

## 1 Notes

- $V(X)$  – vertices of structure  $X$ . Will be written as  $X$  when obvious.
- $a - b$ , when  $a$  and  $b$  are nodes –  $a$  and  $b$  are neighbors.
- $a \cdots b$ , when  $a$  and  $b$  are nodes –  $a$  and  $b$  are not neighbors.
- $a - X$ , when  $a$  is a node and  $X$  is a set of nodes –  $a$  has a neighbor in  $X$ .
- $a \cdots X$ , when  $a$  is a node and  $X$  is a set of nodes –  $a$  has a nonneighbor in  $X$ .
- $a \blacktriangleleft X$ , when  $a$  is a node and  $X$  is a set of nodes –  $a$  is complete to  $X$ .
- $a \ntriangleleft X$ , when  $a$  is a node and  $X$  is a set of nodes –  $a$  is anticomplete to  $X$ .
- $X \blacksquare Y$ , when  $X$  and  $Y$  are set of nodes –  $X$  is complete to  $Y$ .
- $X \not\sqsupset Y$ , when  $X$  and  $Y$  are set of nodes –  $X$  is anticomplete to  $Y$ .
- $[n] = \{1, \dots, n\}$ .
- $L(BS(K_4))$  – a line-graph of a bipartite subdivision of  $K_4$ .
- $a \leftarrow b$  – let  $a$  be equal  $b$ .
- $a : \in X$  – let  $a$  be equal to any element of  $X$
- $a \underline{\vee} b$  –  $a$  xor  $b$

## 2 Algorithms

**COLOR-GOOD-PARTITION**( $G, (K_1, K_2, K_3, L, R), c_1, c_2$ )

**Input:**  $G$  – square-free, Berge graph  
 $(K_1, K_2, K_3, L, R)$  – good partition  
 $c_1, c_2$  – colorings of  $G \setminus R$  and  $G \setminus L$  (possibly NULL)

**Output:**  $\omega(G)$ -coloring of  $G$

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1  $G_1 \leftarrow G \setminus R$ 
2  $G_2 \leftarrow G \setminus L$ 
3 if  $c_1, c_2 = \text{NULL}$  then
4    $c_1 \leftarrow \text{COLOR-GRAPH}(G_1)$ 
5    $c_2 \leftarrow \text{COLOR-GRAPH}(G_2)$ 
6 foreach  $u \in K_1 \cup K_2$  do
7   relabel  $c_2$ , so that  $c_1(u) = c_2(u)$ 
8  $B \leftarrow \{u \in K_3 : c_1(u) \neq c_2(u)\}$ 
9 if  $B = \emptyset$  then return  $c_1 \cup c_2$ 
10 foreach  $h \in [2]$ , distinct colors  $i, j$  do
11    $G_h^{i,j} \leftarrow$  subgraph induced on  $G_h$  by  $\{v \in G_h : c_h(v) \in \{i, j\}\}$ 
12 foreach  $u \in K_3$  do
13    $C_h^{i,j}(u) \leftarrow$  component of  $G_h^{i,j}$  containing  $u$ 
    ASSERT:  $C_h^{c_1(u), c_2(u)}(u) \cap K_2 = \emptyset$ 
14 if  $\exists u \in B, h \in [2] : C_h^{c_1(u), c_2(u)}(u) \cap K_1 = \emptyset$  then
15    $c'_1 \leftarrow c_1$  with colors  $i$  and  $j$  swapped in  $C_1^{i,j}(u)$ 
    ASSERT:  $c'_1$  and  $c_2$  agree on  $K_1 \cup K_2$ 
    ASSERT:  $\forall u \in K_3 \setminus B : c'_1(u) = c_1(u)$ 
    ASSERT:  $c'_1(u) = j = c_2(u)$ 
16   return COLOR-GOOD-PARTITION( $G, K_1, K_2, K_3, L, R, c'_1, c_2$ )
17 else
18    $w \leftarrow$  vertex of  $B$  with most neighbors in  $K_1$ 
    ASSERT:  $\forall u \in B : N(u) \cap K_1 \subset N(w) \cap K_1$ 
19   relabel  $c_1, c_2$ , so that  $c_1(w) = 1, c_2(w) = 2$ 
20    $P \leftarrow$  chordless path  $w - p_1 - \dots - p_k - a$  in  $C_1^{1,2}(w)$  so that
      $k \geq 1, p_1 \in K_3 \cup L, p_2 \dots p_k \in L, a \in K, c_1(a) \in [2]$ 
21    $Q \leftarrow$  chordless path  $w - q_1 - \dots - q_l - a$  in  $C_2^{1,2}(w)$  so that
      $l \geq 1, q_1 \in K_3 \cup R, q_2 \dots q_l \in R, a \in K, c_2(a) \in [2]$ 
22    $i \leftarrow c_1(a)$ 
23    $j \leftarrow 3 - i$ 
    ASSERT: exactly one of the colors 1 and 2 appears in  $K_1$  (as in
    Lemma 2.2.(3))
    ASSERT:  $|P|$  and  $|Q|$  have different parities
    ASSERT:  $p_1 \in K_3 \vee p_2 \in K_3$  (as in Lemma 2.2.(4))
    ASSERT:  $\nexists y \in K_3 : c_1(y) = 2 \wedge c_2(y) = 1$  (as in Lemma 2.2.(5))
24   if  $p_1 \in K_3$  then
     ASSERT:  $c_2(p_1) \notin [2]$ 
25   relabel  $c_2$ , so that  $c_2(p_1) = 3$ 

