1 Notes

- $V(X)$ – vertices of structure X. Will be written as X when obvious.

- $a - b$, when a and b are nodes – a and b are neighbors.

- $a \cdots b$, when a and b are nodes – a and b are not neighbors.

- $a - X$, when a is a node and X is a set of nodes – a has a neighbor in X.

- $a \cdots X$, when a is a node and X is a set of nodes – a has a nonneighbor in X.

- $a \blacktriangleleft X$, when a is a node and X is a set of nodes – a is complete to X.

- $a \ntriangleleft X$, when a is a node and X is a set of nodes – a is anticomplete to X.

- $X \blacksquare Y$, when X and Y are set of nodes – X is complete to Y.

- $X \boxdiagup Y$, when X and Y are set of nodes – X is anticomplete to Y.

- $[n] - \{1, \dots, n\}$.

- $L(BS(K_4))$– a line-graph of a biparite subdivision of K_4.

- $a \leftarrow b$ – let a be equal b.

- $a :\in X$ – let a be equal to any element of X

- $a \veebar b$ – a xor b

2 Algorithms

COLOR-GOOD-PARTITION$(G, K_1, K_2, K_3, L, R, c_1, c_2)$

Input: G – square-free, Berge graph
$\qquad\quad K_1, K_2, K_3, L, R$ – good partition
$\qquad\quad c_1, c_2$ – colorings of $G \setminus R$ and $G \setminus L$ (possibly NULL)

Output: $\omega(G)$-coloring of G

1   $G_1 \leftarrow G \setminus R$
2   $G_2 \leftarrow G \setminus L$
3   **if** $c_1, c_2 =$ NULL **then**
4     $c_1 \leftarrow$ COLOR-GRAPH(G_1)
5     $c_2 \leftarrow$ COLOR-GRAPH(G_2)

6   **foreach** $u \in K_1 \cup K_2$ **do**
7   relabel c_2, so that $c_1(u) = c_2(u)$

8   $B \leftarrow \{u \in K_3 : c_1(u) \neq c_2(u)\}$
9   **if** $B = \emptyset$ **then return** $c_1 \cup c_2$
10   **foreach** $h \in [2]$, distinct colors i, j **do**
11     $G_h^{i,j} \leftarrow$ subgraph induced on G_h by $\{v \in G_h : c_h(v) \in \{1, 2\}\}$

12   **foreach** $u \in K_3$ **do**
13     $C_h^{i,j}(u) \leftarrow$ component of $G_h^{i,j}$ containing u

    ASSERT: $C_h^{c_1(u),c_2(u)}(u) \cap K_2 = \emptyset$

14   **if** $\exists u \in B, h \in [2] : C_h^{c_1(u),c_2(u)}(u) \cap K_1 = \emptyset$ **then**
15     $c_1' \leftarrow c_1$ with colors i and j swapped in $C_1^{i,j}(u)$
     ASSERT: c_1' and c_2 agree on $K_1 \cup K_2$
     ASSERT: $\forall u \in K_3 \setminus B : c_1'(u) = c_1(u)$
     ASSERT: $c_1'(u) = j = c_2(u)$
16   **return** COLOR-GOOD-PARTITION$(G, K_1, K_2, K_3, L, R, c_1', c_2)$

