Decision procedures for a non-Fregean logic SCI

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Test: ąžźćłóęń

1 Sentential logic and the Fregean axiom

2 SCI

2.1 Basic notions

Definition 2.1 (Vocabulary of SCI)**.** The vocabulary of SCI consists of symbols from the following pairwise disjoint sets:

- $\mathtt{V} = \{p, q, r, ...\}$ – a countable infinite set of propositional variables,
- $\{\neg, \rightarrow, \equiv\}$ – the set consisting of the unary operator of negation ($\neg$) and binary operators of implication ($\rightarrow$) and identity ($\equiv$),
- $\{(,)\}$ – the set of auxiliary symbols.

Definition 2.2 (Formulas of SCI)**.** The set of formulas of SCI is defined with the following grammar:

$$\mathtt{FOR} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \rightarrow \varphi) \mid (\varphi \equiv \varphi)$$

where $p \in \mathtt{V}$ is a propositional variable.

The propositional variables will also be called atomic formulas.

From now on whenever we write $p, q, r, ...$ we will mean the atomic formulas. We will omit brackets when it will lead to no misunderstanding. We will write $\varphi \not\equiv \psi$ as a shorthand for $\neg(\varphi \equiv \psi)$.

The set of identities $\mathtt{ID}$ is a set of formulas $\varphi \equiv \psi$ where $\varphi, \psi \in \mathtt{FOR}$. Formulas $\varphi \equiv \varphi$ are the trivial identities.

Definition 2.3 (Subformulas)**.** For a formula $\varphi \in \mathtt{FOR}$ let us define the set of subformulas of φ as:

$$\mathtt{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in \mathtt{V}, \\ \{\varphi\} \cup \mathtt{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \mathtt{SUB}(\psi) \cup \mathtt{SUB}(\vartheta), & \text{if } \varphi = \psi \rightarrow \vartheta, \text{or } \varphi = \psi \equiv \vartheta \end{cases}$$

By $\varphi(\psi/\vartheta)$ we will denote the formula φ with all occurrences of its subformula ψ substituted with ϑ.

Definition 2.4 (Simple formulas)**.** The formula φ is called a simple formula if it has one of the following form:

$$p,\ \neg p,\ p \equiv q,\ p \not\equiv q,\ p \equiv \neg q,\ p \equiv (q \rightarrow r),\ p \equiv (q \equiv r)$$

for $p, q, r \in \mathtt{V}$.

Definition 2.5 (Size of a formula)**.** Given a formula φ, let us define its size $s(\varphi)$:

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta, \text{ or } \varphi = \psi \rightarrow \vartheta. \end{cases}$$

$\mathtt{V}$ is a countable set. Let us take any full ordering of it and mark it as $<$. Let us then extend it by saying that for each $p \in \mathtt{V}$: $p < \neg < (<) < \rightarrow < \equiv$. Now, let us define an ordering of formulas $\prec$ to be a lexicographical ordering with $<$.

If we consider formulas that contain only the negation and implication operators, they form the classical Propositional Calculus. For simplicity, in SCI we'll consider every tautology of the classical Propositional Calculus to be an axiom.

Definition 2.6 (Axiomatization of SCI)**.** SCI is axiomatized with the following axioms:

- Any tautology of the classical Propositional Calculus
- $(Ax1)$ $\varphi \equiv \varphi$,
- $(Ax2)$ $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$,
- $(Ax3)$ $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$,
- $(Ax4)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \theta)))$,
- $(Ax5)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$.

The only inference rule is the *modus ponens* rule:

$$\mathsf{MP} : \frac{\varphi,\ \varphi \rightarrow \psi}{\psi}$$

Definition 2.7 (A thesis of SCI)**.** A SCI-formula φ is *a thesis of* SCI if there exists a finite sequence of formulas $\varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is a thesis of SCI we will denote it by $\vdash \varphi$, say that φ is *provable in* SCI and call the sequence $\varphi_1, ..., \varphi_n$ the *proof of* φ.

Remark 2.1. If we have a proof of some formula φ and take a variable p appearing in this formula, we can easily obtain a proof of any formula $\varphi(p/\psi)$. To do it, we simply take the proof of φ and replace p with ψ in every formula of the proof. We can do it with axioms, because the resulting formula is still an instance of the same given axiom and all the *modus ponens* inferences are still correct.

Definition 2.8 (Derivability in SCI)**.** Given a set of SCI-formulas X, a formula φ is *derivable from* X if there exists a finite sequence of formulas $\varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, is an element of X, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is derivable from X we will denote it by $X \vdash \varphi$ and call the sequence $\varphi_1, ..., \varphi_n$ the *proof of* φ *from* X.

In terms of derivability we can see that a formula is a thesis of SCI if and only if it is derivable from the emptyset.

Theorem 1 (Deduction theorem for SCI)**.** *For all* $X \subseteq \mathtt{FOR}$ *and* $\varphi, \psi \in \mathtt{FOR}$*:* $X \cup \{\varphi\} \vdash \psi$ *if and only if* $X \vdash \varphi \rightarrow \psi$.

Proof. First, let us prove the implication from right to left. Let us assume that $X \vdash \varphi \rightarrow \psi$. Let $\varphi_1, ..., \varphi_n$ be a proof of $\varphi \rightarrow \psi$ in X. Let us set $\varphi_{n+1} = \varphi$ and $\varphi_{n+2} = \psi$. Then $\varphi_{n+1} \in X \cup \{\varphi\}$ and φ_{n+2} is derived from φ_{n+1} and φ_n via the *modus ponens* rule, so the sequence $\varphi_1, ..., \varphi_n, \varphi_{n+1}, \varphi_{n+2}$ is a proof of ψ in $X \cup \{\varphi\}$.

