Decision procedures for a non-Fregean logic SCI

Adrian Siwiec

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Test: ążźćłóęń

1 Sentential logic and the Fregean axiom

2 SCI

2.1 Basic notions

Definition 2.1 (Vocabulary of SCI)**.** The vocabulary of SCI consists of symbols from the following pairwise disjoint sets:

- $\mathtt{V} = \{p, q, r, ...\}$ – a countable infinite set of propositional variables,
- $\{\neg, \rightarrow, \equiv\}$ – the set consisting of the unary operator of negation ($\neg$) and binary operators of implication ($\rightarrow$) and identity ($\equiv$),
- $\{(,)\}$ – the set of auxiliary symbols.

Definition 2.2 (Formulas of SCI)**.** The set of formulas of SCI is defined with the following grammar:

$$\mathtt{FOR} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \rightarrow \varphi) \mid (\varphi \equiv \varphi)$$

where $p \in \mathtt{V}$ is a propositional variable.

The propositional variables will also be called atomic formulas.

From now on whenever we write $p, q, r, ...$ we will mean the atomic formulas. We will omit brackets when it will lead to no misunderstanding. We will write $\varphi \not\equiv \psi$ as a shorthand for $\neg(\varphi \equiv \psi)$.

The set of identities $\mathtt{ID}$ is a set of formulas $\varphi \equiv \psi$ where $\varphi, \psi \in \mathtt{FOR}$. Formulas $\varphi \equiv \varphi$ are the trivial identities.

Definition 2.3 (Subformulas)**.** For a formula $\varphi \in \mathtt{FOR}$ let us define the set of subformulas of φ as:

$$\mathtt{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in \mathtt{V}, \\ \{\varphi\} \cup \mathtt{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \mathtt{SUB}(\psi) \cup \mathtt{SUB}(\vartheta), & \text{if } \varphi = \psi \rightarrow \vartheta, \text{or } \varphi = \psi \equiv \vartheta \end{cases}$$

By $\varphi(\psi/\vartheta)$ we will denote the formula φ with all occurrences of its subformula ψ substituted with ϑ.

Definition 2.4 (Simple formulas)**.** The formula φ is called a simple formula if it has one of the following form:

$$p,\ \neg p,\ p \equiv q,\ p \not\equiv q,\ p \equiv \neg q,\ p \equiv (q \rightarrow r),\ p \equiv (q \equiv r)$$

for $p, q, r \in \mathtt{V}$.

Definition 2.5 (Size of a formula)**.** Given a formula φ, let us define its size $s(\varphi)$:

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta,\ \text{or } \varphi = \psi \rightarrow \vartheta. \end{cases}$$

$\mathtt{V}$ is a countable set. Let us take any full ordering of it and mark it as $<$. Let us then extend it by saying that for each $p \in \mathtt{V}$: $p < \neg < (<) < \to < \equiv$. Now, let us define an ordering of formulas $\prec$ by saying that if $s(\varphi) < s(\psi)$ then $\varphi \prec \psi$ and if $s(\varphi) = s(\psi)$ then $\varphi \prec \psi$ if it is earlier within the lexicographical ordering with $<$.

If we consider formulas that contain only the negation and implication operators, they form the classical Propositional Calculus. For simplicity, in SCI we'll consider every tautology of the classical Propositional Calculus to be an axiom.

Definition 2.6 (Axiomatization of SCI)**.** SCI is axiomatized with the following axioms:

- Any tautology of the classical Propositional Calculus
- $(Ax1)$ $\varphi \equiv \varphi$,
- $(Ax2)$ $(\varphi \equiv \psi) \to (\varphi \to \psi)$,
- $(Ax3)$ $(\varphi \equiv \psi) \to (\neg\varphi \equiv \neg\psi)$,
- $(Ax4)$ $(\varphi \equiv \psi) \to ((\chi \equiv \theta) \to ((\varphi \to \chi) \equiv (\psi \to \theta)))$,
- $(Ax5)$ $(\varphi \equiv \psi) \to ((\chi \equiv \theta) \to ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$.

The only inference rule is the *modus ponens* rule:

$$\mathsf{MP} : \frac{\varphi,\ \varphi \to \psi}{\psi}$$

Definition 2.7 (A thesis of SCI)**.** A SCI-formula φ is *a thesis of* SCI if there exists a finite sequence of formulas $\Psi = \varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is a thesis of SCI we will denote it by $\vdash \varphi$, say that φ is *provable in* SCI and call the sequence Ψ the *proof of* φ.

Remark 2.1. If we have a proof of some formula φ and take a variable p appearing in this formula, we can easily obtain a proof of any formula $\varphi(p/\psi)$. To do it, we simply take the proof of φ and replace p with ψ in every formula of the proof. We can do it with axioms, because the resulting formula is still an instance of the same given axiom and all the *modus ponens* inferences are still correct.

Definition 2.8 (Derivability in SCI)**.** Given a set of SCI-formulas X, a formula φ is *derivable from* X if there exists a finite sequence of formulas $\Psi = \varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, is an element of X, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is derivable from X we will denote it by $X \vdash \varphi$ and call the sequence Ψ the *proof of* φ *from* X.

In terms of derivability we can see that a formula is a thesis of SCI if and only if it is derivable from the empty set.

Remark 2.2. For every $\varphi \in \mathtt{FOR}$ and $X, Y \subseteq \mathtt{FOR}$, if $X \subseteq Y$ and $X \vdash \varphi$, then $Y \vdash \varphi$. This is because the proof of φ from X is also a proof of φ from Y.

Remark 2.3. For every $\varphi, \psi \in \mathtt{FOR}$, $\{\varphi, \neg\varphi\} \vdash \psi$.

Proof. $\varphi \to (\neg\varphi \to \psi)$ is an instance of a PC tautology $\alpha \to (\neg\alpha \to \beta)$, so a sequence $\varphi \to (\neg\varphi \to \psi)$, $\neg\varphi \to \psi$, ψ is a proof of ψ in $\{\varphi, \neg\varphi\}$. □

Remark 2.4. For every $\varphi, \psi \in \mathtt{FOR}, X \subseteq \mathtt{FOR}$, we have that $X \vdash \psi$ if and only if $X \vdash \varphi \to \psi$ and $X \vdash \neg\varphi \to \psi$.