17   **else**
18     $w \leftarrow$ vertex of B with nost neighbors in K_1
     ASSERT: $\forall u \in B : N(u) \cap K_1 \subset N(w) \cap K_1$
19   relabel c_1, c_2, so that $c_1(w) = 1, c_2(w) = 2$
20     $P \leftarrow$ chordless path $w - p_1 - \ldots - p_k - a$ in $C_1^{1,2}(w)$ so that
      $k \geq 1$, $p_1 \in K_3 \cup L$, $p_2 \ldots p_k \in L$, $a \in K$, $c_1(a) \in [2]$
21     $Q \leftarrow$ chordless path $w - q_1 - \ldots - q_l - a$ in $C_2^{1,2}(w)$ so that
      $l \geq 1$, $q_1 \in K_3 \cup R$, $q_2 \ldots q_l \in R$, $a \in K$, $c_2(a) \in [2]$
22     $i \leftarrow c_1(a)$
23     $j \leftarrow 3 - i$
     ASSERT: exactly one of the colors 1 and 2 appears in K_1 (as in
      Lemma 2.2.(3))
     ASSERT: $|P|$ and $|Q|$ have different parities
     ASSERT: $p_1 \in K_3 \vee p_2 \in K_3$ (as in Lemma 2.2.(4))
     ASSERT: $\nexists y \in K_3 : c_1(y) = 2 \wedge c_2(y) = 1$ (as in Lemma 2.2.(5))
24   **if** $p_1 \in K_3$ **then**
      ASSERT: $c_2(p_1) \notin [2]$
25   relabel c_2, so that $c_2(p_1) = 3$

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// **else**   // $\nexists u \in B, h \in [2] : C_h^{c_1(u), c_2(u)}(u) \cap K_1 = \emptyset$

  // **if** $p_1 \in K_3$ **then**

    ASSERT: color 3 does not appear in K_2

    ASSERT: color 3 does not appear in K_1

    ASSERT: $C_2^{j,3}(p_1) \cap K_1 = \emptyset$

26    $c_2' \leftarrow c_2$ with colors j and 3 swapped in $C_2^{j,3}(p_1)$

    ASSERT: $j = 2$

27   **return** Color-Good-Partition$(G, K_1, K_2, K_3, L, R, c_1, c_2')$

28 **else**

29   relabel c_1, so that $c_1(q_1) = 3$

30   **if** 3 *does not appear in* K_1 **then**

       ASSERT: $C_1^{j,3}(q_1) \cap K_1 = \emptyset$

       ASSERT: $j = 1$

31       $c_1' \leftarrow c_1$ with colors j and 3 swapped in $C_1^{j,3}(q_1)$

32   **return** Color-Good-Partition$(G, K_1, K_2, K_3, L, R, c_1', c_2)$

33   **else**

       ASSERT: $q_1 \not\sim \{a, a_3\}$

       ASSERT: $C_1^{i,3}(q_1) \cap K_1 = \emptyset$

       ASSERT: $i = 1$

34       $c_1' \leftarrow c_1$ with colors i and 3 swapped in $C_1^{i,3}(q_1)$

35   **return** Color-Good-Partition$(G, K_1, K_2, K_3, L, R, c_1', c_2)$

GROW-HYPERPRISM(G, G, M, F)   // Lemma 3.3

Input: G – square-free, Berge graph
       $H = (A_1, \ldots B_3)$ – a hyperprism in G
       M – the set of major neighbors of H in G
       F – a minimal component of $G \setminus (H \cup M)$ with a set of attachments
       in H not local.

Output: H' – a larger hyperprism, or
       L – a $L(BS(K_4))$

    $X \leftarrow$ set of attachments of F in H

1 **if** $\exists i : X \cap C_i \neq \emptyset$ **then**
2   relabel strips of H, so that $X \cap C_1 \neq \emptyset$
3      $x_1 :\in X \cap C_1$
       ASSERT: $X \cap S_2 \neq \emptyset$
4      $x_2 :\in X \cap S_2$
5      $R_1 \leftarrow$ 1-rung of H, so that $x_1 \in V(R_1)$
6      $R_2 \leftarrow$ 2-rung of H, so that $x_2 \in V(R_2)$
7      $R_3 \leftarrow$ a 3-rung of H
8      $\forall i \in [3] : a_i, b_i \leftarrow$ ends of R_i, so that $a_i \in A_i, b_i \in B_i$
9      $K \leftarrow$ a prism (R_1, R_2, R_3)
       ASSERT: no vertex in F is major w.r.t. K (as in SPGT 10.5)
10    $f_1 - \ldots - f_n \leftarrow$ a minimal path in F, so that
          $f_1 \blacktriangleleft \{a_2, a_3\}$,
          $f_n - R \setminus \{a_1\}$
          there are no other edges between $\{f_1, \ldots f_n\}$ and $V(K) \setminus \{a_1\}$
       ASSERT: $F = \{f_1, \ldots, f_n\}$
       ASSERT: $f_1 \blacktriangleleft A_3$
11    $A_1' \leftarrow A_1 \cup \{f_1\}$
12    $C_1' \leftarrow C_1 \cup \{f_2, \ldots, f_n\}$
13   **return** $H' \leftarrow (A_1', A_2, \ldots, B_3, C_1', C_2, C_3)$