The proof in the other direction similarly involves modifying the proof, but is slightly more complicated.

Let us assume that $X \cup \{\varphi\} \vdash \psi$ and let $\varphi_1, ..., \varphi_n$ be a proof of ψ in $X \cup \{\varphi\}$. We will describe a procedure to modify it to be a proof of $\varphi \rightarrow \psi$ in X.

If $\varphi = \psi$, we want to prove $\psi \rightarrow \psi$, which is a PC axiom. Otherwise, let's start with an empty sequence Ψ and consider in turn each of φ_i starting with φ_1. We will uphold an invariant such that, if $\varphi_i \neq \varphi$, after considering φ_i, Ψ will be a proof of φ_i or $\varphi \rightarrow \varphi_i$.

- If φ_i is an axiom of SCI, or $\varphi_i \in X$, we append it to Ψ,
- If $\varphi_i = \varphi$, we skip it,
- Otherwise φ_i was inferred by *modus ponens* from some formulas $\alpha, \alpha \rightarrow \varphi_i$, such that α and $\alpha \rightarrow \varphi_i$ were considered before.

– If $\alpha = \varphi$, we don't append anything to Ψ, as $\varphi \to \varphi_i$ is already in it,

– If $\alpha \to \varphi_i = \varphi$. From our invariant we know that within Ψ there is a proof of α or $\varphi \to \alpha$ in Ψ.

  * If Ψ contains a proof of α, we append to Ψ a PC axiom $\alpha \to ((\alpha \to \varphi_i) \to \varphi_i)$, then from α and it by *modus ponens* we infer $(\alpha \to \varphi_i) \to \varphi_i$, which is $\varphi \to \varphi_i$,
  * Otherwise, Ψ contains a proof of $\varphi \to \alpha = (\alpha \to \varphi_i) \to \alpha$. We append to Ψ a PC axiom $((\alpha \to \varphi_i) \to \alpha) \to ((\alpha \to \varphi_i) \to \varphi_i)$, then from $(\alpha \to \varphi_i) \to \alpha$ and it by *modus ponens* we infer $(\alpha \to \varphi_i) \to \varphi_i$ which is $\varphi \to \varphi_i$.

– Otherwise we have a proof of α or $\varphi \to \alpha$ in Ψ and also a proof of $\alpha \to \varphi_i$ or $\varphi \to (\alpha \to \varphi_i)$ in Ψ. So, we have four cases to look at:

  * We have a proof of α and $\alpha \to \varphi_i$ in Ψ. Then we simply infer a proof of φ_i.
  * We have a proof of $\varphi \to \alpha$ and $\alpha \to \varphi_i$. Then we append a PC axiom $(\varphi \to \alpha) \to ((\alpha \to \varphi_i) \to (\varphi \to \varphi_i))$ to Ψ, then from it and $\varphi \to \alpha$ we infer $(\alpha \to \varphi_i) \to (\varphi \to \varphi_i)$ and then from $\alpha \to \varphi_i$ we infer $\varphi \to \varphi_i$.
  * We have a proof of α and $\varphi \to (\alpha \to \varphi_i)$ in Ψ. Then we append a PC axiom $\alpha \to ((\varphi \to (\alpha \to \varphi_i)) \to (\varphi \to \varphi_i))$ and then infer $(\varphi \to (\alpha \to \varphi_i)) \to (\varphi \to \varphi_i)$ and then $\varphi \to \varphi_i$.
  * We have a proof of $\varphi \to \alpha$ and $\varphi \to (\alpha \to \varphi_i)$. Then we append a PC axiom $(\varphi \to \alpha) \to ((\varphi \to (\alpha \to \varphi_i)) \to (\varphi \to \varphi_i))$ and then infer $(\varphi \to (\alpha \to \varphi_i)) \to (\varphi \to \varphi_i)$ and then $\varphi \to \varphi_i$.

It is easy to see that at each step we uphold our invariant so at the end we arrive of a proof of $\varphi \to \psi$ or just a proof of ψ. If we have a proof of ψ we add a PC axiom $\psi \to (\varphi \to \psi)$ and infer $\varphi \to \psi$ to get a proof of $\varphi \to \psi$.

□

Let us now give some semantic definitions:

Definition 2.9 (SCI-model)**.** A model of SCI (or an SCI-model) model is a structure $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$ where:

- $U \neq \emptyset$ is an *universe* of M,
- $\emptyset \neq D \subsetneq U$ is a *set of designated values*,
- $\tilde{\neg}$ is an unary operation on U, such that for all $a \in U$: $\tilde{\neg}a \in D$ if and only if $a \notin D$,
- $\tilde{\to}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\to} b \in D$ if and only if $a \notin D$ or $b \in D$,
- $\tilde{\equiv}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\equiv} b \in D$ if and only if $a = b$.

If an universe U is finite, we'll call a given SCI-model a *finite* SCI*-model*.

