1 Notes

- $V(X)$ – vertices of structure X. Will be written as X when obvious.
- $a - b$, when a and b are nodes – a and b are neighbors.
- $a \cdots b$, when a and b are nodes – a and b are not neighbors.
- $a - X$, when a is a node and X is a set of nodes – a has a neighbor in X.
- $a \cdots X$, when a is a node and X is a set of nodes – a has a nonneighbor in X.
- $a \blacktriangleleft X$, when a is a node and X is a set of nodes – a is complete to X.
- $a \not\triangleleft X$, when a is a node and X is a set of nodes – a is anticomplete to X.
- $X \blacksquare Y$, when X and Y are set of nodes – X is complete to Y.
- $X \not\square Y$, when X and Y are set of nodes – X is anticomplete to Y.
- $[n]$ – $\{1, \ldots, n\}$.
- $L(BS(K_4))$– a line-graph of a biparite subdivision of K_4.
- $a \leftarrow b$ – let a be equal b.
- $a :\in X$ – let a be equal to any element of X
- $a \veebar b$ – a xor b

2 Algorithms

Color-Good-Partition($G, K_1, K_2, K_3, L, R, c_1, c_2$)

Input: G – square-free, Berge graph
    K_1, K_2, K_3, L, R – good partition
    c_1, c_2 – colorings of $G \setminus R$ and $G \setminus L$ (possibly NULL)

Output: $\omega(G)$-coloring of G

$G_1 \leftarrow G \setminus R$
$G_2 \leftarrow G \setminus L$
if $c_1, c_2 =$ NULL **then**
    $c_1 \leftarrow$ Color-Graph(G_1)
    $c_2 \leftarrow$ Color-Graph(G_2)
foreach $u \in K_1 \cup K_2$ **do**
    relabel c_2, so that $c_1(u) = c_2(u)$
$B \leftarrow \{u \in K_3 : c_1(u) \neq c_2(u)\}$
if $B = \emptyset$ **then return** $c_1 \cup c_2$
foreach $h \in [2]$, distinct colors i, j **do**
    $G_h^{i,j} \leftarrow$ subgraph induced on G_h by $\{v \in G_h : c_h(v) \in \{1, 2\}\}$
foreach $u \in K_3$ **do**
    $C_h^{i,j}(u) \leftarrow$ component of $G_h^{i,j}$ containing u
ASSERT: $C_h^{c_1(u),c_2(u)}(u) \cap K_2 = \emptyset$
if $\exists u \in B, h \in [2] : C_h^{c_1(u),c_2(u)}(u) \cap K_1 = \emptyset$ **then**
    $c_1' \leftarrow c_1$ with colors i and j swapped in $C_1^{i,j}(u)$
    ASSERT: c_1' and c_2 agree on $K_1 \cup K_2$
    ASSERT: $\forall u \in K_3 \setminus B : c_1'(u) = c_1(u)$
    ASSERT: $c_1'(u) = j = c_2(u)$
    return Color-Good-Partition($G, K_1, K_2, K_3, L, R, c_1', c_2$)
else
    $w \leftarrow$ vertex of B with nost neighbors in K_1
    ASSERT: $\forall u \in B : N(u) \cap K_1 \subset N(w) \cap K_1$
    relabel c_1, c_2, so that $c_1(w) = 1, c_2(w) = 2$
    $P \leftarrow$ chordless path $w - p_1 - \ldots - p_k - a$ in $C_1^{1,2}(w)$ so that
        $k \geq 1$, $p_1 \in K_3 \cup L$, $p_2 \ldots p_k \in L$, $a \in K$, $c_1(a) \in [2]$
    $Q \leftarrow$ chordless path $w - q_1 - \ldots - q_l - a$ in $C_2^{1,2}(w)$ so that
        $l \geq 1$, $q_1 \in K_3 \cup R$, $q_2 \ldots q_l \in R$, $a \in K$, $c_2(a) \in [2]$
    $i \leftarrow c_1(a)$
    $j \leftarrow 3 - i$
    ASSERT: exactly one of the colors 1 and 2 appears in K_1 (as in Lemma 2.2.(3))
    ASSERT: $|P|$ and $|Q|$ have different parities
    ASSERT: $p_1 \in K_3 \vee p_2 \in K_3$ (as in Lemma 2.2.(4))
    ASSERT: $\nexists y \in K_3 : c_1(y) = 2 \wedge c_2(y) = 1$ (as in Lemma 2.2.(5))
    if $p_1 \in K_3$ **then**
        ASSERT: $c_2(p_1) \notin [2]$
        relabel c_2, so that $c_2(p_1) = 3$

