

# Decision procedures for a non-Fregean logic SCI

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August 13, 2024

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## 1 Sentential logic and the Fregean axiom

## 2 SCI

### 2.1 Basic notions

**Definition 2.1** (Alphabet of SCI). The alphabet of SCI is as follows:

- $V = \{p, q, r, \dots\}$  – a countable infinite set of atomic formulas,
- $\neg$  – unary operator “not”,
- $\rightarrow, \equiv$  – binary operators “implication” and “identity”.

**Definition 2.2** (Language of SCI). The language of SCI is a set of formulas **FOR** defined with the grammar:

$$\text{FOR} \ni \varphi ::= p \mid \neg \varphi \mid (\varphi \rightarrow \varphi) \mid (\varphi \equiv \varphi)$$

where  $p \in V$  is an atomic formula.

From now on whenever we write  $p, q, r, \dots$  we will mean the atomic formulas. We will omit brackets when obvious. We will write  $\varphi \not\equiv \psi$  as a shorthand for  $\neg(\varphi \equiv \psi)$ .

The set of identities **ID** is a set of formulas  $\varphi \equiv \psi$  where  $\varphi, \psi \in \text{FOR}$ . Formulas  $\varphi \equiv \varphi$  are the trivial identities.

**Definition 2.3** (Subformulas). For a formula  $\varphi \in \text{FOR}$  let us define the set of subformulas of  $\varphi$  as:

$$\text{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in V, \\ \{\varphi\} \cup \text{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \text{SUB}(\psi) \cup \text{SUB}(\vartheta), & \text{if } \varphi = \psi \rightarrow \vartheta, \text{ or } \varphi = \psi \equiv \vartheta \end{cases}$$

By  $\varphi(\psi/\vartheta)$  we will denote the formula  $\varphi$  with all occurrences of its subformula  $\psi$  substituted with  $\vartheta$ .

**Definition 2.4** (Simple formulas). The formula  $\varphi$  is called a simple formula if it is equal to one of:

$$p, \neg p, p \equiv q, p \not\equiv q, p \equiv \neg q, p \equiv (q \rightarrow r), p \equiv (q \equiv r)$$

for  $p, q, r \in V$ .

**Definition 2.5** (Size of formula). Given a formula  $\varphi$ , let us define its size  $s(\varphi)$ :

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta, \text{ or } \varphi = \psi \rightarrow \vartheta. \end{cases}$$

Since  $V$  is a countable set we can assume it has a full ordering  $<$ . Let us extend it by saying that for each  $p \in V$ :  $p < \neg$  < “(” < “)” < “ $\rightarrow$ ” < “ $\equiv$ ”. Now, let us define an ordering of formulas  $\prec$  to be a lexicographical ordering with  $<$ .

**Definition 2.6** (Axiomatization of SCI). SCI is axiomatized with the following axioms:

- (Ax1)  $\varphi \rightarrow (\psi \rightarrow \varphi)$
- (Ax2)  $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$

- (Ax3)  $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$
- (Ax4)  $\varphi \equiv \varphi$ ,
- (Ax5)  $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$ ,
- (Ax6)  $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$ ,
- (Ax7)  $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \theta)))$ ,
- (Ax8)  $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$ .

The only inference rule is the *modus ponens* rule:

$$MP : \frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

**Definition 2.7** (A theorem). A formula  $\varphi$  is a *theorem* if there exists a finite set of formulas  $\varphi_1, \dots, \varphi_n$  ( $n \geq 1$ ), such that  $\varphi = \varphi_n$ , and for all  $i \in \{1, \dots, n\}$  the formula  $\varphi_i$  is either an axiom of SCI, or it is inferred from formulas  $\varphi_j, \varphi_k$  ( $j, k < i$ ) via the *modus ponens* rule. If  $\varphi$  is a theorem we'll mark it as  $\vdash \varphi$  and say that it is *provable*.

Let us give an example of a provable formula with its full proof:

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**Definition 2.8** (Decidability). A logic is *decidable* if there exists an effective method to determine whether a given formula of this logic is a theorem.

Let us now give some semantic definitions:

**Definition 2.9** (Model). A SCI model is a structure  $M = (U, D, \neg, \rightarrow, \equiv)$  where:

- $U \neq \emptyset$  is the *universe* of  $M$ ,
- $\emptyset \neq D \subsetneq U$  is the *set of designated values*,
- $\neg : U \rightarrow U$  is the algebraic counterpart of the operator not, such that  $\forall a \in U$ :  $\neg a \in D$  iff  $a \notin D$ ,
- $\rightarrow : U \times U \rightarrow U$  is the algebraic counterpart of the operator of implication, such that  $\forall a, b \in U$ :  $a \rightarrow b \in D$  iff  $a \notin D$  or  $b \in D$ ,
- $\equiv : U \times U \rightarrow U$  is the algebraic counterpart of the operator of identity, such that  $\forall a, b \in U$ :  $a \equiv b$  iff  $a = b$  (that is, iff  $a$  and  $b$  denote the same element of the universe  $U$ ).

**Definition 2.10** (Valuation). Given a model  $M = (U, D, \neg, \rightarrow, \equiv)$ , a valuation in this model is a function  $V : \text{FOR} \rightarrow U$ , such that for all  $\varphi, \psi \in \text{FOR}$ :

- $V(\neg\varphi) = \neg V(\varphi)$
- $V(\varphi \rightarrow \psi) = V(\varphi) \rightarrow V(\psi)$
- $V(\varphi \equiv \psi) = V(\varphi) \equiv V(\psi)$

If  $V(\varphi) = a$  we will call  $a$  the *denotation* of  $\varphi$ .

**Definition 2.11** (Satisfiability of a formula). Given a model  $M = (U, D, \neg, \rightarrow, \equiv)$  and a valuation  $V$ , a formula  $\varphi$  is *satisfied in this valuation* iff its valuation belongs to  $D$ . If a formula  $\varphi$  is satisfied by a valuation  $V$  in a model  $M$  we will mark it by  $M, V \models \varphi$ .

**Definition 2.12** (Truth of a formula). Given a model  $M = (U, D, \neg, \rightarrow, \equiv)$ , a formula is *true in this model* iff it is satisfied in all valuations of this model. If a formula  $\varphi$  is true in a model  $M$  we'll mark it by  $M \models \varphi$ .

**Definition 2.13** (Validity of a formula). A formula is *valid* iff it is true in all models. If a formula  $\varphi$  is valid we'll mark it by  $\models \varphi$ .

Let us see some examples:

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**Theorem 1** (Correctness and fullness of SCI). *For any formula  $\varphi$  the following two are equivalent:*

- $\varphi$  is *provable*,
- $\varphi$  is *valid*.

The proof of the correctness and fullness theorem can be found in [1] (Theorem 1.9).

**Theorem 2** (Decidability of SCI). *SCI is decidable.*

The proof of the decidability theorem can be found in [1] (Corollary 2.4).

## References

- [1] Stephen L. Bloom, Roman Suszko (1972) *Investigations into the Sentential Calculus with Identity*, Notre Dame Journal of Formal Logic, vol. XIII, no 3.