Definition 2.10 (Valuation)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, a valuation in $\mathcal{M}$ is a function $V : \mathtt{FOR} \longrightarrow U$ that assigns a value $V(p) \in U$ for all propositional variables p, and such that for all $\varphi, \psi \in \mathtt{FOR}$:

- $V(\neg\varphi) = \tilde{\neg}V(\varphi)$
- $V(\varphi \to \psi) = V(\varphi) \tilde{\to} V(\psi)$
- $V(\varphi \equiv \psi) = V(\varphi) \tilde{\equiv} V(\psi)$

If $V(\varphi) = a$ we will call a the *denotation of* φ.

Definition 2.11 (Satisfaction of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$ and a valuation V in $\mathcal{M}$, a formula φ is *satisfied in* $\mathcal{M}$ *by* V if and only if $V(\varphi) \in D$. If a formula φ is satisfied in $\mathcal{M}$ by a valuation V, then will denote it by $M, V \models \varphi$.

Definition 2.12 (Truth of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, a formula is *true in* $\mathcal{M}$ if and only if it is satisfied in $\mathcal{M}$ by all valuations of $\mathcal{M}$. If a formula φ is true in $\mathcal{M}$, then we will denote it by $\mathcal{M} \models \varphi$.

Definition 2.13 (Validity of a formula)**.** A formula is *valid in* SCI (or SCI-*valid*) if and only if it is true in all models. If a formula φ is SCI-valid we will denote it by $\models \varphi$.

Theorem 2 (Soundness of SCI)**.** *For every* SCI*-formula* φ*, if* φ *is provable in* SCI*, then* φ *is valid in* SCI*.*

Proof. The proof will have two main parts:

(a) Every axiom of SCI is valid in SCI.

(b) By applying the *modus ponens* rule to valid formulas φ and $\varphi \to \psi$ the inferred formula ψ is also valid.

Let us show (a) first:

- First, we need to prove that every tautology of the classical Propositional Calculus (PC) is valid in SCI.

  Let φ be a tautology of PC. Let us take any SCI-model $\mathcal{M}$ and any SCI-valuation $\mathcal{M}, V$ in it. We want to show that $\mathcal{M}, V \models \varphi$.

  Given $\mathcal{M}, V$, we will construct V' that will be a PC valuation of φ. We can do this, because φ doesn't contain operator $\equiv$.

  For every $\psi \in \mathtt{SUB}(\varphi)$, let us define:

$$V'(\psi) = \begin{cases} 1, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \in D, \\ 0, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \notin D, \\ 1 - V'(\chi), & \text{if } \psi = \neg\chi, \\ max(1 - V'(\chi), V'(\theta)), & \text{if } \psi = \chi \to \theta. \end{cases}$$

  It is easy to see that for every $\psi \in \mathtt{SUB}(\varphi)$, $V'(\psi) = 1$ if and only if $V(\psi) \in D$.

  The V' function is constructed in the same way the valuation function in the PC is constructed, therefore since φ is a tautology of PC, we have that $V'(\varphi) = 1$. So, we have that $V(\varphi) \in D$, what is what we wanted to show.

- Second, we want to show that axioms $(Ax1) - (Ax5)$ are valid in SCI.

  - $(Ax1)$ $\varphi \equiv \varphi$

    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.10, we have that $V(\varphi \equiv \varphi) = V(\varphi)\tilde{\equiv}V(\varphi)$. Based on definition of $\tilde{\equiv}$ from 2.9, we have that $V(\varphi)\tilde{\equiv}V(\varphi)$ is in D if and only if $V(\varphi) = V(\varphi)$, which is trivially the case. So, we have that $\mathcal{M}, V \models \varphi \equiv \varphi$.

  - $(Ax2)$ $(\varphi \equiv \psi) \to (\varphi \to \psi)$

    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.10, we have that $V((\varphi \equiv \psi) \to (\varphi \to \psi)) = V(\varphi \equiv \psi)\tilde{\to}V(\varphi \to \psi)$. There are two cases.

    1° $V(\varphi \equiv \psi) \notin D$. Then, by definition of $\tilde{\to}$ in 2.9, we have that $V(\varphi \equiv \psi)\tilde{\to}V(\varphi \to \psi) \in D$, so $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

    2° $V(\varphi \equiv \psi) \in D$. By definition 2.10 it means, that $V(\varphi)\tilde{\equiv}V(\psi)$, which by definition 2.9 means that $V(\varphi) = V(\psi)$.
    By definition 2.10 $V(\varphi \to \psi)$ is equal to $V(\varphi)\tilde{\to}V(\psi)$. By definition 2.9 $V(\varphi)\tilde{\to}V(\psi)$ is in D if and only if $V(\varphi) \notin D$ or $V(\psi) \in D$, but since $V(\varphi) = V(\psi)$ we have that $V(\varphi)\tilde{\to}V(\psi)$ is in D if and only if $V(\varphi) \notin D$ or $V(\varphi) \in D$, which is trivially the case.

    So, we have that $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

  - Validity of axioms $(Ax3), (Ax4)$ and $(Ax5)$ can be shown in a similar way. (TODO: a może rozpisać dla jasności?)

To show (b), let take any valid formulas φ and $\varphi \to \psi$ and any SCI model and valuation $\mathcal{M}, V$. From validity of $\varphi \to \psi$, we know that $V(\varphi \to \psi) \in D$. From definition 2.10 we have that $D \ni V(\varphi \to \psi) = V(\varphi)\tilde{\to}V(\psi)$. From definition 2.9, since we know that $V(\varphi)\tilde{\to}V(\psi)$ we know that $V(\varphi) \notin D$, or $V(\psi) \in D$. But we have that φ is valid, so $V(\varphi) \in D$, so it must be that $V(\psi) \in D$, so $\mathcal{M}, V \models \psi$. The same holds for any other SCI model and valuation, so $\models \psi$, which is what we wanted to show.

