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# **Perfect Graph Recognition and Coloring**

Master Thesis

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## Abstract

TODOa

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# Definitions

We use standard definitions, sourced from the book by Béla Bollobás *Modern graph theory*, modified and extended as needed.

**Definition 0.1** (graph). A graph  $G$  is an ordered pair of disjoint sets  $(V, E)$  such that  $E$  is the subset of the set  $\binom{V}{2}$  that is of unordered pairs of  $V$ .

We will only consider finite graphs, that is  $V$  and  $E$  are always finite. If  $G$  is a graph, then  $V = V(G)$  is the *vertex set* of  $G$ , and  $E = E(G)$  is the *edge set*. The size of the vertex set of a graph  $G$  will be called the *cardinality* of  $G$ .

An edge  $\{x, y\}$  is said to *join*, or be between vertices  $x$  and  $y$  and is denoted by  $xy$ . Thus  $xy$  and  $yx$  mean the same edge (all our graphs are *unordered*). If  $xy \in E(G)$  then  $x$  and  $y$  are adjacent, connected or neighboring. By  $N(x)$  we will denote the *neighborhood* of  $x$ , that is all vertices  $y$  such that  $xy$  is an edge. If  $xy \notin E(G)$  then  $xy$  is a *nonedge* and  $x$  and  $y$  are *anticonnected*.

Figure shows an example of a graph  $G_0 = (V, E)$  with  $V = \{v_1, v_2, v_3, v_4\}$  and  $E = \{v_1v_2, v_2v_3, v_3v_4\}$ . We will mark edges as solid lines on figures. Nonedges significant to the ongoing reasoning will be marked as dashed lines.

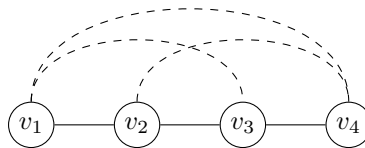


Figure 1: An example graph  $G_0$

**Definition 0.2** (subgraph).  $G' = (V', E')$  is a subgraph of  $G = (V, E)$  if  $V' \subseteq V$  and  $E' \subseteq E$ .

**Definition 0.3** (induced subgraph). If  $G' = (V', E')$  is a subgraph of  $G$  and it contains all edges of  $G$  that join two vertices in  $V'$ , then  $G'$  is said to be induced subgraph of  $G$  and is denoted  $G[V']$ .

Given a graph  $G = (V, E)$  and a set  $X \subseteq V$  by  $G \setminus X$  we will denote a induced subgraph  $G[V \setminus X]$ .

For example  $(\{v_1, v_2, v_3\}, \{v_1v_2\})$  is *not* an induced subgraph of the example graph  $G_0$ , while  $(\{v_1, v_2, v_3\}, \{v_1v_2, v_2v_3\}) = G_0[\{v_1, v_2, v_3\}] = G_0 \setminus \{v_4\}$  is.

**Definition 0.4** ( $X$ -completeness). Given set  $X \subseteq V$ , vertex  $v \notin X$  is  $X$ -complete if it is adjacent to every node  $x \in X$ . A set  $Y \subseteq V$  is  $X$ -complete if  $X \cap Y = \emptyset$  and every node  $y \in Y$  is  $X$ -complete.

**Definition 0.5** (path). A path is a graph  $P$  of the form

$$V(P) = \{x_1, x_2, \dots, x_l\}, \quad E(P) = \{x_1x_2, x_2x_3, \dots, x_{l-1}x_l\}$$

This path  $P$  is usually denoted by  $x_1x_2 \dots x_l$  or  $x_1 - x_2 - \dots - x_l$ . The vertices  $x_1$  and  $x_l$  are the *endvertices* and  $l - 1 = |E(P)|$  is the *length* of the path  $P$ .  $\{x_1, \dots, x_{l-1}\}$  is the *inside* of the path  $P$ , denoted as  $P^*$ .

Graph  $G_0$  is a path of length 3, with the inside  $G_0^* = \{v_2, v_3\}$ . If we would add any edge to  $G_0$  it would stop being a path (sometimes we call such an edge a *chord*).

**Definition 0.6** (connected graph, subset). A graph  $G$  is connected if for every pair  $\{x, y\} \subseteq V(G)$  of distinct vertices, there is a path from  $x$  to  $y$ . A subset  $X \subseteq V(G)$  is connected if the graph  $G[X]$  is connected.

formal enough?

**Definition 0.7** (component). A component of a graph  $G$  is its maximal connected induced subgraph.

**Definition 0.8** (cycle). A cycle is a graph  $C$  of the form

$$V(C) = \{x_1, x_2, \dots, x_l\}, \quad E(C) = \{x_1x_2, x_2x_3, \dots, x_{l-1}x_l, x_lx_1\}$$

This cycle  $C$  is usually denoted by  $x_1x_2 \dots x_lx_1$  or  $x_1 - x_2 - \dots - x_l - x_1$ .  $l = |E(C)|$  is the *length* of the cycle  $C$ . Sometimes we will denote the cycle of length  $l$  as  $C_l$ .