Proof. Left to right. Let Ψ be a proof of ψ in X. By adding to it a formula $\psi \to (\varphi \to \psi)$, which is an instance of a PC tautology $\alpha \to (\beta \to \alpha)$ we can infer $\varphi \to \psi$ arriving at proof of $\varphi \to \psi$ in X. In the same way we can arrive at the proof of $\neg\varphi \to \psi$ in X.

Right to left. We have proofs of $\varphi \to \psi$ and $\neg\varphi \to \psi$ in X. Let Ψ be a sequence resulting from appending one after the other. Let us append to it a formula $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)$, which is an instance of a PC tautology $(\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)$. Then we by *modus ponens* we can infer $(\neg\varphi \to \psi) \to \psi$ and then ψ thus arriving at the proof of ψ in X. □

Remark 2.5 (Deduction theorem for SCI). For all $X \subseteq \mathtt{FOR}$ and $\varphi, \psi \in \mathtt{FOR}$: $X \cup \{\varphi\} \vdash \psi$ if and only if $X \vdash \varphi \to \psi$.

Proof. First, let us prove the implication from right to left. Let us assume that $X \vdash \varphi \to \psi$. Let $\varphi_1, ..., \varphi_n$ be a proof of $\varphi \to \psi$ in X. Let us set $\varphi_{n+1} = \varphi$ and $\varphi_{n+2} = \psi$. Then $\varphi_{n+1} \in X \cup \{\varphi\}$ and φ_{n+2} is derived from φ_{n+1} and φ_n via the *modus ponens* rule, so the sequence $\varphi_1, ..., \varphi_n, \varphi_{n+1}, \varphi_{n+2}$ is a proof of ψ in $X \cup \{\varphi\}$.

The proof in the other direction similarly involves modifying the proof, but is slightly more complicated.

Let us assume that $X \cup \{\varphi\} \vdash \psi$ and let $\varphi_1, ..., \varphi_n$ be a proof of ψ in $X \cup \{\varphi\}$. We will describe a procedure to modify it to be a proof of $\varphi \to \psi$ in X.

We will construct a sequence of formulas Ψ by considering each φ_i in turn. We'll uphold an invariant such that after considering φ_i, Ψ will contain a proof of $\varphi \to \varphi_i$.

- If φ_i is an axiom of SCI, or $\varphi_i \in X$, we append it to Ψ. Then we append to Ψ a formula $\varphi_i \to (\varphi \to \varphi_i)$, which is an instance of a PC tautology $\alpha \to (\beta \to \alpha)$ and by *modus ponens* infer $\varphi \to \varphi_i$.
- If $\varphi_i = \varphi$, we add to Ψ a formula $\varphi \to \varphi$, which is an instance of a PC tautology $\alpha \to \alpha$,
- Otherwise φ_i was inferred by *modus ponens* from some formulas $\vartheta, \vartheta \to \varphi_i$, such that ϑ and $\vartheta \to \varphi_i$ were considered before. By our invariant $\varphi \to \vartheta$ and $\varphi \to (\vartheta \to \varphi_i)$ have their proofs in Ψ already.

  We append to Ψ a formula $(\varphi \to \vartheta) \to ((\varphi \to (\vartheta \to \varphi_i)) \to (\varphi \to \varphi_i))$, which is an instance of a PC tautology $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$ and then by *modus ponens* infer $(\varphi \to (\vartheta \to \varphi_i)) \to (\varphi \to \varphi_i)$ and then $\varphi \to \varphi_i$.

It is easy to see that at each step we uphold our invariant and φ is not directly used in the proof so at the end Ψ is a proof of $\varphi \to \psi$ in X.

□

Let us now give some semantic definitions:

Definition 2.9 (SCI-model). A model of SCI (or an SCI-model) model is a structure $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$ where:

- $U \neq \emptyset$ is an *universe* of M,
- $\emptyset \neq D \subsetneq U$ is a *set of designated values*,
- $\tilde{\neg}$ is an unary operation on U, such that for all $a \in U$: $\tilde{\neg} a \in D$ if and only if $a \notin D$,
- $\tilde{\to}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\to} b \in D$ if and only if $a \notin D$ or $b \in D$,
- $\tilde{\equiv}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\equiv} b \in D$ if and only if $a = b$.

If an universe U is finite, we'll call a given SCI-model a *finite* SCI-*model.*

Definition 2.10 (Valuation). Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, a valuation in $\mathcal{M}$ is a function $V : \mathtt{FOR} \longrightarrow U$ that assigns a value $V(p) \in U$ for all propositional variables p, and such that for all $\varphi, \psi \in \mathtt{FOR}$:

- $V(\neg\varphi) = \tilde{\neg} V(\varphi)$
- $V(\varphi \to \psi) = V(\varphi) \tilde{\to} V(\psi)$
- $V(\varphi \equiv \psi) = V(\varphi) \tilde{\equiv} V(\psi)$

If $V(\varphi) = a$ we will call a the *denotation of* φ.

Definition 2.11 (Satisfaction of a formula). Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$ and a valuation V in $\mathcal{M}$, a formula φ is *satisfied in* $\mathcal{M}$ *by* V if and only if $V(\varphi) \in D$. If a formula φ is satisfied in $\mathcal{M}$ by a valuation V, then will denote it by $M, V \models \varphi$.

Definition 2.12 (Truth of a formula). Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, a formula is *true in* $\mathcal{M}$ if and only if it is satisfied in $\mathcal{M}$ by all valuations of $\mathcal{M}$. If a formula φ is true in $\mathcal{M}$, then we will denote it by $\mathcal{M} \models \varphi$.

Definition 2.13 (Validity of a formula)**.** A formula is *valid in* SCI (or SCI-*valid*) if and only if it is true in all models. If a formula φ is SCI-valid we will denote it by $\models \varphi$.