Now, looking at definition 2.7, for a given provable φ, let us take its proof $\varphi_1, ..., \varphi_n = \varphi$. Every subsequent formula in this proof is either an SCI axiom and thus, by (a), valid, or is inferred by the *modus ponens* rule from valid formulas and thus, by (b), valid. So, φ is valid. □

Theorem 3 (Completeness of SCI). *For every* SCI*-formula* φ*, if* φ *is valid in* SCI*, then* φ *is provable in* SCI*.*

Proof. To show that every valid SCI-formula is SCI-provable is, by contraposition, to show that every non-SCI-provable formula is non-SCI-valid.

To do this, we will construct a SCI-model and a valuation in which all, and only, SCI-provable formulas are satisfied. Then, it will be shown that for every formula that is not SCI-provable, this model and this valuation will be a witness of its non-SCI-validity, thus proving completeness of SCI

Wydaje mi się, że pomieszałem coś w notatkach z tego dowodu. Tak jak to napisałem to chyba nie może działać.

Na wykładzie co prawda dla każdego niesprzecznego zbioru formuł X konstruowaliśmy model, w którym były prawdziwe wszystkie formuły $X \vdash \varphi$, ale dla samego dowodu pełności i tak stawialiśmy na koniec (chyba?) $X = \emptyset$, wiec uprościłem.

Próbowałem to porównać z [Suszko, Bloom 1972], ale tam ten dowód jest bardzo lakonicznie opisany.

A może bierze się niedowodliwą formułę i dla niej konkretnie buduje model (lub wartościowanie), które ją inwaliduje?

First, let us define a relation R on SCI-formulas:

$$\varphi R \psi \text{ if and only if} \vdash \varphi \equiv \psi$$

Note that for clarity we're using infix notation for relations. We want to show that R is an equivalence relation.

- R is reflexive. It is trivially true from axiom $(Ax1)$.
- R is symmetrical, that is for all $\varphi, \psi \in \texttt{FOR}, \varphi R \psi$ if and only if $\psi R \varphi$.

  From example 3.1 below and remark 2.1 we can prove that for all $\varphi, \psi, \vdash (\varphi \equiv \psi) \to (\psi \equiv \varphi)$. If $\varphi R \psi$, then $\vdash \varphi \equiv \psi$, so by *modus ponens* we can infer $\vdash \psi \equiv \varphi$, so $\psi R \varphi$. In the same way we can show that if $\psi R \varphi$ then $\varphi R \psi$, so R is symmetrical.
- R is transitive, that is for all $\varphi, \psi, \vartheta \in \texttt{FOR}$, if $\varphi R \psi$ and $\psi R \vartheta$ then $\varphi R \vartheta$.

  From example 3.3 below and remark 2.1, we can prove that for all $\varphi, \psi, \vartheta \in \texttt{FOR}$, $\vdash (\psi \equiv \vartheta) \to ((\varphi \equiv \psi) \to (\varphi \equiv \vartheta))$. Let us assume that $\varphi R \psi$ and $\psi R \vartheta$. This gives us that $\vdash \varphi \equiv \psi$ and $\vdash \psi \equiv \vartheta$. Applying *modus ponens* twice we get $\vdash \varphi \equiv \vartheta$, so $\varphi R \vartheta$.

So, R is an equivalence relation. Let us now show that it is a congruence, that is:

- For all $\varphi, \psi \in \texttt{FOR}$, if $\varphi R \psi$, then $\neg\varphi R \neg\psi$. This is trivially true from axiom $(Ax3)$.
- For all $\varphi, \psi, \vartheta, \chi \in \texttt{FOR}$, if $\varphi R \psi$ and $\vartheta R \chi$, then $\varphi \to \vartheta R \psi \to \chi$. This is easily proven from $(Ax4)$.
- For all $\varphi, \psi, \vartheta, \chi \in \texttt{FOR}$, if $\varphi R \psi$ and $\vartheta R \chi$, then $\varphi \equiv \vartheta R \psi \equiv \chi$. This is easily proven from $(Ax5)$.

Now that we know that R is a congruent equivalence relation, we can define a structure: $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, such that:

- $U = \{|\varphi|_R : \varphi \in \texttt{FOR}\}$ (by $|\varphi|_R$ we mean an equivalence class of R of which φ is a member),
- $D = \{|\varphi|_R : \varphi \in \texttt{FOR}, \vdash \varphi\}$
- $\tilde{\neg}$ is an unary function on equivalence classes of R, such that $\tilde{\neg}|\varphi|_R \stackrel{def}{=} |\neg\varphi|_R$
- $\tilde{\to}$ is a binary function on equivalence classes of R, such that $|\varphi|_R \tilde{\to} |\psi|_R \stackrel{def}{=} |\varphi \to \psi|_R$
- $\tilde{\equiv}$ is a binary function on equivalence classes of R, such that $|\varphi|_R \tilde{\equiv} |\psi|_R \stackrel{def}{=} |\varphi \equiv \psi|_R$

Let us add some comments:

- For any SCI-formulas φ, ψ, such that $\varphi \equiv \psi$, $\vdash \varphi$ if and only if $\vdash \psi$. This follows from $(Ax2)$. Because of that, $|\varphi|_R = |\psi|_R$ and so D is well defined.
- From congruence of R, we know that for any $\varphi, \psi \in \texttt{FOR}$ if $|\varphi|_R = |\psi|_R$, then $|\neg\varphi|_R = |\neg\psi|_R$, so $\tilde{\neg}$ is a well defined function.
- Similarly, form the congruence of R we can see that $\tilde{\to}$ and $\tilde{\equiv}$ are well defined functions.