Notice, that a cycle is not a path (nor is a path a cycle). If we add an edge  $v_1v_4$  to the path  $G_0$  it becomes an even cycle  $C_4$ .

**Definition 0.9** (hole). A hole is a cycle of length at least four.

If a path, a cycle or a hole has an odd length, it will be called *odd*. Otherwise, it will be called *even*.

**Definition 0.10** (complement). A complement of a graph  $G = (V, E)$  is a graph  $\overline{G} = (V, \binom{V}{2} \setminus E)$ , that is two vertices  $x, y$  are adjacent in  $\overline{G}$  iff they are not adjacent in  $G$ .

**Definition 0.11** (anticonnected graph, subset). A graph  $G$  is anticonnected if  $\overline{G}$  is connected. A subset  $X$  is anticonnected if  $\overline{G}[X]$  is connected.

**Definition 0.12** (anticomponent). An anticomponent of a graph  $G$  is an induced subgraph whose complement is a component in  $G$ .

**Definition 0.13** (clique). A complete graph or a clique is a graph of the form  $G = (V, \binom{V}{2})$ , that is every two vertices are connected. We will denote a clique of  $n$  vertices as  $K_n$ .

**Definition 0.14** (anticlique). An anticlique is a graph in which there are no edges. We will also call anticliques independent sets.

In a similar fashion, given a graph  $G = (V, E)$ , a subset of its vertices  $V' \subseteq V$  will be called *independent* (in the context of  $G$ ) iff  $G[V']$  is an anticlique.

**Definition 0.15** (clique number). *A clique number of a graph  $G$ , denoted as  $\omega(G)$ , is a cardinality of its largest induced clique.*

**Definition 0.16** (coloring). *Given a graph  $G$ , its coloring is a function  $c : V(G) \rightarrow \mathbb{N}^+$ , such that for every edge  $xy \in E(G)$ ,  $c(x)$  is different from  $c(y)$ . A  $k$ -coloring of  $G$  (if exists) is a coloring, such that for all vertices  $x \in V(G)$ ,  $c(x) \leq k$ .*

**Definition 0.17** (chromatic number). *A chromatic number of a graph  $G$ , denoted as  $\chi(G)$ , is a smallest natural number  $k$ , for which there exists a  $k$ -coloring of  $G$ .*

**Definition 0.18** (Berge graph). *A graph  $G$  is Berge if both  $G$  and  $\overline{G}$  have no odd hole.*

**Definition 0.19** (perfect graph). *A graph  $G$  is perfect if for its every induced subgraph  $G'$  we have  $\chi(G') = \omega(G')$*

**Definition 0.20** (C-major vertices). *Given a shortest odd hole  $C$  in  $G$ , a node  $v \in V(G) \setminus V(C)$  is  $C$ -major if the set of its neighbors in  $C$  is not contained in any 3-node path of  $C$ .*

*a picture of this, clean odd hole, amenable hole*

**Definition 0.21** (clean odd hole). *An odd hole  $C$  of  $G$  is clean if no vertex in  $G$  is  $C$ -major.*

**Definition 0.22** (cleaner). *Given a shortest odd hole  $C$  in  $G$ , a subset  $X \subseteq V(G)$  is a cleaner for  $C$  if  $X \cap V(C) = \emptyset$  and every  $C$ -major vertex belongs to  $X$ .*

Let us notice, that if  $X$  is a cleaner for  $C$  then  $C$  is a clean hole in  $G \setminus X$ .

**Definition 0.23** (near-cleaner). *Given a shortest odd hole  $C$  in  $G$ , a subset  $X \subseteq V(G)$  is a near-cleaner for  $C$  if  $X$  contains all  $C$ -major vertices and  $X \cap V(C)$  is a subset of vertex set of some 3-node path of  $C$ .*

**Definition 0.24** (amenable odd hole). *An odd hole  $C$  of  $G$  is amenable if it is a shortest odd hole in  $G$ , it is of length at least 7 and for every anticonnected set  $X$  of  $C$ -major vertices there is a  $X$ -complete edge in  $C$ .*

# Chapter 1

## Perfect Graphs

Given a graph  $G$ , let us consider a problem of coloring it using as few colors as possible. If  $G$  contains a clique  $K$  as a subgraph, we must use at least  $|V(K)|$  colors to color it. This gives us a lower bound for a chromatic number  $\chi(G)$  – it is always greater or equal to the cardinality of the largest clique  $\omega(G)$ . The reverse is not always true, in fact we can construct a graph with no triangle and requiring arbitrarily large numbers of colors (e.g. construction by Mycielski [11]).