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24 0.4// else .4 //  $\nexists u \in B, h \in [2] : C_h^{c_1(u), c_2(u)}(u) \cap K_1 = \emptyset$ 
25 0.4// if .4  $p_1 \in K_3$  0.4then
    ASSERT: color 3 does not appear in  $K_2$ 
    ASSERT: color 3 does not appear in  $K_1$ 
    ASSERT:  $C_2^{j,3}(p_1) \cap K_1 = \emptyset$ 
26  $c'_2 \leftarrow c_2$  with colors  $j$  and 3 swapped in  $C_2^{j,3}(p_1)$ 
    ASSERT:  $j = 2$ 
27 return COLOR-GOOD-PARTITION( $G, K_1, K_2, K_3, L, R, c_1, c'_2$ )
28 else
29   relabel  $c_1$ , so that  $c_1(q_1) = 3$ 
30   if 3 does not appear in  $K_1$  then
31     ASSERT:  $C_1^{j,3}(q_1) \cap K_1 = \emptyset$ 
32     ASSERT:  $j = 1$ 
33      $c'_1 \leftarrow c_1$  with colors  $j$  and 3 swapped in  $C_1^{j,3}(q_1)$ 
34     return COLOR-GOOD-PARTITION( $G, K_1, K_2, K_3, L, R, c'_1,$ 
35        $c_2$ )
36   else
37     ASSERT:  $q_1 \notin \{a, a_3\}$ 
38     ASSERT:  $C_1^{i,3}(q_1) \cap K_1 = \emptyset$ 
39     ASSERT:  $i = 1$ 
40      $c'_1 \leftarrow c_1$  with colors  $i$  and 3 swapped in  $C_1^{i,3}(q_1)$ 
41     return COLOR-GOOD-PARTITION( $G, K_1, K_2, K_3, L, R, c'_1,$ 
42        $c_2$ )

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// Lemma 3.3

**GROW-HYPERPRISM**( $G, G, M, F$ )

**Input:**  $G$  – square-free, Berge graph  
 $H = (A_1, \dots, B_3)$  – a hyperprism in  $G$   
 $M$  – the set of major neighbors of  $H$  in  $G$   
 $F$  – a minimal component of  $G \setminus (H \cup M)$  with a set of attachments in  $H$  not local.

