
M – set of major neighbors of H

Output: A good partition of G

$Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$
relabel strips of H, so that $M \cup A_1$ and $M \cup B_1$ are cliques
$F_1 \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_1 \cup B_1 \cup C_1$ $\}$
ASSERT: M is a clique
ASSERT: $M \cup A_i$ is a clique for at least two values of i
ASSERT: $M \cup B_j$ is a clique for at least two values of j
$K_1 \leftarrow A_1$, $K_2 \leftarrow M$, $K_3 \leftarrow B_1$
$R \leftarrow C_1 \cup F_1 \cup Z$
$L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$
return (K_1, K_2, K_3, L, R)

Good-Partition-From-Odd-Hyperprism(G, H, M)

Input: G – square-free, Berge graph containing no $L(BS(K_4))$
$H = (A_1, \ldots, B_3)$ – maximal odd hyperprism in G
M – set of major neighbors of H

Output: A good partition of G

$Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$
relabel strips of H, so that $A_1 \not\square B_1$ and $A_2 \not\square B_2$
ASSERT: $C_1 \neq \emptyset, C_2 \neq \emptyset$
$\forall_{i \in [3]} F_i \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_i \cup B_i \cup C_i$ $\}$
$F_B \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z \cup F_1 \cup F_2 \cup F_3\}$ that attaches to $B_1 \cup B_2 \cup B_3$ $\}$
TODO: F_i, F_A, F_B are from algIV, make sure it is correct
ASSERT: At least two of A_i and at least two of B_i are cliques
ASSERT: M is complete to at least two of A_i and at least two of B_i
ASSERT: M is a clique
ASSERT: For at least two i : $A_i \cup M$ is a clique
ASSERT: For at least two j : $A_j \cup M$ is a clique
choose h, so that $M \cup A_h$ and $M \cup B_h$ are cliques
if $h = 1 \vee h = 2$ **then** `//` `# TODO:` `make sure` $h = 2$ `is ok`

    $K_1 \leftarrow A_1$, $K_2 \leftarrow M$, $K_3 \leftarrow B_1$
    $R \leftarrow C_1 \cup F_1 \cup Z$
    $L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$
    return (K_1, K_2, K_3, L, R)

else

    relabel H so that $M \cup A_1$ and $M \cup B_2$ are cliques
    $K_1 \leftarrow B_2 \cup B_3$, $K_2 \leftarrow M$, $K_3 \leftarrow A_1 \cup A_3$
    $L \leftarrow B_1 \cup C_1 \cup F_1 \cup F_B$
    $R \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup L\}$
    return (K_1, K_2, K_3, L, R)

2.1 ALG VI

2.1.1 Notes

- $T_{uv} := N_u \cap N_v$
- Strip S_{uv} is *rich* iff $S_{uv} \setminus T_{uv} \neq \emptyset$

Good-Partition-From-J-Strip-System$(G, J, (S, N), M)$

Input: G – square-free, Berge graph
J – a maximal 3-connected graph with appearance in G
(S, N) – a maximal J-strip system
M – a set of major vertices w.r.t. (S, N)

Output: A good partition of G

$S^*_{uv} \leftarrow S_{uv} \cup$ (components of $G \setminus V(S, N)$ that attach in S_{uv} only)
$T_{uv} \leftarrow N_u \cap N_v$
ASSERT: $T_{uv} \blacksquare N_u \setminus N_{uv}$, $T_{uv} \blacksquare N_v \setminus N_{vu}$
ASSERT: $M \cup T_{uv}$ is a clique
if $\exists S_{uv}$ *– a rich strip in* (S, N) **then**
  if $\exists S_{uv}$ *– a rich strip in* (S, N)*, such that* $M \cup (N_u \setminus N_{uv})$ *and* $M \cup (N_v \setminus N_{vu})$ *are cliques* **then**
    $K_1 \leftarrow N_u \setminus N_{uv}$, $K_2 \leftarrow M \cup T_{uv}$, $K_3 \leftarrow N_v \setminus N_{vu}$
    $L \leftarrow (S^*_{uv} \setminus T_{uv}) \cup$ (components of $G \setminus V(S, N)$ that attach only to N_u and those that attach only to N_v)
    $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$
    return (K_1, K_2, K_3, L, R)
  else
    $S_{uv} \leftarrow$ a rich strip in (S, N), such that $M \cup (N_u \setminus N_{uv})$ is not a clique and $M \cup (N_v \setminus N_{vu})$ is a clique
    $K_1 \leftarrow N_{uv} \setminus T_{uv}$, $K_2 \leftarrow M \cup T_{uv}$, $K_3 \leftarrow N_v \setminus N_{vu}$
    $R \leftarrow (S^*_{uv} \setminus N_u) \cup$ (components of $G \setminus V(S, U)$ that attach only to N_v)
    $L \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup R)$
    return (K_1, K_2, K_3, L, R)
else
  ASSERT: $\forall uv \in E(J) : S_{uv} = T_{uv}, S_{uv}$ is a clique
  $S_{uv} \leftarrow$ any strip
  $K_1 \leftarrow N_u \setminus S_{uv}$, $K_2 \leftarrow M$, $K_3 \leftarrow N_v \setminus S_{uv}$
  $L \leftarrow S^*_{uv} \cup$ (components of $G \setminus V(S, N)$ that attach only to N_u and only to N_v)
  $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$
  return (K_1, K_2, K_3, L, R)