14 **else**
15   relabel strips of H, so that there is $\{x_1 :\in A_1, x_2 :\in A_2\} \subset X$ that is
       not local
16   find a path $x - f_1 - \ldots - f_n - x_2$
       ASSERT: $F = \{f_1, \ldots f_n\}$
17   **if** n is even and H is even, or n is odd and H is odd **then**
         ASSERT: $f_1 - a_3 \veebar f_n - b_3$
18   **if** $f_1 - a_3$ **then**
19          $H' \leftarrow$ mirrored H – every A_i and B_i are swapped
          # TODO: check if M and F are OK
20   **return** GROW-HYPERPRISM(G, H', M, F)
21   **else**
22   **if** $f_n \blacktriangleleft B_2 \cup B_3$ **then**
23             $B_1' \leftarrow B_1 \cup \{f_n\}$
24             $C_1' \leftarrow C_1 \cup \{f_1, \ldots, f_{n-1}\}$

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21   // **else** // $\forall_{i\in[3]} X \cap C_i = \emptyset$

22   // **if** n is even and H is even, or n is odd and H is odd **then**

23   // **else** // $f_n - b_3$

24   // **if** $f_n \blacktriangleleft B_2 \cup B_3$ **then**

25   **return** $H' \leftarrow \begin{pmatrix} A_1 & C_1' & B_1' \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{pmatrix}$

26   **else**

27           $\forall_{i\in[3]} : A_i' \leftarrow$ neighbors of f_1 in A_i

28           $\forall_{i\in[3]} : A_i'' \leftarrow A_i \setminus A_i'$

29           $\forall_{i\in[3]} : B_i'' \leftarrow$ neighbors of f_n in B_i

30           $\forall_{i\in[3]} : B_i' \leftarrow B_i \setminus B_i''$

          ASSERT: Every i-rung is between A_i' and B_i' or A_i'' and B_i''

31           $\forall_{i\in[3]} : C_i' \leftarrow$ union of interiors of i-rings between A_i' and B_i'

32           $\forall_{i\in[3]} : C_i'' \leftarrow$ union of interiors of i-rings between A_i'' and B_i''

          ASSERT: $C_i = C_i' \cup C_i''$, $C_i' \cap C_i'' = \emptyset$

          ASSERT: $A_i' \cup C_i' \boxtimes C_i'' \cup B_i''$, $A_i'' \cup C_i'' \boxtimes C_i \cup B_i$

          ASSERT: $A_i' \blacksquare A_i''$, $B_i' \blacksquare B_i''$

          ASSERT: $A_1', A_2'', A_3', A_3'' \neq \emptyset$

33           $H' \leftarrow \begin{pmatrix} A_1' & C_1' & B_1' \\ A_2' \cup A_3' & C_2' \cup C_3' & B_2' \cup B_3' \\ \bigcup_i A_i'' \cup \{f_1\} & \bigcup_i C_i'' \cup \{f_2,\ldots,f_n\} & \bigcup_i B_i'' \end{pmatrix}$

34   **return** H'

35   **else**

36     $a_1 \leftarrow$ neighbor of f_1 in A_1

37     $R_1 \leftarrow$ 1-rung with end a_1

38     $b_1 \leftarrow$ the other end of R_1

39     $b_2 \leftarrow$ neighbor of f_2 in B_2

40     $R_2 \leftarrow$ 2-rung with end b_2

41     $a_2 \leftarrow$ the other end of R_2

    ASSERT: $b_1 \in X$, $a_2 \in X$

    ASSERT: $(b_1 - f_1 \wedge a_2 - f_n) \veebar (b_1 - f_n \wedge a_2 - f_1)$

42   **if** $f_1 - b_1$ **then**

      ASSERT: H is odd

43       $R_3 \leftarrow$ any 3-rung with ends a_3, b_3, such that $\{a_3, b_3\} \boxtimes \{f_1, f_n\}$