Theorem 1 (Soundness of SCI)**.** *For every* SCI*-formula* φ*, if* φ *is provable in* SCI*, then* φ *is valid in* SCI*.*

Proof. The proof will have two main parts:

(a) Every axiom of SCI is valid in SCI.

(b) By applying the *modus ponens* rule to valid formulas φ and $\varphi \to \psi$ the inferred formula ψ is also valid.

Let us show (a) first:

- First, we need to prove that every tautology of the classical Propositional Calculus (PC) is valid in SCI.

  Let φ be a tautology of PC. Let us take any SCI-model $\mathcal{M}$ and any SCI-valuation $\mathcal{M}, V$ in it. We want to show that $\mathcal{M}, V \models \varphi$.

  Given $\mathcal{M}, V$, we will construct V' that will be a PC valuation of φ. We can do this, because φ doesn't contain operator $\equiv$.

  For every $\psi \in \mathtt{SUB}(\varphi)$, let us define:

$$V'(\psi) = \begin{cases} 1, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \in D, \\ 0, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \notin D, \\ 1 - V'(\chi), & \text{if } \psi = \neg\chi, \\ max(1 - V'(\chi), V'(\theta)), & \text{if } \psi = \chi \to \theta. \end{cases}$$

  It is easy to see that for every $\psi \in \mathtt{SUB}(\varphi)$, $V'(\psi) = 1$ if and only if $V(\psi) \in D$.

  The V' function is constructed in the same way the valuation function in the PC is constructed, therefore since φ is a tautology of PC, we have that $V'(\varphi) = 1$. So, we have that $V(\varphi) \in D$, what is what we wanted to show.

- Second, we want to show that axioms $(Ax1) - (Ax5)$ are valid in SCI.

  – $(Ax1)$ $\varphi \equiv \varphi$

    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.10, we have that $V(\varphi \equiv \varphi) = V(\varphi) \tilde{\equiv} V(\varphi)$. Based on definition of $\tilde{\equiv}$ from 2.9, we have that $V(\varphi) \tilde{\equiv} V(\varphi)$ is in D if and only if $V(\varphi) = V(\varphi)$, which is trivially the case. So, we have that $\mathcal{M}, V \models \varphi \equiv \varphi$.

  – $(Ax2)$ $(\varphi \equiv \psi) \to (\varphi \to \psi)$

    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.10, we have that $V((\varphi \equiv \psi) \to (\varphi \to \psi)) = V(\varphi \equiv \psi) \tilde{\to} V(\varphi \to \psi)$. There are two cases.

    1° $V(\varphi \equiv \psi) \notin D$. Then, by definition of $\tilde{\to}$ in 2.9, we have that $V(\varphi \equiv \psi) \tilde{\to} V(\varphi \to \psi) \in D$, so $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

    2° $V(\varphi \equiv \psi) \in D$. By definition 2.10 it means, that $V(\varphi) \tilde{\equiv} V(\psi)$, which by definition 2.9 means that $V(\varphi) = V(\psi)$.
    By definition 2.10 $V(\varphi \to \psi)$ is equal to $V(\varphi) \tilde{\to} V(\psi)$. By definition 2.9 $V(\varphi) \tilde{\to} V(\psi)$ is in D if and only if $V(\varphi) \notin D$ or $V(\psi) \in D$, but since $V(\varphi) = V(\psi)$ we have that $V(\varphi) \tilde{\to} V(\psi)$ is in D if and only if $V(\varphi) \notin D$ or $V(\varphi) \in D$, which is trivially the case.

    So, we have that $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

  – Validity of axioms $(Ax3), (Ax4)$ and $(Ax5)$ can be shown in a similar way. (TODO: a może rozpisać dla jasności?)

To show (b), let us take any valid formulas φ and $\varphi \to \psi$ and any SCI model and valuation $\mathcal{M}, V$. From validity of $\varphi \to \psi$, we know that $V(\varphi \to \psi) \in D$. From definition 2.10 we have that $D \ni V(\varphi \to \psi) = V(\varphi) \tilde{\to} V(\psi)$. From definition 2.9, since we know that $V(\varphi) \tilde{\to} V(\psi)$ we know that $V(\varphi) \notin D$, or $V(\psi) \in D$. But we have that φ is valid, so $V(\varphi) \in D$, so it must be that $V(\psi) \in D$, so $\mathcal{M}, V \models \psi$. The same holds for any other SCI model and valuation, so $\models \psi$, which is what we wanted to show.

Now, looking at definition 2.7, for a given provable φ, let us take its proof $\varphi_1, ..., \varphi_n = \varphi$. Every subsequent formula in this proof is either an SCI axiom and thus, by (a), valid, or is inferred by the *modus ponens* rule from valid formulas and thus, by (b), valid. So, φ is valid. □

Theorem 2 (Completeness of SCI)**.** *For every* SCI*-formula* φ*, if* φ *is valid in* SCI*, then* φ *is provable in* SCI*.*

Proof. To show that every valid SCI-formula is SCI-provable is, by contraposition, to show that every non-SCI-provable formula is non-SCI-valid.

Let us take a non-SCI-provable formula $\hat{\varphi}$. We will construct an SCI-model and a valuation in which $\hat{\varphi}$ is not true, thus showing that it is not SCI-valid.

First, let us define a maximal consistent set of formulas of which $\hat{\varphi}$ will not be an element of. Let $\varphi_1, \varphi_2, ...$ be all formulas of $\mathtt{FOR}$ in order of $\prec$, and then let us define:

- $d_0 = \emptyset$
- For $i \in \mathbb{N}_+$:
$$d_i = \begin{cases} d_{i-1} \cup \{\varphi_i\}, & \text{if } d_{i-1} \cup \{\varphi_i\} \nvdash \hat{\varphi} \\ d_{i-1}, & \text{otherwise} \end{cases}$$
- $d = \bigcup d_i$

Let us show some properties of d_i and d:

Remark 2.6. $d \nvdash \hat{\varphi}$. Since $\hat{\varphi}$ is not SCI-provable we know that $d_0 \nvdash \hat{\varphi}$, and each subsequent d_i is constructed in such a way that $d_i \nvdash \hat{\varphi}$.