Do graphs that admit coloring using only  $\omega(G)$  color are "simpler" to further analyze? Not necessarily so. Given a graph  $G = (V, E)$ ,  $|V| = n$ , let us construct a graph  $G'$  equal to the union of  $G$  and a clique  $K_n$ . We can see that indeed  $\chi(G') = n = \omega(G')$ , but it gives us no indication of the structure of  $G$  or  $G'$ .

A definition of perfect graphs (definition 0.19) states that given a graph  $G$ , the chromatic number and cardinality of the largest clique of *its every induced subgraph* should be equal. The notion of perfect graphs was first introduced by Berge in 1961 [2] and it indeed captures some of the idea of graph being "simple" – in all perfect graphs the coloring problem, maximum (weighted) clique problem, and maximum (weighted) independent set problem can be solved in polynomial time [8]. Other classical NP-complete problems are still NP-complete in perfect graphs e.g. Hamiltonian path [10], maximum cut problem [3] or dominating set problem [7].

The most fundamental problem – the problem of recognizing perfect graphs – was open since its posing in 1961 until recently. Its solution, a polynomial algorithm recognizing perfect graphs is a union of the strong perfect graph theorem (section 1.1) stating that a graph is perfect if and only if it is Berge (definition 0.18) and an algorithm for recognizing Berge graphs in polynomial time (section 1.2).

The first step to solve the problem of recognizing perfect graphs was the (*weak*) *perfect graph theorem* first conjured by Berge in 1961 [2] and then proven by Lovász in 1972 [9].

**Theorem 1.1** (Perfect graph theorem). *A graph is perfect if and only if its*

all these citations are about subclasses e.g. bipartite graphs. Should we mention some subclasses?



complement graph is also perfect.

Should we give some proof of that here? Maybe based on proof in [6]

## 1.1 Strong Perfect Graph Theorem

Odd holes are not perfect, since their chromatic number is 3 and their largest cliques are of size 2. It is also easy to see, that an odd antihole of size  $n$  has a chromatic number of  $\frac{n+1}{2}$  and largest cliques of size  $\frac{n-1}{2}$ . A graph with no odd hole and no odd antihole is called *Berge* (definition 0.18) after Claude Berge who studied perfect graphs.

In 1961 Berge conjectured that a graph is perfect iff it contains no odd hole and no odd antihole in what has become known as a strong perfect graph conjecture. In 2001 Chudnovsky et al. have proven it and published the proof in an over 150 pages long paper “The strong perfect graph theorem” [5]. The following overview of the proof will be based on this paper and on an article with the same name by Cornuéjols [6].

**Theorem 1.2** (Strong perfect graph theorem). *A graph is perfect if and only if it is Berge.*

How long and detailed overview of the proof should we provide?

Znakomite pytanie. Generalnie ponieważ dowód jest drobiazgowy i trikowy, to wystarczy ”z lotu ptaka” tj. to, co Chudnovsky i Seymour piszą w pracach popularnych i referują w wystąpieniach gościnnych typu te nagrania na YT. Głębiej nie ma sensu. Warto podkreślić, że technika dowodu i sam algorytm są zależne na dużo głębszym poziomie niż widać tj. ktoś mając dowód Strong Perfect Graph Theorem pewnie nie rozgryzłby algorytmu i vice versa.

## 1.2 Recognizing Berge Graphs

The following is based on the paper by Maria Chudnovsky et al. “Recognizing Berge Graphs”. We will not provide full proof of its correctness, but will aim to show the intuition behind the algorithm.

Berge graph recognition algorithm could be divided into two parts: first we check if either  $G$  or  $\overline{G}$  contain any of a number of simple structures as a induced subgraph (1.2.1). If they do, we output that graph is not Berge and stop. Else, we check if there is a near-cleaner for a shortest odd hole (1.2.2).

### 1.2.1 Simple structures

Pyramids

A *pyramid* in  $G$  is an induced subgraph formed by the union of a triangle<sup>1</sup>  $\{b_1, b_2, b_3\}$ , three paths  $\{P_1, P_2, P_3\}$  and another vertex  $a$ , so that:

- $\forall_{1 \leq i \leq 3} P_i$  is a path between  $a$  and  $b_i$
- $\forall_{1 \leq i < j \leq 3} a$  is the only vertex in both  $P_i$  and  $P_j$  and  $b_i b_j$  is the only edge between  $V(P_i) \setminus \{a\}$  and  $V(P_j) \setminus \{a\}$ .
- $a$  is adjacent to at most one of  $\{b_1, b_2, b_3\}$ .

We will say that  $a$  can be *linked onto* the triangle  $\{b_1, b_2, b_3\}$  *via* the paths  $P_1, P_2, P_3$ . Let us notice, that a pyramid is determined by its paths  $P_1, P_2, P_3$ .

It is easy to see that every graph containing a pyramid contains an odd hole – at least two of the paths  $P_1, P_2, P_3$  will have the same parity.

