



# A class of perfectly contractile graphs

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## Abstract

We consider the class  $\mathcal{A}$  of graphs that contain no odd hole, no antihole, and no “prism” (a graph consisting of two disjoint triangles with three disjoint paths between them). We prove that every graph  $G \in \mathcal{A}$  different from a clique has an “even pair” (two vertices that are not joined by a chordless path of odd length), as conjectured by Everett and Reed [“Even pairs”, in: J.L. Ramírez-Alfonsín, B.A. Reed (eds.), *Perfect Graphs*, Wiley Interscience, New York, 2001]. Our proof is a polynomial-time algorithm that produces an even pair with the additional property that the contraction of this pair yields a graph in  $\mathcal{A}$ . This entails a polynomial-time algorithm, based on successively contracting even pairs, to color optimally every graph in  $\mathcal{A}$ . This generalizes several results concerning some classical families of perfect graphs.

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## 1. Introduction

A graph  $G$  is *perfect* if every induced subgraph  $G'$  of  $G$  satisfies  $\chi(G') = \omega(G')$ , where  $\chi(G')$  is the chromatic number of  $G'$  and  $\omega(G')$  is the maximum clique size in  $G'$ . Berge [1–3] introduced perfect graphs and conjectured that *a graph is perfect if and only if it does not contain as an induced subgraph an odd hole or an odd antihole of length at least 5*,

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where a *hole* is a chordless cycle with at least four vertices and an *antihole* is the complement of a hole. We follow the tradition of calling *Berge graph* any graph that contains no odd hole and no odd antihole of length at least 5. This famous question (the Strong Perfect Graph Conjecture) was the object of much research (see [20]), until it was finally proved by Chudnovsky et al. [6]: *every Berge graph is perfect*. Moreover, a polynomial-time algorithm was devised to decide if a graph is Berge (hence perfect); it is due to Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [5,9,7].

Despite those breakthroughs, some conjectures about Berge graphs remain open. An *even pair* in a graph  $G$  is a pair  $\{x, y\}$  of non-adjacent vertices having the property that every chordless path between them has even length (number of edges). Given two vertices  $x, y$  in a graph  $G$ , the operation of *contracting* them means removing  $x$  and  $y$  and adding one vertex with edges to every vertex of  $G \setminus \{x, y\}$  that is adjacent in  $G$  to at least one of  $x, y$ ; we denote by  $G/xy$  the graph that results from this operation. Fonlupt and Uhry [11] proved that *if  $G$  is a perfect graph and  $\{x, y\}$  is an even pair in  $G$ , then the graph  $G/xy$  is perfect and has the same chromatic number as  $G$* . In particular, given a  $\chi(G/xy)$ -coloring  $c$  of the vertices of  $G/xy$ , one can easily obtain a  $\chi(G)$ -coloring of the vertices of  $G$  as follows: keep the color for every vertex different from  $x, y$ ; assign to  $x$  and  $y$  the color assigned by  $c$  to the contracted vertex. This idea could be the basis for a conceptually simple coloring algorithm for Berge graphs: as long as the graph has an even pair, contract any such pair; when there is no even pair find a coloring  $c$  of the contracted graph and, applying the procedure above repeatedly, derive from  $c$  a coloring of the original graph. The algorithm for recognizing Berge graphs mentioned at the end of the preceding paragraph can be used to detect an even pair in a Berge graph  $G$ ; indeed, it is easy to see that two non-adjacent vertices  $a, b$  form an even pair in  $G$  if and only if the graph obtained by adding a vertex adjacent only to  $a$  and  $b$  is Berge. Thus, given a Berge graph  $G$ , one can try to color its vertices by keeping contracting even pairs until none can be found. Then some questions arise: what are the Berge graphs with no even pair? What are, on the contrary, the graphs for which a sequence of even-pair contractions leads to graphs that are trivially easy to color?

As a first step towards getting a better grasp on these questions, Bertschi [4] proposed the following definitions. A graph  $G$  is *even-contractile* if either  $G$  is a clique or there exists a sequence  $G_0, \dots, G_k$  of graphs such that  $G = G_0$ , for  $i = 0, \dots, k-1$  the graph  $G_i$  has an even pair  $\{x_i, y_i\}$  such that  $G_{i+1} = G_i/x_i y_i$ , and  $G_k$  is a clique. A graph  $G$  is *perfectly contractile* if every induced subgraph of  $G$  is even-contractile. This class is of interest because it turns out that many families of graphs that are considered classical are perfectly contractile (see Section 4). Everett and Reed proposed a conjecture aiming at a characterization of perfectly contractile graphs. To understand it, one more definition is needed: say that a graph is a *prism* if it consists of two vertex-disjoint triangles (cliques of size 3) with three vertex-disjoint paths between them, and with no other edge than those in the two triangles and in the three paths. (Prisms were called stretchers in [17,16].) Say that a prism is *odd* if all these three paths have odd length. For example the graph  $\overline{C}_6$  that is the complement of a hole of length 6 is an odd prism. See Fig. 1.

**Conjecture 1.1** (Everett et al. [10] and Reed [21]). *A graph is perfectly contractile if and only if it contains no odd hole, no antihole of length at least 5, and no odd prism.*

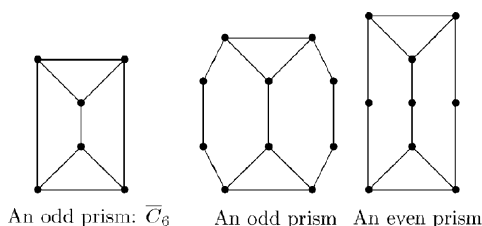


Fig. 1. Some prisms.

The if part of this conjecture remains open. The only if part is not hard to establish, but it requires some careful checking; this was done formally in [17]. A weaker form was also proposed by Everett and Reed:

**Conjecture 1.2** (Everett et al. [10] and Reed [21]). *If a graph contains no odd hole, no antihole of length at least 5, and no prism then it is perfectly contractile.*

Here we will prove this second conjecture. For this purpose, some definitions and notation must be introduced.

Let  $G = (V, E)$  be a graph. Its complementary graph is denoted by  $\overline{G}$ . The subgraph induced by any  $X \subseteq V$  is denoted by  $G[X]$ . We say that a vertex  $u$  *sees* a vertex  $v$  when  $u, v$  are adjacent, else we say that  $u$  *misses*  $v$ . For any  $T \subseteq V$ , we let  $N(T)$  denote the set of vertices of  $G \setminus T$  that see at least one vertex of  $T$ . A vertex of  $V \setminus T$  is called  $T$ -complete if it sees all vertices of  $T$ ; then  $C(T)$  denotes the set of  $T$ -complete vertices of  $V \setminus T$ . We call  $T$ -edge any edge whose two vertices are  $T$ -complete.

A non-empty set  $T \subseteq V$  is *interesting* if  $\overline{G}[T]$  is connected (in short we will say that  $T$  is co-connected) and  $G[C(T)]$  is not a clique (we view the empty set as a clique, so  $|C(T)| \geq 2$ ). Note that a graph  $G$  may fail to have any interesting set; in that case it must be in particular that the neighborhood  $N(v)$  of every vertex  $v$  induces a clique ( $v$  is *simplicial*), for otherwise  $\{v\}$  would be interesting; this means that every connected component of  $G$  is a clique, i.e.,  $G$  is a disjoint union of cliques.

Paths in a graph may be described in two different ways: either we list their vertices in order ( $P = ab \cdots cd$ ); or, if  $x, y$  are two vertices of a given path  $P$ , we may use  $P[x, y]$  or  $P[y, x]$  to denote the subpath of  $P$  whose endvertices are  $x, y$ . The length of a path is its number of edges. An edge between two vertices that are not consecutive along the path is a *chord*, and a path that has no chord is *chordless*.

A *snake*  $S$  is a graph that consists of four disjoint chordless paths  $S_1 = a \cdots a'$ ,  $S_2 = b \cdots b'$ ,  $S_3 = c \cdots c'$ ,  $S_4 = d \cdots d'$ , where  $S_1, S_2$  may have length 0 but  $S_3, S_4$  have length at least 1, and such that the edge-set of  $S$  is  $E(S_1) \cup E(S_2) \cup E(S_3) \cup E(S_4) \cup \{a'c, a'd, cd, b'c', b'd', c'd'\}$ . Note that  $a'cd$  and  $b'c'd'$  are triangles in  $S$ . Vertices  $a$  and  $b$  are the endvertices of the snake, and we may also say that  $S$  is an  $(a, b)$ -snake. See Fig. 2. A snake is *proper* if at least one of  $S_1, S_2$  has length at least 1. An even pair  $\{a, b\}$  in a graph  $G$  is *special* if the graph  $G$  contains no proper  $(a, b)$ -snake.

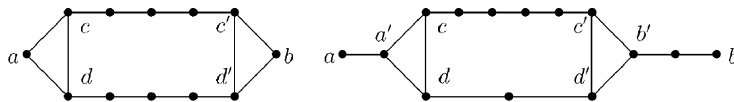


Fig. 2. Some  $(a, b)$ -snakes; the second one is proper.

Let  $\mathcal{A}$  be the class of graphs that contain no odd hole, no antihole of length at least 5, and no prism. Our main result is the following.

**Theorem 1.3.** *In a graph  $G \in \mathcal{A}$  that is not a disjoint union of cliques, let  $T$  be any interesting set. Then  $C(T)$  contains a special even pair of  $G$ .*

The proof of this theorem will be given in Section 3. Some technical lemmas will be given in Section 2.