Good-Partition-From-Special-Strip-System$(G, J, (S, N), M)$

Input: G – square-free, Berge graph
(S, N) – a special K_4 strip system
M – a set of major vertices w.r.t. (S, N)

Output: A good partition of G

$\forall_{i,j\in[4]} O_{ij} \leftarrow$ set of vertices in $V(G) \setminus V(S,N)$ that are complete to $(N_i \cup N_j) \setminus S_{ij}$ and anticomplete to $V(S,N) \setminus (N_i \cup N_j \cup S_{ij})$

if $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ *and* $(N_2 \setminus N_{12}) \cup M \cup O_{12}$ *are cliques* **then**
    $K_1 \leftarrow N_1 \setminus N_{12}$, $K_2 \leftarrow O_{12} \cup M$, $K_3 \leftarrow N_2 \setminus N_{12}$
    $L \leftarrow$ union of those components of $G \setminus (K_1 \cup K_2 \cup K_3)$ that contain vertices of S_{12}
    $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$
    return (K_1, K_2, K_3, L, R)

else
    if $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ *is a clique* **then**
        relabel 1 and 2 in J so that $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ is not a clique
    ASSERT: $N_{12} \cup M \cup O_{12}$ is a clique
    if $N_{21} \cup M \cup O_{12}$ *is a clique* **then**
        $X \leftarrow N_{21}$
    else
        $X \leftarrow N_2 \setminus N_{21}$
    $K_1 \leftarrow N_{12}$, $K_2 \leftarrow M \cup O_{12}$, $K_3 \leftarrow X$
    $L \leftarrow$ component of $G \setminus (K_1 \cup K_2 \cup K_3)$ that contains $N_1 \setminus N_{12}$
    $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$
    return (K_1, K_2, K_3, L, R)

Find-Special-K4-Strip-System($G, J, (S, N), m$)

Input: G – square-free, Berge graph
J – a 3-connected graph with appearance in G
(S, N) – a J-strip system
$m \in G \setminus (S, N)$ that is major w.r.t. some choice of rungs of (S, N) but not w.r.t. (S, N)

Output: (S', N') – A special K_4-strip system, or
(S'', N'') – a bigger J-strip system

$X \leftarrow N(m)$
$M \leftarrow$ vertices of $G \setminus V(S, N)$ major w.r.t. (S, N)
$M^* \leftarrow$ vertices of $G \setminus V(S, N)$ major w.r.t. some choice of rungs
ASSERT: $J = K_4$ (as in SPGT 8.4)
$V(J) \leftarrow [4]$
$\forall_{i \neq j \in [4]}$: choose rungs R_{ij}, R_{ij}, forming line graphs $L(H)$ and $L(H')$ so that X saturates $L(H)$ but does not saturate $L(H')$
ASSERT: $R_{ij} \neq R'_{ij} \iff \{i, j\} = [2]$
$r_{ij}, r_{ji} \leftarrow$ ends of each R_{ij}
$r'_{ij}, r'_{ji} \leftarrow$ ends of each R'_{ij}
$\forall_{i \in [4]} T_i \leftarrow \{r_{ij}, j \in [4] \setminus \{i\}\}$
$\forall_{i \in [4]} T'_i \leftarrow \{r'_{ij}, j \in [4] \setminus \{i\}\}$
ASSERT: X has at least two members in each $T_1, \ldots T_4$
ASSERT: There is T'_i that contains at most one member of X
ASSERT: $T_3 = T'_3, T_4 = T'_4$
relabel 1 and 2 in J, so that $|X \cap T_1| = 2$ and $|X \cap T'_1| = 1$
ASSERT: $r_{12} \in X, r'_{12} \notin X$
ASSERT: $r_{13} \in X \veebar r_{14} \in X$
relabel 3 and 4 in J, so that $r_{13} \in X, r_{14} \notin X$
ASSERT: R_{34} is even and $[X \cap V(L(H'))] \setminus V(R_{34}) = \{r_{31}, r_{32}, r_{41}, r_{42}\}$ (as in 6.3.(3))
ASSERT: R_{14} has odd length, $r_{21} \in X$ (as in 6.3.(4))
ASSERT: R_{12} has length 0, every 12-rung has even length (as in 6.3.(5))
ASSERT: R_{24} has length 0 and R_{23} has odd length (as in 6.3.(6))
ASSERT: Every 34-rung has non-zero even length (as in 6.3.(7))
$\forall_{i \neq j \in [4]} O_{ij} \leftarrow$ set of vertices that are not major w.r.t $L(H')$ and are complete to $(T'_i \cup T'_j) \setminus R'_{ij}$
ASSERT: $r_{12} = r_{21} \in O_{12}$, $m \in O_{34}$
ASSERT: $O_{34} = M^* \setminus M$
$(S', N') \leftarrow$ strip system obtained from (S, N) by replacing S_{12} with $S_{12} \setminus O_{12}$
if $\exists_{rungR}$: *adding R to S'_{12} produces enlargement of* (S, N) **then**
  return (S'', N'') – an enlargement of (S, N)
else
  return (S', N') – a special K_4 strip system