// **else** // $\nexists u \in B, h \in [2] : C_h^{c_1(u),c_2(u)}(u) \cap K_1 = \emptyset$

// **if** $p_1 \in K_3$ **then**

ASSERT: color 3 does not appear in K_2

ASSERT: color 3 does not appear in K_1

ASSERT: $C_2^{j,3}(p_1) \cap K_1 = \emptyset$

$c_2' \leftarrow c_2$ with colors j and 3 swapped in $C_2^{j,3}(p_1)$

ASSERT: $j = 2$

return Color-Good-Partition(G, K_1, K_2, K_3, L, R, c_1, c_2')

else

relabel c_1, so that $c_1(q_1) = 3$

if 3 *does not appear in* K_1 **then**

ASSERT: $C_1^{j,3}(q_1) \cap K_1 = \emptyset$

ASSERT: $j = 1$

$c_1' \leftarrow c_1$ with colors j and 3 swapped in $C_1^{j,3}(q_1)$

return Color-Good-Partition(G, K_1, K_2, K_3, L, R, c_1', c_2)

else

ASSERT: $q_1 \ntriangleleft \{a, a_3\}$

ASSERT: $C_1^{i,3}(q_1) \cap K_1 = \emptyset$

ASSERT: $i = 1$

$c_1' \leftarrow c_1$ with colors i and 3 swapped in $C_1^{i,3}(q_1)$

return Color-Good-Partition(G, K_1, K_2, K_3, L, R, c_1', c_2)

Grow-Hyperprism(G, G, M, F) // Lemma 3.3

Input: G – square-free, Berge graph

$H = (A_1, \dots B_3)$ – a hyperprism in G

M – the set of major neighbors of H in G

F – a minimal component of $G \setminus (H \cup M)$ with a set of attachments in H not local.

Output: H' – a larger hyperprism, or

L – a $L(BS(K_4))$

$X \leftarrow$ set of attachments of F in H

if $\exists i : X \cap C_i \neq \emptyset$ **then**

  relabel strips of H, so that $X \cap C_1 \neq \emptyset$

  $x_1 :\in X \cap C_1$

  ASSERT: $X \cap S_2 \neq \emptyset$

  $x_2 :\in X \cap S_2$

  $R_1 \leftarrow$ 1-rung of H, so that $x_1 \in V(R_1)$

  $R_2 \leftarrow$ 2-rung of H, so that $x_2 \in V(R_2)$

  $R_3 \leftarrow$ a 3-rung of H

  $\forall i \in [3] : a_i, b_i \leftarrow$ends of R_i, so that $a_i \in A_i, b_i \in B_i$

  $K \leftarrow$ a prism (R_1, R_2, R_3)

  ASSERT: no vertex in F is major w.r.t. K (as in SPGT 10.5)

  $f_1 - \dots - f_n \leftarrow$ a minimal path in F, so that

    $f_1 \blacktriangleleft \{a_2, a_3\}$,

    $f_n - R \setminus \{a_1\}$

    there are no other edges between $\{f_1, \dots f_n\}$ and $V(K) \setminus \{a_1\}$