Now we will want to show that $\mathcal{M}$ is an SCI-model. From the definition 2.9 we want to show that:

- $U \neq \emptyset$ – this is trivial, $|p|_R \in U$
- $\emptyset \neq D$ – this is true, because for example $\vdash p \equiv p$, so $|p \equiv p|_R \in D$
- $D \subsetneq U$ – $D \subseteq U$ is trivial from the definition of D. $D \neq U$, because there are formulas which are not valid (e.g. $\neg(p \equiv p)$), which from theorem 2 are not provable.
- For all $a \in U : \tilde{\neg} a \in D$ if and only if $a \notin D$
  Czy to nie jest fałsz? Weźmy $a = |p \equiv q|$.
  Chcemy pokazać: $\tilde{\neg}|p \equiv q| \in D$ iff $|p \equiv q| \notin D$.
  Prawa strona: Prawdziwa, bo $\nvdash |p \equiv q|$
  Lewa strona: $\tilde{\neg}|p \equiv q| \overset{\text{def.}\tilde{\neg}}{\Longleftrightarrow} |\neg(p \equiv q)|$. Ale: $\nvdash \neg(p \equiv q)$, więc $|\neg(p \equiv q)| \notin D$, więc lewa nieprawdziwa.
- For all $a, b \in U : a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$.
  TODO.
- For all $a, b \in U : a \tilde{\equiv} b \in D$ if and only if $a = b$.
  TODO.

So, $\mathcal{M}$ is a SCI-model. Now, let us define a valuation V for all $\varphi \in \texttt{FOR}$:

$$V(\varphi) = |\varphi|_R$$

We need to show that V is indeed a valuation:

- TODO (chyba proste wszystko)

Now, for the final proof of completeness let us take a non-provable formula φ. Since $\nvdash \varphi$ we have that $|\varphi|_R \notin D$, so $V(\varphi) \notin D$. So, we have found a model and a valuation $\mathcal{M}, V$, such that $\mathcal{M}, V \not\models \varphi$, so φ is not valid.
W notatkach mam: "

- Mamy dane X, φ: $X \nvdash \varphi$
- budujemy model $\mathcal{M}^X, V^X$, t.że $X \vdash \psi \Rightarrow |\psi|_{R_X} \in D^X$
- Ale oczywiście skoro $X \nvdash \varphi$ to $|\varphi|_{R_X} \notin D^X$, więc $X \not\models \varphi$

"
Więc X nic tutaj nie pomaga, stawiając $X = \emptyset$ dostaję to co jest u góry.

□

Definition 2.14 (Decidability)**.** A logic is *decidable* if there exists an effective method, that is, an algorithm that will always terminate, to determine whether any formula of this logic is a theorem.

We will want to show that SCI is decidable. To do this, first let us show the following:

Theorem 4 (Finite model property of SCI)**.** *For any $\varphi \in \texttt{FOR}$, if there exists $\mathcal{M}, V$, such that $\mathcal{M}, V \models \varphi$, then there exists a finite* SCI*-model $\mathcal{M}', V'$, such that $\mathcal{M}', V' \models \varphi$.*

Proof. We are given $\mathcal{M}, V$ and $\varphi \in \texttt{FOR}$, such that $\mathcal{M}, v \models \varphi$. Let us define $\mathcal{M}' = (U', D', \tilde{\neg}', \tilde{\rightarrow}')$ and V':

- $U' = \{V(\psi) : \psi \in \texttt{SUB}(\varphi)\} \cup \{1, 0\}$ – let us create $1, 0$, such that $1, 0 \notin U$,
- $D' = \{V(\psi) : \psi \in \texttt{SUB}(\varphi), V(\psi) \in D\} \cup \{1\}$
- $\tilde{\neg}' a = \begin{cases} \tilde{\neg} a, & \text{if exists } \psi \text{ such that } a = V(\psi) \text{ and } \neg\psi \in \texttt{SUB}(\varphi) \\ 0, & \text{otherwise, if } a \in D' \\ 1, & \text{otherwise, if } a \notin D' \end{cases}$
- $a \tilde{\rightarrow}' b = \begin{cases} a \tilde{\rightarrow} b, & \text{if exist } \psi, \vartheta \text{ such that } a = V(\psi), b = V(\vartheta) \text{ and } \psi \rightarrow \vartheta \in \texttt{SUB}(\varphi) \\ 1, & \text{otherwise, if } a \notin D' \text{ or } b \in D' \\ 0, & \text{otherwise, if } a \in D' \text{ and } b \notin D' \end{cases}$

- $a \tilde{\equiv}' b = TODO \begin{cases} a \tilde{\equiv} b, & \text{if } a = v(\psi), b = v(\vartheta) \text{ and } \psi \equiv \vartheta \in \mathtt{SUB}(\varphi) \\ 1, & \text{otherwise, if } a = b, \\ 0, & \text{otherwise.} \end{cases}$
- $V'(\psi) = \begin{cases} V(\psi), & \text{if } \psi \in \mathtt{SUB}(\varphi) \\ 1, & \text{if } \psi \notin \mathtt{SUB}(\varphi) \text{ and } V(\psi) \in D \\ 0, & \text{if } \psi \notin \mathtt{SUB}(\varphi) \text{ and } V(\psi) \notin D \end{cases}$

