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Problem z własnością skończonego modelu

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The next theorem shows that Cn has the "finite model property". It is used to prove the recursive decidability of $\mathrm{Cn}(\emptyset)$. For any formula φ, let DES(φ) be the number of subformulas of φ.

2.3 Theorem. *If φ is satisfiable in some model, then φ is satisfiable in a finite model M with* $|M| \leq \mathrm{DES}(\varphi) + 1$.

Proof: Suppose φ is satisfiable in the model $N = \langle \mathfrak{A}, B\rangle$ by the valuation h. Suppose $\mathfrak{A} = \langle A, \dot{\neg}, \dot{\to}, \dot{\equiv}\rangle$, and let $\varphi_1, \varphi_2, \ldots, \varphi_n = \varphi$ be all subformulas of φ (so $n = \mathrm{DES}(\varphi)$). Let C_0 be the set $\{h(\varphi_1), \ldots, h(\varphi_n)\}$. Note that $h(\varphi_n) = h(\varphi) \in B$, since h satisfies φ by hypothesis. If C_0 contains an element of $A - B$, let $C = C_0$. Otherwise, let $C = C_0 \cup \{O\}$, where O is any element of $A - B$. Denote $h(\varphi_n)$ by 1. We let $D = C \cap B$, and we will so define the operations $\mathring{\neg}$, $\mathring{\to}$, $\mathring{\equiv}$ on C such that D becomes a prime, normal filter in C. For a, b in C, define (where # is $\to$ or $\equiv$):

$$\mathring{\neg} a = \begin{cases} \dot{\neg} a & \text{if } \dot{\neg} a \in C; \\ 1 & \text{if } \dot{\neg} a \in B - C; \\ O & \text{otherwise} \end{cases}$$

$$a \mathbin{\mathring{\#}} b = \begin{cases} a \mathbin{\mathring{\#}} b & \text{if } a \mathbin{\dot{\#}} b \in C; \\ 1 & \text{if } a \mathbin{\dot{\#}} b \in B - C; \\ O & \text{otherwise} \end{cases}$$

That D is a prime, normal filter in the SCI-algebra $\mathfrak{C} = \langle C, \mathring{\neg}, \mathring{\to}, \mathring{\equiv}\rangle$ follows from the facts that N is a model (i.e. B is a prime, normal filter in $\mathfrak{A}$) *and* (for a, $b \in C$)

$$\mathring{\neg} a \in D \Longleftrightarrow \dot{\neg} a \in B;$$
$$a \mathbin{\mathring{\to}} b \in D \Longleftrightarrow a \mathbin{\dot{\to}} b \in B;$$
$$a \mathbin{\mathring{\equiv}} b \in D \Longleftrightarrow a \mathbin{\dot{\equiv}} b \in B.$$

Hence $M = \langle \mathfrak{C}, D\rangle$ is a finite model, and $|M| \leq 1 + \mathrm{DES}(\varphi)$.
Finally, let h' be the valuation of $\mathfrak{C}$ defined by

$$h'(p) = \begin{cases} h(p) & \text{if } p \text{ is a subformula of } \varphi \\ 1 & \text{otherwise} \end{cases}$$

From the way the functions in $\mathfrak{C}$ are defined, it is easily checked that $h'(\varphi) = h(\varphi) = 1$, so that h' satisfies φ in M, q.e.d..

Trivially this estimate is best possible since the smallest model in which the formula consisting of a single variable p is satisfiable has two elements; i.e. $1 + \mathrm{DES}(p)$ elements. But this does not tell the whole story: e.g. if φ contains no negation signs, then φ is satisfiable in the two element model. We do not discuss these matters further here. Clearly from **2.3**, we have:

2.4 *Corollary. There is an effective procedure to determine, given a formula φ, whether φ is a logical theorem.*

The decidability of the classical sentential calculus may be proved using the so-called truth table method, which may be considered as an application of the following well-known theorem:

Translacja symboli Suszki

Suszko	moja notacja
$\mathfrak{A}$	$\mathcal{M}$
A	U
B	D
h	V
C	U'
D	D'
h'	V'

Definicja skończonego modelu:

- $1 = V(\varphi), 0 =$ dowolny element z $U \setminus D$.
- $U' = \{V(\varphi_1), ..., V(\varphi_n) = 1, 0\}$ ($0 =$ pewne $V(\varphi_i)$, jeśli tylko $\exists V(\varphi_i) \in U' \setminus D'$)
- $D' = U' \cap D$
- For $a \in U'$: $\tilde{\neg}' a = \begin{cases} \tilde{\neg} a, & \text{if } \tilde{\neg} a \in U' \\ 1, & \text{if } \tilde{\neg} a \in D \setminus U' \\ 0, & \text{otherwise} \end{cases}$
- $V'(\psi) = \begin{cases} V(\psi), & \text{if } \psi \in \texttt{SUB}(\varphi) \\ 1, & \text{otherwise} \end{cases}$

Trywialny kontrprzykład

$V'(\neg\varphi) = 1$ – z def V', bo $\neg\varphi \notin \texttt{SUB}(\varphi)$

$V'(\neg\neg\varphi) = 1$ – z def V', bo $\neg\varphi \notin \texttt{SUB}(\varphi)$

Więc V' to nie waluacja, bo nieprawda, że $V'(\neg\psi) \in D$ iff $V'(\psi) \notin D$.

Ale to łatwo naprawić robiąc $V'(\psi) \in \{1, 0\}$ gdy $\psi \notin \texttt{SUB}(\varphi)$.