44   **return** $V(R_1) \cup V(R_2) \cup V(R_3) \cup \{f_1, \ldots, f_n\}$ - a $L(BS(K_4))$

      # TODO: Is it valid input for part of ALG I?

// **else** // n is odd and H is even, or n is even and H is odd

45   **else** // $f_1 - a_2$

46      $\forall_{i\in[3]} : A_i' \leftarrow A_i \cap X,\ A_i'' \leftarrow A_i \setminus X$

47      $\forall_{i\in[3]} : B_i' \leftarrow B_i \cap X,\ B_i'' \leftarrow B_i \setminus X$

48      $\forall_{i\in[3]} : C_i' \leftarrow$ union of i-rungs between A_i' and B_i'

49      $\forall_{i\in[3]} : C_i'' \leftarrow$ union of i-rungs between A_i'' and B_i''

     ASSERT: $C_i = C_i' \cup C_i'',\ C_i' \cap C_i'' = \emptyset$

50   **if** f_1 *is complete to at least two of* A_i **then**

51   relabel strips of H, so that f_1 is complete to A_1 and A_2

       ASSERT: f_n is complete to B_1 and B_2

       ASSERT: $n > 1$ (as in SPGT 10.5 OK for odd H?)

52   **return** $\begin{pmatrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 \cup \{f_1\} & C_3 \cup \{f_2,\dots,f_{n-1}\} & B_3 \cup \{f_n\} \end{pmatrix}$

53   **else**

       ASSERT: $A_i' \ \blacksquare \ A_i''$ # TODO: OK odd H?

       ASSERT: $B_i' \ \blacksquare \ B_i''$ # TODO: OK odd H?

54   **return**

$$\begin{pmatrix} A_1' & C_1' & B_1' \\ A_2' \cup A_3' & C_2 \cup C_3' & B_2' \cup C_3' \\ \bigcup_i A_i'' \cup \{f_1\} & \bigcup_i C_i'' \cup \{f_2,\dots,f_{n-1}\} & \bigcup_i B_i'' \cup \{f_n\} \end{pmatrix}$$

GOOD-PARTITION-FROM-EVEN-HYPERPRISM(G, H, M)

Input: G – square-free, Berge graph containing no $L(BS(K_4))$

$\qquad H = (A_1, \ldots, B_3)$ – maximal even hyperprism in G

$\qquad M$ – set of major neighbors of H

Output: A good partition of G

1 $Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$

2 relabel strips of H, so that $M \cup A_1$ and $M \cup B_1$ are cliques

3 $F_1 \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_1 \cup B_1 \cup C_1 \}$

ASSERT: M is a clique

ASSERT: $M \cup A_i$ is a clique for at least two values of i

ASSERT: $M \cup B_j$ is a clique for at least two values of j

4 $K_1 \leftarrow A_1$, $K_2 \leftarrow M$, $K_3 \leftarrow B_1$

5 $R \leftarrow C_1 \cup F_1 \cup Z$

6 $L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$

7 **return** (K_1, K_2, K_3, L, R)

GOOD-PARTITION-FROM-EVEN-HYPERPRISM(G, H, M)

Input: G – square-free, Berge graph containing no $L(BS(K_4))$

$\quad\quad H = (A_1, \ldots, B_3)$ – maximal odd hyperprism in G

$\quad\quad M$ – set of major neighbors of H

Output: A good partition of G

1 $Z \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{V(H) \cup M\}$ with no attachments in $H\}$

2 relabel strips of H, so that $A_1 \not\sqsubset B_1$ and $A_2 \not\sqsubset B_2$
$\quad$ ASSERT: $C_1 \neq \emptyset, C_2 \neq \emptyset$

3 $\forall_{i \in [3]} F_i \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z\}$ that attaches to $A_i \cup B_i \cup C_i\}$