Special even pairs are valuable because of the following result (see [10]):

**Theorem 1.4** (Everett et al. [10]). *Let  $G$  be a graph and  $\{x, y\}$  be an even pair of  $G$ . Then*

1. *If  $G$  contains no odd hole, then  $G/xy$  contains no odd hole.*
2. *If  $G$  contains no antihole of length at least 5, then  $G/xy$  contains no antihole of length at least 5 except possibly  $\overline{C}_6$ .*
3. *If  $G$  contains no prism and no proper  $(x, y)$ -snake, then  $G/xy$  contains no prism.*

Note that  $\overline{C}_6$  (the complement of the 6-vertex hole) is itself a prism.

**Corollary 1.5.** *Every graph  $G \in \mathcal{A}$  either is a clique or has a special even pair. Every graph  $G \in \mathcal{A}$  is perfectly contractile.*

**Proof.** If  $G$  is a disjoint union of cliques and not itself a clique, then any two non-adjacent vertices  $x, y$  form a special even pair. If  $G$  is not a disjoint union of cliques, apply Theorem 1.3 to get a special even pair  $\{x, y\}$ . In either case, Theorem 1.4 implies that  $G/xy$  is in  $\mathcal{A}$ , which by induction entails the second sentence of the corollary.  $\square$

## 2. Some technical lemmas

First, we recall a nice lemma due to Roussel and Rubio [22], also proved independently by Chudnovsky, Robertson, Seymour, Thomas, and Thomassen (but not published) and used intensively in [6]. We propose here our own proof of that lemma, which we believe is simpler and shorter than Roussel and Rubio's. Then we derive some lemmas of a similar flavor.

For any chordless path  $P = xx' \cdots y'y$  of length at least 2 in  $G$ , let  $P^*$  denote the interior of  $P$ , i.e., the path  $P \setminus \{x, y\}$ . Following [6], we say that a pair  $\{u, v\}$  of non-adjacent vertices of  $V \setminus P$  is a *leap* for  $P$  if  $N(u) \cap P = \{x, x', y\}$  and  $N(v) \cap P = \{x, y', y\}$ . Note that in that case  $P^* \cup \{u, v\}$  is a chordless path of the same length as  $P$ .

**Lemma 2.1** (Roussel and Rubio [22]). *In a Berge graph  $G = (V, E)$ , let  $P, T \subset V$  be disjoint sets such that  $P$  induces a chordless path,  $T$  induces a co-connected subgraph, and the endvertices of  $P$  are  $T$ -complete. Then one of the following four outcomes holds:*

0.  $P$  has even length and has an even number of  $T$ -edges.
1.  $P$  has odd length and has an odd number of  $T$ -edges.
2.  $P$  has odd length at least 3 and there is a leap for  $P$  in  $T$ .
3.  $P$  has length 3 and its two internal vertices are the endvertices of a chordless odd path of  $\overline{G}$  whose interior is in  $T$ .

**Proof.** We prove the lemma by induction on  $|P \cup T|$ . If  $P$  has length 0 or 1 then we have outcome 0 or 1. So let us assume that  $P$  has length at least 2. Put  $P = xx' \cdots y'y$ . We distinguish between two cases.

*Case 1: There is no  $T$ -complete vertex in  $P^*$ :* If  $P$  has even length, we have outcome 0. So we may assume that  $P$  has odd length. If  $|T| = 1$ , then  $T \cup P$  induces an odd hole. So  $|T| \geq 2$ . Let us suppose that outcomes 2 and 3 do not hold for  $P$ . Therefore, and by induction, we know that for every co-connected proper subset  $U$  of  $T$  there is an odd number of  $U$ -edges in  $P$ . Note that, for any  $t \in T$ , any  $T \setminus \{t\}$ -complete vertex of  $P^*$  misses  $t$ .

*Case 1.1:  $T$  induces a stable set:* Let  $t$  be any vertex of  $T$ . We claim that  $N(t) \cap P = \{x, x', y\}$  or  $N(t) \cap P = \{x, y', y\}$ . Call  $t$ -segment of  $P$  any subpath of  $P$  whose endvertices see  $t$  and whose internal vertices miss  $t$ . Since  $x, y$  see  $t$ ,  $P$  is edge-wise partitioned into its  $t$ -segments. Also, we know that there is an odd number of  $T \setminus \{t\}$ -edges in  $P$ . So there is a  $t$ -segment  $P[r, s]$  of  $P$  that has an odd number of  $T \setminus \{t\}$ -edges. Assume that  $x, r, s, y$  appear in this order along  $P$ . Note that  $P[r, s]$  has length at least 2, for otherwise one of  $r, s$  would be a  $T$ -complete vertex in  $P^*$ . So  $P[r, s] \cup \{t\}$  induces an even hole and  $P[r, s]$  has even length. Let  $r', s'$ , respectively, be the  $T \setminus \{t\}$ -complete vertices of  $P[r, s]$  closest to  $r$  and to  $s$ . Note that  $P[r', s']$  contains all the  $T \setminus \{t\}$ -edges of  $P[r, s]$ , so the induction hypothesis, applied to  $P[r', s']$  and  $T \setminus \{t\}$ , implies that  $P[r', s']$  has odd length. Thus, exactly one of the paths  $P[r, r']$  and  $P[s', s]$  has odd length. Assume that  $P[r, r']$  has odd length. So  $r \neq r'$  and  $x \neq r$ .

If  $r'$  misses  $y$ , then the odd path  $P_1 = P[r', r] \cup \{t, y\}$  is chordless. The endvertices  $r', y$  of  $P_1$  are  $T \setminus \{t\}$ -complete, and  $P_1$  has no  $T \setminus \{t\}$ -edge since  $P_1^*$  contains no  $T \setminus \{t\}$ -complete vertex. Thus, by the induction hypothesis, we must have outcome 2 or 3 for  $P_1$  and  $T \setminus \{t\}$ ; but these are impossible since  $t$  has no neighbor in  $T \setminus \{t\}$ . So  $r'$  must see  $y$ , meaning that  $s' = s = y$  and  $r' = y'$ .

If  $r$  misses  $x$ , then the odd path  $P_2 = P[r', r] \cup \{t, x\}$  is chordless. The endvertices  $r', x$  of  $P_2$  are  $T \setminus \{t\}$ -complete, and  $P_2$  has no  $T \setminus \{t\}$ -edge since  $P_2^*$  contains no  $T \setminus \{t\}$ -complete vertex. Thus, by the induction hypothesis, we must have outcome 2 or 3 for  $P_2$  and  $T \setminus \{t\}$ ; but these again are impossible since  $t$  has no neighbor in  $T \setminus \{t\}$ . So,  $r$  must see  $x$ , meaning that  $r = x'$ . Thus we have  $N(t) \cap P = \{x, x', y\}$ . Likewise if  $P[s', s]$  has odd length then  $N(t) \cap P = \{x, y', y\}$ . So the above claim is proved, for every  $t \in T$ .

Now, since  $y'$  is not  $T$ -complete there is a vertex  $u \in T$  such that  $N(u) \cap P = \{x, x', y\}$ , and since  $x'$  is not  $T$ -complete there is a vertex  $v \in T$  such that  $N(v) \cap P = \{x, y', y\}$ . Thus,  $\{u, v\}$  is a leap in  $T$  for  $P$ .

*Case 1.2:  $T$  does not induce a stable set:* Let  $Q = u \cdots v$  be a longest path of  $\overline{G}[T]$ . So  $Q$  has length at least 2 (since  $T$  is not a stable set), and  $T \setminus \{u\}$  and  $T \setminus \{v\}$  are co-connected sets. So we know that  $P$  has an odd number of  $T \setminus \{u\}$ -edges and an odd number of  $T \setminus \{v\}$ -edges. Note that a  $T \setminus \{u\}$ -edge and a  $T \setminus \{v\}$ -edge have no common vertex, for otherwise there would be a  $T$ -complete vertex in  $P^*$ . In particular, all  $T \setminus \{u\}$ -edges and  $T \setminus \{v\}$ -edges are different.

Suppose that  $Q$  has even length. Let  $x_u x'_u$  be a  $T \setminus \{u\}$ -edge of  $P$  and  $y'_v y_v$  be a  $T \setminus \{v\}$ -edge of  $P$  such that, without loss of generality,  $x, x_u, x'_u, y'_v, y_v, y$  appear in this order on  $P$ . If  $x'_u$  misses  $y'_v$  then  $\{x'_u, y'_v\} \cup Q$  induces an odd antihole. If  $x \neq x_u$  then  $\{x_u, y'_v\} \cup Q$  induces an odd antihole. If  $y_v \neq y$  then  $\{x'_u, y_v\} \cup Q$  induces an odd antihole. It follows that  $P = x_u x'_u y'_v y_v$ , but then  $P \cup Q$  induces an odd antihole. Thus  $Q$  has odd length (at least 3).

Suppose that  $T \setminus \{u, v\}$  is not co-connected. Since  $T \setminus \{u\}$  and  $T \setminus \{v\}$  are co-connected, there exists a vertex  $w$  in a connected component of  $\overline{G}[T] \setminus \{u, v\}$  that does not contain  $Q^*$  and such that  $w$  sees in  $\overline{G}$  at least one of  $u, v$ ; but then  $Q \cup \{w\}$  induces in  $\overline{G}[T]$  either a chordless path longer than  $Q$  or an odd hole, a contradiction. So  $T \setminus \{u, v\}$  is co-connected.