**Output:**  $H'$  – a larger hyperprism, or  
 $L$  – a  $L(BS(K_4))$

$X \leftarrow$  set of attachments of  $F$  in  $H$

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1  if  $\exists i : X \cap C_i \neq \emptyset$  then
2      relabel strips of  $H$ , so that  $X \cap C_1 \neq \emptyset$ 
3       $x_1 \in X \cap C_1$ 
4      ASSERT:  $X \cap S_2 \neq \emptyset$ 
5       $x_2 \in X \cap S_2$ 
6       $R_1 \leftarrow$  1-rung of  $H$ , so that  $x_1 \in V(R_1)$ 
7       $R_2 \leftarrow$  2-rung of  $H$ , so that  $x_2 \in V(R_2)$ 
8       $R_3 \leftarrow$  a 3-rung of  $H$ 
9       $\forall i \in [3] : a_i, b_i \leftarrow$  ends of  $R_i$ , so that  $a_i \in A_i, b_i \in B_i$ 
10      $K \leftarrow$  a prism  $(R_1, R_2, R_3)$ 
11     ASSERT: no vertex in  $F$  is major w.r.t.  $K$  (as in SPGT 10.5)
12      $f_1 - \dots - f_n \leftarrow$  a minimal path in  $F$ , so that
13          $f_1 \blacktriangleleft \{a_2, a_3\}$ ,
14          $f_n - R \setminus \{a_1\}$ 
15         there are no other edges between  $\{f_1, \dots, f_n\}$  and  $V(K) \setminus \{a_1\}$ 
16     ASSERT:  $F = \{f_1, \dots, f_n\}$ 
17     ASSERT:  $f_1 \blacktriangleleft A_3$ 
18      $A'_1 \leftarrow A_1 \cup \{f_1\}$ 
19      $C'_1 \leftarrow C_1 \cup \{f_2, \dots, f_n\}$ 
20     return  $H' \leftarrow (A'_1, A_2, \dots, B_3, C'_1, C_2, C_3)$ 
21 else
22     relabel strips of  $H$ , so that there is  $\{x_1 \in A_1, x_2 \in A_2\} \subset X$  that is
23     not local
24     find a path  $x - f_1 - \dots - f_n - x_2$ 
25     ASSERT:  $F = \{f_1, \dots, f_n\}$ 
26     if  $n$  is even and  $H$  is even, or  $n$  is odd and  $H$  is odd then
27         ASSERT:  $f_1 - a_3 \vee f_n - b_3$ 
28         if  $f_1 - a_3$  then
29              $H' \leftarrow$  mirrored  $H$  – every  $A_i$  and  $B_i$  are swapped
30             # TODO: check if  $M$  and  $F$  are OK
31             return GROW-HYPERPRISM( $G, H', M, F$ )
32         else
33             if  $f_n \blacktriangleleft B_2 \cup B_3$  then
34                  $B'_1 \leftarrow B_1 \cup \{f_n\}$ 
35                  $C'_1 \leftarrow C_1 \cup \{f_1, \dots, f_{n-1}\}$ 

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21 0.4// else .4//  $\forall_{i \in [3]} X \cap C_i = \emptyset$ 
22 0.4// if .4n is even and  $H$  is even, or  $n$  is odd and  $H$  is odd
23 0.4then
24   0.4// else .4//  $f_n - b_3$ 
25   0.4// if .4 $f_n \blacktriangleleft B_2 \cup B_3$  0.4then
26     
$$\text{return } H' \leftarrow \begin{pmatrix} A_1 & C'_1 & B'_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{pmatrix}$$

27   else
28      $\forall_{i \in [3]} : A'_i \leftarrow \text{neighbors of } f_1 \text{ in } A_i$ 
29      $\forall_{i \in [3]} : A''_i \leftarrow A_i \setminus A'_i$ 
30      $\forall_{i \in [3]} : B'_i \leftarrow \text{neighbors of } f_n \text{ in } B_i$ 
31      $\forall_{i \in [3]} : B''_i \leftarrow B_i \setminus B'_i$ 
32     ASSERT: Every  $i$ -rung is between  $A'_i$  and  $B'_i$  or  $A''_i$  and  $B''_i$ 
33      $\forall_{i \in [3]} : C'_i \leftarrow \text{union of interiors of } i\text{-rings between } A'_i \text{ and } B'_i$ 
34      $\forall_{i \in [3]} : C''_i \leftarrow \text{union of interiors of } i\text{-rings between } A''_i \text{ and } B''_i$ 
35     ASSERT:  $C_i = C'_i \cup C''_i, C'_i \cap C''_i = \emptyset$ 
36     ASSERT:  $A'_i \cup C'_i \sqsupset C''_i \cup B''_i, A''_i \cup C''_i \sqsupset C_i \cup B_i$ 
37     ASSERT:  $A'_i \blacksquare A''_i, B'_i \blacksquare B''_i$ 
38     ASSERT:  $A'_1, A'_2, A'_3, A''_3 \neq \emptyset$ 
39     
$$H' \leftarrow \begin{pmatrix} A'_1 & C'_1 & B'_1 \\ A'_2 \cup A'_3 & C'_2 \cup C'_3 & B'_2 \cup B'_3 \\ \bigcup_i A''_i \cup \{f_1\} & \bigcup_i C''_i \cup \{f_2, \dots, f_n\} & \bigcup_i B''_i \end{pmatrix}$$

