Decision procedures for a non-Fregean logic SCI

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Test: ążźćłóęń

1 Sentential logic and the Fregean axiom

2 SCI

2.1 Basic notions

Definition 2.1 (Alphabet of SCI)**.** The alphabet of SCI is as follows:

- $\mathtt{V} = \{p, q, r, ...\}$ – a countable infinite set of atomic formulas,
- $\neg$ – unary operator "not",
- $\to, \equiv$ – binary operators "implication" and "identity".

Definition 2.2 (Language of SCI)**.** The language of SCI is a set of formulas $\mathtt{FOR}$ defined with the grammar:

$$\mathtt{FOR} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid (\varphi \equiv \varphi)$$

where $p \in \mathtt{V}$ is an atomic formula.

From now on whenever we write $p, q, r, ...$ we will mean the atomic formulas. We will omit brackets when obvious. We will write $\varphi \not\equiv \psi$ as a shorthand for $\neg(\varphi \equiv \psi)$.

The set of identities $\mathtt{ID}$ is a set of formulas $\varphi \equiv \psi$ where $\varphi, \psi \in \mathtt{FOR}$. Formulas $\varphi \equiv \varphi$ are the trivial identities.

Definition 2.3 (Subformulas)**.** For a formula $\varphi \in \mathtt{FOR}$ let us define the set of subformulas of φ as:

$$\mathtt{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in V, \\ \{\varphi\} \cup \mathtt{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \mathtt{SUB}(\psi) \cup \mathtt{SUB}(\vartheta), & \text{if } \varphi = \psi \to \vartheta, \text{or } \varphi = \psi \equiv \vartheta \end{cases}$$

By $\varphi(\psi/\vartheta)$ we will denote the formula φ with all occurrences of its subformula ψ substituted with ϑ.

Definition 2.4 (Simple formulas)**.** The formula φ is called a simple formula if it is equal to one of:

$$p,\ \neg p,\ p \equiv q,\ p \not\equiv q,\ p \equiv \neg q,\ p \equiv (q \to r),\ p \equiv (q \equiv r)$$

for $p, q, r \in \mathtt{V}$.

Definition 2.5 (Size of formula)**.** Given a formula φ, let us define its size $s(\varphi)$:

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta, \text{ or } \varphi = \psi \to \vartheta. \end{cases}$$

Since $\mathtt{V}$ is a countable set we can assume it has a full ordering $<$. Let us extend it by saying that for each $p \in \mathtt{V}$: $p <$ "$\neg$" $<$ "(" $<$ ")" $<$ "$\to$" $<$ "$\equiv$". Now, let us define a ordering of formulas $\prec$ to be a lexicographical ordering with $<$.

Definition 2.6 (Axiomatization of SCI)**.** SCI is axiomatized with the following axioms:

- $(Ax1)\ \varphi \to (\psi \to \varphi)$
- $(Ax2)\ (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

- $(Ax3)$ $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
- $(Ax4)$ $\varphi \equiv \varphi$,
- $(Ax5)$ $(\varphi \equiv \psi) \to (\varphi \to \psi)$,
- $(Ax6)$ $(\varphi \equiv \psi) \to (\neg\varphi \equiv \neg\psi)$,
- $(Ax7)$ $(\varphi \equiv \psi) \to ((\chi \equiv \theta) \to ((\varphi \to \chi) \equiv (\psi \to \theta)))$,
- $(Ax8)$ $(\varphi \equiv \psi) \to ((\chi \equiv \theta) \to ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$.

The only inference rule is the *modus ponens* rule:

$$MP : \frac{\varphi,\ \varphi \to \psi}{\psi}$$

Definition 2.7 (A theorem)**.** A formula φ is *a theorem* if there exists a finite set of formulas $\varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is a theorem we'll mark it as $\vdash \varphi$ and say that it is *provable*.

Let us give an example of a provable formula with its full proof:
TODO

Definition 2.8 (Decidability)**.** A logic is *decidable* if there exists an effective method to determine whether a given formula of this logic is a theorem.

Let us now give some semantic definitions:

Definition 2.9 (Model)**.** A SCI model is a structure $M = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$ where:

- $U \neq \emptyset$ is the *universe* of M,
- $\emptyset \neq D \subsetneq U$ is the *set of designated values*,
- $\tilde{\neg} : U \longrightarrow U$ is the algebraic counterpart of the operator not, such that $\forall a \in U$: $\tilde{\neg}a \in D$ iff $a \notin D$,
- $\tilde{\to} : U \times U \longrightarrow U$ is the algebraic counterpart of the operator of implication, such that $\forall a, b \in U$: $a \tilde{\to} b \in D$ iff $a \notin D$ or $b \in D$,
- $\tilde{\equiv} : U \times U \longrightarrow U$ is the algebraic counterpart of the operator of identity, such that $\forall a, b \in U$: $a \tilde{\equiv} b$ iff $a = b$ (that is, iff a and b denote the same element of the universe U).

Definition 2.10 (Valuation)**.** Given a model $M = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, a valuation in this model is a function $V : \texttt{FOR} \longrightarrow U$, such that for all $\varphi, \psi \in \texttt{FOR}$:

- $V(\neg\varphi) = \tilde{\neg}V(\varphi)$
- $V(\varphi \to \psi) = V(\varphi)\tilde{\to}V(\psi)$
- $V(\varphi \equiv \psi) = V(\varphi)\tilde{\equiv}V(\psi)$

If $V(\varphi) = a$ we will call a the *denotation of* φ.

Definition 2.11 (Satisfiability of a formula)**.** Given a model $M = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$ and a valuation V, a formula φ is *satisfied in this valuation* iff its valuation belongs to D. If a formula φ is satisfied by a valuation V in a model M we will mark it by $M, V \models \varphi$.

Definition 2.12 (Truth of a formula)**.** Given a model $M = (U, D)(U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, a formula is *true in this model* iff it is satisfied in all valuations of this model. If a formula φ is true in a model M we'll mark it by $M \models \varphi$.

Definition 2.13 (Validity of a formula)**.** A formula is *valid* iff it is true in all models. If a formula φ is valid we'll mark it by $\models \varphi$.

Let us see some examples:
TODO

Theorem 1 (Correctness and fullness of SCI)**.** *For any formula φ the following two are equivalent:*

- *φ is provable,*
- *φ is valid.*

The proof of the correctness and fullness theorem can be found in [1] (Theorem 1.9).

Theorem 2 (Dedidability of SCI)**.** *SCI is decidable.*

The proof of the decidability theorem can be found in [1] (Corollary 2.4).

References

[1] Stephen L. Bloom, Roman Suszko (1972) *Investigations into the Sentential Calculus with Identity*, Notre Dame Journal of Formal Logic, vol. XIII, no 3.