Now we know that there is an odd number of  $T \setminus \{u, v\}$ -edges in  $P$ . Recall that  $P$  has an odd number of  $T \setminus \{u\}$ -edges, an odd number of  $T \setminus \{v\}$ -edges, and that these are different, so these account for an even number of  $T \setminus \{u, v\}$ -edges; thus  $P$  has at least one  $T \setminus \{u, v\}$ -edge  $x''y''$  that is neither a  $T \setminus \{u\}$ -edge nor a  $T \setminus \{v\}$ -edge. We may assume that  $x, x'', y'', y$  appear in this order along  $P$  and that  $y'' \in P^*$ . So  $y''$  misses one of  $u, v$ , say  $v$ . Then  $y''$  sees  $u$ , for otherwise  $Q \cup \{y''\}$  would induce an odd antihole. Then  $x''$  misses  $u$ , for otherwise  $x''y''$  would be a  $T \setminus \{v\}$ -edge. Then  $x''$  sees  $v$ , for otherwise  $Q \cup \{x''\}$  would induce an odd antihole. Then  $x'' = x'$  for otherwise  $Q \cup \{x'', y'', x\}$  would induce an odd antihole, and similarly  $y'' = y'$ . So  $P = xx''y''y$  and  $Q \cup \{x'', y''\}$  is a chordless odd path of  $\overline{G}$ , and we have outcome 3.

*Case 2: There is a  $T$ -complete vertex in  $P^*$ :* Let  $z$  be such a vertex. By induction, we can apply the lemma to the path  $P_{xz} = P[x \cdots z]$  and  $T$ . If  $P_{xz}$  is odd and there is a leap  $\{u, v\}$  for  $P_{xz}$  in  $T$ , then  $P_{xz}^* \cup \{u, v, y\}$  induces an odd hole. If  $P_{xz}$  has length 3 and its two internal vertices are the endvertices of a chordless odd path  $M$  of  $\overline{G}$  whose interior is in  $T$ , then  $M \cup \{y\}$  induces an odd antihole. So it must be that the number of  $T$ -edges in  $P_{xz}$  and the length of  $P_{xz}$  have the same parity. The same holds for  $P[z \cdots y]$ . Then the number of  $T$ -edges in  $P$  and the length of  $P$  have the same parity, and we have outcome 0 or 1.  $\square$

Let  $\mathcal{A}'$  be the class of graphs that contain no odd hole, no antihole of length at least 5, and no odd prism. Clearly  $\mathcal{A} \subset \mathcal{A}'$ .

**Lemma 2.2.** *In a graph  $G = (V, E) \in \mathcal{A}'$ , let  $P, T \subset V$  be disjoint sets such that  $P$  induces a chordless path,  $T$  induces a co-connected subgraph, and the endvertices of  $P$  are in  $C(T)$ . Then the number of  $T$ -edges in  $P$  has the same parity as the length of  $P$ . In particular, if  $P$  has odd length then some internal vertex of  $P$  is in  $C(T)$ .*

**Proof.** Apply Lemma 2.1. Observe that in outcome 2 of Lemma 2.1, the set  $P \cup \{u, v\}$  induces an odd prism, and that in outcome 3, letting  $M$  denote a chordless odd path of  $\overline{G}$

whose interior is in  $T$  and whose endvertices are the two internal vertices of  $P$ ,  $P \cup M$  induces an antihole of length at least 6 in  $G$ . Thus we must have outcome 0 or 1.  $\square$

**Lemma 2.3.** *In a graph  $G = (V, E) \in \mathcal{A}'$ , let  $H, T \subset V$  be disjoint sets such that  $H$  induces a hole,  $T$  induces a co-connected subgraph, and at least two non-adjacent vertices of  $H$  are in  $C(T)$ . Then the number of  $T$ -edges in  $H$  is even.*

**Proof.** Let  $x, y$  be two non-adjacent vertices of  $H \cap C(T)$ . Then the two  $(x, y)$ -paths contained in  $H$  have the same parity, so Lemma 2.2 implies that the numbers of  $T$ -edges they contain have the same parity. Thus,  $H$  has an even number of  $T$ -edges.  $\square$

In a graph  $G$ , three paths  $P_1, P_2, P_3$  are said to form a *pyramid* with triangle  $x_1x_2x_3$  and apex  $x$  (where  $x, x_1, x_2, x_3$  are four distinct vertices) if for each  $i = 1, 2, 3$ ,  $P_i$  is a chordless  $(x_i, x)$ -path,  $x_1, x_2, x_3$  induce a triangle, at least two of the paths have length at least 2, and the pyramid has no other edge than those in the three paths and in the triangle. If a graph  $G$  contains a pyramid then two of the three paths have the same parity and so, since at least one of them has length at least 2, the union of these two paths induces an odd hole. Thus

**Lemma 2.4.** *In a graph that contains no odd hole, there is no pyramid.*

**Lemma 2.5.** *In a graph  $G = (V, E) \in \mathcal{A}$ , let  $S, T \subset V$  be disjoint sets such that  $S$  induces an  $(a, b)$ -snake,  $T$  induces a co-connected subgraph, and  $a, b \in C(T)$ . Then for each triangle of  $S$  at least two vertices of the triangle are in  $C(T)$ .*

**Proof.** We use the notation for snakes given above. We first claim that every  $t \in T$  sees at least two vertices of the triangle  $a'cd$ . Let  $S_1^t$  be a chordless  $(a', t)$ -path contained in  $S_1 \cup \{t\}$ , and  $S_2^t$  be a chordless  $(b', t)$ -path contained in  $S_2 \cup \{t\}$ . Put  $H = S_3 \cup S_4$ .

If  $t$  has no neighbor in  $H$  then  $H \cup S_1^t \cup S_2^t$  induces a prism (which has the same two triangles as  $S$ ), a contradiction.

Suppose that  $t$  has exactly one neighbor  $h$  in  $H$ . If  $h \notin \{c, d\}$ , then  $H \cup S_1^t$  induces a pyramid with triangle  $a'cd$  and apex  $h$ , and if  $h \in \{c, d\}$ , then  $H \cup S_2^t$  induces a pyramid with triangle  $b'c'd'$  and apex  $h$ , in either case a contradiction to Lemma 2.4.

Suppose that  $t$  has exactly two neighbors  $h, h'$  in  $H$  and these are adjacent. Call  $c''$  (resp.,  $d''$ ) the neighbor of  $c$  along  $S_3$  (resp., of  $d$  along  $S_4$ ). If the pair  $\{h, h'\}$  is none of the two pairs  $\{c, c''\}, \{d, d''\}$  then  $H \cup S_1^t$  induces a prism (with triangles  $a'cd, thh'$ ). If  $\{h, h'\} = \{c, c''\}$ , then either  $S_1^t \cup \{c\} \cup S_4 \cup S_2^t$  induces a pyramid with triangle  $a'cd$  and apex  $t$ , a contradiction, or  $t$  sees  $a'$  (thus  $t$  sees two of  $a', c, d$  as desired). If  $\{h, h'\} = \{d, d''\}$ , similarly  $t$  sees  $a'$  and  $d$ .

Finally, suppose that  $t$  sees two non-adjacent vertices of  $H$ . Let  $h$  and  $h'$ , respectively, be the vertices of  $H \cap N(t)$  closest to  $c$  along  $H \setminus \{d\}$  (resp., closest to  $d$  along  $H \setminus \{c\}$ ). Call  $H[c, h]$  the chordless  $(c, h)$ -path in  $H \setminus \{d\}$ , and call  $H[d, h']$  the chordless  $(d, h')$ -path in  $H \setminus \{c\}$ . Then, since  $S_1^t \cup H[c, h] \cup H[d, h']$  cannot induce a pyramid (with triangle  $a'cd$  and apex  $t$ ), it must be that at least two of the three paths  $S_1^t, H[c, h] \cup \{t\}, H[d, h'] \cup \{t\}$  have length 1, so  $t$  sees at least two of  $a', c, d$ .



Suppose now that the lemma fails for the first triangle of the snake, that is, there exist vertices  $\alpha, \beta \in \{a', c, d\}$  and vertices  $x, y \in T$ , such that  $x$  misses  $\alpha$  (and thus sees  $\beta$ ) and  $y$  misses  $\beta$  (and thus sees  $\alpha$ ). Since  $T$  is co-connected, there is a chordless  $(x, y)$ -path  $R$  in  $\overline{G}(T)$ , and we can choose  $x, y$  so that  $R$  is as short as possible; it follows that all the internal vertices of  $R$  see both  $\alpha, \beta$  in  $G$ . But now  $R \cup \{\alpha, \beta\}$  induces an antihole of length at least 5 in  $G$ , a contradiction. Thus at least two vertices among  $a', c, d$  are in  $C(T)$ . The same holds for the other triangle of  $S$ .  $\square$

**Lemma 2.6.** *In a graph  $G = (V, E) \in \mathcal{A}$ , let  $H, P, T \subset V$  be pairwise disjoint sets such that  $H$  induces an even hole,  $P$  induces a chordless  $(x, y)$ -path,  $T$  induces a co-connected subgraph, and there are two disjoint edges  $ab, cd$  of  $H$  such that the set of edges between  $H$  and  $P$  is  $\{ax, bx\}$  and  $c, d, y$  are in  $C(T)$ . Then at least one of  $a, b$  is in  $C(T)$ .*

**Proof.** Assume that  $a, c, d, b$  lie in this order along  $H$ , and call  $P_1$  the chordless  $(a, c)$ -path contained in  $H \setminus \{b, d\}$  and  $P_2$  the chordless  $(b, d)$ -path contained in  $H \setminus \{a, c\}$ .