40   return  $H'$ 
41 else
42    $a_1 \leftarrow \text{neighbor of } f_1 \text{ in } A_1$ 
43    $R_1 \leftarrow \text{1-rung with end } a_1$ 
44    $b_1 \leftarrow \text{the other end of } R_1$ 
45    $b_2 \leftarrow \text{neighbor of } f_2 \text{ in } B_2$ 
46    $R_2 \leftarrow \text{2-rung with end } b_2$ 
47    $a_2 \leftarrow \text{the other end of } R_2$ 
48   ASSERT:  $b_1 \in X, a_2 \in X$ 
49   ASSERT:  $(b_1 - f_1 \wedge a_2 - f_n) \vee (b_1 - f_n \wedge a_2 - f_1)$ 
50   if  $f_1 - b_1$  then
51     ASSERT:  $H$  is odd
52      $R_3 \leftarrow \text{any 3-rung with ends } a_3, b_3, \text{ such that}$ 
53      $\{a_3, b_3\} \sqsupset \{f_1, f_n\}$ 
54     return  $V(R_1) \cup V(R_2) \cup V(R_3) \cup \{f_1, \dots, f_n\}$  - a  $L(BS(K_4))$ 
55   # TODO: Is it valid input for part of ALG I?

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0.4 // else .4 //  $\forall_{i \in [3]} X \cap C_i = \emptyset$ 
0.4 // else .4 //  $n$  is odd and  $H$  is even, or  $n$  is even and  $H$  is odd
45   else //  $f_1 - a_2$ 
46      $\forall_{i \in [3]} : A'_i \leftarrow A_i \cap X, A''_i \leftarrow A_i \setminus X$ 
47      $\forall_{i \in [3]} : B'_i \leftarrow B_i \cap X, B''_i \leftarrow B_i \setminus X$ 
48      $\forall_{i \in [3]} : C'_i \leftarrow$  union of  $i$ -rungs between  $A'_i$  and  $B'_i$ 
49      $\forall_{i \in [3]} : C''_i \leftarrow$  union of  $i$ -rungs between  $A''_i$  and  $B''_i$ 
    ASSERT:  $C_i = C'_i \cup C''_i, C'_i \cap C''_i = \emptyset$ 
50   if  $f_1$  is complete to at least two of  $A_i$  then
51     relabel strips of  $H$ , so that  $f_1$  is complete to  $A_1$  and  $A_2$ 
    ASSERT:  $f_n$  is complete to  $B_1$  and  $B_2$ 
    ASSERT:  $n > 1$  (as in SPGT 10.5 OK for odd  $H$ ?)
52     return  $\begin{pmatrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 \cup \{f_1\} & C_3 \cup \{f_2, \dots, f_{n-1}\} & B_3 \cup \{f_n\} \end{pmatrix}$ 
53   else
    ASSERT:  $A'_i \blacksquare A''_i$  # TODO: OK odd  $H$ ?
    ASSERT:  $B'_i \blacksquare B''_i$  # TODO: OK odd  $H$ ?
54   return  $\begin{pmatrix} A'_1 & C'_1 & B'_1 \\ A'_2 \cup A'_3 & C'_2 \cup C'_3 & B'_2 \cup C'_3 \\ \bigcup_i A''_i \cup \{f_1\} & \bigcup_i C''_i \cup \{f_2, \dots, f_{n-1}\} & \bigcup_i B''_i \cup \{f_n\} \end{pmatrix}$ 

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**GOOD-PARTITION-FROM-EVEN-HYPERPRISM**( $G, H, M$ )

**Input:**  $G$  – square-free, Berge graph containing no  $L(BS(K_4))$   
 $H = (A_1, \dots, B_3)$  – maximal even hyperprism in  $G$   
 $M$  – set of major neighbors of  $H$

**Output:** A good partition of  $G$

- 1  $Z \leftarrow \bigcup \{V(C) : C \text{ is a component of } G \setminus \{V(H) \cup M\} \text{ with no attachments in } H\}$
- 2 relabel strips of  $H$ , so that  $M \cup A_1$  and  $M \cup B_1$  are cliques
- 3  $F_1 \leftarrow \bigcup \{V(C) : C \text{ is a component of } G \setminus \{H \cup M \cup Z\} \text{ that attaches to } A_1 \cup B_1 \cup C_1\}$   
 ASSERT:  $M$  is a clique  
 ASSERT:  $M \cup A_i$  is a clique for at least two values of  $i$   
 ASSERT:  $M \cup B_j$  is a clique for at least two values of  $j$
- 4  $K_1 \leftarrow A_1, K_2 \leftarrow M, K_3 \leftarrow B_1$
- 5  $R \leftarrow C_1 \cup F_1 \cup Z$
- 6  $L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$
- 7 **return**  $(K_1, K_2, K_3, L, R)$

