1 Notes

- $V(X)$ – vertices of structure X. Will be written as X when obvious.
- $a - b$, when a and b are nodes – a and b are neighbors.
- $a - X$, when a is a node and X is a set of nodes – a is complete to X.
- $a \cdots b$, when a and b are nodes – a and b are not neighbors.
- $a \cdots X$, when a is a node and X is a set of nodes – a is anticomplete to X.
- $[n]$ – $\{1, \ldots, n\}$.
- $L(BS(K_4))$– a line-graph of a biparite subdivision of K_4.

2 Algorithms

Color-Good-Partition($G, K_1, K_2, K_3, L, R, c_1, c_2$)

Input: G – square-free, Berge graph
K_1, K_2, K_3, L, R – good partition
c_1, c_2 – colorings of $G \setminus R$ and $G \setminus L$ (possibly NULL)

Output: $\omega(G)$-coloring of G

$G_1 \leftarrow G \setminus R$
$G_2 \leftarrow G \setminus L$
if $c_1, c_2 =$ NULL **then**
  $c_1 \leftarrow$ Color-Graph(G_1)
  $c_2 \leftarrow$ Color-Graph(G_2)
foreach $u \in K_1 \cup K_2$ **do**
  relabel c_2, so that $c_1(u) = c_2(u)$
$B \leftarrow \{u \in K_3 : c_1(u) \neq c_2(u)\}$
if $B = \emptyset$ **then return** $c_1 \cup c_2$
foreach $h \in [2]$, distinct colors i, j **do**
  $G_h^{i,j} \leftarrow$ subgraph induced on G_h by $\{v \in G_h : c_h(v) \in \{1, 2\}\}$
foreach $u \in K_3$ **do**
  $C_h^{i,j}(u) \leftarrow$ component of $G_h^{i,j}$ containing u
ASSERT: $C_h^{c_1(u),c_2(u)}(u) \cap K_2 = \emptyset$
if $\exists u \in B, h \in [2] : C_h^{c_1(u),c_2(u)}(u) \cap K_1 = \emptyset$ **then**
  $c_1' \leftarrow c_1$ with colors i and j swapped in $C_1^{i,j}(u)$
  ASSERT: c_1' and c_2 agree on $K_1 \cup K_2$
  ASSERT: $\forall u \in K_3 \setminus B : c_1'(u) = c_1(u)$
  ASSERT: $c_1'(u) = j = c_2(u)$
  return Color-Good-Partition($G, K_1, K_2, K_3, L, R, c_1', c_2$)
else
  $w \leftarrow$ vertex of B with nost neighbors in K_1
  ASSERT: $\forall u \in B : N(u) \cap K_1 \subset N(w) \cap K_1$
  relabel c_1, c_2, so that $c_1(w) = 1, c_2(w) = 2$
  $P \leftarrow$ chordless path $w - p_1 - \ldots - p_k - a$ in $C_1^{1,2}(w)$ so that $k \geq 1$, $p_1 \in K_3 \cup L$, $p_2 \ldots p_k \in L$, $a \in K$, $c_1(a) \in [2]$
  $Q \leftarrow$ chordless path $w - q_1 - \ldots - q_l - a$ in $C_2^{1,2}(w)$ so that $l \geq 1$, $q_1 \in K_3 \cup R$, $q_2 \ldots q_l \in R$, $a \in K$, $c_2(a) \in [2]$
  $i \leftarrow c_1(a)$
  $j \leftarrow 3 - i$
  ASSERT: exactly one of the colors 1 and 2 appears in K_1 (as in Lemma 2.2.(3))
  ASSERT: $|P|$ and $|Q|$ have different parities
  ASSERT: $p_1 \in K_3 \vee p_2 \in K_3$ (as in Lemma 2.2.(4))
  ASSERT: $\nexists y \in K_3 : c_1(y) = 2 \wedge c_2(y) = 1$ (as in Lemma 2.2.(5))
  if $p_1 \in K_3$ **then**
    ASSERT: $c_2(p_1) \notin [2]$
    relabel c_2, so that $c_2(p_1) = 3$

// **else** // $\nexists u \in B, h \in [2] : C_h^{c_1(u),c_2(u)}(u) \cap K_1 = \emptyset$

// **if** $p_1 \in K_3$ **then**

ASSERT: color 3 does not appear in K_2

ASSERT: color 3 does not appear in K_1

ASSERT: $C_2^{j,3}(p_1) \cap K_1 = \emptyset$

$c_2' \leftarrow c_2$ with colors j and 3 swapped in $C_2^{j,3}(p_1)$

ASSERT: $j = 2$

return Color-Good-Partition(G, K_1, K_2, K_3, L, R, c_1, c_2')

else

relabel c_1, so that $c_1(q_1) = 3$

if 3 *does not appear in* K_1 **then**

ASSERT: $C_1^{j,3}(q_1) \cap K_1 = \emptyset$

ASSERT: $j = 1$

$c_1' \leftarrow c_1$ with colors j and 3 swapped in $C_1^{j,3}(q_1)$

return Color-Good-Partition(G, K_1, K_2, K_3, L, R, c_1', c_2)

else

ASSERT: $q_1 \cdots \{a, a_3\}$

ASSERT: $C_1^{i,3}(q_1) \cap K_1 = \emptyset$

ASSERT: $i = 1$

$c_1' \leftarrow c_1$ with colors i and 3 swapped in $C_1^{i,3}(q_1)$

return Color-Good-Partition(G, K_1, K_2, K_3, L, R, c_1', c_2)

Good-Partition-From-Even-Hyperprism(G, H, M)

Input: G – square-free, Berge graph containing no $L(BS(K_4))$
$H = (A_1, \ldots, B_3)$ – maximal even hyperprism in G
M – set of major neighbors of H

Output: A good partition of G

$Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$
relabel strips of H, so that $M \cup A_1$ and $M \cup B_1$ are cliques
$F_1 \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_1 \cup B_1 \cup C_1$ $\}$
ASSERT: M is a clique
ASSERT: $M \cup A_i$ is a clique for at least two values of i
ASSERT: $M \cup B_j$ is a clique for at least two values of j
$K_1 \leftarrow A_1$, $K_2 \leftarrow M$, $K_3 \leftarrow B_1$
$R \leftarrow C_1 \cup F_1 \cup Z$
$L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$
return (K_1, K_2, K_3, L, R)