Decision procedures for a non-Fregean logic SCI

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Test: ążźćłóęń

1 Sentential logic and the Fregean axiom

2 SCI

2.1 Basic notions

Definition 2.1 (Vovabulary of SCI). The vocabulary of SCI consists of symbols from the following pairwise disjoint sets:

- $\mathtt{V} = \{p, q, r, ...\}$ – a countable infinite set of propositional variables,
- $\{\neg, \to, \equiv\}$ – the set consisting of the unary operator of negation ($\neg$) and binary operators of implication ($\to$) and identity ($\equiv$),
- $\{(,)\}$ – the set of auxiliary symbols.

Definition 2.2 (Formulas of SCI). The set of formulas of SCI is defined with the following grammar:

$$\mathtt{FOR} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid (\varphi \equiv \varphi)$$

where $p \in \mathtt{V}$ is a propositional variable.

The propositional variables will also be called atomic formulas.

From now on whenever we write $p, q, r, ...$ we will mean the atomic formulas. We will omit brackets when it will lead to no misunderstanding. We will write $\varphi \not\equiv \psi$ as a shorthand for $\neg(\varphi \equiv \psi)$.

The set of identities $\mathtt{ID}$ is a set of formulas $\varphi \equiv \psi$ where $\varphi, \psi \in \mathtt{FOR}$. Formulas $\varphi \equiv \varphi$ are the trivial identities.

Definition 2.3 (Subformulas). For a formula $\varphi \in \mathtt{FOR}$ let us define the set of subformulas of φ as:

$$\mathtt{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in \mathtt{V}, \\ \{\varphi\} \cup \mathtt{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \mathtt{SUB}(\psi) \cup \mathtt{SUB}(\vartheta), & \text{if } \varphi = \psi \to \vartheta, \text{or } \varphi = \psi \equiv \vartheta \end{cases}$$

By $\varphi(\psi/\vartheta)$ we will denote the formula φ with all occurrences of its subformula ψ substituted with ϑ.

Definition 2.4 (Simple formulas). The formula φ is called a simple formula if it has one of the following form:

$$p,\ \neg p,\ p \equiv q,\ p \not\equiv q,\ p \equiv \neg q,\ p \equiv (q \to r),\ p \equiv (q \equiv r)$$

for $p, q, r \in \mathtt{V}$.

Definition 2.5 (Size of a formula). Given a formula φ, let us define its size $s(\varphi)$:

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta,\ \text{or } \varphi = \psi \to \vartheta. \end{cases}$$

$\mathtt{V}$ is a countable set. Let us take any full ordering of it and mark it as $<$. Let us then extend it by saying that for each $p \in \mathtt{V}$: $p < \neg < (<) < \rightarrow < \equiv$. Now, let us define a ordering of formulas $\prec$ to be a lexicographical ordering with $<$.

If we consider formulas that contain only the negation and implication operators, they form a classical Propositional Calculus. For simplicity, in SCI we'll consider every tautology of the classical Propositional Calculus to be an axiom.

Definition 2.6 (Axiomatization of SCI)**.** SCI is axiomatized with the following axioms:

- Any tautology of the classical Propositional Calculus
- $(Ax1)$ $\varphi \equiv \varphi$,
- $(Ax2)$ $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$,
- $(Ax3)$ $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$,
- $(Ax4)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \theta)))$,
- $(Ax5)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$.

The only inference rule is the *modus ponens* rule:

$$\mathsf{MP} : \frac{\varphi,\ \varphi \rightarrow \psi}{\psi}$$

Definition 2.7 (A thesis of SCI)**.** A formula φ is *a thesis of* SCI if there exists a finite sequence of formulas $\varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is a thesis of SCI we will denote it by $\vdash \varphi$, say that φ is *provable in* SCI and call the sequence $\varphi_1, ..., \varphi_n$ the *proof of* φ.

Let us now give some semantic definitions:

Definition 2.8 (SCI-model)**.** A model of SCI (or an SCI-model) model is a structure $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ where:

- $U \neq \emptyset$ is an *universe* of M,
- $\emptyset \neq D \subsetneq U$ is a *set of designated values*,
- $\tilde{\neg}$ is an unary operation on U, such that for all $a \in U$: $\tilde{\neg}a \in D$ if and only if $a \notin D$,
- $\tilde{\rightarrow}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$,
- $\tilde{\equiv}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\equiv} b \in D$ if and only if $a = b$ (that is, if and only if a and b denote the same element of the universe U).