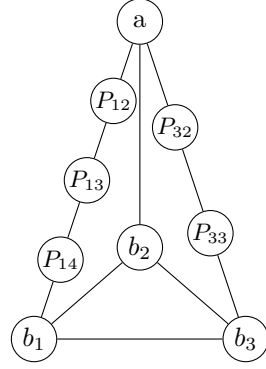


Figure 1.1: An example of a pyramid.

**Finding Pyramids** First, let us enumerate all 6-tuples  $b_1, b_2, b_3, s_1, s_2, s_3$  such that:

- $\{b_1, b_2, b_3\}$  is a triangle
- for  $1 \leq i < j \leq 3$ ,  $b_i, s_i$  is disjoint from  $b_j, s_j$  and  $b_i b_j$  is the only edge between them
- there is a vertex  $a$  adjacent to all of  $s_1, s_2, s_3$  and to at most one of  $b_1, b_2, b_3$ , such that if  $a$  is adjacent to  $b_i$ , then  $s_i b_i$ .

There are  $O(|V(G)|^6)$  6-tuples, and it takes  $O(|V(G)|)$  time to check each one. For each such 6-tuple we follow with the rest of the algorithm.

We define  $M = V(G) \setminus \{b_1, b_2, b_3, s_1, s_2, s_3\}$ . Now, for each  $m \in M$ , we set  $S_1(m)$  equal to the shortest path between  $s_1$  and  $m$  such that  $s_2, s_3, b_2, b_3$  have no neighbors in its interior, if such a path exists. We set  $S_2$  and  $S_3$  similarly. Then similarly we set  $T_1(m)$  to be the shortest path between  $m$  and  $b_1$ , such that  $s_2, s_3, b_2, b_3$  have no neighbors in its interior, if such a path exists. We do similar for  $T_2$  and  $T_3$ . It takes  $O(|V(G)|^2)$  time to calculate paths  $T_i(m)$  for all  $i$  and  $m$ .

Now, we will calculate all possible paths  $P_i$ . For each  $m \in M \cup \{b_1\}$  we will define a path  $P_1(m)$  and paths  $P_2(m), P_3(m)$  will be defined in a similar manner.

If  $s_1 = b_1$  let  $P_1(b_1)$  be the one-vertex path with vertex  $b_1$ , and let  $P_1(m)$  be undefined for each  $m \in M$ .

If  $s_1 \neq b_1$ , then  $P_1(b_1)$  is undefined and for all  $m \in M$  we will check if all the following are true:

<sup>1</sup>A triangle is a clique  $K_3$ .

- $m$  is nonadjacent to all of  $b_2, b_3, s_2, s_3$
- $S_1(m)$  and  $T_1(m)$  both exist
- $V(S_1(m) \cap T_1(m)) = \{m\}$
- there are no edges between  $V(S_1(m) \setminus m)$  and  $V(T_1(m) \setminus m)$

If so, then we assign a path  $s_1 - S_1(m) - m - T_1(m) - b_1$  to  $P_1(m)$ , otherwise we let  $P_1(m)$  be undefined. It takes  $O(|V(G)|^2)$  to check this, given  $m$ . We assign  $P_2$  and  $P_3$  in a similar manner. Total time of finding all  $P_i(m)$  paths for a given 6-tuple is  $O(|V(G)|^3)$ .

Now we want to check if there is a triple  $m_1, m_2, m_3$ , so that  $P_1(m_1), P_2(m_2), P_3(m_3)$  form a pyramid. A most obvious approach of enumerating them all would be too slow, so we do it carefully.

For  $1 \leq i < j \leq 3$ , we say that  $(m_i, m_j)$  is a *good*  $(i, j)$ -pair, iff  $m_i \in M \cup \{b_i\}$ ,  $m_j \in M \cup \{b_j\}$ ,  $P_i(m_i), P_j(m_j)$  both exist and the sets  $V(P_i(m_i)), V(P_j(m_j))$  are both disjoint and  $b_i b_j$  is the only edge between them.

We show how to find the list of all good  $(1, 2)$ -pairs, with similar algorithm for all other good  $(i, j)$ -pairs. For each  $m_1 \in M \cup \{b_1\}$ , we find the set of all  $m_2$  such that  $(m_1, m_2)$  is a good  $(1, 2)$ -pair as follows.

If  $P_1(m_1)$  does not exist, there are no such good pairs. If it exists, color black the vertices of  $M$  that either belong to  $P_1(m_1)$  or have a neighbor in  $P_1(m_1)$ . Color all other vertices white. (We can do this in  $O(|V(G)|^2)$ ) Then for each  $m_2 \in M \cup \{b_2\}$ , test whether  $P_2(m_2)$  exists and contains no black vertices. We do this for all  $m_1$  and get a set of all  $(1, 2)$ -good pairs. In similar way we calculate all good  $(1, 3)$ -pair and  $(2, 3)$ -pairs (in  $O(|V(G)|^3)$  time).