Growing-J-Strip$(G, J, (S, N))$

Input: G – square-free, Berge graph
J – a 3-connected graph with appearance in G
(S, N) – a J-strip system
TODO: make sure def of J is correct

Output: J' and a maximal J'-strip system, or a special strip system

$M \leftarrow$ vertices of $G \setminus V(S, N)$ that are major on some choice of Rungs of (S, N)
# TODO: M like ALGI
if $\exists m : m$ *is not major on some choice of rungs of* (S, N) **then**
  $OUT \leftarrow$ Find-Special-K4-Strip-System$(G, J, (S, N), m)$
  if OUT *is a special strip system* **then**
    return OUT
  else
    return Growing-J-Strip(G, OUT)
else if $\exists F$: *F is a component of $G \setminus (V(S, N) \cup M)$, such that no member of F is major w.r.t. (S, N) and set of attachments of F on H is not local* **then**
  ASSERT: (as in 6.2, or actually SPGT 8.5)
  $F \leftarrow$ minimal (component?) with this property
  if $\exists v \in V(J) : X \subset \bigcup(S_{uv} : uv \in E(J))$ **then**
    # TODO: what is X?
    $x :\in X \cap S_{uv} \setminus N_v$, for some $uv \in E(J)$
    # TODO: brackets?
    $x' :\in X \cap S_{u'v}$, for some $u'v \in E(J), u' \neq u$
    ASSERT: $\{x, x'\}$ is not local w.r.t. (S, N)
    $L(H) \leftarrow \forall_{i,j \in E(J)}$ choose ij-rung R_{ij}, so that $x \in V(R_{uv}), x' \in V(R_{u'v})$
    ASSERT: $\{x, x'\}$ is not local w.r.t. $L(H)$
    $D \leftarrow$ a branch of H with ends d, u: $\delta_H(d) \setminus E(D) = (X \cap E(H)) \setminus E(D)$
    $P \leftarrow$ a path with ends p_1, p_2, so that:
      $p_1 \blacktriangleleft N_v \setminus N_{vu}$ and no other vertex of P has neighbors in $N_v \setminus N_{uv}$
      $p_2 - x$ and no other vertex of P has neighbors in $S_{uv} \setminus N_v$
    $(S', N') \leftarrow$ add p_1 to N_v and F to S_{uv}
    return Growing-J-Strip $(G, J, (S', N'))$
  else
    $K \leftarrow \{uv \in E(J) : X \cap S_{uv} \neq \emptyset\}$
    ASSERT: There are two disjoint edges in K (as in SPGT 8.5.(3))
    F is a vertex set of a path $\leftarrow f_1 - \ldots - f_n$
    ASSERT: Every choice of rungs is broad

// **if** $\exists F \ldots$ **then**

// **else** // $\nexists v \in V(J) : X \subset \bigcup(S_{uv} : uv \in E(J))$

ASSERT: every choice of rungs has the same traversal. (Hard to assert)

$ij \leftarrow$ the traversal edge

$A_1 \leftarrow N_i \setminus S_{ij}$, $A_2 \leftarrow N_j \setminus S_{ij}$

ASSERT: $X \cap (V(S, N) \setminus S_{ij}) = A_1 \cup A_2$

if $n = 1$ **then**

$(S', N') \leftarrow$ add f_1 to N_i, N_j, S_{ij}

return GROWING-J-STRIP $(G, J, (S', N'))$

else

$x_1 :\in A_1, x_2 :\in A_2$, so that x_1 and x_2 are in disjoint strips

ASSERT: $x_1 - f_1 \veebar x_1 - f_n$

if $x_1 - f_n$ **then**

relabel $f_1 - \ldots - f_n$ front to back

$(S', N') \leftarrow$ add f_1 to N_i, f_n to N_j and F to S_{ij}

return GROWING-J-STRIP $(G, J, (S', N'))$

else

return $J, (S, N)$ – a maximal J-strip