Remark 2.7. For all $\varphi_i \in \mathtt{FOR}$: $\varphi_i \in d$ if and only if $d \vdash \varphi_i$. Left to right is trivial. Right to left: since we know that $d \nvdash \hat{\varphi}$ and we have that $d \vdash \varphi_i$ we have that $d \cup \{\varphi_i\} \nvdash \hat{\varphi}$, so $d_{i-1} \cup \{\varphi_i\} \nvdash \hat{\varphi}$, so $d_i = d_{i-1} \cup \{\varphi_i\}$.

Remark 2.8. For all $\varphi_i \in \mathtt{FOR}$ such that $\vdash \varphi_i$, $\varphi_i \in d$. This is trivially true from remark 2.7.

Let us now define a relation R on SCI-formulas:

$$\varphi R \psi \text{ if and only if } \varphi \equiv \psi \in d$$

Note that for clarity we're using infix notation for relations. We now want to show that R is an equivalence relation.

- R is reflexive. Trivial from axiom $(Ax1)$ and remark 2.8.
- R is symmetrical, that is for all $\varphi, \psi \in \mathtt{FOR}, \varphi R \psi$ if and only if $\psi R \varphi$.

  From example 3.1 below and remark 2.1 we can prove that for all $\varphi, \psi \in \mathtt{FOR}$, $\vdash (\varphi \equiv \psi) \to (\psi \equiv \varphi)$. If $\varphi R \psi$, then $d \vdash \varphi \equiv \psi$, so by *modus ponens* we can infer $d \vdash \psi \equiv \varphi$, so $\psi R \varphi$. In the same way we can show that if $\psi R \varphi$ then $\varphi R \psi$, so R is symmetrical.
- R is transitive, that is for all $\varphi, \psi, \vartheta \in \mathtt{FOR}$, if $\varphi R \psi$ and $\psi R \vartheta$ then $\varphi R \vartheta$.

  From example 3.3 below and remark 2.1, we can prove that for all $\varphi, \psi, \vartheta \in \mathtt{FOR}$, $\vdash (\psi \equiv \vartheta) \to ((\varphi \equiv \psi) \to (\varphi \equiv \vartheta))$. Let us assume that $\varphi R \psi$ and $\psi R \vartheta$. This gives us that $d \vdash \varphi \equiv \psi$ and $d \vdash \psi \equiv \vartheta$. Applying *modus ponens* twice we get $d \vdash \varphi \equiv \vartheta$, so $\varphi R \vartheta$.

So, R is an equivalence relation. Let us now show that it is a congruence, that is:

- For all $\varphi, \psi \in \mathtt{FOR}$, if $\varphi R \psi$, then $\neg\varphi R \neg\psi$. This is trivially true from axiom $(Ax3)$ and remark 2.8.
- For all $\varphi, \psi, \vartheta, \chi \in \mathtt{FOR}$, if $\varphi R \psi$ and $\vartheta R \chi$, then $\varphi \to \vartheta R \psi \to \chi$. This is easily proven from $(Ax4)$ and remark 2.8.
- For all $\varphi, \psi, \vartheta, \chi \in \mathtt{FOR}$, if $\varphi R \psi$ and $\vartheta R \chi$, then $\varphi \equiv \vartheta R \psi \equiv \chi$. This is easily proven from $(Ax5)$ and remark 2.8.

Now that we know that R is a congruent equivalence relation, let us define:

- $U = \{|\varphi|_R : \varphi \in \mathtt{FOR}\}$ (by $|\varphi|_R$ we mean an equivalence class of R of which φ is a member),
- $D = \{|\varphi|_R : \varphi \in d\}$
- $\tilde{\neg}$ is an unary function on equivalence classes of R, such that $\tilde{\neg}|\varphi|_R \stackrel{def}{=} |\neg\varphi|_R$
- $\tilde{\to}$ is a binary function on equivalence classes of R, such that $|\varphi|_R \tilde{\to} |\psi|_R \stackrel{def}{=} |\varphi \to \psi|_R$

- $\tilde{\equiv}$ is a binary function on equivalence classes of R, such that $|\varphi|_R \tilde{\equiv} |\psi|_R \stackrel{def}{=} |\varphi \equiv \psi|_R$

Let us add some comments:

- For any SCI-formulas φ, ψ, such that $\varphi \equiv \psi$, $d \vdash \varphi$ if and only if $d \vdash \psi$. This follows from $(Ax2)$. Because of that, $|\varphi|_R = |\psi|_R$ and so D is well defined.
- From congruence of R, we know that for any $\varphi, \psi \in \mathtt{FOR}$ if $|\varphi|_R = |\psi|_R$, then $|\neg\varphi|_R = |\neg\psi|_R$, so $\tilde{\neg}$ is a well defined function.
- Similarly, form the congruence of R we can see that $\tilde{\rightarrow}$ and $\tilde{\equiv}$ are well defined functions.

Now we will want to show that $\mathcal{M}$ is an SCI-model. From the definition 2.9 we want to show that:

- $U \neq \emptyset$ – this is trivial, $|p|_R \in U$
- $\emptyset \neq D$ – this is true, because for example $\vdash p \equiv p$, so from remark 2.8 $|p \equiv p|_R \in D$
- $D \subsetneq U$ – $D \subseteq U$ is trivial from the definition of D. $D \neq U$, because $\hat{\varphi} \notin d$, because $\{\hat{\varphi}\} \vdash \hat{\varphi}$.
- For all $a \in U : \tilde{\neg}a \in D$ if and only if $a \notin D$.

  Let us take any $a \in U$. We know that there is a formula $\varphi \in \mathtt{FOR}$, such that $a = |\varphi|_R$. Let us take any such φ. From the definition of $\tilde{\neg}$ we know that $\tilde{\neg}|\varphi|_R = |\neg\varphi|_R$.

  We will show that $\neg\varphi \in d$ if and only if $\varphi \notin d$. Let us assume that $\varphi = \varphi_i$ and $\neg\varphi = \varphi_j$. Let's assume for now that $i < j$.