  ASSERT: $F = \{f_1, \dots, f_n\}$

  ASSERT: $f_1 \blacktriangleleft A_3$

  $A'_1 \leftarrow A_1 \cup \{f_1\}$

  $C'_1 \leftarrow C_1 \cup \{f_2, \dots, f_n\}$

  return $H' \leftarrow (A'_1, A_2, \dots, B_3, C'_1, C_2, C_3)$

else

  relabel strips of H, so that there is $\{x_1 :\in A_1, x_2 :\in A_2\} \subset X$ that is not local

  find a path $x - f_1 - \dots - f_n - x_2$

  ASSERT: $F = \{f_1, \dots f_n\}$

  if n is even and H is even, or n is odd and H is odd **then**

    ASSERT: $f_1 - a_3 \veebar f_n - b_3$

    if $f_1 - a_3$ **then**

      $H' \leftarrow$ mirrored H – every A_i and B_i are swapped

      # TODO: check if M and F are OK

      return Grow-Hyperprism(G, H', M, F)

    else

      if $f_n \blacktriangleleft B_2 \cup B_3$ **then**

        $B'_1 \leftarrow B_1 \cup \{f_n\}$

        $C'_1 \leftarrow C_1 \cup \{f_1, \dots, f_{n-1}\}$

// **else** // $\forall_{i \in [3]} X \cap C_i = \emptyset$
// **if** n is even and H is even, or n is odd and H is odd **then**
// **else** // $f_n - b_3$
// **if** $f_n \blacktriangleleft B_2 \cup B_3$ **then**

return $H' \leftarrow \begin{pmatrix} A_1 & C'_1 & B'_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{pmatrix}$

else

$\forall_{i \in [3]} : A'_i \leftarrow$ neighbors of f_1 in A_i
$\forall_{i \in [3]} : A''_i \leftarrow A_i \setminus A'_i$
$\forall_{i \in [3]} : B''_i \leftarrow$ neighbors of f_n in B_i
$\forall_{i \in [3]} : B'_i \leftarrow B_i \setminus B''_i$
ASSERT: Every i-rung is between A'_i and B'_i or A''_i and B''_i
$\forall_{i \in [3]} : C'_i \leftarrow$ union of interiors of i-rings between A'_i and B'_i
$\forall_{i \in [3]} : C''_i \leftarrow$ union of interiors of i-rings between A''_i and B''_i
ASSERT: $C_i = C'_i \cup C''_i$, $C'_i \cap C''_i = \emptyset$
ASSERT: $A'_i \cup C'_i \not\sqsubset C''_i \cup B''_i$, $A''_i \cup C''_i \not\sqsubset C_i \cup B_i$
ASSERT: $A'_i \blacksquare A''_i$, $B'_i \blacksquare B''_i$
ASSERT: $A'_1, A''_2, A'_3, A''_3 \neq \emptyset$

$H' \leftarrow \begin{pmatrix} A'_1 & C'_1 & B'_1 \\ A'_2 \cup A'_3 & C'_2 \cup C'_3 & B'_2 \cup B'_3 \\ \bigcup_i A''_i \cup \{f_1\} & \bigcup_i C''_i \cup \{f_2, \ldots, f_n\} & \bigcup_i B''_i \end{pmatrix}$

return H'

else

$a_1 \leftarrow$ neighbor of f_1 in A_1
$R_1 \leftarrow$ 1-rung with end a_1
$b_1 \leftarrow$ the other end of R_1
$b_2 \leftarrow$ neighbor of f_2 in B_2
$R_2 \leftarrow$ 2-rung with end b_2
$a_2 \leftarrow$ the other end of R_2
ASSERT: $b_1 \in X$, $a_2 \in X$
ASSERT: $(b_1 - f_1 \wedge a_2 - f_n) \veebar (b_1 - f_n \wedge a_2 - f_1)$
if $f_1 - b_1$ **then**
ASSERT: H is odd
$R_3 \leftarrow$ any 3-rung with ends a_3, b_3, such that $\{a_3, b_3\} \not\sqsubset \{f_1, f_n\}$
return $V(R_1) \cup V(R_2) \cup V(R_3) \cup \{f_1, \ldots, f_n\}$ - a $L(BS(K_4))$
TODO: Is it valid input for part of ALG I?