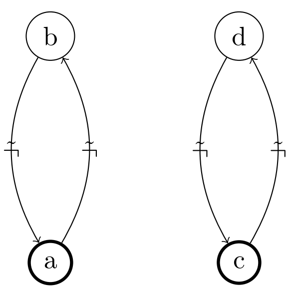
Poprawka i ogólny problem z definicją $V'(\psi)$ w stylu: $\psi \notin SUB(\varphi) \Rightarrow V'(\psi) \in \{1, 0\}$

$$V'(\psi) = \begin{cases} V(\psi), & \text{if } \psi \in \texttt{SUB}(\varphi) \\ 1, & \text{otherwise, if } V(\psi) \in D \\ 0, & \text{otherwise, if } V(\psi) \notin D \end{cases}$$

Kontrprzykład, pokazujący że V' nie jest waluacją:

Mamy dane:

- $\varphi = \neg(p \equiv \neg p)$
- $U = \{a, b, c, d, ...\}$,
- $D = \{a, c, ...\}$,
- $V(\varphi) = a \in D$
- $V(p \equiv \neg p) = b \notin D$
- $V(p) = c \in D$
- $V(\neg p) = d \in D$
- $V(\neg\varphi) = b \notin D$, $V(\neg\neg\varphi) = a \in D$, ...
- $\tilde{\neg} a = b$, $\tilde{\neg} b = a$
- $\tilde{\neg} c = d$, $\tilde{\neg} d = c$



Dostaniemy:

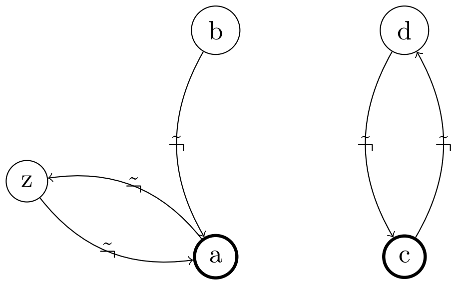
- $U' = \{a, b, c, d\}$
- $D' = \{a, c\}$
- $1 = a$. Przyjmijmy $0 = d$
- $\tilde{\neg}' a = b, \tilde{\neg}' b = a$
- $V'(\varphi) = a$
- $V'(\neg\varphi) = 0 = d$
- Więc $\tilde{\neg}' V'(\varphi) = \tilde{\neg}' a = b \neq 0 = V'(\neg\varphi)$, więc V' nie jest waluacją.

Próba prostej poprawki V'

$$V'(\psi) = \begin{cases} V(\psi), & \text{if } V(\psi) \in U' \\ 1, & \text{otherwise, if } V(\psi) \in D \\ 0, & \text{otherwise, if } V(\psi) \notin D \end{cases}$$

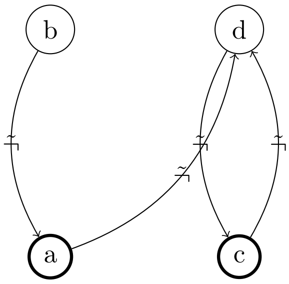
Kontrprzykład, pokazujący że V' nie jest waluacją:
Mamy dane:

- $\varphi = \neg(p \equiv \neg p)$
- $U = \{a, b, c, d, z, ...\}$
- $D = \{a, c, ...\}$
- $\tilde{\neg} a = z$, $\tilde{\neg} z = a$
- $\tilde{\neg} b = a$
- $\tilde{\neg} c = d$, $\tilde{\neg} d = c$
- $V(p) = c, V(\neg p) = d, V(p \equiv \neg p) = b$
- $V(\varphi) = a$
- $V(\neg\varphi) = z$
- $V(\neg\neg\varphi) = a$



I dostajemy:

- $U' = \{a, b, c, d\}$
- $D = \{a, c\}$
- $1 = a$
- Weźmy, że $0 = d$
- $\tilde{\neg}' b = a$
- $\tilde{\neg}' a = 0 = d$
- $\tilde{\neg}' d = c$



- $\tilde{\neg}' c = d$
- $V'(\varphi) = a$
- $V'(\neg\varphi) = 0 = d$
- $V'(\neg\neg\varphi) = a$
- I dostajemy: $\tilde{\neg}' V'(\neg\varphi) = \tilde{\neg}' 0 = 1 = c$
- Ale: $V'(\neg\neg\varphi) = a$, więc $\tilde{\neg}' V'(\neg\varphi) \neq V'(\neg\neg\varphi)$, więc V' nie jest wartościowaniem.

Druga próba poprawki V'

$$V'(\psi) = \begin{cases} V(\psi), & \text{if } \psi \in \mathtt{SUB}(\varphi)' \\ 1, & \text{otherwise, if } \psi = p \in D \\ 0, & \text{otherwise, if } \psi = p \notin D \\ \tilde{\neg}' V'(\vartheta), & \text{otherwise, if } \psi = \neg\vartheta \\ V'(\vartheta) \tilde{\rightarrow}' V'(\chi), & \text{otherwise, if } \psi = \vartheta \rightarrow \chi \\ V'(\vartheta) \tilde{\equiv}' V'(\chi), & \text{otherwise, if } \psi = \vartheta \equiv \chi \end{cases}$$

Chodzi o to, że V' jest zdefiniowane indukcyjnie, po $s(\psi)$, korzystając z V' dla mniejszych ψ. Widać, że:

- $V'(\psi) \in U'$
- $\tilde{\neg}' V'(\psi) = V'(\neg\psi)$