  There are two cases:

  - $\varphi_i \in d_i$. We want to show that φ_j was not added to d_j. From remark 2.3 we know that $\{\varphi, \neg\varphi\} \vdash \hat{\varphi}$ and so $d_{i-1} \cup \{\varphi_j\} \vdash \hat{\varphi}$. $d_{i-1} \subseteq d_{j-1}$ so $d_{j-1} \cup \{\varphi_j\} \vdash \hat{\varphi}$, so $\varphi_j \notin d_j$.
  - $\varphi_i \notin d_i$. We want to show that φ_j was added to d_j, that is that $d_{j-1} \cup \{\varphi_j\} \nvdash \hat{\varphi}$.

    Let us assume the opposite, that $d_{j-1} \cup \{\neg\varphi\} \vdash \hat{\varphi}$. From the deduction theorem we have that $d_{j-1} \vdash \neg\varphi \rightarrow \hat{\varphi}$. Since $\varphi_i \notin d_i$ we know that $d_{i-1} \cup \{\varphi\} \vdash \hat{\varphi}$. Because $d_{i-1} \subseteq d_{j-1}$ we have that $d_{j-1} \cup \{\varphi\} \vdash \hat{\varphi}$ and again from the deduction theorem $d_{j-1} \vdash \varphi \rightarrow \hat{\varphi}$.

    Let us take the proof of $\neg\varphi \rightarrow \hat{\varphi}$ in d_{j-1} and append to it the proof of $\varphi \rightarrow \hat{\varphi}$ creating a sequence Ψ. Then let us append to Ψ a formula $(\varphi \rightarrow \hat{\varphi}) \rightarrow ((\neg\varphi \rightarrow \hat{\varphi}) \rightarrow \hat{\varphi})$, which is an instance of a PC tautology $(\alpha \rightarrow \beta) \rightarrow ((\neg\alpha \rightarrow \beta) \rightarrow \beta)$. Then, by *modus ponens* we can infer $(\neg\varphi \rightarrow \hat{\varphi}) \rightarrow \hat{\varphi}$ and then $\hat{\varphi}$ thus proving that $d_{j-1} \vdash \hat{\varphi}$. But this is a contradiction, because we know that $d_{j-1} \nvdash \hat{\varphi}$. So our assumption that $d_{j-1} \cup \{\neg\varphi\} \vdash \hat{\varphi}$ was false, so $\varphi_j \in d_j$.

  We can see that if $i > j$ the proof will be analogous. Na pewno? Może rozpisać dla pewności?

- For all $a, b \in U : a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$.

  Let us take any $a, b \in U$. We know that there exist $\varphi, \psi \in \mathtt{FOR}$, such that $|\varphi|_R = a, |\psi|_R = b$. From the definition of $\tilde{\rightarrow}$ we have that $|\varphi|_R \tilde{\rightarrow} |\psi|_R = |\varphi \rightarrow \psi|_R$. So, we want to show that $|\varphi \rightarrow \psi|_R \in D$ if and only if $|\varphi|_R \notin D$ or $|\psi|_R \in D$, so to show that $\varphi \rightarrow \psi \in d$ if and only if $\varphi \notin d$ or $\psi \in d$. Let us assume that $\varphi = \varphi_i$, $\psi = \varphi_j$ and $\varphi \rightarrow \psi = \varphi_k$. Because $s(\varphi) < s(\varphi \rightarrow \psi)$ and $s(\psi) < s(\varphi \rightarrow \psi)$ we know that $i < k$ and $j < k$ which will slightly simplify the proof. There are three cases:

  - $\psi \in d$. We know that $d_{j-1} \cup \{\psi\} \nvdash \hat{\varphi}$ and want to show that $d_{k-1} \cup \{\varphi \rightarrow \psi\} \nvdash \hat{\varphi}$.

    Let us assume the opposite, that $d_{k-1} \cup \{\varphi \rightarrow \psi\} \vdash \hat{\varphi}$. By the deduction theorem we have that $d_{k-1} \vdash (\varphi \rightarrow \psi) \rightarrow \hat{\varphi}$. Let Ψ be the proof of $(\varphi \rightarrow \psi) \rightarrow \hat{\varphi}$ in d_{k-1}. Let us append to Ψ a formula $((\varphi \rightarrow \psi) \rightarrow \hat{\varphi}) \rightarrow (\psi \rightarrow \hat{\varphi})$. Then by modus ponens we can infer $\psi \rightarrow \hat{\varphi}$, so we have a proof of $\psi \rightarrow \hat{\varphi}$ in d_{k-1}. By the deduction theorem $d_{k-1} \cup \{\psi\} \vdash \hat{\varphi}$. But we know that $\psi \in d_{j-1}$ and because $d_{j-1} \subseteq d_{k-1}$, we know that $d_{k-1} \cup \{\psi\} = d_{k-1}$. So we have arrived at $d_{k-1} \vdash \hat{\varphi}$, which is a contradiction. So we know that $d_{k-1} \cup \{\varphi \rightarrow \psi\} \nvdash \hat{\varphi}$ so $\varphi \rightarrow \psi \in d$.
  - $\varphi \notin d$. The case when $\psi \in d$ is described above, so let us assume that $\psi \notin d$. We know that $d_{i-1} \cup \{\varphi\} \vdash \hat{\varphi}$ and $d_{j-1} \cup \{\psi\} \vdash \hat{\varphi}$, so by the deduction theorem we know that $d_{i-1} \vdash \varphi \rightarrow \hat{\varphi}$ and $d_{j-1} \vdash \psi \rightarrow \hat{\varphi}$, and because $i, j < k$ we know that $d_{k-1} \vdash \varphi \rightarrow \hat{\varphi}$ and $d_{k-1} \vdash \psi \rightarrow \hat{\varphi}$. We want to show that $\varphi \rightarrow \psi \in d$, that is $d_{k-1} \cup \{\varphi \rightarrow \psi\} \nvdash \hat{\varphi}$.

    Let us assume the opposite, that $d_{k-1} \cup \{\varphi \rightarrow \psi\} \vdash \hat{\varphi}$. By deduction theorem we have that $d_{k-1} \vdash (\varphi \rightarrow \psi) \rightarrow \hat{\varphi}$.