4 $F_B \leftarrow \bigcup\{V(C) : C$ is a component of $G \setminus \{H \cup M \cup Z \cup F_1 \cup F_2 \cup F_3\}$ that attaches to $B_1 \cup B_2 \cup B_3\}$
$\quad$ # TODO: F_i, F_A, F_B are from algIV, make sure it is correct
$\quad$ ASSERT: At least two of A_i and at least two of B_i are cliques
$\quad$ ASSERT: M is complete to at least two of A_i and at least two of B_i
$\quad$ ASSERT: M is a clique
$\quad$ ASSERT: For at least two i : $A_i \cup M$ is a clique
$\quad$ ASSERT: For at least two j : $A_j \cup M$ is a clique

5 choose h, so that $M \cup A_h$ and $M \cup B_h$ are cliques

6 **if** $h = 1 \vee h = 2$ **then** // # TODO: make sure $h = 2$ is ok

7 $\quad\quad K_1 \leftarrow A_1, K_2 \leftarrow M, K_3 \leftarrow B_1$

8 $\quad\quad R \leftarrow C_1 \cup F_1 \cup Z$

9 $\quad\quad L \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup R\}$

10 $\quad\quad$ **return** (K_1, K_2, K_3, L, R)

11 **else**

12 $\quad\quad$ relabel H so that $M \cup A_1$ and $M \cup B_2$ are cliques

13 $\quad\quad K_1 \leftarrow B_2 \cup B_3, K_2 \leftarrow M, K_3 \leftarrow A_1 \cup A_3$

14 $\quad\quad L \leftarrow B_1 \cup C_1 \cup F_1 \cup F_B$

15 $\quad\quad R \leftarrow G \setminus \{K_1 \cup K_2 \cup K_3 \cup L\}$

16 $\quad\quad$ **return** (K_1, K_2, K_3, L, R)

2.1 ALG VI

2.1.1 Notes

- $T_{uv} := N_u \cap N_v$

- Strip S_{uv} is *rich* iff $S_{uv} \setminus T_{uv} \neq \emptyset$

Good-Partition-From-J-Strip-System$(G, J, (S, N), M)$

Input: G – square-free, Berge graph

        J – a maximal 3-connected graph with appearance in G

        (S, N) – a maximal J-strip system

        M – a set of major vertices w.r.t. (S, N)

Output: A good partition of G

1 $S_{uv}^* \leftarrow S_{uv} \cup ($ components of $G \setminus V(S, N)$ that attach in S_{uv} only $)$

2 $T_{uv} \leftarrow N_u \cap N_v$

    ASSERT: $T_{uv} \blacksquare N_u \setminus N_{uv}$, $T_{uv} \blacksquare N_v \setminus N_{vu}$

    ASSERT: $M \cup T_{uv}$ is a clique

3 **if** $\exists S_{uv}$ – a rich strip in (S, N) **then**

4   **if** $\exists S_{uv}$ – a rich strip in (S, N), such that $M \cup (N_u \setminus N_{uv})$ and $M \cup (N_v \setminus N_{vu})$ are cliques **then**

5         $K_1 \leftarrow N_u \setminus N_{uv}$, $K_2 \leftarrow M \cup T_{uv}$, $K_3 \leftarrow N_v \setminus N_{vu}$

6         $L \leftarrow (S_{uv}^* \setminus T_{uv}) \cup ($ components of $G \setminus V(S, N)$ that attach only to N_u and those that attach only to N_v $)$

7         $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$

8   **return** (K_1, K_2, K_3, L, R)

9   **else**

10        $S_{uv} \leftarrow$ a rich strip in (S, N), such that $M \cup (N_u \setminus N_{uv})$ is not a clique and $M \cup (N_v \setminus N_{vu})$ is a clique

11        $K_1 \leftarrow N_{uv} \setminus T_{uv}$, $K_2 \leftarrow M \cup T_{uv}$, $K_3 \leftarrow N_v \setminus N_{vu}$