Now, we examine all triples  $m_1, m_2, m_3$  such that  $m_i \in M \cup \{b_i\}$  and test whether  $(m_i, m_j)$  is a good  $(i, j)$ -pair. If we find a triple such that all three pairs are good, we output that  $G$  contains a pyramid and stop.

If after examining all choices of  $b_1, b_2, b_3, s_1, s_2, s_3$  we find no pyramid, output that  $G$  contains no pyramid. Since there are  $O(|V(G)|^6)$  such choices and it takes a time of  $O(|V(G)|^3)$  to analyze each one, the total time is  $O(|V(G)|^9)$ .

some proofs

## Jewels

Five vertices  $v_1, \dots, v_5$  and a path  $P$  form a *jewel* iff:

- $v_1, \dots, v_5$  are distinct vertices.
- $v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_5, v_5 v_1$  are edges.
- $v_1 v_3, v_2 v_4, v_1, v_4$  are nonedges.
- $P$  is a path between  $v_1$  and  $v_4$ , such that  $v_2, v_3, v_5$  have no neighbors in its inside.

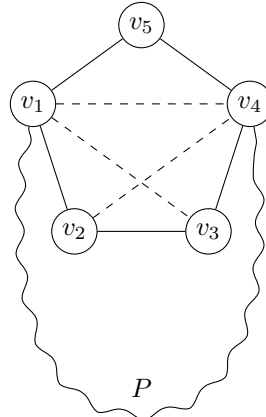


Figure 1.2: An example of a jewel.

Most obvious way to find a jewel would be to enumerate all choices of  $v_1, \dots, v_5$ , check if a choice is correct and if it is try to find a path  $P$  as required. This gives us a time of  $O(|V|^7)$ . We could speed it up to  $O(|V|^6)$  with more careful algorithm, but since whole algorithm takes time  $O(|V|^9)$  and our testing showed that time it takes to test for jewels is negligible we decided against it.

### Configurations of type $\mathcal{T}_1$

A configuration of type  $\mathcal{T}_1$  is a hole of length 5. To find it we simply iterate all choices of paths of length of 4, and check if there exists a fifth vertex to complete the hole. See section 2.1.1 for more implementation details.

### Configurations of type $\mathcal{T}_2$

A configuration of type  $\mathcal{T}_2$  is a tuple  $(v_1, v_2, v_3, v_4, P, X)$ , such that:

- $v_1 v_2 v_3 v_4$  is a path in  $G$ .
- $X$  is an anticomponent of the set of all  $\{v_1, v_2, v_4\}$ -complete vertices.
- $P$  is a path in  $G \setminus (X \cup \{v_2, v_3\})$  between  $v_1$  and  $v_4$  and no vertex in  $P^*$  is  $X$ -complete or adjacent to  $v_2$  or adjacent to  $v_3$ .

Checking if configuration of type  $\mathcal{T}_2$  exists in our graph is straightforward: we enumerate all paths  $v_1 \dots v_4$ , calculate set of all  $\{v_1, v_2, v_4\}$ -complete vertices and its anticomponents. Then, for each anticomponent  $X$  we check if required path  $P$  exists.

To prove that existence of  $\mathcal{T}_2$  configuration implies that the graph is not Berge, we will need the following Roussel-Rubio lemma:

**Lemma 1.3** (Roussel-Rubio Lemma [12, 4]). *Let  $G$  be Berge,  $X$  be an anticonnected subset of  $V(G)$ ,  $P$  be an odd path  $p_1 \dots p_n$  in  $G \setminus X$  with length at least 3, such that  $p_1$  and  $p_n$  are  $X$ -complete and  $p_2, \dots, p_{n-1}$  are not. Then:*

- *$P$  is of length at least 5 and there exist nonadjacent  $x, y \in X$ , such that there are exactly two edges between  $x, y$  and  $P^*$ , namely  $x p_2$  and  $y p_{n-1}$ ,*
- *or  $P$  is of length 3 and there is an odd antipath joining internal vertices of  $P$  with interior in  $X$ .*

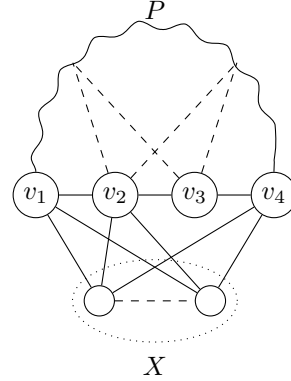


Figure 1.3: An example of a  $\mathcal{T}_2$ .

We may use this lemma quite often, might want to provide proof if so.