Let Ψ be a sequence we get by appending the proof of $\varphi \to \hat{\varphi}$ in d_{k-1} and the proof of $\psi \to \hat{\varphi}$ in d_{k-1} to the proof of $(\varphi \to \psi) \to \hat{\varphi}$ in d_{k-1}. Then, let us append to Ψ the formula $((\varphi \to \psi) \to \hat{\varphi}) \to ((\varphi \to \hat{\varphi}) \to ((\psi \to \hat{\varphi}) \to \hat{\varphi}))$, which is an instance of a PC tautology $((\alpha \to \beta) \to \gamma) \to ((\alpha \to \gamma) \to ((\beta \to \gamma) \to \gamma))$. Then by *modus ponens* we can infer $(\varphi \to \hat{\varphi}) \to ((\psi \to \hat{\varphi}) \to \hat{\varphi})$, then we can infer $(\psi \to \hat{\varphi}) \to \hat{\varphi}$ and finally we can infer $\hat{\varphi}$ thus proving that $d_{k-1} \vdash \hat{\varphi}$ which is a contradiction. So we know that $d_{k-1} \cup \{\varphi \to \psi\} \nvdash \hat{\varphi}$ so $\varphi \to \psi \in d$.

– $\varphi \in d, \psi \notin d$. We want to show that $\varphi \to \psi \notin d$. As we have shown above, from $\varphi \in d$ we know that $\neg\varphi \notin d$. Let $\neg\varphi = \varphi_l$. We know that $l < k$.

  So, we have that $d_{l-1} \cup \{\neg\varphi\} \vdash \hat{\varphi}$, $d_{j-1} \cup \{\psi\} \vdash \hat{\varphi}$ and want to show that $d_{k-1} \cup \{\varphi \to \psi\} \vdash \hat{\varphi}$. By the deduction theorem we have that $d_{l-1} \vdash \neg\varphi \to \hat{\varphi}$, $d_{j-1} \vdash \psi \to \hat{\varphi}$ and want to show that $d_{k-1} \vdash (\varphi \to \psi) \to \hat{\varphi}$.

  Let Ψ be a proof of $\neg\varphi \to \hat{\varphi}$ in d_{l-1} with a proof of $\psi \to \hat{\varphi}$ in d_{j-1} appended to it. Then let us append to Ψ a formula $(\neg\varphi \to \hat{\varphi}) \to ((\psi \to \hat{\varphi}) \to ((\varphi \to \psi) \to \hat{\varphi}))$ which is an instance of a PC theorem $(\neg\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \to \beta) \to \gamma))$. Then by *modus ponens* we can infer $(\psi \to \hat{\varphi}) \to ((\varphi \to \psi) \to \hat{\varphi})$ and $(\varphi \to \psi) \to \hat{\varphi}$ arriving at the proof of $(\varphi \to \psi) \to \hat{\varphi}$ in d_{k-1}, thus showing that $\varphi \to \psi \notin d$.

- For all $a, b \in U : a \tilde{\equiv} b \in D$ if and only if $a = b$.

  Let us take any $a, b \in U$. We know that there exist $\varphi, \psi \in \mathtt{FOR}$ such that $|\varphi|_R = a, |\psi|_R = b$. From the definition of $\tilde{\equiv}$ we have that $|\varphi|_R \tilde{\equiv} |\psi|_R \in D$ if and only if $|\varphi \equiv \psi|_R \in D$, that is if and only if $\varphi \equiv \psi \in d$.

  On the other hand, $a = b$ if and only if $|\varphi|_R = |\psi|_R$, which from the definition of R means that $d \vdash \varphi \equiv \psi$, which from remark 2.7 is the case if and only if $\varphi \equiv \psi \in d$.

So, $\mathcal{M}$ is a SCI-model. Now, let us define a valuation V for all $\varphi \in \mathtt{FOR}$:

$$V(\varphi) = |\varphi|_R$$

We need to show that V is indeed a valuation, that is that for all $\varphi, \psi \in \mathtt{FOR}$:

- $V(\neg\varphi) = \tilde{\neg} V(\varphi)$. From the definition of V, we want to show that $|\neg\varphi|_R = \tilde{\neg}|\varphi|_R$, but that is true from the definition of $\tilde{\neg}$.
- $V(\varphi \to \psi) = V(\varphi) \tilde{\to} V(\psi)$. Same as above, from definitions of V and $\tilde{\to}$.
- $V(\varphi \equiv \psi) = V(\varphi) \tilde{\equiv} V(\psi)$. Same as above, from definitions of V and $\tilde{\equiv}$.

Now, for the final proof of completeness let us take a non-provable formula φ. Since $\nvdash \varphi$ we have that $|\varphi|_R \notin D$, so $V(\varphi) \notin D$. So, we have found a model and a valuation $\mathcal{M}, V$, such that $\mathcal{M}, V \not\models \varphi$, so φ is not valid. □

Definition 2.14 (Decidability)**.** A logic is *decidable* if there exists an effective method, that is, an algorithm that will always terminate, to determine whether a given formula of this logic is a theorem.

We will want to show that SCI is decidable. To do this, first let us show the following:

Theorem 3 (Finite model property of SCI)**.** *For any $\varphi \in \mathtt{FOR}$, if there exists $\mathcal{M}, V$, such that $\mathcal{M}, V \models \varphi$, then there exists a finite SCI-model $\mathcal{M}', V'$, such that $\mathcal{M}', V' \models \varphi$.*

Proof. Ten dowód nie jest poprawny, dyskusja w osobnym PDFie We are given $\mathcal{M}, V$ and $\varphi \in \mathtt{FOR}$, such that $\mathcal{M}, v \models \varphi$. Let us define $\mathcal{M}' = (U', D', \tilde{\neg}', \tilde{\to}')$ and V':