If an universe U is finite, we'll call a given SCI-model a *finite* SCI-model.

Definition 2.9 (Valuation)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a valuation in $\mathcal{M}$ is a function $V : \mathtt{FOR} \longrightarrow U$ that assigns a value $V(p) \in U$ for all propositional variables p, and such that for all $\varphi, \psi \in \mathtt{FOR}$:

- $V(\neg\varphi) = \tilde{\neg}V(\varphi)$
- $V(\varphi \rightarrow \psi) = V(\varphi) \tilde{\rightarrow} V(\psi)$
- $V(\varphi \equiv \psi) = V(\varphi) \tilde{\equiv} V(\psi)$

If $V(\varphi) = a$ we will call a the *denotation of* φ.

Definition 2.10 (Satisfaction of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ and a valuation V in $\mathcal{M}$, a formula φ is *satisfied in* $\mathcal{M}$ *by* V if and only if $V(\varphi) \in D$. If a formula φ is satisfied in $\mathcal{M}$ by a valuation V, then will denote it by $M, V \models \varphi$.

Definition 2.11 (Truth of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a formula is *true in* $\mathcal{M}$ if and only if it is satisfied in $\mathcal{M}$ by all valuations of $\mathcal{M}$. If a formula φ is true in $\mathcal{M}$, then we will denote it by $\mathcal{M} \models \varphi$.

Definition 2.12 (Validity of a formula)**.** A formula is *valid in* SCI (or SCI-*valid*) if and only if it is true in all models. If a formula φ is SCI-valid we will denote it by $\models \varphi$.

Let us see some examples:
TODO

Theorem 1 (Soundness of SCI)**.** *For every* SCI*-formula* φ*, if* φ *is provable in* SCI*, then* φ *is valid in* SCI*.*

Proof. The proof will have two main parts:

(a) Every axiom of SCI is valid in SCI.

(b) By applying the *modus ponens* rule to valid formulas φ and $\varphi \to \psi$ the inferred formula ψ is also valid.

Let us show (a) first:

- First, we need to prove that every tautology of the classical Propositional Calculus (PC) is valid in SCI.

  Let φ be a tautology of PC. Let us take any SCI-model $\mathcal{M}$ and any SCI-valuation $\mathcal{M}, V$ in it. We want to show that $\mathcal{M}, V \models \varphi$.

  Given $\mathcal{M}, V$, we will construct V' that will be a PC valuation of φ. We can do this, because φ doesn't contain operator $\equiv$.

  For every $\psi \in \mathtt{SUB}(\varphi)$, let us define:

$$V'(\psi) = \begin{cases} 1, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \in D, \\ 0, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \notin D, \\ 1 - V'(\chi), & \text{if } \psi = \neg\chi, \\ max(1 - V'(\chi), V'(\theta)), & \text{if } \psi = \chi \to \theta. \end{cases}$$

  It is easy to see that for every $\psi \in \mathtt{SUB}(\varphi)$, $V'(\psi) = 1$ if and only if $V(\psi) \in D$.

  The V' function is constructed in the same way the valuation function in the PC is constructed, therefore since φ is a tautology of PC, we have that $V'(\varphi) = 1$. So, we have that $V(\varphi) \in D$, whis is what we wanted to show.

- Second, we want to show that axioms $(Ax1) - (Ax5)$ are valid in SCI.

  - $(Ax1)$ $\varphi \equiv \varphi$
    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.9, we have that $V(\varphi \equiv \varphi) = V(\varphi)\tilde{\equiv}V(\varphi)$. Based on definition of $\tilde{\equiv}$ from 2.8, we have that $V(\varphi)\tilde{\equiv}V(\varphi)$ is in D if and only if $V(\varphi) = V(\varphi)$, which is trivially the case. So, we have that $\mathcal{M}, V \models \varphi \equiv \varphi$.
  - $(Ax2)$ $(\varphi \equiv \psi) \to (\varphi \to \psi)$
    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.9, we have that $V((\varphi \equiv \psi) \to (\varphi \to \psi)) = V(\varphi \equiv \psi)\tilde{\to}V(\varphi \to \psi)$. There are two cases.