Now, we shall prove the following:

**Lemma 1.4.** *If  $G$  contains configuration of type  $\mathcal{T}_2$  then  $G$  is not Berge.*

*Proof.* Let  $(v_1, v_2, v_3, v_4, P, X)$  be a configuration of type  $\mathcal{T}_2$ . Let us assume that  $G$  is not Berge and consider the following:

- If  $P$  is even, then  $v_1, v_2, v_3, v_4, P, v_1$  is an odd hole,
- If  $P$  is of length 3. Let us name its vertices  $v_1, p_2, p_3, v_4$ . It follows from Lemma 1.3, that there exists an odd antipath between  $p_2$  and  $p_3$  with interior in  $X$ . We can complete it with  $v_2p_2$  and  $v_2p_3$  into an odd antihole.
- If  $P$  is odd with the length of at least 5, it follows from Lemma 1.3 that we have  $x, y \in X$  with only two edges to  $P$  being  $xp_2$  and  $yp_{n-1}$ . This gives us an odd hole:  $v_2, x, p_2, \dots, p_{n-1}, y, v_2$ .

□

I merged a couple of proofs from [5], check in the morning if this is correct.

check in the morning

### Configurations of type $\mathcal{T}_3$

A configuration of type  $\mathcal{T}_3$  is a sequence  $v_1, \dots, v_6$ ,  $P$ ,  $X$ , such that:

- $v_1, \dots, v_6$  are distinct vertices.
- $v_1v_2, v_3v_4, v_1v_4, v_2v_3, v_3v_5, v_4v_6$  are edges, and  $v_1v_3, v_2v_4, v_1v_5, v_2v_5, v_1v_6, v_2v_6, v_4v_5$  are nonedges.
- $X$  is an anticomponent of the set of all  $\{v_1, v_2, v_5\}$ -complete vertices, and  $v_3, v_4$  are not  $X$ -complete.
- $P$  is a path of  $G \setminus (X \cup \{v_1, v_2, v_3, v_4\})$  between  $v_5$  and  $v_6$  and no vertex in  $P^*$  is  $X$ -complete or adjacent to  $v_1$  or adjacent to  $v_2$ .
- If  $v_5v_6$  is an edge, then  $v_6$  is not  $X$ -complete.

The following algorithm with running time of  $O(|V(G)|^6)$  checks whether  $G$  contains a configuration of type  $\mathcal{T}_3$ :

For each triple  $v_1, v_2, v_5$  of vertices such that  $v_1v_2$  is an edge and  $v_1v_5, v_2v_5$  are nonedges find the set  $Y$  of all  $\{v_1, v_2, v_5\}$ -complete vertices. For each anticomponent  $X$  of  $Y$  find the maximal connected subset  $F'$  containing

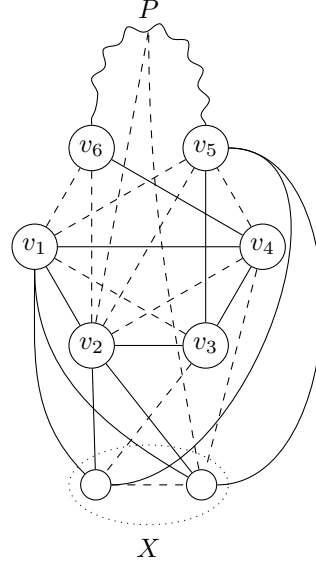


Figure 1.4: An example of a  $\mathcal{T}_3$ .

$v_5$  such that  $v_1, v_2$  have no neighbors in  $F'$  and no vertex of  $F' \setminus \{v_5\}$  is  $X$ -complete. Let  $F$  be the union of  $F'$  and the set of all  $X$ -complete vertices that have a neighbor in  $F'$  and are nonadjacent to all of  $v_1, v_2$  and  $v_5$ .

Then, for each choice of  $v_4$  that is adjacent to  $v_1$  and not to  $v_2$  and  $v_5$  and has a neighbor in  $F$  (call it  $v_6$ ) and a nonneighbor in  $X$ , we test whether there is a vertex  $v_3$ , adjacent to  $v_2, v_4, v_5$  and not to  $v_1$ , with a nonneighbor in  $X$ . If there is such a vertex  $v_3$ , find  $P$  – a path from  $v_6$  to  $v_5$  with interior in  $F'$  and return that  $v_1, \dots, v_6, P, X$  is a configuration of type  $\mathcal{T}_3$ . If we exhaust our search and find none, report that graph does not contain it.

To see that the algorithm below has a running time of  $O(|V(G)|^6)$ , let us note that for each triple  $v_1, v_2, v_5$  we examine, of which there are  $O(|V(G)|^3)$ , there are linear many choices of  $X$ , each taking  $O(|V(G)|^2)$  time to process and generating a linear many choices of  $v_4$  which take a linear time to process in turn. This gives us the total running time of  $O(|V(G)|^6)$ .

We will skip the proof that each graph containing  $\mathcal{T}_3$  is not Berge. See section 6.7 of [4] for the proof.

### 1.2.2 Amenable holes.

**Theorem 1.5.** *Let  $G$  be a graph, such that  $G$  and  $\overline{G}$  contain no Pyramid, no Jewel and no configuration of types  $\mathcal{T}_1, \mathcal{T}_2$  or  $\mathcal{T}_3$ . Then every shortest hole in  $G$  is amenable.*

The proof of this theorem is quite technical and we will not discuss it here. See section 8 of [4] for the proof.