Suppose that the lemma does not hold: there exists a vertex  $u \in T \setminus N(a)$  and a vertex  $v \in T \setminus N(b)$ . Let  $a_u$  be the vertex of  $P_1 \cap N(u)$  closest to  $a$ , let  $b_u$  be the vertex of  $P_2 \cap N(u)$  closest to  $b$ , and let  $x_u$  be the vertex of  $P \cap N(u)$  closest to  $x$ . If  $a_u = c$  and  $b_u = d$  then  $P_1 \cup P_2 \cup P[x, x_u] \cup \{u\}$  induce a prism (with triangles  $abx$  and  $ucd$ ), a contradiction. If either  $a_u \neq c$  or  $b_u \neq d$ , then the three paths  $P_1[a, a_u]$ ,  $P_2[b, b_u]$ ,  $P[x, x_u]$  have no edge between them (other than the edges of the triangle  $abx$ ), so, by Lemma 2.4, at least two of them have length 0; this means that  $u$  sees  $b$  and  $x$ . Similarly  $v$  sees  $a$  and  $x$ .

Since  $T$  is co-connected, there exists a chordless  $(u, v)$ -path  $R$  in  $\overline{G}(T)$ . We choose  $u, v$  that minimize the length of  $R$ , so the internal vertices of  $R$  (if any) see both  $a, b$ . If some vertex  $w \in \{y, c, d\}$  misses both  $a, b$  then  $R \cup \{a, b, w\}$  induces an antihole of length at least 5, a contradiction. In the remaining case we must have  $y = x, ac \in E, bd \in E$ ; but then  $R \cup \{a, b, c, d, y\}$  induces an antihole of length at least 7.  $\square$

**Lemma 2.7.** *In a graph  $G = (V, E) \in \mathcal{A}$ , let  $H, P, T \subset V$  be such that  $H$  induces a hole,  $P$  induces a chordless  $(x, y)$ -path,  $T$  induces a co-connected subgraph disjoint from  $H \cup P$ ,  $H \cup P$  is connected, and there are adjacent vertices  $u, v \in H$  such that  $x \neq u, v$  and  $u, v, x \in C(T)$ . Then, either some vertex of  $P$  sees at least one of  $u, v$ , or some vertex of  $H \setminus \{u, v, x\}$  is in  $C(T)$ .*

**Proof.** Note that  $H$  and  $P$  are not assumed to be disjoint. We may even have  $P \subset H$ .

Suppose that the lemma does not hold, and consider a counterexample with  $|H \cup P|$  minimal.

If  $x \in H$ , then, by Lemma 2.3,  $H$  has an even number of  $T$ -edges; thus  $H$  has a  $T$ -edge different from  $uv$ , so there is a vertex of  $H \setminus \{u, v, x\}$  in  $C(T)$ , and the triple  $(H, P, T)$  is not a counterexample to the lemma. Therefore we may assume  $x \notin H$ .

Let  $x'$  be the vertex of  $P$  that has a neighbor in  $H$  and is closest to  $x$  along  $P$ . So  $P[x, x'] \cap H = \emptyset$ . Let  $u'$  (resp.,  $v'$ ) be the vertex of  $H \cap N(x')$  closest to  $u$  along  $H \setminus \{v\}$  (resp., to  $v$  along  $H \setminus \{u\}$ ). By the assumption,  $u' \neq u$  and  $v' \neq v$ . Call  $H[u, u']$  the path from  $u$  to  $u'$  in  $H \setminus \{v\}$ , and call  $H[v, v']$  the path from  $v$  to  $v'$  in  $H \setminus \{u\}$ .



Suppose  $u' = v'$ . The paths  $P[x, x'] \cup H[u', u]$  and  $P[x, x'] \cup H[v', v]$  are chordless and one of them is odd. By Lemma 2.2, this odd path must have an odd number of  $T$ -edges. Hence, there is at least one vertex  $x''$  of  $C(T)$  in  $P[x, x'] \cup H \setminus \{u, v, x\}$ . If  $x''$  is in  $H$ , the triple  $(H, P, T)$  is not a counterexample. So  $x''$  is in  $P[x, x'] \setminus \{x\}$ ; but then the hole  $H$ , the path  $P[x'', x']$ , and the set  $T$  form a counterexample with  $|H \cup P[x'', x']| < |H \cup P|$ , a contradiction.

Suppose  $u' \neq v'$  and  $u'v' \in E$ . Then we can apply Lemma 2.6 to the hole  $H$ , the path  $P[x, x']$ , and the set  $T$ : so at least one of  $u', v'$  is in  $C(T)$ , a contradiction.

Suppose  $u' \neq v'$  and  $u'v' \notin E$ . Consider the hole  $H'$  induced by  $H[u, u'] \cup H[v, v'] \cup \{x'\}$ . Then  $H', P[x, x'], T$  form a counterexample to the lemma with  $|H' \cup P[x, x']| < |H \cup P|$ , a contradiction.  $\square$

Recall that a graph  $G$  is *weakly triangulated* [12] if  $G$  and  $\overline{G}$  contain no hole of length at least 5.

**Lemma 2.8.** *In a weakly triangulated graph  $G = (V, E)$ , let  $P, T \subset V$  be disjoint sets such that  $P$  induces a chordless  $(x, y)$ -path of length at least 3,  $T$  induces a co-connected subgraph, and  $x, y$  are in  $C(T)$ . Then at least one internal vertex of  $P$  is in  $C(T)$ .*

**Proof.** Observe that no vertex  $t \in T$  misses two consecutive vertices  $u, v$  of  $P$ , for otherwise  $P \cup \{t\}$  contains a hole, of length at least 5, containing  $u, v, t$ . Now let  $v$  be an internal vertex of  $P$  that sees the most vertices of  $T$ . If  $v \in C(T)$  we are done, so assume that there is a vertex  $t \in T \setminus N(v)$ . Call  $u, w$  the neighbors of  $v$  on  $P$ , with  $u \in P[x, v], w \in P[v, y]$ . Then both  $tu, tw$  are edges, by the observation. We may assume that  $u \neq x$  (else, by symmetry,  $w \neq y$ ). By the choice of  $v$ , and since  $t$  sees  $u$  and misses  $v$ , there is a vertex  $t' \in T$  that sees  $v$  and misses  $u$ . Since  $T$  is co-connected, there is a chordless  $(t, t')$ -path  $R$  in  $\overline{G}[T]$ , and we choose  $t, t'$  so that  $R$  is as short as possible; so, and by the observation, the internal vertices of  $R$  see both  $u, v$ . If  $u$  misses  $x$  then  $R \cup \{u, v, x\}$  induces an antihole of length at least 5. So  $u$  sees  $x$ . Likewise,  $v$  sees  $y$ , for otherwise  $R \cup \{u, v, y\}$  induces an antihole of length at least 5. But then  $R \cup \{x, u, v, y\}$  induces an antihole of length at least 6, a contradiction.  $\square$

### 3. Proof of Theorem 1.3

We now prove Theorem 1.3. Let  $G = (V, E)$  be a graph that contains no odd hole, no antihole of length at least 5, and no prism and that is not a disjoint union of cliques. We proceed by induction on  $|V|$ . The smallest possible value is  $|V| = 3$ , in which case  $G$  is a three-vertex path and the desired conclusion is obvious. Now let us assume  $|V| \geq 4$ . Along this proof we will also make several remarks, for further reference, concerning the complexity of finding various sets and paths relevant to the proof.

First, observe that we need only prove the theorem for (inclusion-wise) *maximal* interesting sets. For if  $T$  is an interesting set in  $G$ , then for any maximal interesting set  $T'$  with  $T \subset T'$  we have  $C(T') \subseteq C(T)$ , thus any special even pair of  $G$  in  $C(T')$  is also in  $C(T)$ .

So let  $T$  be a maximal interesting set. We observe that for every vertex  $z \in V \setminus (T \cup C(T))$ , the set  $N(z) \cap C(T)$  induces a clique in  $G$ , for otherwise  $T \cup \{z\}$  would be a larger interesting set because  $T \cup \{z\}$  is co-connected (since  $T$  is co-connected and  $z \notin C(T)$ ) and  $C(T \cup \{z\}) = C(T) \cap N(z)$ .

**Remark 1.** One can determine a maximal interesting set in polynomial time. Start from any non-simplicial vertex  $v$  and put  $T := \{v\}$ . (It may take time  $O(|V||E|)$  to find such a  $v$ .) As long as there exists a vertex  $w \in V \setminus (T \cup C(T))$  such that  $N(w) \cap C(T)$  does not induce a clique in  $G$ , put  $T := T \cup \{w\}$  and iterate. At termination  $T$  is a maximal interesting set. There may be  $O(|V|)$  iterations, each taking time  $O(|V||E|)$ , so the time to find such a set  $T$  is  $O(|V|^2|E|)$ .

Let us call *outer path* any chordless path whose endvertices are non-adjacent vertices of  $C(T)$  and whose internal vertices are in  $V \setminus (T \cup C(T))$ .

Observe that *there is no outer path of odd length*, by Lemma 2.2. Moreover, if  $P$  is any outer path of even length, then its length is at least 4, for if it was 2 the internal vertex  $z$  of  $P$  would be such that  $N(z) \cap C(T)$  is not a clique, since it contains the endvertices of  $P$ , a contradiction to the maximality of  $T$ . We now distinguish between two cases.