    1° $V(\varphi \equiv \psi) \notin D$. Then, by definition of $\tilde{\to}$ in 2.8, we have that $V(\varphi \equiv \psi)\tilde{\to}V(\varphi \to \psi) \in D$, so $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

    2° $V(\varphi \equiv \psi) \in D$. By definition 2.9 it means, that $V(\varphi)\tilde{\equiv}V(\psi)$, which by definition 2.8 means that $V(\varphi) = V(\psi)$.
    By definition 2.9 $V(\varphi \to \psi)$ is equal to $V(\varphi)\tilde{\to}V(\psi)$. By definition 2.8 $V(\varphi)\tilde{\to}V(\psi)$ is in D if and only if $V(\varphi) \notin D$ or $V(\psi) \in D$, but since $V(\varphi) = V(\psi)$ we have that $V(\varphi)\tilde{\to}V(\psi)$ is in D if and only if $V(\varphi) \notin D$ or $V(\varphi) \in D$, which is trivially the case.

    So, we have that $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.
  - Validity of axioms $(Ax3), (Ax4)$ and $(Ax5)$ can be shown in a similar way. (TODO: a może rozpisać dla jasności?)

To show (b), let take any valid formulas φ and $\varphi \to \psi$ and any SCI model and valuation $\mathcal{M}, V$. From validity of $\varphi \to \psi$, we know that $V(\varphi \to \psi) \in D$. From definition 2.9 we have that $D \ni V(\varphi \to \psi) = V(\varphi) \tilde{\to} V(\psi)$. From definition 2.8, since we know that $V(\varphi) \tilde{\to} V(\psi)$ we know that $V(\varphi) \notin D$, or $V(\psi) \in D$. But we have that φ is valid, so $V(\varphi) \in D$, so it must be that $V(\psi) \in D$, so $\mathcal{M}, V \models \psi$. The same holds for any other SCI model and valuation, so $\models \psi$, which is what we wanted to show.

Now, looking at definition 2.7, for a given provable φ, let us take its proof $\varphi_1, ..., \varphi_n = \varphi$. Every subsequent formula in this proof is either an SCI axiom and thus, by (a), valid, or is inferred by the *modus ponens* rule from valid formulas and thus, by (b) valid. So, φ is valid. □

Theorem 2 (Completeness of SCI)**.** *For every* SCI*-formula* φ*, if* φ *is valid in* SCI*, then* φ *is provable in* SCI*.*

Proof. To show that every valid SCI-formula is SCI-provable is, by contraposition, to show that every non-SCI-provable formula is non-SCI-valid.

To do this, we will construct a SCI-model and a valuation in which every SCI-provable formula is satisfied (this model will be similar in style to the Herbrandt model). Then, it will be shown that for every formula that is not SCI-provable, this model and this valuation will be a witness of its non-SCI-validity, thus proving completeness of SCI

Czy to jest w ogóle poprawne odtworzenie intuicji dowodu z wykładu? Na wykładzie co prawda dla każdego niesprzecznego zbioru formuł X konstruowaliśmy model, w którym były prawdziwe wszystkie formuły $X \vdash \varphi$, ale dla samego dowodu pełności i tak stawialiśmy na koniec (chyba?) $X = \emptyset$, wiec uprościłem.

First, let us define a relation R on SCI-formulas:

$$\varphi R \psi \text{ if and only if} \vdash \varphi \equiv \psi$$

Note that for clarity we're using infix notation for relations. We want to show that R is an equivalence relation.

(a) R is reflexive. It is trivially from axiom $(Ax1)$.

(b) R is symmetrical, that is for all $\varphi, \psi \in \texttt{FOR}, \varphi R \psi$ if and only if $\psi R \varphi$.

As we have shown in example TODO, we can prove that for all $\varphi, \psi, \vdash (\varphi \equiv \psi) \to (\psi \equiv \varphi)$. If $\varphi R \psi$, then $\vdash \varphi \equiv \psi$, so by *modus ponens* we can infer $\vdash \psi \equiv \varphi$, so $\psi R \varphi$. In the same way we can show that if $\psi R \varphi$ then $\varphi R \psi$, so R is symmetrical.

(c) R is transitive, that is for all $\varphi, \psi, \vartheta \in \texttt{FOR}$, if $\varphi R \psi$ and $\psi R \vartheta$ then $\varphi R \vartheta$.

As we have shown in example TODO, we can prove that for all $\varphi, \psi, \vartheta \in \texttt{FOR}$, $\vdash (\varphi \equiv \psi) \to ((\psi \equiv \vartheta) \to (\varphi \equiv \vartheta))$. Let us assume that $\varphi R \psi$ and $\psi R \vartheta$. This gives us that $\vdash \varphi \equiv \psi$ and $\vdash \psi \equiv \vartheta$. Applying *modus ponens* twice we get $\vdash \varphi \equiv \vartheta$, so $\varphi R \vartheta$.