- $U' = \{V(\psi) : \psi \in \mathtt{SUB}(\varphi)\} \cup \{1, 0\}$ – let us create $1, 0$, such that $1, 0 \notin U$,
- $D' = \{V(\psi) : \psi \in \mathtt{SUB}(\varphi), V(\psi) \in D\} \cup \{1\}$
- $\tilde{\neg}' a = \begin{cases} \tilde{\neg} a, & \text{if } a \in U \text{ and } \tilde{\neg} a \in U' \\ 0, & \text{otherwise, if } a \in D' \\ 1, & \text{otherwise, if } a \notin D' \end{cases}$
- $a \tilde{\to}' b = \begin{cases} a \tilde{\to} b, & \text{if } a, b \in U \text{ and } a \tilde{\to} b \in U' \\ 1, & \text{otherwise, if } a \notin D' \text{ or } b \in D' \\ 0, & \text{otherwise, if } a \in D' \text{ and } b \notin D' \end{cases}$

- $a \tilde{\equiv}' b = \begin{cases} a \tilde{\equiv} b, & \text{if } a, b \in U \text{ and } a \tilde{\equiv} b \in U' \\ 1, & \text{otherwise, if } a = b, \\ 0, & \text{otherwise, if } a \neq b. \end{cases}$
- $V'(\psi) = \begin{cases} V(\psi), & \text{if } V(\psi) \in U' \\ 1, & \text{otherwise, if } V(\psi) \in D \\ 0, & \text{otherwise, if } V(\psi) \notin D \end{cases}$

Let us explicate some trivial properties:

Remark 2.9. For all $a \in U'$, $a \in D'$ if and only if $a \in D$ or $a = 1$.

Remark 2.10. For all $\psi \in \texttt{FOR}$, $V'(\psi) \in D'$ if and only if $V(\psi) \in D$.

We need to show that $\mathcal{M}'$ is an SCI-model, V' is a valuation and that $V'(\varphi) \in D'$:

- $\emptyset \neq U' \supsetneq D' \neq \emptyset$ – trivially from $0, 1 \in U', 1 \in D'$, $0 \notin D'$.
- For all $a \in U'$, $\tilde{\neg}' a \in D'$ if and only if $a \notin D'$ – there are cases:

  1° $a \in \{1, 0\}$. It is easy to see that $\tilde{\neg}' 1 = 0$ and $\tilde{\neg}' 0 = 1$.

  2° $a \in U' \setminus \{1, 0\}$ and $\tilde{\neg} a \in U'$.
  Since $\mathcal{M}$ is a model, we know that $\tilde{\neg} a \in D$ if and only if $a \notin D$. From remark 2.9 we get $\tilde{\neg} a \in D'$ if and only if $a \notin D'$ and from the definition of $\tilde{\neg}'$ we're in the case where $\tilde{\neg}' a = \tilde{\neg} a$, so we get $\tilde{\neg}' a \in D'$ if and only if $a \notin D'$.

  3° Otherwise, and $a \in D'$. Then $\tilde{\neg}' a = 0 \notin D'$.

  4° Otherwise, and $a \notin D'$. Then $\tilde{\neg}' a = 1 \in D'$.

- For all $\psi \in \texttt{FOR}$, $V'(\neg\psi) = \tilde{\neg}' V'(\psi)$ – there are cases:

  1° $V(\neg\psi) \in U'$ and $V(\psi) \in U'$. From the definition of V' we have that $V'(\neg\psi) = V(\neg\psi)$ and $V'(\psi) = V(\psi)$. So we want to show that $V(\neg\psi) = \tilde{\neg}' V(\psi)$, but from the definition of $\tilde{\neg}'$, since $V(\psi) \in U'$, we know that $\tilde{\neg}' V(\psi) = \tilde{\neg} V(\psi)$, so we want to show that $V(\neg\psi) = \tilde{\neg} V(\psi)$, which is the case, because $\mathcal{M}$ is a model.

  2° $V(\neg\psi) \in U'$ and $V(\psi) \notin U'$.

  2.1° $V(\psi) \in D$.
  Then from the definition of V' we have that $V'(\psi) = 1$ so $\tilde{\neg}' V'(\psi) = 0$.
  On the other hand: $V'(\neg\psi) = V(\neg\psi) \neq 0$

  2° $V(\neg\psi) \notin U'$ and $V(\neg\psi) \in D$. From the definition of V', $V'(\neg\psi) = 1$. Since $V(\neg\psi) \in D$ and V is a valuation, we know that $V(\psi) \notin D$, so $V'(\psi) \notin D'$. So, from the definition of $\tilde{\neg}'$ we have that $\tilde{\neg}' V'(\psi) = 1$.

  3° $V(\neg\psi) \notin U'$ and $V(\neg\psi) \notin D$. Analogously to 2°.

- For all $a, b \in U'$, $a \tilde{\to}' b \in D'$ if and only if $a \notin D'$ or $b \in D'$ – there are cases:

  1° $a \in \{1, 0\}$ or $b \in \{1, 0\}$. Directly from the definition of $\tilde{\to}'$ we get:
  - $0 \tilde{\to}' b = 1 \in D'$
  - $1 \tilde{\to}' b \in D'$ if and only if $b \in D'$
  - $a \tilde{\to}' 0 \in D'$ if and only if $a \notin D'$
  - $a \tilde{\to}' 1 = 1 \in D'$

  2° $a, b \in U' \setminus \{1, 0\}$ and $a \tilde{\to}' b \in U'$. Similarly to $\tilde{\neg}'$, since $\mathcal{M}$ is a model, we know that $a \tilde{\to} b \in D$ if and only if $a \notin D$ or $b \in D$. From remark 2.9 we get $a \tilde{\to} b \in D'$ if and only if $a \notin D'$ or $b \in D'$ and from the definition of $\tilde{\to}'$ we're in the case where $a \tilde{\to}' b = a \tilde{\to} b$ so we get $a \tilde{\to}' b \in D'$ if and only if $a \notin D'$ or $b \in D'$.

  3° $a, b \in U' \setminus \{1, 0\}$ and $a \notin D'$ or $b \in D'$. Then $a \tilde{\to}' b = 1 \in D'$.

  4° $a, b \in U' \setminus \{1, 0\}$ and $a \in D'$ and $b \notin D'$. Then $a \tilde{\to}' b = 0 \notin D'$.