*Case 1: There is no outer path at all.*

Let  $\{a, b\}$  be any special even pair of the graph  $G[C(T)]$ . Recall that  $G[C(T)]$  is not a clique; so, such a pair exists, by the induction hypothesis if  $G[C(T)]$  is not a disjoint union of cliques, trivially if  $G[C(T)]$  is a disjoint union of cliques. Consider any chordless  $(a, b)$ -path  $P$  in  $G$ . If  $P$  has a vertex  $t \in T$  then  $P = atb$ , so it has length 2. If  $P \cap T = \emptyset$ , it must be that  $P$  lies entirely in  $C(T)$ , for otherwise  $P$  would contain an outer path; so  $P$  has even length. Therefore  $\{a, b\}$  is an even pair of  $G$ . Moreover, if there exists a proper  $(a, b)$ -snake  $S$  in  $G$  (with the above notation for snakes), then the two  $(a, b)$ -paths  $P_1 = S_1 \cup S_3 \cup S_2$  and  $P_2 = S_1 \cup S_4 \cup S_2$  have length at least 4, and so  $P_1, P_2$  lie entirely in  $C(T)$  by the same argument as precedently with  $P$ ; so  $S$  lies entirely in  $C(T)$ , a contradiction. It follows that  $\{a, b\}$  is a special even pair of  $G$ .

*Case 2: There exists an (even) outer path.*

Let  $\alpha z_1 \cdots z_n \beta$  be a *minimal* outer path, that is, an outer path such that there is no outer path whose interior is strictly contained in  $z_1 \cdots z_n$ . Note that  $n$  is odd and that  $n \geq 3$  as pointed out above. Put  $Z = z_1 \cdots z_n$ . Define

$$A = \{v \in C(T) \mid vz_1 \in E, vz_i \notin E \ (i = 2, \dots, n)\},$$

$$B = \{v \in C(T) \mid vz_n \in E, vz_i \notin E \ (i = 1, \dots, n-1)\}.$$

Note that  $A$  is not empty, because  $\alpha \in A$ , and that  $A$  induces a clique, because  $A \subseteq N(z_1) \cap C(T)$ . Likewise  $B$  is a non-empty clique. Clearly,  $A \cap B = \emptyset$ . Moreover, there is no edge  $uv$  with  $u \in A, v \in B$ , for otherwise  $u, z_1, \dots, z_n, v$  would induce an odd hole.

**Remark 2.** Finding a minimal outer path (or concluding that there is no outer path at all) and, if there is any, finding the corresponding sets  $A, B$  can be done in polynomial time. Indeed it suffices, for every pair of non-adjacent vertices  $u, v \in C(T)$ , to look for a shortest  $(u, v)$ -path in  $G \setminus (T \cup C(T) \setminus \{u, v\})$ , and to take the shortest of them all (if any) over all pairs  $u, v$ . Looking for a shortest path takes time  $O(|E|)$  for each pair  $u, v$ . In total, since  $T$

may have size  $O(|V|)$ , finding a minimal outer path or concluding that there is none may take time  $O(|V|^2|E|)$ .

We now show that some well-chosen vertices  $a \in A$ ,  $b \in B$  form a special even pair of  $G$ . This is established in Lemmas 3.1–3.5. The outline of the proof from here on is quite similar to that in [16].

**Lemma 3.1.**  $C(T) \cap N(Z) \subseteq A \cup B \cup C(T \cup A \cup B)$ .

**Proof.** Pick any  $w \in C(T) \cap N(Z)$ ; so there exists an edge  $z_i w$  with  $z_i \in Z$  ( $1 \leq i \leq n$ ). If  $w \in C(T \cup A \cup B)$  we are done, so let us assume that there is a vertex  $u \in A \cup B$  with  $uw \notin E$ , say  $u \in A$ . Let  $i$  be the smallest integer with  $z_i w \in E$ . Then  $uz_1 \cdots z_i w$  is an outer path whose interior is contained in  $z_1 \cdots z_n$ . The minimality of  $\alpha z_1 \cdots z_n \beta$  implies  $i = n$ , and so  $w \in B$ .  $\square$

**Lemma 3.2.** Consider any odd chordless  $(u, v)$ -path  $P = uu' \cdots v'v$ , with  $u \in A$  and  $v \in B$ . Then exactly one of  $u' \in A$  or  $v' \in B$  holds.

**Proof.** We prove the lemma by induction on the length of  $P$ . Note that  $P$  has length at least 3, since there is no edge between  $A$  and  $B$ ; also  $P$  clearly contains no vertex of  $T$  and no vertex of  $C(T \cup A \cup B)$ .

Suppose that neither  $u' \in A$  nor  $v' \in B$  holds. We will show that this leads to a contradiction. We claim that

$$\text{The only edges from } Z \text{ to } P \cap C(T) \text{ are } z_1 u \text{ and } z_n v. \quad (1)$$

For if  $zw$  is an edge with  $z \in Z$  and  $w \in P \cap C(T)$ , then, since  $w \notin C(T \cup A \cup B)$  as observed above, and by Lemma 3.1, we have  $w \in A \cup B$ . Since  $A$  is a clique, the case  $w \in A$  means either  $w = u$  (so  $zw = z_1 u$  as claimed) or  $w = u'$  (so  $u' \in A$ , which we have excluded). The case  $w \in B$  is similar. Thus (1) holds.

Let us mark those vertices of  $P$  that have a neighbor in  $Z$  (in particular, the vertices of  $P \cap Z$  are marked). Call  $Z$ -segment of  $P$  any subpath of  $P$ , of length at least 1, whose endvertices are marked and whose internal vertices are not marked. By (1) the marked vertices of  $P \setminus \{u, v\}$  are all in  $P \setminus C(T)$ . Since  $u, v$  are marked,  $P$  is (edge-wise) partitioned into its  $Z$ -segments. Also some internal vertex of  $P$  must be marked, for otherwise  $Z \cap P = \emptyset$  and  $Z \cup P$  induces an odd hole; so  $P$  has at least two  $Z$ -segments.

By Lemma 2.2, we know that  $P$  has an odd number of  $T$ -edges.

It follows from the conclusion of the preceding two paragraphs that there exists a  $Z$ -segment  $Q$  of  $P$  that contains an odd number of  $T$ -edges, and that  $Q$  does not contain both  $u, v$ , say  $Q$  does not contain  $v$ . Call  $w, x$  the endvertices of  $Q$ , and call  $w', x'$ , respectively, the vertices of  $Q \cap C(T)$  closest to  $w$  and to  $x$  along  $Q$ , so that  $u, w, w', x', x, v$  lie in that order along  $P$ . We have  $w' \neq x'$  since  $Q$  contains at least one  $T$ -edge. For the sake of clarity, note that  $Q \cap Z = \emptyset$ ; indeed, if a vertex of  $Q$  was in  $Z$  then it would be marked and so would its neighbor in  $Q$ ; thus we would have  $Q = wx = w'x'$ ; but then one of  $w', x'$  would be in  $Z \cap C(T)$ , a contradiction. Also note that if  $w' = w$  then this vertex is in  $P \cap C(T) \cap N(Z)$

so, by (1),  $w' = w = u$ . On the other hand  $x' = x$  is not possible as  $v \notin Q$ . It follows that  $Q$  has length at least 2.

By the definition of marked vertices, there exists a subpath  $Z'$  of  $Z$  with endvertices  $z_i, z_j$  such that  $z_i w, z_j x$  are edges (each of  $i < j, i = j, i > j$  is possible). We choose  $Z'$  minimal with that property; so its internal vertices, if any, miss both  $w, x$ , and consequently  $H = Q \cup Z'$  is a hole of length at least 4, and of course it is an even hole. Note that there is no vertex of  $C(T)$  in  $H \setminus Q[w', x']$ , by the definition of  $w', x'$  and because  $Z \subseteq V \setminus (T \cup C(T))$ . Moreover, the  $T$ -edges in  $H$  are exactly the  $T$ -edges in  $Q$ .

If  $w', x'$  are not consecutive along  $P$ , then  $H$  has two non-adjacent vertices of  $C(T)$  and yet it has an odd number of  $T$ -edges, a contradiction to Lemma 2.3. So  $w', x'$  must be consecutive along  $P$ .

Put  $k = \max\{i, j\}$  ( $k \geq 1$ ). Define a path  $Y$  by setting  $Y = Z[z_{k+1}, z_n] \cup \{v\}$  if  $k < n$  and  $Y = \{v\}$  if  $k = n$ . Note that  $H \cup Y$  is connected since  $z_k$  is a vertex of  $H$  adjacent to  $Y$ . We claim that every vertex  $z \in Y$  misses both  $w', x'$ . Indeed,  $v$  itself does miss both because  $w', x', x, v$  are four distinct vertices in that order along  $P$ ; if  $z \in Z[z_{k+1}, z_n]$  then  $z$  misses  $x'$  because  $x'$  is not marked; moreover, if  $z$  sees  $w'$ , then  $w'$  is marked, so  $w' = w = u$ , so  $w' \in P \cap C(T)$ , but then the edge  $zw'$  contradicts (1). So the claim holds. Now the triple  $(H, Y, T)$  contradicts Lemma 2.7, as there is no vertex of  $C(T)$  in  $H \setminus \{w', x'\}$ . This completes the proof that either  $u' \in A$  or  $v' \in B$  holds.