**GOOD-PARTITION-FROM-ODD-HYPERPRISM**( $G, H, M$ )

**Input:**  $G$  – square-free, Berge graph containing no  $L(BS(K_4))$   
 $H = (A_1, \dots, B_3)$  – maximal odd hyperprism in  $G$   
 $M$  – set of major neighbors of  $H$

**Output:** A good partition of  $G$

- 1  $Z \leftarrow \bigcup \{V(C) : C \text{ is a component of } G \setminus \{V(H) \cup M\} \text{ with no attachments in } H\}$
- 2 relabel strips of  $H$ , so that  $A_1 \sqsupset B_1$  and  $A_2 \sqsupset B_2$   
 ASSERT:  $C_1 \neq \emptyset, C_2 \neq \emptyset$
- 3  $\forall_{i \in [3]} F_i \leftarrow \bigcup \{V(C) : C \text{ is a component of } G \setminus \{H \cup M \cup Z\} \text{ that attaches to } A_i \cup B_i \cup C_i\}$
- 4  $F_B \leftarrow \bigcup \{V(C) : C \text{ is a component of } G \setminus \{H \cup M \cup Z \cup F_1 \cup F_2 \cup F_3\} \text{ that attaches to } B_1 \cup B_2 \cup B_3\}$   
 # **TODO:**  $F_i, F_A, F_B$  are from algIV, make sure it is correct  
 ASSERT: At least two of  $A_i$  and at least two of  $B_i$  are cliques  
 ASSERT:  $M$  is complete to at least two of  $A_i$  and at least two of  $B_i$   
 ASSERT:  $M$  is a clique  
 ASSERT: For at least two  $i$  :  $A_i \cup M$  is a clique  
 ASSERT: For at least two  $j$  :  $A_j \cup M$  is a clique
- 5 choose  $h$ , so that  $M \cup A_h$  and  $M \cup B_h$  are cliques
- 6 **if**  $h = 1 \vee h = 2$  **then** // # **TODO:** make sure  $h = 2$  is ok
  - 7  $K_1 \leftarrow A_1, K_2 \leftarrow M, K_3 \leftarrow B_1$
  - 8  $R \leftarrow C_1 \cup F_1 \cup Z$
  - 9  $L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$
  - 10 **return**  $(K_1, K_2, K_3, L, R)$
- 11 **else**
  - 12 relabel  $H$  so that  $M \cup A_1$  and  $M \cup B_2$  are cliques
  - 13  $K_1 \leftarrow B_2 \cup B_3, K_2 \leftarrow M, K_3 \leftarrow A_1 \cup A_3$
  - 14  $L \leftarrow B_1 \cup C_1 \cup F_1 \cup F_B$
  - 15  $R \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup L\}$
  - 16 **return**  $(K_1, K_2, K_3, L, R)$

## 2.1 ALG VI

### 2.1.1 Notes

- $T_{uv} := N_u \cap N_v$
- Strip  $S_{uv}$  is *rich* iff  $S_{uv} \setminus T_{uv} \neq \emptyset$