12        $R \leftarrow (S_{uv}^* \setminus N_u) \cup ($ components of $G \setminus V(S, U)$ that attach only to N_v $)$

13        $L \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup R)$

14   **return** (K_1, K_2, K_3, L, R)

15 **else**

      ASSERT: $\forall uv \in E(J) : S_{uv} = T_{uv}$, S_{uv} is a clique

16      $S_{uv} \leftarrow$ any strip

17      $K_1 \leftarrow N_u \setminus S_{uv}$, $K_2 \leftarrow M$, $K_3 \leftarrow N_v \setminus S_{uv}$

18      $L \leftarrow S_{uv}^* \cup ($ components of $G \setminus V(S, N)$ that attach only to N_u and only to N_v $)$

19      $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$

20   **return** (K_1, K_2, K_3, L, R)

GOOD-PARTITION-FROM-J-STRIP-SYSTEM$(G, J, (S, N), M)$

Input: G – square-free, Berge graph

$\quad\quad$ (S, N) – a special K_4 strip system

$\quad\quad$ M – a set of major vertices w.r.t. (S, N)

Output: A good partition of G

1 $\forall_{i,j \in [4]} O_{ij} \leftarrow$ set of vertices in $V(G) \setminus V(S, N)$ that are complete to $(N_i \cup N_j) \setminus S_{ij}$ and anticomplete to $V(S, N) \setminus (N_i \cup N_j \cup S_{ij})$

2 **if** $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ *and* $(N_2 \setminus N_{12}) \cup M \cup O_{12}$ *are cliques* **then**

3 $\quad$ $K_1 \leftarrow N_1 \setminus N_{12}$, $K_2 \leftarrow O_{12} \cup M$, $K_3 \leftarrow N_2 \setminus N_{12}$

4 $\quad$ $L \leftarrow$ union of those components of $G \setminus (K_1 \cup K_2 \cup K_3)$ that contain vertices of S_{12}

5 $\quad$ $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$

6 $\quad$ **return** (K_1, K_2, K_3, L, R)

7 **else**

8 $\quad$ relabel G so that $(N_1 \setminus N_{12}) \cup M \cup O_{12}$ is not a clique

$\quad$ ASSERT: $N_{12} \cup M \cup O_{12}$ is a clique

9 $\quad$ **if** $N_{21} \cup M \cup O_{12}$ *is a clique* **then**

10 $\quad\quad$ $X \leftarrow N_{21}$

11 $\quad$ **else**

12 $\quad\quad$ $X \leftarrow N_2 \setminus N_{21}$

13 $\quad$ $K_1 \leftarrow N_{12}$, $K_2 \leftarrow M \cup O_{12}$, $K_3 \leftarrow X$

14 $\quad$ $L \leftarrow$ component of $G \setminus (K_1 \cup K_2 \cup K_3)$ that contains $N_1 \setminus N_{12}$

15 $\quad$ $R \leftarrow G \setminus (K_1 \cup K_2 \cup K_3 \cup L)$

16 $\quad$ **return** (K_1, K_2, K_3, L, R)

Find-Special-K4-Strip-System$(G, J, (S, N), m)$

Input: G – square-free, Berge graph

J – a 3-connected graph with appearance in G

(S, N) – a J-strip system

$m \in G \setminus (S, N)$ that is major w.r.t. some choice of rungs of (S, N)

but not w.r.t. (S, N)

Output: (S', N') – A special K_4-strip system, or

(S'', N'') – a bigger J-strip system

GROWING-J-STRIP$(G, J, (S, N))$

Input: G – square-free, Berge graph
$\qquad$ J – a 3-connected graph with appearance in G
$\qquad$ (S, N) – a J-strip system
$\qquad$ # TODO: make sure def of J is correct
Output: J' and a maximal J'-strip system, or a special strip system

1 $M \leftarrow$ vertices of $G \setminus V(S, N)$ that are major on some choice of Rungs of (S, N)
$\quad$ # TODO: M like ALGI

2 **if** $\exists m : m$ *is not major on some choice of rungs of* (S, N) **then**