// **else** // $\forall_{i\in[3]} X \cap C_i = \emptyset$

// **else** // n is odd and H is even, or n is even and H is odd

else // $f_1 - a_2$

$\forall_{i\in[3]} : A'_i \leftarrow A_i \cap X,\ A''_i \leftarrow A_i \setminus X$

$\forall_{i\in[3]} : B'_i \leftarrow B_i \cap X,\ B''_i \leftarrow B_i \setminus X$

$\forall_{i\in[3]} : C'_i \leftarrow$ union of i-rungs between A'_i and B'_i

$\forall_{i\in[3]} : C''_i \leftarrow$ union of i-rungs between A''_i and B''_i

ASSERT: $C_i = C'_i \cup C''_i,\ C'_i \cap C''_i = \emptyset$

if f_1 *is complete to at least two of* A_i **then**

relabel strips of H, so that f_1 is complete to A_1 and A_2

ASSERT: f_n is complete to B_1 and B_2

ASSERT: $n > 1$ (as in SPGT 10.5 OK for odd H?)

return $\begin{pmatrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 \cup \{f_1\} & C_3 \cup \{f_2, \dots, f_{n-1}\} & B_3 \cup \{f_n\} \end{pmatrix}$

else

ASSERT: $A'_i \blacksquare A''_i$ # TODO: OK odd H?

ASSERT: $B'_i \blacksquare B''_i$ # TODO: OK odd H?

return

$$\begin{pmatrix} A'_1 & C'_1 & B'_1 \\ A'_2 \cup A'_3 & C_2 \cup C'_3 & B'_2 \cup C'_3 \\ \bigcup_i A''_i \cup \{f_1\} & \bigcup_i C''_i \cup \{f_2, \dots, f_{n-1}\} & \bigcup_i B''_i \cup \{f_n\} \end{pmatrix}$$

Good-Partition-From-Even-Hyperprism(G, H, M)

Input: G – square-free, Berge graph containing no $L(BS(K_4))$
$H = (A_1, \ldots, B_3)$ – maximal even hyperprism in G
M – set of major neighbors of H

Output: A good partition of G

$Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$
relabel strips of H, so that $M \cup A_1$ and $M \cup B_1$ are cliques
$F_1 \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_1 \cup B_1 \cup C_1$ $\}$
ASSERT: M is a clique
ASSERT: $M \cup A_i$ is a clique for at least two values of i
ASSERT: $M \cup B_j$ is a clique for at least two values of j
$K_1 \leftarrow A_1$, $K_2 \leftarrow M$, $K_3 \leftarrow B_1$
$R \leftarrow C_1 \cup F_1 \cup Z$
$L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$
return (K_1, K_2, K_3, L, R)

Good-Partition-From-Even-Hyperprism(G, H, M)

Input: G – square-free, Berge graph containing no $L(BS(K_4))$

$H = (A_1, \dots, B_3)$ – maximal odd hyperprism in G

M – set of major neighbors of H

Output: A good partition of G

$Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$

relabel strips of H, so that $A_1 \not\sqsubset B_1$ and $A_2 \not\sqsubset B_2$

ASSERT: $C_1 \neq \emptyset, C_2 \neq \emptyset$

$\forall_{i \in [3]} F_i \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_i \cup B_i \cup C_i$ $\}$

$F_B \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z \cup F_1 \cup F_2 \cup F_3\}$ that attaches to $B_1 \cup B_2 \cup B_3$ $\}$

\# TODO: F_i, F_A, F_B are from algIV, make sure it is correct

ASSERT: At least two of A_i and at least two of B_i are cliques

ASSERT: M is complete to at least two of A_i and at least two of B_i

ASSERT: M is a clique

ASSERT: For at least two $i :$ $A_i \cup M$ is a clique

ASSERT: For at least two $j :$ $A_j \cup M$ is a clique

choose h, so that $M \cup A_h$ and $M \cup B_h$ are cliques

if $h = 1 \vee h = 2$ **then** // \# TODO: make sure $h = 2$ is ok

    $K_1 \leftarrow A_1$, $K_2 \leftarrow M$, $K_3 \leftarrow B_1$

    $R \leftarrow C_1 \cup F_1 \cup Z$

    $L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$

    return (K_1, K_2, K_3, L, R)

else

    relabel H so that $M \cup A_1$ and $M \cup B_2$ are cliques

    $K_1 \leftarrow B_2 \cup B_3$, $K_2 \leftarrow M$, $K_3 \leftarrow A_1 \cup A_3$

    $L \leftarrow B_1 \cup C_1 \cup F_1 \cup F_B$

    $R \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup L\}$

    return (K_1, K_2, K_3, L, R)