**GOOD-PARTITION-FROM-J-STRIP-SYSTEM**( $G, J, (S, N), M$ )

**Input:**  $G$  – square-free, Berge graph

$J$  – a maximal 3-connected graph with appearance in  $G$

$(S, N)$  – a maximal  $J$ -strip system

$M$  – a set of major vertices w.r.t.  $(S, N)$

**Output:** A good partition of  $G$

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1  $S_{uv}^* \leftarrow S_{uv} \cup (\text{components of } G \setminus V(S, N) \text{ that attach in } S_{uv} \text{ only})$ 
2  $T_{uv} \leftarrow N_u \cap N_v$ 
  ASSERT:  $T_{uv} \blacksquare N_u \setminus N_{uv}, T_{uv} \blacksquare N_v \setminus N_{vu}$ 
  ASSERT:  $M \cup T_{uv}$  is a clique
3 if  $\exists S_{uv}$  – a rich strip in  $(S, N)$  then
4   if  $\exists S_{uv}$  – a rich strip in  $(S, N)$ , such that  $M \cup (N_u \setminus N_{uv})$  and
      $M \cup (N_v \setminus N_{vu})$  are cliques then
5      $K_1 \leftarrow N_u \setminus N_{uv}, K_2 \leftarrow M \cup T_{uv}, K_3 \leftarrow N_v \setminus N_{vu}$ 
6      $L \leftarrow (S_{uv}^* \setminus T_{uv}) \cup (\text{components of } G \setminus V(S, N) \text{ that attach only}$ 
       to  $N_u$  and those that attach only to  $N_v$ )
7      $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$ 
8     return  $(K_1, K_2, K_3, L, R)$ 
9   else
10     $S_{uv} \leftarrow$  a rich strip in  $(S, N)$ , such that  $M \cup (N_u \setminus N_{uv})$  is not a
       clique and  $M \cup (N_v \setminus N_{vu})$  is a clique
11     $K_1 \leftarrow N_{uv} \setminus T_{uv}, K_2 \leftarrow M \cup T_{uv}, K_3 \leftarrow N_v \setminus N_{vu}$ 
12     $R \leftarrow (S_{uv}^* \setminus N_u) \cup (\text{components of } G \setminus V(S, U) \text{ that attach only}$ 
       to  $N_v$ )
13     $L \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup R)$ 
14    return  $(K_1, K_2, K_3, L, R)$ 
15 else
  ASSERT:  $\forall uv \in E(J) : S_{uv} = T_{uv}, S_{uv}$  is a clique
16  $S_{uv} \leftarrow$  any strip
17  $K_1 \leftarrow N_u \setminus S_{uv}, K_2 \leftarrow M, K_3 \leftarrow N_v \setminus S_{uv}$ 
18  $L \leftarrow S_{uv}^* \cup (\text{components of } G \setminus V(S, N) \text{ that attach only to } N_u \text{ and}$ 
   only to  $N_v$ )
19  $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$ 
20 return  $(K_1, K_2, K_3, L, R)$ 

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**GOOD-PARTITION-FROM-SPECIAL-STRIP-SYSTEM**( $G, J, (S, N), M$ )

