Decision procedures for a non-Fregean logic SCI

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Test: ążźćłóęń

1 Sentential logic and the Fregean axiom

2 SCI

2.1 Basic notions

Definition 2.1 (Vovabulary of SCI)**.** The vocabulary of SCI consists of symbols from the following pairwise disjoint sets:

- $\mathtt{V} = \{p, q, r, ...\}$ – a countable infinite set of propositional variables,
- $\{\neg, \to, \equiv\}$ – the set consisting of the unary operator of negation ($\neg$) and binary operators of implication ($\to$) and identity ($\equiv$),
- $\{(,)\}$ – the set of auxiliary symbols.

Definition 2.2 (Formulas of SCI)**.** The set of formulas of SCI is defined with the following grammar:

$$\mathtt{FOR} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid (\varphi \equiv \varphi)$$

where $p \in \mathtt{V}$ is a propositional variable.

The propositional variables will also be called atomic formulas.

From now on whenever we write $p, q, r, ...$ we will mean the atomic formulas. We will omit brackets when it will lead to no misunderstanding. We will write $\varphi \not\equiv \psi$ as a shorthand for $\neg(\varphi \equiv \psi)$.

The set of identities $\mathtt{ID}$ is a set of formulas $\varphi \equiv \psi$ where $\varphi, \psi \in \mathtt{FOR}$. Formulas $\varphi \equiv \varphi$ are the trivial identities.

Definition 2.3 (Subformulas)**.** For a formula $\varphi \in \mathtt{FOR}$ let us define the set of subformulas of φ as:

$$\mathtt{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in \mathtt{V}, \\ \{\varphi\} \cup \mathtt{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \mathtt{SUB}(\psi) \cup \mathtt{SUB}(\vartheta), & \text{if } \varphi = \psi \to \vartheta, \text{or } \varphi = \psi \equiv \vartheta \end{cases}$$

By $\varphi(\psi/\vartheta)$ we will denote the formula φ with all occurrences of its subformula ψ substituted with ϑ.

Definition 2.4 (Simple formulas)**.** The formula φ is called a simple formula if it has one of the following form:

$$p,\ \neg p,\ p \equiv q,\ p \not\equiv q,\ p \equiv \neg q,\ p \equiv (q \to r),\ p \equiv (q \equiv r)$$

for $p, q, r \in \mathtt{V}$.

Definition 2.5 (Size of a formula)**.** Given a formula φ, let us define its size $s(\varphi)$:

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta, \text{ or } \varphi = \psi \to \vartheta. \end{cases}$$

$\mathtt{V}$ is a countable set. Let us take any full ordering of it and mark it as $<$. Let us then extend it by saying that for each $p \in \mathtt{V}$: $p < \neg < (<) < \rightarrow < \equiv$. Now, let us define a ordering of formulas $\prec$ to be a lexicographical ordering with $<$.

If we consider formulas that contain only the negation and implication operators, they form a classical Propositional Calculus. For simplicity, in SCI we'll consider every tautology of the classical Propositional Calculus to be an axiom.

Definition 2.6 (Axiomatization of SCI)**.** SCI is axiomatized with the following axioms:

- Any tautology of the classical Propositional Calculus
- $(Ax1)$ $\varphi \equiv \varphi$,
- $(Ax2)$ $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$,
- $(Ax3)$ $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$,
- $(Ax4)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \theta)))$,
- $(Ax5)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$.

The only inference rule is the *modus ponens* rule:

$$\mathsf{MP} : \frac{\varphi,\ \varphi \rightarrow \psi}{\psi}$$

Definition 2.7 (A thesis of SCI)**.** A formula φ is *a thesis of* SCI if there exists a finite sequence of formulas $\varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is a thesis of SCI we will denote by $\vdash \varphi$ and say that φ is *provable in* SCI.

Let us now give some semantic definitions:

Definition 2.8 (SCI-model)**.** A model of SCI (or an SCI-model) model is a structure $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ where:

- $U \neq \emptyset$ is an *universe* of M,
- $\emptyset \neq D \subsetneq U$ is a *set of designated values*,
- $\tilde{\neg}$ is an unary operation on U, such that for all $a \in U$: $\tilde{\neg}a \in D$ if and only if $a \notin D$,
- $\tilde{\rightarrow}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$,
- $\tilde{\equiv}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\equiv} b$ if and only if $a = b$ (that is, if and only if a and b denote the same element of the universe U).

Definition 2.9 (Valuation)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a valuation in $\mathcal{M}$ is a function $V : \mathtt{FOR} \longrightarrow U$ that assigns a value $V(p) \in U$ for all propositional variables p, and such that for all $\varphi, \psi \in \mathtt{FOR}$:

- $V(\neg\varphi) = \tilde{\neg}V(\varphi)$
- $V(\varphi \rightarrow \psi) = V(\varphi) \tilde{\rightarrow} V(\psi)$
- $V(\varphi \equiv \psi) = V(\varphi) \tilde{\equiv} V(\psi)$

If $V(\varphi) = a$ we will call a the *denotation of* φ.

Definition 2.10 (Satisfaction of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ and a valuation V in $\mathcal{M}$, a formula φ is *satisfied in* $\mathcal{M}$ *by* V if and only if $V(\varphi) \in D$. If a formula φ is satisfied in $\mathcal{M}$ by a valuation V, then will denote it by $M, V \models \varphi$.

Definition 2.11 (Truth of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a formula is *true in* $\mathcal{M}$ if and only if it is satisfied in $\mathcal{M}$ by all valuations of $\mathcal{M}$. If a formula φ is true in $\mathcal{M}$, then we will denote it by $\mathcal{M} \models \varphi$.

Definition 2.12 (Validity of a formula)**.** A formula is *valid in* SCI (or SCI-*valid*) if and only if it is true in all models. If a formula φ is SCI-valid we will denote it by $\models \varphi$.

Let us see some examples:
TODO

Theorem 1 (Soundness and completeness of SCI)**.** *For every* SCI*-formula* φ*, the following conditions are equivalent:*

- φ *is provable in* SCI*,*
- φ *is valid in* SCI*.*

The proof of the correctness and fullness theorem can be found in [1] (Theorem 1.9).
TODO: podać ideę dowodu

Definition 2.13 (Decidability)**.** A logic is *decidable* if there exists an effective method to determine whether a given formula of this logic is a theorem.

TODO: definicja rozstrzygalności

Theorem 2 (Decidability of SCI)**.** SCI *is decidable.*

The proof of the decidability theorem can be found in [1] (Corollary 2.4).
TODO: dowód.

3 TODO

Let us give an example of a provable formula with its full proof:
TODO

References

[1] Stephen L. Bloom, Roman Suszko (1972) *Investigations into the Sentential Calculus with Identity*, Notre Dame Journal of Formal Logic, vol. XIII, no 3.