Finally, suppose that both  $u' \in A$  and  $v' \in B$  hold. The path  $P' = P[u', v']$  has odd length and, since there is no edge between  $A$  and  $B$ , this length is at least 3. Put  $P' = u'u'' \dots v''v'$ . By the induction hypothesis applied to  $P'$ , one of  $u'' \in A$  or  $v'' \in B$  holds; but this contradicts the fact that  $A$  and  $B$  are cliques and  $P$  is chordless. So exactly one of  $u' \in A$  and  $v' \in B$  holds. This completes the proof of the lemma.  $\square$

We continue with the proof of Theorem 1.3. Define a relation  $<_A$  on  $A$  by setting  $u <_A u'$  if and only if there exists an odd chordless path from  $u$  to a vertex of  $B$  such that  $u'$  is the second vertex of that path.

**Lemma 3.3.** *The relation  $<_A$  is antisymmetric.*

**Proof.** Suppose on the contrary that there are vertices  $u, v \in A$ , such that  $u <_A v$  and  $v <_A u$ . So there exists an odd chordless path  $P_u = u_0 \dots u_p$  such that  $u = u_0, v = u_1, u_p \in B, p \geq 3, p$  odd, and there exists an odd chordless path  $P_v = v_0 \dots v_q$  such that  $v = v_0, u = v_1, v_q \in B, q \geq 3, q$  odd. Possibly  $u_p = v_q$ , and otherwise  $u_p$  sees  $v_q$  since  $B$  is a clique. We observe that

$$\text{No vertex of } P_u \cup P_v \setminus \{u, v\} \text{ is in } A, \quad (2)$$

because such a vertex misses one of  $u, v$  and  $A$  is a clique. Also,

$$\text{No vertex of } P_u \cup P_v \setminus \{u_p, v_q\} \text{ is in } B; \quad (3)$$

indeed, since  $B$  is a clique, a vertex of  $(P_u \cup P_v \setminus \{u_p, v_q\}) \cap B$  can be only  $u_{p-1}$  or  $v_{q-1}$ . However, if  $u_{p-1}$  is in  $B$  then  $P_u[u_1, u_{p-1}]$  is an odd chordless path from  $A$  to  $B$ , and Lemma 3.2 implies either  $u_2 \in A$  (but this contradicts (2)) or  $u_{p-2} \in B$  (but this contradicts that  $B$  is a clique). The case  $v_{q-1} \in B$  is similar. So (3) is established.

Let  $r$  be the smallest integer such that a vertex  $u_r \in P_u \setminus \{u_0, u_1\}$  has a neighbor in  $P_v$ , and let  $s$  be the smallest integer such that  $u_r v_s$  is an edge, with  $2 \leq s \leq q$ . Such integers exist since  $v_q$  itself has a neighbor in  $P_u$ . Note that  $u_r \notin P_v$  and that the vertices  $u_1, \dots, u_r, v_1, \dots, v_s$  induce a hole  $H$ , so  $r, s$  have the same parity, and  $u_r, v_s$  are different and adjacent.

Now we claim that we can assume that

$$\begin{aligned} &\text{Either (a) } r = p \text{ and } s = q, \text{ or (b) } r < p, s < q \text{ and } P_u[u_{r+1}, u_p] \\ &= P_v[v_{s+1}, v_q]. \end{aligned} \quad (4)$$

To prove this, let  $t$  be the largest integer such that  $u_r v_t$  is an edge, with  $2 \leq s \leq t \leq q$ .

If  $t - s$  is even then  $P_u[u_1, u_r] \cup P_v[v_t, v_q]$  is an odd chordless path from  $A$  to  $B$ . Its second vertex is  $u_2$ , and its penultimate vertex  $w$  is either  $v_{q-1}$  (if  $t < q$ ) or  $u_r$  (if  $t = q$ ). By Lemma 3.2 applied to that path, we have either  $u_2 \in A$  (but this contradicts (2)) or  $w \in B$ . By (3) the latter is possible only if  $w = u_r = u_p$  (so  $t = q$ ); in that case  $P_v[v_1, v_s] \cup \{u_p\}$  is an odd chordless path from  $A$  to  $B$ , so, by Lemma 3.2, we must have either  $v_2 \in A$  (but this contradicts (2)) or  $v_s \in B$ ; this is possible only if  $v_s = v_q$ . Thus we obtain case (a) of (4).

If  $t - s$  is odd and  $t \geq s + 3$  then  $P_v[v_1, v_s] \cup \{u_r\} \cup P_v[v_t, v_q]$  is an odd chordless path from  $A$  to  $B$ . Its second vertex is  $v_2$ , and its penultimate vertex  $w$  is either  $v_{q-1}$  (if  $t < q$ ) or  $u_r$  (if  $t = q$ ). By Lemma 3.2 applied to that path, we must have either  $v_2 \in A$  (but this contradicts (2)) or  $w \in B$ . By (3) the latter is possible only if  $w = u_r = u_p$  (so  $t = q$ ), thus  $r, t$  are odd, but this is impossible because  $t - s$  is odd and  $r - s$  is even.

The remaining possibility is when  $t = s + 1$ . Then the path  $P_u[u_0, u_r] \cup P_v[v_{s+1}, v_q]$  is odd and chordless, so it can play the role of  $P_u$ , and we are in case (b). So (4) is proved.

Let  $P$  be a chordless path defined as follows. In case (a), set  $P = Z[z_2, z_n]$ . In case (b), set  $P$  to be a chordless  $(z_2, u_{r+1})$ -path contained in  $Z[z_2, z_n] \cup P_u[u_{r+1}, u_p]$  (which induces a connected subgraph of  $G$ ). We observe that Lemma 2.7 can be applied to the triple  $(H, P, \{z_1\})$ , indeed:  $H$  is a hole;  $P$  is a chordless path;  $\{z_1\}$  induces a co-connected subgraph disjoint from  $H \cup P$ ;  $H \cup P$  induces a connected subgraph; every vertex  $z \in P$  misses both  $u_1, v_1$  (when  $z \in Z[z_2, z_n]$  this is because  $u_1, v_1 \in A$ , when  $z \in P_u[u_{r+1}, u_p]$  this is because  $P_u, P_v$  are chordless paths and  $r, s \geq 2$ ); and  $u_1, v_1, z_2$  are in  $C(\{z_1\})$ . Now Lemma 2.7 implies that some vertex of  $H \setminus \{u_1, v_1\}$  is in  $C(\{z_1\}) = N(z_1)$ . So  $z_1$  has at least 3 neighbors in  $H$ . Call  $z_1$ -segment any subpath of  $H$ , of length at least 1, whose endvertices are adjacent to  $z_1$  and whose internal vertices are not. The conclusion of this paragraph is that  $H$  is (edge-wise) partitioned into at least three  $z_1$ -segments.

Now let us consider the vertices of  $(H \setminus \{u_1, v_1\}) \cap C(T)$ . In case (a), both  $u_r, v_s$  are in  $C(T)$ . In case (b), we can apply Lemma 2.6 to the hole  $H$ , the path  $P_u[u_{r+1}, u_p]$  and the set  $T$ , with respect to the edges  $u_1 v_1$  and  $u_r v_s$ ; Lemma 2.6 implies that at least one of  $u_r, v_s$  is in  $C(T)$ . Thus, in either case (a) or (b),  $H$  contains at least three vertices of  $C(T)$ . By Lemma 2.3, we can conclude that  $H$  has an even number of  $T$ -edges.

Observe that  $u_1 v_1$  is a  $z_1$ -segment that contains one  $T$ -edge. It follows from this and the conclusion of the preceding two paragraphs that there exists a  $z_1$ -segment  $Q$  of  $H$ , different from  $u_1 v_1$ , that contains an odd number of  $T$ -edges. Call  $x, y$  the endvertices of  $Q$ , and call  $x', y'$ , respectively, the first and last vertex of  $Q \cap C(T)$ , so that  $u_1, x, x', y', y, v_1$  lie in this order along  $H$ . Since  $H$  has at least three  $z_1$ -segments, we have either  $x \neq u_1$  or  $y \neq v_1$ , so let us assume up to symmetry that  $y \neq v_1$ .

If  $Q$  has length 1, we have  $Q = xy = x'y'$ , so  $y \in N(z_1) \cap C(T)$ ; Lemma 3.1 implies  $y \in A \cup B \cup C(T \cup A \cup B)$ ; but  $y \in A \cup C(T \cup A \cup B)$  is impossible because  $y$  misses  $u_1$ , and  $y \in B$  is impossible because  $y \in N(z_1)$ . So  $Q$  has length at least 2. Then  $Q \cup \{z_1\}$  induces a hole  $H_1$ , and Lemma 2.3 applied to the pair  $H_1, T$  implies that  $Q[x', y']$  has length 1.