**Input:**  $G$  – square-free, Berge graph

$(S, N)$  – a special  $K_4$  strip system

$M$  – a set of major vertices w.r.t.  $(S, N)$

**Output:** A good partition of  $G$

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1  $\forall_{i,j \in [4]} O_{ij} \leftarrow$  set of vertices in  $V(G) \setminus V(S, N)$  that are complete to
    $(N_i \cup N_j) \setminus S_{ij}$  and anticomplete to  $V(S, N) \setminus (N_i \cup N_j \cup S_{ij})$ 
2 if  $(N_1 \setminus N_{12}) \cup M \cup O_{12}$  and  $(N_2 \setminus N_{12}) \cup M \cup O_{12}$  are cliques then
3    $K_1 \leftarrow N_1 \setminus N_{12}, K_2 \leftarrow O_{12} \cup M, K_3 \leftarrow N_2 \setminus N_{12}$ 
4    $L \leftarrow$  union of those components of  $G \setminus (K_1 \cup K_2 \cup K_3)$  that contain
   vertices of  $S_{12}$ 
5    $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$ 
6   return  $(K_1, K_2, K_3, L, R)$ 
7 else
8   if  $(N_1 \setminus N_{12}) \cup M \cup O_{12}$  is a clique then
9     relabel 1 and 2 in  $J$  so that  $(N_1 \setminus N_{12}) \cup M \cup O_{12}$  is not a clique
    ASSERT:  $N_{12} \cup M \cup O_{12}$  is a clique
10  if  $N_{21} \cup M \cup O_{12}$  is a clique then
11     $X \leftarrow N_{21}$ 
12  else
13     $X \leftarrow N_2 \setminus N_{21}$ 
14   $K_1 \leftarrow N_{12}, K_2 \leftarrow M \cup O_{12}, K_3 \leftarrow X$ 
15   $L \leftarrow$  component of  $G \setminus (K_1 \cup K_2 \cup K_3)$  that contains  $N_1 \setminus N_{12}$ 
16   $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$ 
17  return  $(K_1, K_2, K_3, L, R)$ 
```

**FIND-SPECIAL-K4-STRIP-SYSTEM**( $G, J, (S, N), m$ )

**Input:**  $G$  – square-free, Berge graph

$J$  – a 3-connected graph with appearance in  $G$

$(S, N)$  – a  $J$ -strip system

$m \in G \setminus (S, N)$  that is major w.r.t. some choice of rungs of  $(S, N)$   
but not w.r.t.  $(S, N)$

**Output:**  $(S', N')$  – A special  $K_4$ -strip system, or

$(S'', N'')$  – a bigger  $J$ -strip system

- 1  $X \leftarrow N(m)$
- 2  $M \leftarrow$  vertices of  $G \setminus V(S, N)$  major w.r.t.  $(S, N)$
- 3  $M^* \leftarrow$  vertices of  $G \setminus V(S, N)$  major w.r.t. some choice of rungs  
ASSERT:  $J = K_4$  (as in SPGT 8.4)
- 4  $V(J) \leftarrow [4]$
- 5  $\forall_{i \neq j \in [4]}$ : choose rungs  $R_{ij}, R'_{ij}$ , forming line graphs  $L(H)$  and  $L(H')$  so  
that  $X$  saturates  $L(H)$  but does not saturate  $L(H')$   
ASSERT:  $R_{ij} \neq R'_{ij} \iff \{i, j\} = [2]$
- 6  $r_{ij}, r_{ji} \leftarrow$  ends of each  $R_{ij}$
- 7  $r'_{ij}, r'_{ji} \leftarrow$  ends of each  $R'_{ij}$
- 8  $\forall_{i \in [4]} T_i \leftarrow \{r_{ij}, j \in [4] \setminus \{i\}\}$
- 9  $\forall_{i \in [4]} T'_i \leftarrow \{r'_{ij}, j \in [4] \setminus \{i\}\}$   
ASSERT:  $X$  has at least two members in each  $T_1, \dots, T_4$   
ASSERT: There is  $T'_i$  that contains at most one member of  $X$   
ASSERT:  $T_3 = T'_3, T_4 = T'_4$
- 10 relabel 1 and 2 in  $J$ , so that  $|X \cap T_1| = 2$  and  $|X \cap T'_1| = 1$   
ASSERT:  $r_{12} \in X, r'_{12} \notin X$   
ASSERT:  $r_{13} \in X \vee r_{14} \in X$
- 11 relabel 3 and 4 in  $J$ , so that  $r_{13} \in X, r_{14} \notin X$   
ASSERT:  $R_{34}$  is even and  $[X \cap V(L(H'))] \setminus V(R_{34}) = \{r_{31}, r_{32}, r_{41}, r_{42}\}$   
(as in 6.3.(3))  
ASSERT:  $R_{14}$  has odd length,  $r_{21} \in X$  (as in 6.3.(4))  
ASSERT:  $R_{12}$  has length 0, every 12-rung has even length (as in 6.3.(5))  
ASSERT:  $R_{24}$  has length 0 and  $R_{23}$  has odd length (as in 6.3.(6))  
ASSERT: Every 34-rung has non-zero even length (as in 6.3.(7))
- 12  $\forall_{i \neq j \in [4]} O_{ij} \leftarrow$  set of vertices that are not major w.r.t  $L(H')$  and are  
complete to  $(T'_i \cup T'_j) \setminus R'_{ij}$   
ASSERT:  $r_{12} = r_{21} \in O_{12}, m \in O_{34}$   
ASSERT:  $O_{34} = M^* \setminus M$
- 13  $(S', N') \leftarrow$  strip system obtained from  $(S, N)$  by replacing  $S_{12}$  with  
 $S_{12} \setminus O_{12}$
- 14 **if**  $\exists_{\text{rung } R}$ : adding  $R$  to  $S'_{12}$  produces enlargement of  $(S, N)$  **then**
- 15     **return**  $(S'', N'')$  – an enlargement of  $(S, N)$
- 16 **else**
- 17     **return**  $(S', N')$  – a special  $K_4$  strip system



**GROWING-J-STRIP**( $G, J, (S, N)$ )

**Input:**  $G$  – square-free, Berge graph  
 $J$  – a 3-connected graph with appearance in  $G$   
 $(S, N)$  – a  $J$ -strip system  
*# TODO:* make sure def of  $J$  is correct

**Output:**  $J'$  and a maximal  $J'$ -strip system, or a special strip system