We need to show that $\mathcal{M}'$ is an SCI-model, V' is a valuation and that $V'(\varphi) = V(\varphi)$:

- $\emptyset \neq U' \supsetneq D' \neq \emptyset$ – trivially from $0, 1 \in U', 1 \in D'$, $0 \notin D'$.
- $\tilde{\neg}' a \in D'$ if and only if $a \notin D'$ – there are cases:

  1° There exists ψ, such that $a = V(\psi)$ and $\neg\psi \in \mathtt{SUB}(\varphi)$. Then from the definition of $\tilde{\neg}'$ we have $\tilde{\neg}' a = \tilde{\neg} a$.

  Since $\mathcal{M}$ is a model we know that $\tilde{\neg} a \in D$ if and only if $a \notin D$, and since $\neg\psi \in \mathtt{SUB}(\varphi)$, from the definition of D', we have $\tilde{\neg} a \in D'$ if and only if $a \notin D'$ and finally, since $\tilde{\neg}' a = \tilde{\neg} a$ we have that $\tilde{\neg}' a \in D'$ if and only if $a \notin D'$.

  2° Otherwise, and $a \in D'$. Then $\tilde{\neg}' a = 0 \notin D'$.

  3° Otherwise, and $a \notin D'$. Then $\tilde{\neg}' a = 1 \in D'$.

- For all $\psi \in \mathtt{FOR}$, $V'(\neg\psi) = \tilde{\neg}' V'(\psi)$ – there are cases:

  1° $\neg\psi \in \mathtt{SUB}(\varphi)$. Then from the definition of $\tilde{\neg}'$ we have that $\tilde{\neg}' V'(\psi) = V'(\neg\psi)$.

  2° $\neg\psi \notin \mathtt{SUB}(\varphi)$.

  Kontrprzykład:

  - weźmy ψ: $\neg\psi \notin \mathtt{SUB}(\varphi)$
  - Z definicji V': $V'(\neg\psi) = 1$ lub 0, w zależności czy $V(\psi) \in D$.
  - Niech $V'(\psi) = b$
  - Niech istnieje χ, takie że: $\neg\chi \in \mathtt{SUB}(\varphi)$, $V'(\chi) = b$, $V'(\neg\chi) = c \notin \{1, 0\}$
  - wtedy, z definicji $\tilde{\neg}' V'(\psi) = \tilde{\neg}' b = c \notin \{1, 0\}$. Sprzeczność.

  Konkretny kontrprzykład:

  - $\varphi := \neg(p \equiv \neg p)$
  - $U := \{a, b\}$
  - $D := \{b\}$
  - $V(p) := V(p \equiv \neg p) := a$
  - $V(\neg p) := V(\neg(p \equiv \neg p)) := b$
  - Konstruujemy $\mathcal{M}', V'$ jak wyżej
  - $\psi := \neg(p \equiv \neg p)$
  - $V'(\neg\psi) = 0$, bo $\neg\psi \notin \mathtt{SUB}(\varphi)$
  - $V'(\psi) = a$
  - $\tilde{\neg}' a = \tilde{\neg} a = b$, bo istnieje $\chi = p$ t.że $a = V(\chi)$ i $\neg\chi \in \mathtt{SUB}(\varphi)$
  - więc $\tilde{\neg}' V'(\psi) = b \neq 0$ – sprzeczność.

  Trzeba chyba skomplikować definicję $V'(\psi)$ rozbijając ψ. Np. jeśli $\exists\chi : \neg\chi = \psi, \chi \in \mathtt{SUB}(\varphi)$ to $V'(\psi) = \tilde{\neg}' V'(\chi)$. Wtedy V' przestaje mieć tą ładną właściwość, że $\psi \notin \mathtt{SUB}(\varphi) \Rightarrow V'(\psi) \in \{1, 0\}$, np. w przykładzie u góry $V'(\neg\neg(p \equiv \neg p))$ będzie równe a. Ale nie ma to wpływu na liczność modelu i wydaje mi się że będzie się wszystko zgadzać. A może da się jakoś prościej?

- $a \tilde{\rightarrow}' b \in D'$ if and only if $a \notin D'$ or $b \in D'$ – there are cases:

  1° TODO There exists ψ and ϑ such that $a = v(\psi), b = v(\vartheta)$ and $\psi \rightarrow \vartheta \in \mathtt{SUB}(\varphi)$. Then from the definition of $\tilde{\rightarrow}'$ we have that $a \tilde{\rightarrow}' b \in D' \iff a \tilde{\rightarrow} b \in D$, from the definition of $\tilde{\rightarrow}$ we have that $a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$, which from the definition of D' is if and only if $a \notin D'$ or $b \in D'$.

  2° TODO There are no such ψ and ϑ and $a \notin D$. Then, from the definition of $\tilde{\rightarrow}'$ we have that $a \tilde{\rightarrow}' b = 1 \in D'$ and from the definition of D' we have that $a \notin D'$.

  3° TODO There are no such ψ and ϑ and $b \in D$. Then again, from the definition of $\tilde{\rightarrow}'$ we have that $a \tilde{\rightarrow}' b = 1 \in D'$ and from the definition of D' we have that $b \in D'$.