So, R is an equivalence relation. Let us now show that it is a congruence, that is:

(a) For all $\varphi, \psi \in \texttt{FOR}$, if $\varphi R \psi$, then $\neg\varphi R \neg\psi$. This is trivially true from axiom $(Ax3)$.

(b) For all $\varphi, \psi, \vartheta, \chi \in \texttt{FOR}$, if $\varphi R \psi$ and $\vartheta R \chi$, then $\varphi \to \vartheta R \psi \to \chi$. This is easily proven from $(Ax4)$.

(c) For all $\varphi, \psi, \vartheta, \chi \in \texttt{FOR}$, if $\varphi R \psi$ and $\vartheta R \chi$, then $\varphi \equiv \vartheta R \psi \equiv \chi$. This is easily proven from $(Ax5)$.

Now that we know that R is a congruent equivalence relation, we can define a structure: $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, such that:

- $U = \{|\varphi|_R : \varphi \in \texttt{FOR}\}$ (by $|\varphi|_R$ we mean an equivalence class of R of which φ is a member),
- $D = \{|\varphi|_R : \varphi \in \texttt{FOR}, \vdash \varphi\}$
- $\tilde{\neg}$ is an unary function on equivalence classes of R, such that $\tilde{\neg}|\varphi|_R = |\neg\varphi|_R$
- $\tilde{\to}$ is a binary function on equivalence classes of R, such that $|\varphi|_R \tilde{\to} |\psi|_R = |\varphi \to \psi|_R$
- $\tilde{\equiv}$ is a binary function on equivalence classes of R, such that $|\varphi|_R \tilde{\equiv} |\psi|_R = |\varphi \equiv \psi|_R$

Let us add some comments:

- For any SCI-formulas φ, ψ, such that $\varphi \equiv \psi$, $\vdash \varphi$ if and only if $\vdash \psi$. This follows from $(Ax2)$. Because of that, $|\varphi|_R = |\psi|_R$ and so D is well defined.

- From congruence of R, we know that for any $\varphi, \psi \in \texttt{FOR}$ if $|\varphi|_R = |\psi|_R$, then $|\neg\varphi|_R = |\neg\psi|_R$, so $\tilde{\neg}$ is a well defined function.
- Similarly, form the congruence of R we can see that $\tilde{\to}$ and $\tilde{\equiv}$ are well defined functions.

Now we will want to show that $\mathcal{M}$ is an SCI-model. From the definition 2.8 we want to show that:

- $U \neq \emptyset$ – this is trivial,
- $\emptyset \neq D$ – this is true, because for example $\vdash p \equiv p$
- $D \subsetneq U$ – $D \subseteq U$ is trivial from the definition of D. $D \neq U$, because there are formulas which are not valid (e.g. $\neg(p \equiv p)$), which from theorem 1 are not provable.
- For all $a \in U : \tilde{\neg}a \in D$ if and only if $a \notin D$

  Czy to nie jest fałsz? Weźmy $a = |p \equiv q|$.

  Chcemy pokazać: $\tilde{\neg}|p \equiv q| \in D$ iff $|p \equiv q| \notin D$.

  Prawa strona: Prawdziwa, bo $\nvdash |p \equiv q|$

  Lewa strona: $\tilde{\neg}|p \equiv q| \overset{\text{def.}\tilde{\neg}}{\Longleftrightarrow} |\neg(p \equiv q)|$. Ale: $\nvdash \neg(p \equiv q)$, więc $|\neg(p \equiv q)| \notin D$, więc lewa nieprawdziwa.
- For all $a, b \in U : a\tilde{\to}b \in D$ if and only if $a \notin D$ or $b \in D$.

  TODO.
- For all $a, b \in U : a\tilde{\equiv}b \in D$ if and only if $a = b$.

  TODO.

$\square$

Definition 2.13 (Decidability)**.** A logic is *decidable* if there exists an effective method, that is, an algorithm that will always terminate, to determine whether any formula of this logic is a theorem.