With theorem 1.5 we can describe the rest of the algorithm.

**Algorithm 1.1** (List possible near cleaners, 9.2 of [4])

*Input:* A graph  $G$ .

*Output:*  $O(|V(G)|^5)$  subsets of  $V(G)$ , such that if  $C$  is an amenable hole in  $G$ , then one of the subsets is a near-cleaner for  $C$ .

*Begin.*

We will call a triple  $(a, b, c)$  of vertices *relevant* if  $a, b$  are distinct (possibly  $c \in \{a, b\}$ ) and  $G[\{a, b, c\}]$  is an independent set.

Given a relevant triple  $(a, b, c)$  we can compute the following:

- $r(a, b, c) \leftarrow$  the cardinality of the largest anticomponent of  $N(a, b)$ , that contains a nonneighbor of  $c$ , or 0, if  $c$  is  $N(a, b)$ -complete.
- $Y(a, b, c) \leftarrow$  the union of all anticomponents of  $N(a, b)$  that have cardinality strictly greater than  $r(a, b, c)$ .
- $W(a, b, c) \leftarrow$  the anticomponent of  $N(a, b) \cup \{c\}$  that contains  $c$ .
- $Z(a, b, c) \leftarrow$  the set of all  $Y(a, b, c) \cup W(a, b, c)$ -complete vertices.
- $X(a, b, c) \leftarrow Y(a, b, c) \cup Z(a, b, c)$ .

For every two adjacent vertices  $u, v$  compute the set  $N(u, v)$  and list all such sets. For each relevant triple  $(a, b, c)$  compute the set  $X(a, b, c)$  and list all such sets.

Output all subsets of  $V(G)$  that are the union of a set from the first list and a set from the second list. *End*

To prove the correctness of algorithm 1.1 we will need the following theorem.

**Theorem 1.6** (9.1 of [4]). *Let  $C$  be a shortest odd hole in  $G$ , with length at least 7. Then there is a relevant triple  $(a, b, c)$  of vertices such that*

- *the set of all  $C$ -major vertices not in  $X(a, b, c)$  is anticonnected*
- *$X(a, b, c) \cap V(C)$  is a subset of the vertex set of some 3-vertex path of  $C$ .*

*Proof.*

do we want it here? 1-2 pages long

□

Let us suppose that  $C$  is an amenable hole in  $G$ . By theorem 1.6, there is a relevant triple  $(a, b, c)$  satisfying that theorem. Let  $T$  be the set of all  $C$ -major vertices not in  $X(a, b, c)$ . From theorem 1.6 we get that  $T$  is anticonnected. Since  $C$  is amenable, there is an edge  $uv$  of  $C$  that is  $T$ -complete, and therefore  $T \subseteq N(u, v)$ . But then  $N(u, v) \cup X(a, b, c)$  is a near-cleaner for  $C$ . Therefore the output of the algorithm 1.1 is correct.

**Algorithm 1.2** (Test possible near cleaner, 5.1 of [4])

*Input:* A graph  $G$  containing no simple bad structure, and a subset  $X \subseteq V(G)$ .

*Output:* Determines one of the following:

define this somewhere

- $G$  has an odd hole
- There is no shortest odd hole  $C$  such that  $X$  is a near-cleaner for  $C$ .

*Begin.*

For every pair  $x, y \in V(G)$  of distinct vertices find shortest path  $R(x, y)$  between  $x, y$  with no internal vertex in  $X$ . If there is one, let  $r(x, y)$  be its length, if not, let  $r(x, y)$  be infinite.

For all  $y_1 \in V(G) \setminus X$  and all 3-vertex paths  $x_1 - x_2 - x_3$  of  $G \setminus y_1$  we check the following:

- $R(x_1, y_1), R(x_2, y_2)$  both exist – define  $y_2$  as the neighbor of  $y_1$  in  $R(x_2, y_1)$ .
- $r(x_2, y_1) = r(x_1, y_1) + 1 = r(x_1, y_2)$  ( $= n$  say)
- $r(x_3, y_1), r(x_3, y_2) \geq n$

If there is such a choice of  $x_1, x_2, x_3, y_1$  then we output that there is an odd hole. If not, we report that there is no shortest odd hole  $C$  such that  $X$  is a near-cleaner for  $C$ . *End*

Below we will prove that if the algorithm 1.2 reports an odd hole in  $G$ , there indeed is one. The proof of the correctness of the other possible result is more complicated, see section 4 and theorem 5.1 of [4].

*Proof.* Let us suppose that there is a choice of  $x_1, x_2, x_3, y_1$  satisfying the three conditions in the algorithm 1.2 and let  $y_2$  and  $n$  be defined as in there. We claim that  $G$  contains an odd hole.