- ... TODO

$\square$

Theorem 5 (Decidability of SCI)**.** SCI *is decidable.*

Proof. Let us take any SCI-formula φ. We want to know if φ is a SCI-theorem. From fullness and completeness of SCI we know that φ is a theorem if and only if $\neg\varphi$ has no SCI-model and valuation $\mathcal{M}, V$, such that $\mathcal{M}, V \models \neg\varphi$. But from theorem 4 we know that if there is $\mathcal{M}, V$, such that $\mathcal{M}, V \models \neg\varphi$, then there is a finite model $\mathcal{M}'$ and V' such that $\mathcal{M}', V' \models \neg\varphi$. Furthermore, from the proof of theorem 4 we can see that $|U'| \leq |\mathtt{SUB}(\varphi)|$ ($|X|$ here meaning the size of a set X).

For any finite SCI-model $\mathcal{M}'$ and SCI-valuation V' we can construct an isomorphic model $\mathcal{M}''$ with $U'' = \{1, 2, ..., |U'|\}$ by setting any bijection $f : U' \to U''$ and then setting the rest of the model and V'' accordingly. So, we know that if there is $\mathcal{M}, V$ such that $\mathcal{M}, V \models \neg\varphi$, then and only then there is a $\mathcal{M}'', V''$, such that $\mathcal{M}'', V'' \models \neg\varphi$ and $U'' = \{1, 2, ..., |\mathtt{SUB}(\neg\varphi)|\}$. There is a finite amount of such finite models.

We can have a procedure to set D'' to be any of $2^{|\mathtt{SUB}(\neg\varphi)|}$ subsets of U'', then for any $a \in U''$ to set $\tilde{\neg}a$ to be any element of U'', then for any $a, b \in U''$ to set $a\tilde{\to}b$ to be any element of U'', then for any $a, b \in U''$ to set $a\tilde{\equiv}b$ to be any element of U'' and then to set every variable of $\neg\varphi$ to be any element of U'' and finally to check, whether such constructed $\mathcal{M}'', V''$ is a SCI-model and a correct valuation. If it happens to be the case, then we have found a SCI-model invalidating φ. If after checking every such combination we have not found any SCI-model invalidating φ, we know that φ is valid, and so that it is a theorem.

If we set $n = s(\varphi)$, then this procedure will take $O(n^{2^n})$ operations, so it is very far from practical, but it is nevertheless finite, so we have a finite procedure to decide whether φ is a SCI-theorem.

$\square$

3 Deduction in SCI

3.1 Deduction systems

The axiomatization of SCI we presented in definition 2.6 is an example of a Hilbert-style deductive system. Such systems consist of a set of axioms and inference rules (often *modus ponens* is the only inference rule) that can be used to deduce theses, as explained in definition 2.7.

Hilbert-style systems are historically first and most fundamental deductive systems. For every thesis of such systems, there is a sequence of formulas all the way back to the axioms, from which a resulting thesis is inferred. But trying to prove some specific formula can be very challenging as it requires creative usages of the axioms so that the given formula is inferred at the end. Those proof systems give no clue how to do that.

The same is true for the deduction system presented in definitions 2.6 and 2.7. From theorem 5 we know that SCI is decidable, but the Hilbert-style deductive system gives no procedure to determine whether a given formula is a tautology. From theorem 5 we know there is a finite procedure to do that, but we are still missing a procedure that could be used in practice.

Let us show some examples to illustrate these points:

Example 3.1. Let us show a full proof of a formula: $(p \equiv q) \to (q \equiv p)$

1. $p \equiv p$ $\quad (Ax1)$
2. $(p \equiv q) \to ((p \equiv p) \to ((p \equiv p) \equiv (q \equiv p)))$ $\quad (Ax5)$
3. $((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))$ $\quad (Ax2)$
4. $(p \equiv p) \to (((p \equiv q) \to ((p \equiv p) \to ((p \equiv p) \equiv (q \equiv p)))) \to ((((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))) \to ((p \equiv q) \to (q \equiv p))))$

   PC axiom $\varphi \to ((\psi \to (\varphi \to \vartheta)) \to ((\vartheta \to (\varphi \to \chi)) \to (\psi \to \chi)))$
   with $\varphi = p \equiv p$
   $\psi = p \equiv q$
   $\vartheta = q \equiv p$
   $\chi = (p \equiv p) \equiv (q \equiv p)$.
5. $((p \equiv q) \to ((p \equiv p) \to ((p \equiv p) \equiv (q \equiv p)))) \to ((((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))) \to ((p \equiv q) \to (q \equiv p)))$ $\quad \mathsf{MP}(1, 4)$

6. $(((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))) \to ((p \equiv q) \to (q \equiv p))$ MP$(2, 5)$

7. $(p \equiv q) \to (q \equiv p)$ MP$(3, 6)$

As can be seen, although the finished proof isn't very long, it would be very difficult to come up with the precise axioms to use.

Example 3.2. Let us take a formula $p \equiv q$. We could be deducing many theses from our axioms, but at no point will we deduce it, nor $p \not\equiv q$, because those formulas aren't tautologies. But we cannot arrive at this conclusion using just our Hilbert-style system.

Example 3.3. Let us show a full proof of a formula $(q \equiv r) \to ((p \equiv q) \to (p \equiv r))$.

