Decision procedures for a non-Fregean logic SCI

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Test: ążźćłóęń

1 Sentential logic and the Fregean axiom

2 SCI

2.1 Basic notions

Definition 2.1 (Vovabulary of SCI). The vocabulary of SCI consists of symbols from the following pairwise disjoint sets:

- $V = \{p, q, r, ...\}$ – a countable infinite set of propositional variables,

- $\{\neg, \rightarrow, \equiv\}$ – the set consisting of the unary operator of negation ($\neg$) and binary operators of implication ($\rightarrow$) and identity ($\equiv$),

- $\{(,)\}$ – the set of auxiliary symbols.

Definition 2.2 (Formulas of SCI). The set of formulas of SCI is defined with the following grammar:

$$\text{FOR} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \rightarrow \varphi) \mid (\varphi \equiv \varphi)$$

where $p \in V$ is a propositional variable.

The propositional variables will also be called atomic formulas.

From now on whenever we write $p, q, r, ...$ we will mean the atomic formulas. We will omit brackets when it will lead to no misunderstanding. We will write $\varphi \not\equiv \psi$ as a shorthand for $\neg(\varphi \equiv \psi)$.

The set of identities ID is a set of formulas $\varphi \equiv \psi$ where $\varphi, \psi \in \text{FOR}$. Formulas $\varphi \equiv \varphi$ are the trivial identities.

Definition 2.3 (Subformulas). For a formula $\varphi \in \text{FOR}$ let us define the set of subformulas of φ as:

$$\text{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in V, \\ \{\varphi\} \cup \text{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \text{SUB}(\psi) \cup \text{SUB}(\vartheta), & \text{if } \varphi = \psi \rightarrow \vartheta, \text{or } \varphi = \psi \equiv \vartheta \end{cases}$$

By $\varphi(\psi/\vartheta)$ we will denote the formula φ with all occurrences of its subformula ψ substituted with ϑ.

Definition 2.4 (Simple formulas). The formula φ is called a simple formula if it has one of the following form:

$$p, \ \neg p, \ p \equiv q, \ p \not\equiv q, \ p \equiv \neg q, \ p \equiv (q \rightarrow r), \ p \equiv (q \equiv r)$$

for $p, q, r \in V$.

Definition 2.5 (Size of a formula). Given a formula φ, let us define its size $s(\varphi)$:

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta, \text{ or } \varphi = \psi \rightarrow \vartheta. \end{cases}$$

Since V is a countable set we can assume it has a full ordering $<$. Let us extend it by saying that for each $p \in V$: $p <$ "$\neg$" $<$ "(" $<$ ")" $<$ "$\rightarrow$" $<$ "$\equiv$". Now, let us define a ordering of formulas $\prec$ to be a lexicographical ordering with $<$.

Definition 2.6 (Axiomatization of SCI). SCI is axiomatized with the following axioms:

- $(Ax1)$ $\varphi \rightarrow (\psi \rightarrow \varphi)$

- $(Ax2)$ $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$

- $(Ax3)$ $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$

- $(Ax4)$ $\varphi \equiv \varphi$,

- $(Ax5)$ $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$,

- $(Ax6)$ $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$,

- $(Ax7)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \theta)))$,

- $(Ax8)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$.

The only inference rule is the *modus ponens* rule:

$$MP : \frac{\varphi, \ \varphi \rightarrow \psi}{\psi}$$

Definition 2.7 (A theorem). A formula φ is *a theorem* if there exists a finite set of formulas $\varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is a theorem we'll mark it as $\vdash \varphi$ and say that it is *provable*.

Let us give an example of a provable formula with its full proof:
    TODO

Definition 2.8 (Decidability). A logic is *decidable* if there exists an effective method to determine whether a given formula of this logic is a theorem.

Let us now give some semantic definitions:

Definition 2.9 (Model). A SCI model is a structure $M = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ where:

- $U \neq \emptyset$ is the *universe* of M,

- $\emptyset \neq D \subsetneq U$ is the *set of designated values*,

- $\tilde{\neg} : U \longrightarrow U$ is the algebraic counterpart of the operator not, such that $\forall a \in U$: $\tilde{\neg}a \in D$ iff $a \notin D$,

- $\tilde{\rightarrow} : U \times U \longrightarrow U$ is the algebraic counterpart of the operator of implication, such that $\forall a, b \in U$: $a\tilde{\rightarrow}b \in D$ iff $a \notin D$ or $b \in D$,

- $\tilde{\equiv} : U \times U \longrightarrow U$ is the algebraic counterpart of the operator of identity, such that $\forall a, b \in U$: $a\tilde{\equiv}b$ iff $a = b$ (that is, iff a and b denote the same element of the universe U).

Definition 2.10 (Valuation). Given a model $M = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a valuation in this model is a function $V : \text{FOR} \longrightarrow U$, such that for all $\varphi, \psi \in \text{FOR}$:

- $V(\neg\varphi) = \tilde{\neg}V(\varphi)$

- $V(\varphi \rightarrow \psi) = V(\varphi)\tilde{\rightarrow}V(\psi)$

- $V(\varphi \equiv \psi) = V(\varphi)\tilde{\equiv}V(\psi)$

If $V(\varphi) = a$ we will call a the *denotation of* φ.

Definition 2.11 (Satisfiability of a formula). Given a model $M = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ and a valuation V, a formula φ is *satisfied in this valuation* iff its valuation belongs to D. If a formula φ is satisfied by a valuation V in a model M we will mark it by $M, V \models \varphi$.

Definition 2.12 (Truth of a formula). Given a model $M = (U, D)(U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a formula is *true in this model* iff it is satisfied in all valuations of this model. If a formula φ is true in a model M we'll mark it by $M \models \varphi$.

Definition 2.13 (Validity of a formula)**.** A formula is *valid* iff it is true in all models. If a formula φ is valid we'll mark it by $\models \varphi$.

Let us see some examples:
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Theorem 1 (Correctness and fullness of SCI)**.** *For any formula φ the following two are equivalent:*

- φ *is provable,*

- φ *is valid.*

The proof of the correctness and fullness theorem can be found in [1] (Theorem 1.9).

Theorem 2 (Dedidability of SCI)**.** SCI *is decidable.*

The proof of the decidability theorem can be found in [1] (Corollary 2.4).

References

[1] Stephen L. Bloom, Roman Suszko (1972) *Investigations into the Sentential Calculus with Identity*, Notre Dame Journal of Formal Logic, vol. XIII, no 3.