3 $\quad$ $OUT \leftarrow$ FIND-SPECIAL-K4-STRIP-SYSTEM$(G, J, (S, N), m)$

4 $\quad$ **if** OUT *is a special strip system* **then**

5 $\quad\quad$ **return** OUT

6 $\quad$ **else**

7 $\quad\quad$ **return** GROWING-J-STRIP(G, OUT)

8 **else if** $\exists F :$ F *is a component of* $G \setminus (V(S, N) \cup M)$, *such that no member of* F *is major w.r.t.* (S, N) *and set of attachments of* F *on* H *is not local* **then**

$\quad$ ASSERT: (as in 6.2, or actually SPGT 8.5)

9 $\quad$ $F \leftarrow$ minimal (component?) with this property

10 $\quad$ **if** $\exists v \in V(J) : X \subset \bigcup(S_{uv} : uv \in E(J))$ **then**

11 $\quad\quad$ $x :\in X \cap S_{uv} \setminus N_v$, for some $uv \in E(J)$
$\quad\quad$ # TODO: brackets?

12 $\quad\quad$ $x' :\in X \cap S_{u'v}$, for some $u'v \in E(J), u' \neq u$
$\quad\quad$ ASSERT: $\{x, x'\}$ is not local w.r.t. (S, N)

13 $\quad\quad$ $L(H) \leftarrow \forall_{i,j \in E(J)}$ choose ij-rung R_{ij}, so that
$\quad\quad$ $x \in V(R_{uv}), x' \in V(R_{u'v})$
$\quad\quad$ ASSERT: $\{x, x'\}$ is not local w.r.t. $L(H)$

14 $\quad\quad$ $D \leftarrow$ a branch of H with ends d, u:
$\quad\quad$ $\delta_H(d) \setminus E(D) = (X \cap E(H)) \setminus E(D)$

15 $\quad\quad$ $P \leftarrow$ a path with ends p_1, p_2, so that:
$\quad\quad\quad$ $p_1 \blacktriangleleft N_v \setminus N_{vu}$ and no other vertex of P has neighbors in $N_v \setminus N_{uv}$
$\quad\quad\quad$ $p_2 - x$ and no other vertex of P has neighbors in $S_{uv} \setminus N_v$

16 $\quad\quad$ $(S', N') \leftarrow$ add p_1 to N_v and F to S_{uv}

17 $\quad\quad$ **return** GROWING-J-STRIP $(G, J, (S', N'))$

18 $\quad$ **else**

19 $\quad\quad$ $K \leftarrow \{uv \in E(J) : X \cap S_{uv} \neq \emptyset\}$
$\quad\quad$ ASSERT: There are two disjoint edges in K (as in SPGT 8.5.(3))

20 $\quad\quad$ F is a vertex set of a path $\leftarrow f_1 - \ldots - f_n$
$\quad\quad$ ASSERT: Every choice of rungs is broad

          ASSERT: every choice of rungs has the same traversal. (Hard to assert)

        21          $ij \leftarrow$ the traversal edge

        22          $A_1 \leftarrow N_i \setminus S_{ij}$, $A_2 \leftarrow N_j \setminus S_{ij}$

              ASSERT: $X \cap (V(S, N) \setminus S_{ij}) = A_1 \cup A_2$

        23   **if** $n = 1$ **then**

        24              $(S', N') \leftarrow$ add f_1 to N_i, N_j, S_{ij}

        25   **return** Growing-J-Strip $(G, J, (S', N'))$

        26   **else**

        27              $x_1 :\in A_1, x_2 :\in A_2$, so that x_1 and x_2 are in disjoint strips

                  ASSERT: $x_1 - f_1 \veebar x_1 - f_n$

        28   **if** $x_1 - f_n$ **then**

        29   relabel $f_1 - \dots - f_n$ front to back

        30              $(S', N') \leftarrow$ add f_1 to N_i, f_n to N_j and F to S_{ij}

        31   **return** Growing-J-Strip $(G, J, (S', N'))$

    32 **else**

    33   **return** $J, (S, N)$ – a maximal J-strip