```

1  $M \leftarrow$  vertices of  $G \setminus V(S, N)$  that are major on some choice of Rungs of  $(S, N)$ 
  # TODO:  $M$  like ALGI
2 if  $\exists m : m$  is not major on some choice of rungs of  $(S, N)$  then
3    $OUT \leftarrow$  FIND-SPECIAL-K4-STRIP-SYSTEM( $G, J, (S, N), m$ )
4   if  $OUT$  is a special strip system then
5     return  $OUT$ 
6   else
7     return GROWING-J-STRIP( $G, OUT$ )
8 else if  $\exists F : F$  is a component of  $G \setminus (V(S, N) \cup M)$ , such that no
   member of  $F$  is major w.r.t.  $(S, N)$  and set of attachments of  $F$  on
    $H(?)$  is not local then
   ASSERT: (as in 6.2, or actually SPGT 8.5)
9    $F \leftarrow$  minimal (component?) with this property
10  if  $\exists v \in V(J) : X \subset \bigcup (S_{uv} : uv \in E(J))$  then
11    # TODO: what is  $X$ ?
12     $x := X \cap S_{uv} \setminus N_v$ , for some  $uv \in E(J)$ 
13    # TODO: brackets?
14     $x' := X \cap S_{u'v}$ , for some  $u'v \in E(J), u' \neq u$ 
15    ASSERT:  $\{x, x'\}$  is not local w.r.t.  $(S, N)$ 
16     $L(H) \leftarrow \forall_{i,j \in E(J)}$  choose  $ij$ -rung  $R_{ij}$ , so that
17     $x \in V(R_{uv}), x' \in V(R_{u'v})$ 
18    ASSERT:  $\{x, x'\}$  is not local w.r.t.  $L(H)$ 
19     $D \leftarrow$  a branch of  $H$  with ends  $d, u$ :
20     $\delta_H(d) \setminus E(D) = (X \cap E(H)) \setminus E(D)$ 
21    # TODO: How can we get  $H$  from  $L(H)$ 
22     $P \leftarrow$  a path with ends  $p_1, p_2$ , so that:
23     $p_1 \blacktriangleleft N_v \setminus N_{vu}$  and no other vertex of  $P$  has neighbors in
24     $N_v \setminus N_{uv}$ 
25     $p_2 - x$  and no other vertex of  $P$  has neighbors in  $S_{uv} \setminus N_v$ 
26     $(S', N') \leftarrow$  add  $p_1$  to  $N_v$  and  $F$  to  $S_{uv}$ 
27    return GROWING-J-STRIP ( $G, J, (S', N')$ )
28  else
29     $K \leftarrow \{uv \in E(J) : X \cap S_{uv} \neq \emptyset\}$ 
30    ASSERT: There are two disjoint edges in  $K$  (as in SPGT
31    8.5.(3))
32     $F$  is a vertex set of a path  $\leftarrow f_1 - \dots - f_n$ 
33    ASSERT: Every choice of rungs is broad

```



```

0.4 // if  $\exists F \dots$  0.4 then
0.4 // else  $\nexists v \in V(J) : X \subset \bigcup (S_{uv} : uv \in E(J))$ 
    ASSERT: every choice of rungs has the same traversal. (Hard to
    assert)
21    $ij \leftarrow$  the traversal edge
22    $A_1 \leftarrow N_i \setminus S_{ij}, A_2 \leftarrow N_j \setminus S_{ij}$ 
    ASSERT:  $X \cap (V(S, N) \setminus S_{ij}) = A_1 \cup A_2$ 
23   if  $n = 1$  then
24      $(S', N') \leftarrow$  add  $f_1$  to  $N_i, N_j, S_{ij}$ 
25     return GROWING-J-STRIP ( $G, J, (S', N')$ )
26   else
27      $x_1 \in A_1, x_2 \in A_2$ , so that  $x_1$  and  $x_2$  are in disjoint strips
    ASSERT:  $x_1 - f_1 \vee x_1 - f_n$ 
28     if  $x_1 - f_n$  then
29       relabel  $f_1 - \dots - f_n$  front to back
30      $(S', N') \leftarrow$  add  $f_1$  to  $N_i$ ,  $f_n$  to  $N_j$  and  $F$  to  $S_{ij}$ 
31     return GROWING-J-STRIP ( $G, J, (S', N')$ )
32 else
33   return  $J, (S, N)$  – a maximal  $J$ -strip

```