Let  $R(x_1, y_1) = p_1 - \dots - p_n$ , and let  $R(x_2, y_1) = q_1 - \dots - q_{n+1}$ , where  $p_1 = x_1$ ,  $p_n = q_{n+1} = y_1$ ,  $q_1 = x_2$  and  $q_n = y_2$ . From the definition of  $R(x_1, y_1)$  and  $R(x_2, y_1)$ , none of  $p_2, \dots, p_{n-1}, q_2, \dots, q_n$  belong to  $X$ . Also, from the definition of  $y_1$ ,  $y_1 \notin X$ .

Since  $r(x_1, y_1) = r(x_2, y_1) - 1$  and since  $x_1, x_2$  are nonadjacent it follows that  $x_2$  does not belong to  $R(x_1, y_1)$  and  $x_1$  does not belong to  $R(x_2, y_1)$ . Since  $r(x_3, y_1), r(x_3, y_2) \geq n (= r(x_1, y_2))$  it follows that  $x_3$  does not belong to  $R(x_1, y_1)$  or to  $R(x_2, y_1)$ , and has no neighbors in  $R(x_1, y_1) \cup R(x_2, y_1)$  other than  $x_1, x_2$ . Since  $r(x_1, y_2) = n$  we get that  $y_2$  does not belong to  $R(x_1, y_1)$ .

We claim first that the insides of paths  $R(x_1, y_1)$  and  $R(x_2, y_1)$  have no common vertices. For suppose that there are  $2 \leq i \leq n-1$  and  $2 \leq j \leq n$  that  $p_i = q_j$ . Then the subpaths of these two paths between  $p_i, y_1$  are both subpaths of the shortest paths, and therefore have the same length, that is  $j = i+1$ . So  $p_1 - \dots - p_i - q_{j+1} - \dots - q_n$  contains a path between  $x_1, y_2$  of length  $\leq n-2$ , contradicting that  $r(x_1, y_2) = n$ . So  $R(x_1, y_1)$  and  $R(x_2, y_1)$  have no common vertex except  $y_1$ .

If there are no edges between  $R(x_1, y_1) \setminus y_1$  and  $R(x_2, y_1) \setminus y_1$  then the union of these two paths and a path  $x_1 - x_3 - x_2$  form an odd hole, so the answer is correct.

Suppose that  $p_i q_j$  is an edge for some  $1 \leq i \leq n-1$  and  $1 \leq j \leq n$ . We claim  $i \geq j$ . If  $j = 1$  this is clear so let us assume  $j > 1$ . Then there is a path between  $x_1, y_2$  within  $\{p_1, \dots, p_i, q_j, \dots, q_n\}$  which has length  $\leq n-j+1$  and has no internal vertex in  $X$  (since  $j > 1$ ); and since  $r(x_1, y_2) = n$ , it follows that  $n-j+1 \geq n$ , that is,  $i \geq j$  as claimed.

Now, since  $x_1, x_2$  are nonadjacent  $i \geq 2$ . But also  $r(x_2, y_1) \geq n$  and so  $j+n-i \geq n$ , that is  $j \geq i$ . So we get  $i = j$ . Let us choose  $i$  minimum, then  $x_3 - x_1 - \dots - p_i - q_i - \dots - x_2 - x_3$  is an odd hole, which was what we wanted.  $\square$

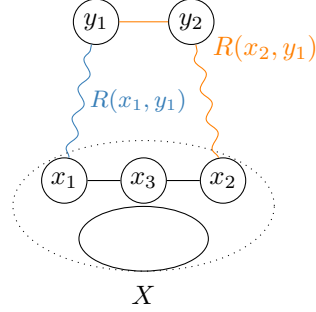


Figure 1.5: An odd hole is found



## Chapter 2

# Recognizing Berge Graphs

### 2.1 Implementation

Anything interesting about algo/data structure?

#### 2.1.1 Optimizations

Bottlenecks in performance (next path, are vectors distinct etc).

In our graph preprocessing we have a pointer to next edge in order to speed up generating next path.

We used callgrind to get idea of methods crucial for time.

In general enumerating all paths is crucial. As is checking if vector has distinct values.

Jewels – we iterate all possibly chordal paths and check if they are ok - much faster

$\mathcal{T}_1$  – we iterate all paths of length 4 and check if there exists a fifth vertex to complete the hole - much faster than iterating vertices.

Validity tests - unit tests, tests of bigger parts, testing vs known answer and vs naive.

### 2.2 Parallelism with CUDA (?)

TODO

## 2.3 Experiments

Naive algorithm - brief description, bottlenecks optimizations (makes huge difference).

Description of tests used.

Results and Corollary - almost usable algorithm.

## Chapter 3

# Coloring Berge Graphs

### 3.1 Ellipsoid method

Description.

Implementation.

Experiments and results.

### 3.2 Combinatorial Method

Cite the paper.

On its complexity - point to appendix for pseudo-code.

# Appendices

## Appendix A

# Perfect Graph Coloring algorithm

TODO

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