1. $(((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))) \to ((q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))))$

   PC axiom $\varphi \to (\psi \to \varphi)$
   with $\varphi = ((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))$
   $\psi = q \equiv r$

2. $((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))$ $(Ax2)$

3. $(q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r)))$ MP$(2, 1)$

4. $((q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r)))) \to (((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))) \to ((q \equiv r) \to ((p \equiv q) \to (p \equiv r))))$

   PC axiom $(\varphi \to (\psi \to \vartheta)) \to ((\varphi \to \psi) \to (\varphi \to \vartheta))$
   with $\varphi = q \equiv r$
   $\psi = (p \equiv q) \equiv (p \equiv r)$
   $\vartheta = (p \equiv q) \to (p \equiv r)$

5. $((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))) \to ((q \equiv r) \to ((p \equiv q) \to (p \equiv r)))$ MP$(3, 4)$

6. $p \equiv p$ $(Ax1)$

7. $(p \equiv p) \to ((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r)))$ $(Ax5)$

8. $(q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))$ MP$(6, 7)$

9. $(q \equiv r) \to ((p \equiv q) \to (p \equiv r))$ MP$(8, 5)$

In 1974 Michaels presented a differet proof system for SCI, in the style of Gentzen's sequent calculus [2]. It was refined by Wasilewska and presented in 1976 as a decision procedure [3]. Although the procedure presented there ws proved to terminate for every input formula, it was too complicated and, if implemented, would have too big time comlexity to be practical. In 2018 Chlebowski further modified and discussed the sequent calculus system, but his result was not a decision procedure.

In 2007 Golińska-Pilarek presented a proof system in the style of Rasiowa-Sikorski dual tableau [5] it was expanded and refined in [6] and [7], but it was not yet a terminating decision procedure.

An important difference between both the above proof systems and the Hilbert style system showed before is that the former start with the formula we want to prove or disprove and at each step offer a finite amount of possible rules to apply. Even though they were not a decision procedures, they could be helpful when manually proving some formulas. Finally, they were important steps at arriving at the full decision procedure of SCI.

The a full decision procedure $\mathsf{T}_{\mathsf{SCI}}$ was presented in 2021 by Golińska-Pilarek, Huuskonen and Zawidzki [8]. Also in 2021 the dual tableau system mentioned above was modified by the same authors to become a terminating decision procedure $\mathsf{DT}_{\mathsf{SCI}}$ [9].

Let us now take a brief look at both $\mathsf{T}_{\mathsf{SCI}}$ and $\mathsf{DT}_{\mathsf{SCI}}$, starting with the latter as it is easier to present.

3.2 $\mathsf{DT}_{\mathsf{SCI}}$

TODO

$$(\neg^+)\quad \frac{w^+ : \neg\varphi}{v^- : \varphi} \qquad\qquad (\neg^-)\quad \frac{w^- : \neg\varphi}{v^+ : \varphi}$$

$$(\to^+)\quad \frac{w^+ : \varphi \to \psi}{\begin{array}{c|c|c} v^- : \varphi & v^- : \varphi & v^+ : \varphi \\ u^- : \psi & u^+ : \psi & u^+ : \psi \end{array}} \qquad\qquad (\to^-)\quad \frac{w^- : \varphi \to \psi}{\begin{array}{c} v^+ : \varphi \\ u^- : \psi \end{array}}$$

$$(\equiv^+)\quad \frac{w^+ : \varphi \equiv \psi}{\begin{array}{c|c} v^+ : \varphi & v^- : \varphi \\ u^+ : \psi & u^- : \psi \\ v^+ = u^+ & v^- = u^- \end{array}} \qquad (\equiv^-)\quad \frac{w^- : \varphi \equiv \psi}{\begin{array}{c|c|c|c} v^+ : \varphi & v^+ : \varphi & v^- : \varphi & v^- : \varphi \\ u^+ : \psi & u^- : \psi & u^+ : \psi & u^- : \psi \\ v^+ \neq u^+ & & & v^- \neq u^- \end{array}}$$

$$(\equiv^\neg)\quad \frac{\begin{array}{c} \varphi \approx \psi \\ u : \neg\varphi \\ y : \neg\psi \end{array}}{u = y} \qquad (\equiv^\to)\quad \frac{\begin{array}{c} \varphi \approx \psi \\ \chi \approx \vartheta \\ x : \varphi \to \chi \\ z : \psi \to \vartheta \end{array}}{x = z} \qquad (\equiv^\equiv)\quad \frac{\begin{array}{c} \varphi \approx \psi \\ \chi \approx \vartheta \\ x : \varphi \equiv \chi \\ z : \psi \equiv \vartheta \end{array}}{x = z} \qquad (\mathtt{F})\quad \frac{\begin{array}{c} w : \varphi \\ v : \varphi \end{array}}{w = v}$$

$$(\mathtt{sym})\quad \frac{w = v}{v = w} \qquad (\mathtt{tran})\quad \frac{\begin{array}{c} w = v \\ v = u \end{array}}{w = u} \qquad (\bot_1)\quad \frac{\begin{array}{c} w = v \\ w \neq v \end{array}}{\bot} \qquad (\bot_2)\quad \frac{w^+ = v^-}{\bot}$$

Figure 1: Deduction rules of the $\mathsf{T}_{\mathsf{SCI}}$ system.

3.3 $\mathsf{T}_{\mathsf{SCI}}$

4 $\mathsf{T}^*_{\mathsf{SCI}}$

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