Suppose  $x \neq u_1$ . Then Lemma 2.7, applied to the triple  $(H_1, u_1, T)$ , implies that  $u_1$  sees  $x'$ , so  $x = x' = u_2$ . But then  $u_2$  is a vertex of  $N(z_1) \cap C(T)$ , which contradicts Lemma 3.1. So  $x = u_1$ , which implies  $x' = u_1$  and  $y' = u_2$ . Consider the path  $P_2 = Z[z_2, z_n] + v_q$ . We can apply Lemma 2.7 to the triple  $(H_1, P_2, T)$ , which implies that a vertex of  $P_2$  sees one of  $u_1, u_2$ . However, no vertex of  $P_2$  can see  $u_1$ , by the definition of  $A$  and  $P_u$ , and no vertex of  $P_2 \setminus v_q$  can see  $u_2$ , for otherwise we have  $u_2 \in N(Z) \cap C(T)$ , which contradicts Lemma 3.1. Thus it must be that  $v_q$  sees  $u_2$ ; this means that  $r = 2$  and either  $s = q - 1$  or  $s = q$ . In fact we cannot have  $s = q$ , for that would imply  $\{u_2, v_q\} \subseteq B$  and  $u_1 u_2$  would be an edge between  $A$  and  $B$ . So we have  $s = q - 1$ . Note that  $v_s$  is in  $Q$  and  $Q[v_s, y]$  has even length since  $H_1$  is an even hole. But then  $Q[v_s, y] \cup \{u_1, z_1, v_q\}$  induces a chordless odd path, of length at least three, whose endvertices  $u_1, v_q$  are  $T$ -complete and whose interior vertices are not, contradicting Lemma 2.2. This completes the proof of Lemma 3.3.  $\square$

**Lemma 3.4.** *The relation  $<_A$  is transitive.*

**Proof.** Let  $u, v, w$  be three vertices of  $A$  such that  $u <_A v <_A w$ . Since  $v <_A w$ , there exists an odd chordless path  $P = v_0 v_1 \cdots v_q$  with  $v_0 = v, v_1 = w, v_q \in B, q$  odd,  $q \geq 3$ .

If  $u$  has no neighbor along  $P[v_2, v_q]$  then  $\{u\} \cup P[v_1, v_q]$  induces an odd chordless path to  $B$ , implying  $u <_A w$  as desired. Now assume that  $u$  has a neighbor  $v_i$  along  $P[v_2, v_q]$ , and let  $i$  be the largest such integer ( $2 \leq i \leq q$ ). We have  $i < q$  as there is no edge between  $A$  and  $B$ .

If  $i$  is odd ( $3 \leq i \leq q - 2$ ), then  $\{u\} \cup P[v_i, v_q]$  is an odd chordless path with  $u \in A$  and  $v_q \in B$ ; applying Lemma 3.2 to this path, we have either  $v_i \in A$  or  $v_{q-1} \in B$ . The former is impossible because  $A$  is a clique; so  $v_{q-1} \in B$ . But then  $\{v_0, u\} \cup P[v_i, v_{q-1}]$  induces an odd chordless path to a vertex in  $B$ , which implies  $v <_A u$ , contradicting Lemma 3.3.

If  $i$  is even ( $2 \leq i \leq q - 1$ ), then  $\{v_0, u\} \cup P[v_i, v_q]$  is an odd chordless path to a vertex in  $B$ , again implying  $v <_A u$  and contradicting Lemma 3.3. This completes the proof of the lemma.  $\square$

**Remark 3.** Determining the orders  $<_A$  and  $<_B$  and their maximal elements can be done in polynomial time. To do this, for any three vertices  $u, v \in A, w \in B$ , we look for a chordless  $(v, w)$ -path in  $G \setminus [(B \setminus \{w\}) \cup (N(u) \setminus \{v\})]$ . If such a path exists it must be even (by Lemma 3.2) and its existence implies  $u <_A v$ . If no such path exists for any  $w \in B$ , we have  $u \not<_A v$  again by Lemma 3.2. For given  $u, v \in A, w \in B$ , looking for such a path takes time  $O(|E|)$ , so, since  $A, B$  may both have size  $O(|V|)$ , the determination of  $<_A$  and  $<_B$  takes time  $O(|V|^3|E|)$ .

**Lemma 3.5.** *Let  $a$  be any maximal vertex of  $(A, <_A)$  and  $b$  be any maximal vertex of  $(B, <_B)$ . Then  $\{a, b\}$  is a special even pair of  $G$ .*

**Proof.** Suppose that there exists an odd chordless  $(a, b)$ -path  $Q = aa' \cdots b'b$ . This path has length at least 3 because there is no edge between  $A$  and  $B$ . Lemma 3.2 implies either  $a' \in A$  (so  $a <_A a'$ ) or  $b' \in B$  (so  $b <_B b'$ ), and in either case the choice of  $a$  or  $b$  is contradicted. So  $\{a, b\}$  is an even pair.

Now suppose that there exists a proper  $(a, b)$ -snake  $S$  (with the same notation as in the definition of snakes). We may assume that  $S_1$  has length at least 1, and we call  $a_1$  the neighbor of  $a'$  along  $S_1$ . Since no vertex of  $S$  sees both  $a, b$ , no vertex of  $S$  is in  $T$ . Note that  $S_1 \cup S_3 \cup S_2$  and  $S_1 \cup S_4 \cup S_2$  induce even paths since  $\{a, b\}$  is an even pair. Call  $H$  the hole induced by  $S_3 \cup S_4$ .

We claim that

$$\text{No vertex of } S \setminus \{a, b\} \text{ is in } N(Z) \cap C(T). \quad (5)$$

For suppose that there is a vertex  $u$  of  $N(Z) \cap C(T)$  in  $S \setminus \{a, b\}$ . By Lemma 3.1,  $u \in A \cup B \cup C(T \cup A \cup B)$ . But  $u \in C(T \cup A \cup B)$  is impossible because no vertex of  $S$  sees both  $a, b$ . Therefore  $u \in A \cup B$ . If  $u \in A$ ,  $u$  must be the neighbor of  $a$  along  $S_1$  (since  $A$  is a clique); then  $(S_1 \setminus \{a\}) \cup S_3 \cup S_2$  induces an odd chordless path  $P_u$  from  $u \in A$  to  $b$  (recall that  $S_1 \cup S_3 \cup S_2$  is an even chordless path); since the neighbor of  $u$  along  $P_u$  is not in  $A$  (because  $A$  is a clique), Lemma 3.2 implies that the neighbor of  $b$  along  $P_u$  is in  $B$ ; but this means that  $b$  is not maximal in  $(B, <_B)$ , a contradiction. If  $u \in B$ ,  $u$  must be either the neighbor of  $b$  along  $S_2$  (if  $S_2$  has length at least 1) or one of  $c', d'$  (if  $S_2$  has length 0), but in either case, an argument similar to the case when  $u \in A$  holds. Thus (5) is proved.

By Lemma 2.5 applied to  $S, T$ , we know that

$$\text{At least two of } a', c, d \text{ and two of } b', c', d' \text{ are in } C(T). \quad (6)$$

Since  $a \neq a'$  (but  $b = b'$  is possible), Facts (5) and (6) imply

If one of  $a', c, d$  is in  $N(Z)$ , then it is in  $N(Z) \setminus C(T)$  and the other two are in  $C(T) \setminus N(Z)$ .

If one of  $c', d'$  is in  $N(Z)$ , then it is in  $N(Z) \setminus C(T)$  and the other is in  $C(T) \setminus N(Z)$ , and  $b' \in C(T)$ .

(7)

Moreover,  $a' \notin Z$  for otherwise one of  $c, d$  would be in  $N(Z) \cap C(T)$ .

Now we define a path  $P$  as follows: let  $a''$  be the vertex of  $N(Z)$  closest to  $a_1$  along  $S_1 \setminus \{a'\}$ , and let  $b''$  be the vertex of  $N(Z)$  closest to  $b'$  along  $S_2$  (vertices  $a'', b''$  exist because of  $a, b$ ). Pick  $z_i \in Z \cap N(a'')$  and  $z_j \in Z \cap N(b'')$  such that the path  $Z[z_i, z_j]$  is as short as possible (each of  $i < j, i = j, i > j$  is possible). Put  $P = S_1[a_1, a''] \cup Z[z_i, z_j] \cup S_2[b'', b']$ ; so  $P$  is a chordless  $(a_1, b')$ -path. By Lemma 2.7 applied to the triple  $(H, P, \{a'\})$ , some vertex  $z$  of  $P$  sees one of  $c, d$ ; we can pick  $z$  closest to  $a_1$  along  $P$  and assume up to symmetry that  $z$  sees  $c$ . The definition of  $S$  implies  $z \notin S_1[a_1, a''] \cup S_2$ , so  $z \in Z[z_i, z_j]$ . By (7), we have  $c \in N(z) \setminus C(T)$  and  $a', d \in C(T) \setminus N(Z)$ . Thus  $a'$  has no neighbor along  $P \setminus \{a_1\}$ , for such a neighbor could only be in  $Z[z_i, z_j]$  (by the definition of  $S$ ), and then we would have  $a' \in N(Z)$ , a contradiction. In other words,

$$P \cup \{a'\} \text{ is a chordless path.} \quad (8)$$



By Lemma 2.7 applied to the triple  $(H, P[z, b'] \setminus \{b'\}, \{b'\})$ , some vertex  $y$  of the path  $P[z, b'] \setminus \{b'\}$  sees one of  $c', d'$ . By the definition of  $S$  and  $P$ , we have  $y \in P[z, z_j] = Z[z, z_j]$ . Thus, and by (7), we know that

$$\begin{aligned} &\text{Exactly one of } c', d' \text{ is in } C(T) \setminus N(Z), \text{ the other is in } N(Z) \setminus C(T), \\ &\text{and } b' \in C(T). \end{aligned} \quad (9)$$

We note that

$$cc' \notin E. \quad (10)$$

For suppose  $cc' \in E$ . If  $c' \in C(T)$ , then  $a'cc'$  is an outer path of length two, a contradiction. If  $c' \notin C(T)$ , then, by (9),  $a'cc'b'$  is an odd outer path, a contradiction. So (10) holds.