We will want to show that SCI is decidable. To do this, first let us show the following:

Theorem 3 (Finite model property of SCI)**.** *For any $\varphi \in \texttt{FOR}$, if there exists $\mathcal{M}, V$, such that $\mathcal{M}, V \models \varphi$, then exists a finite SCI-model $\mathcal{M}', V'$, such that $\mathcal{M}', V' \models \varphi$.*

Proof. We are given $\mathcal{M}, V$ and $\varphi \in FOR$, such that $\mathcal{M}, v \models \varphi$. Let us define $\mathcal{M}' = (U', F', \tilde{\neg}', \tilde{\to}')$ and V':

- $U' = \{v(\psi) : \psi \in \texttt{SUB}(\varphi)\} \cup \{1, 0\}$ – let us create $1, 0$, such that $1, 0 \notin U$,
- $D' = \{v(\psi) : \psi \in \texttt{SUB}(\varphi), v(\psi) \in D\} \cup \{1\}$,
- $\tilde{\neg}'a = \begin{cases} \tilde{\neg}a, & \text{if } a = v(\psi) \text{ and } \neg\psi \in \texttt{SUB}(\varphi) \\ 0, & \text{otherwise, if } a \in D' \cup \{1\}, \\ 1, & \text{otherwise,} \end{cases}$
- $a\tilde{\to}'b = \begin{cases} a\tilde{\to}b, & \text{if } a = v(\psi), b = v(\vartheta) \text{ and } \psi \to \vartheta \in \texttt{SUB}(\varphi) \\ 1, & \text{otherwise, if } a \notin D, \text{ or } b \in D, \\ 0, & \text{otherwise,} \end{cases}$
- $a\tilde{\equiv}'b = \begin{cases} a\tilde{\equiv}b, & \text{if } a = v(\psi), b = v(\vartheta) \text{ and } \psi \equiv \vartheta \in \texttt{SUB}(\varphi) \\ 1, & \text{otherwise, if } a = b, \\ 0, & \text{otherwise.} \end{cases}$

We need to show that $\mathcal{M}'$ is an SCI-model. Following the definition 2.8:

- $\emptyset \neq U' \supsetneq D' \neq \emptyset$ – trivially from $0, 1 \in U', 1 \in D'$.
- $\tilde{\neg}'a \in D'$ if and only if $a \notin D'$ – there are cases:

  1° There exists ψ, such that $a = v(\psi)$ and $\neg\psi \in \texttt{SUB}(\varphi)$. Then from the definition of $\tilde{\neg}'$ we have $\tilde{\neg}'a \in D' \iff \tilde{\neg}a \in D$, from the definition of $\tilde{\neg}$ we have $\tilde{\neg}a \in D \iff a \notin D$ and from the definition of D' (because $a = v(\psi) \in \texttt{SUB}(\varphi)$) we have $a \notin D \iff a \notin D'$.

2° Otherwise, and $a \in D \cup \{1\}$. If $a = 1$, then of course $a \in D'$ and $\tilde{\neg}'a = 0 \notin D$. But there is another case: $a \in D' \setminus \{1\}$

3° Otherwise. We know that $a \in D \cup \{1\}$. If $a = 1$, then of course $a \in D'$ and $\tilde{\neg}'a = 0 \notin D$. If $a \in D'$, then $\tilde{\neg}'a = 0 \notin D'$.

- $a \tilde{\to}' b \in D'$ if and only if $a \notin D'$ or $b \in D'$ – there are cases:

  1° There exists ψ and ϑ such that $a = v(\psi), b = v(\vartheta)$ and $\psi \to \vartheta \in \mathtt{SUB}(\varphi)$. Then from the definition of $\tilde{\to}'$ we have that $a \tilde{\to}' b \in D' \iff a \tilde{\to} b \in D$, from the definition of $\tilde{\to}$ we have that $a \tilde{\to} b \in D$ if and only if $a \notin D$ or $b \in D$, which from the definition of D' is if and only if $a \notin D'$ or $b \in D'$.

  2° There are no such ψ and ϑ and $a \notin D$. Then, from the definition of $\tilde{\to}'$ we have that $a \tilde{\to}' b = 1 \in D'$ and from the definition of D' we have that $a \notin D'$.

  3° There are no such ψ and ϑ and $b \in D$. Then again, from the definition of $\tilde{\to}'$ we have that $a \tilde{\to}' b = 1 \in D'$ and from the definition of D' we have that $b \in D'$.

□

Theorem 4 (Decidability of SCI)**.** SCI *is decidable.*

The proof of the decidability theorem can be found in [1] (Corollary 2.4).

TODO: dowód.

3 Deduction in SCI

3.1 Deduction systems

TODO: Let us give an example of a provable formula with its full proof:

References

[1] Stephen L. Bloom, Roman Suszko (1972) *Investigations into the Sentential Calculus with Identity*, Notre Dame Journal of Formal Logic, vol. XIII, no 3.