Call  $H_1$  the cycle induced by  $P[a_1, z] \cup \{a', c\}$ , which has length at least 4. By (8) and by the choice of  $z$ ,  $H_1$  is a hole. Call  $S'_4$  the path  $S_4 \cup \{c'\}$ . We claim that

$$\text{There is no edge between } P[a_1, z] \text{ and } S'_4. \quad (11)$$

For suppose that there is an edge  $xw$  with  $x \in P[a_1, z]$  and  $w \in S'_4$ . We have  $w \neq d$  since  $d \notin N(Z)$  and  $d$  has no neighbor on  $S_1 \setminus \{a'\}$ . No vertex of  $S'_4 \setminus \{d\}$  is adjacent to  $a'$  or  $c$  by the definition of  $S$  and by (10). But then the triple  $(H_1, S'_4 \setminus \{d\}, \{d\})$  violates Lemma 2.7. So (11) holds.

We know that some vertex  $z'$  of  $P[z, z_j]$  has a neighbor  $\delta$  in  $S'_4$  (because of  $y$ ), so we can pick  $z'$  closest to  $z$  along  $P[z, z_j] = Z[z, z_j]$  and pick  $\delta \in S'_4 \cap N(z')$  closest to  $d$  along  $S'_4$ . Thus  $P[z, z'] \cup S'_4[\delta, d]$  is a chordless path. We have  $\delta \neq d$  since  $z' \in Z$  and  $d \notin N(Z)$ . Let us consider the cycle  $H_2$  induced by  $P[a_1, z'] \cup S'_4[\delta, d] \cup \{a'\}$ , which has length at least 4. Suppose that  $H_2$  has a chord. The definition of  $S$ ,  $P$ ,  $z'$ ,  $\delta$  and the fact that  $P \cup \{a'\}$  and  $P[z, z'] \cup S'_4[\delta, d]$  are chordless imply that the only possible chords in  $H_2$  are of the type  $wx$  with  $w \in S_4$  and  $x \in Z[z_i, z]$ . But this is forbidden by (11). So  $H_2$  is an even hole. Now, along  $H_2$ , vertex  $c$  has three neighbors  $a', d, z$ ; hence, by Lemma 2.3,  $H_2$  has an even number of  $c$ -edges (edges whose two endvertices are neighbors of  $c$ ). One of these is  $a'd$ . Obviously there is no  $c$ -edge along  $S'_4[d, \delta]$ . Also  $\delta z'$  cannot be a  $c$ -edge, for that would imply  $\delta = c'$  and  $cc' \in E$ , contradicting (10). Thus all the  $c$ -edges of  $H_2$  different from  $a'd$  (and there is an odd number of these) lie in  $P[z, z']$ . Call  $z''$  the neighbor of  $c$  closest to  $z'$  along  $P[z, z']$ , so that all the  $c$ -edges of  $H_2$  different from  $a'd$  lie in  $P[z, z'']$ . By Lemma 2.2 applied to  $P[z, z'']$  and  $\{c\}$ , we obtain that:

$$P[z, z''] \text{ has odd length.} \quad (12)$$

By Lemma 2.3, the number of  $T$ -edges in  $H_1$  is either 1 or even. Suppose it is 1; so the vertices of  $H_1 \cap C(T)$  are  $a'$  and  $a_1$  since  $c \notin C(T)$ . By Lemma 2.7 applied to the triple  $(H_1, P[b', z] \setminus \{z\}, T)$ , some vertex of  $P[b', z] \setminus \{z\}$  sees one of  $a', a_1$ ; but this contradicts (8). So:

$$H_1 \text{ has an even number of } T\text{-edges.} \quad (13)$$

In view of (9), let us call  $\gamma'$  whichever of  $c', d'$  is in  $C(T)$  and define a path  $S''_4$  as follows. If  $\delta = c'$  (so  $\gamma' = d'$ ), we put  $S''_4 = c'd'$ . If  $\delta \in S_4$  then let  $\delta'$  be the neighbor of  $z'$  along

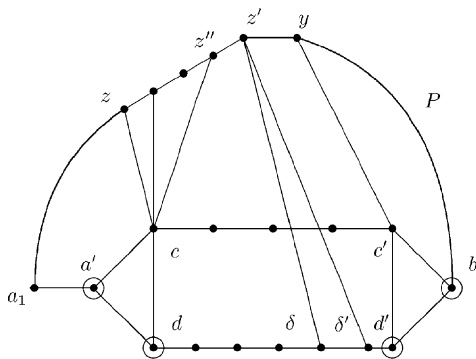


Fig. 3. Lemma 3.5 in case  $d' \in C(T)$ . Circled vertices are in  $C(T)$ .

$S_4$  that is closest to  $d'$ , and put  $S_4'' = S_4'[\delta', \gamma']$ . Now consider the path  $Q_1$  induced by  $\{a', c\} \cup P[z'', z'] \cup S_4''$  and the path  $Q_2$  induced by  $\{a'\} \cup P[a_1, z'] \cup S_4''$ . These paths are chordless by (8), (10), (11) and the definition of  $z', z'', \delta'$ . Their endvertices are  $a'$  and  $\gamma'$ , which are both in  $C(T)$ . Since  $H_1$  is an even hole and  $P[z, z'']$  has odd length, the paths  $Q_1, Q_2$  have lengths of different parity. The numbers of  $T$ -edges in  $Q_1$  and in  $Q_2$  have the same parity, because  $H_1$  has an even number of  $T$ -edges and  $cz$  is not a  $T$ -edge (as  $c \notin C(T)$ ), and because  $P[z, z'']$  contains no  $T$ -edge (as  $P[z, z''] \subseteq Z$ ). Thus one of  $Q_1, Q_2$  violates Lemma 2.2, a contradiction.  $\square$

The conclusion of Lemma 3.5 completes the proof of Theorem 1.3.  $\square$

## 4. Some consequences

### 4.1. Complexity and optimal colorings

Along the proof of Theorem 1.3 we made three remarks on the polynomial complexity of finding various sets, paths or orders related to that proof. In total these remarks show that the proof of Theorem 1.3 is a polynomial-time algorithm which, given any graph  $G = (V, E)$  that is not a clique, either returns a special even pair of  $G$  or answers that  $G$  contains an odd hole, an antihole of length at least 5 or a prism. We note that Case 1 of the proof leads to an iterative call of the algorithm on a subgraph of  $G$ ; this may happen  $O(|V|)$  time. On the other hand, Case 2 does not lead to iterating and therefore happens only once during the execution of the whole algorithm. Thus, a rough estimate of the complexity of this algorithm is  $O(|V|^3|E|)$ .

As suggested in the Introduction, we can use this algorithm to color every graph  $G$  in  $\mathcal{A}$  using no more than  $\omega(G)$  colors. If  $G$  is not a disjoint union of cliques, we use the above algorithm to get a special even pair  $\{x, y\}$  of  $G$ , then iterate the procedure with the graph  $G/xy$ . If  $G$  is a disjoint union of cliques an  $\omega(G)$ -coloring can be produced trivially. As there are at most  $O(|V|)$  iterations, this coloring algorithm has time complexity  $O(|V|^4|E|)$ .

## 4.2. Subclasses of $\mathcal{A}$

The class  $\mathcal{A}$  contains several families of perfect graphs for which the existence of an even pair was already proved, in a specific way for each such family [13–15,18,23]; we will not recall the definition of all these families here, see the survey [10]. Let us, however make a few remarks about two such families.

1. A graph  $G$  is *perfectly orderable* if it admits a *perfect ordering*, that is, an ordering of its vertices such that, for every induced subgraph  $G'$  of  $G$ , using the greedy coloring method on the vertices of  $G'$  along the induced ordering produces an optimal coloring of  $G'$ . Chvátal [8] introduced perfectly orderable graphs and proved that they are perfect. Hertz and de Werra [15] showed that every perfectly orderable graph  $G$  is perfectly contractile by proving that if  $G$  is not a clique it has an even pair  $\{x, y\}$  such that  $G/xy$  is also perfectly orderable. Their proof assumes that a perfect ordering for  $G$  is given. However, determining if a graph is perfectly orderable is an NP-complete problem [19], so there is probably no efficient way to find an even pair in a perfectly orderable graph using that method if no perfect ordering is given. Our result bypasses this difficulty. A drawback is that if  $G$  is perfectly orderable, not a clique, and  $\{a, b\}$  is the even pair produced by our algorithm, we cannot certify that  $G/ab$  is also perfectly orderable, only that it is in  $\mathcal{A}$ .
2. A graph  $G$  is *weakly triangulated* if  $G$  and  $\overline{G}$  contain no hole of length at least 5. Hayward [12] proved that weakly triangulated graphs are perfect, and later Hayward et al. [13] proved that they are perfectly contractile using the following definition and theorem. A *2-pair* is a pair of vertices  $\{u, v\}$  such that all chordless  $(u, v)$ -paths have length 2.

**Theorem 4.1** (Hayward et al. [13]). *If  $G$  is a weakly triangulated graph and not a clique, then:*

1.  $G$  has a 2-pair;
2. For any 2-pair  $\{a, b\}$  of  $G$ ,  $G/ab$  is weakly triangulated.

We show here an alternate proof of the first item of Theorem 4.1. If  $G$  is a disjoint union of cliques then any two non-adjacent vertices form a 2-pair. If  $G$  is not a disjoint union of cliques, we can mimick the proof of Theorem 1.3: consider any maximal interesting set  $T$ ; by Lemma 2.8, there is no outer path with respect to  $T$ , so we are in Case 1; in Case 1, we can assume that the induction hypothesis provides a 2-pair of  $G[C(T)]$ ; then the same arguments as in Case 1 imply that it is also a 2-pair of  $G$ . In other words, when the input graph  $G$  is weakly triangulated, our algorithm produces a 2-pair of  $G$ .  $\square$

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