- For all $\psi, \vartheta \in \texttt{FOR}$, $V'(\psi \to \vartheta) = V'(\psi) \tilde{\to}' V'(\vartheta)$ – there are cases:

  1° aa

- ... TODO

$\square$

Theorem 4 (Decidability of SCI). SCI *is decidable.*

Proof. Let us take any SCI-formula φ. We want to know if φ is a SCI-theorem. From fullness and completeness of SCI we know that φ is a theorem if and only if $\neg\varphi$ has no SCI-model and valuation $\mathcal{M}, V$, such that $\mathcal{M}, V \models \neg\varphi$. But from theorem 3 we know that if there is $\mathcal{M}, V$, such that $\mathcal{M}, V \models \neg\varphi$, then there is a finite model $\mathcal{M}'$ and V' such that $\mathcal{M}', V' \models \neg\varphi$. Furthermore, from the proof of theorem 3 we can see that $|U'| \leq |\mathtt{SUB}(\varphi)|$ ($|X|$ here meaning the size of a set X).

For any finite SCI-model $\mathcal{M}'$ and SCI-valuation V' we can construct an isomorphic model $\mathcal{M}''$ with $U'' = \{1, 2, ..., |U'|\}$ by setting any bijection $f : U' \to U''$ and then setting the rest of the model and V'' accordingly. So, we know that if there is $\mathcal{M}, V$ such that $\mathcal{M}, V \models \neg\varphi$, then and only then there is a $\mathcal{M}'', V''$, such that $\mathcal{M}'', V'' \models \neg\varphi$ and $U'' = \{1, 2, ..., |\mathtt{SUB}(\neg\varphi)|\}$. There is a finite amount of such finite models.

We can have a procedure to set D'' to be any of $2^{|\mathtt{SUB}(\neg\varphi)|}$ subsets of U'', then for any $a \in U''$ to set $\tilde{\neg} a$ to be any element of U'', then for any $a, b \in U''$ to set $a \tilde{\to} b$ to be any element of U'', then for any $a, b \in U''$ to set $a \tilde{\equiv} b$ to be any element of U'' and then to set every variable of $\neg\varphi$ to be any element of U'' and finally to check, whether such constructed $\mathcal{M}'', V''$ is a SCI-model and a correct valuation. If it happens to be the case, then we have found a SCI-model invalidating φ. If after checking every such combination we have not found any SCI-model invalidating φ, we know that φ is valid, and so that it is a theorem.

If we set $n = s(\varphi)$, then this procedure will take $O(n^{2^n})$ operations, so it is very far from practical, but it is nevertheless finite, so we have a finite procedure to decide whether φ is a SCI-theorem.

□

3 Deduction in SCI

3.1 Deduction systems

The axiomatization of SCI we presented in definition 2.6 is an example of a Hilbert-style deductive system. Such systems consist of a set of axioms and inference rules (often *modus ponens* is the only inference rule) that can be used to deduce theses, as explained in definition 2.7.

Hilbert-style systems are historically first and most fundamental deductive systems. For every thesis of such systems, there is a sequence of formulas all the way back to the axioms, from which a resulting thesis is inferred. But trying to prove some specific formula can be very challenging as it requires creative usages of the axioms so that the given formula is inferred at the end. Those proof systems give no clue how to do that.

The same is true for the deduction system presented in definitions 2.6 and 2.7. From theorem 4 we know that SCI is decidable, but the Hilbert-style deductive system gives no procedure to determine whether a given formula is a tautology. From theorem 4 we know there is a finite procedure to do that, but we are still missing a procedure that could be used in practice.

Let us show some examples to illustrate these points:

Example 3.1. Let us show a full proof of a formula: $(p \equiv q) \to (q \equiv p)$

1. $p \equiv p$ $\quad (Ax1)$
2. $(p \equiv q) \to ((p \equiv p) \to ((p \equiv p) \equiv (q \equiv p)))$ $\quad (Ax5)$
3. $((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))$ $\quad (Ax2)$
4. $(p \equiv p) \to (((p \equiv q) \to ((p \equiv p) \to ((p \equiv p) \equiv (q \equiv p)))) \to ((((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))) \to ((p \equiv q) \to (q \equiv p))))$

   PC tautology $\varphi \to ((\psi \to (\varphi \to \vartheta)) \to ((\vartheta \to (\varphi \to \chi)) \to (\psi \to \chi)))$
   with $\varphi = p \equiv p$
   $\psi = p \equiv q$
   $\vartheta = q \equiv p$
   $\chi = (p \equiv p) \equiv (q \equiv p)$.

5. $((p \equiv q) \to ((p \equiv p) \to ((p \equiv p) \equiv (q \equiv p)))) \to ((((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))) \to ((p \equiv q) \to (q \equiv p)))$ $\quad \mathsf{MP}(1,4)$
6. $(((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))) \to ((p \equiv q) \to (q \equiv p))$ $\quad \mathsf{MP}(2,5)$
7. $(p \equiv q) \to (q \equiv p)$ $\quad \mathsf{MP}(3,6)$

As can be seen, although the finished proof isn't very long, it would be very difficult to come up with the precise axioms to use.

Example 3.2. Let us take a formula $p \equiv q$. We could be deducing many theses from our axioms, but at no point will we deduce it, nor $p \not\equiv q$, because those formulas aren't tautologies. But we cannot arrive at this conclusion using just our Hilbert-style system.

Example 3.3. Let us show a full proof of a formula $(q \equiv r) \to ((p \equiv q) \to (p \equiv r))$.

1. $(((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))) \to ((q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))))$

   PC tautology $\varphi \to (\psi \to \varphi)$
   with $\varphi = ((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))$
   $\psi = q \equiv r$

2. $((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))$ $(Ax2)$
3. $(q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r)))$ MP$(2,1)$
4. $((q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r)))) \to (((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))) \to ((q \equiv r) \to ((p \equiv q) \to (p \equiv r))))$

   PC tautology $(\varphi \to (\psi \to \vartheta)) \to ((\varphi \to \psi) \to (\varphi \to \vartheta))$
   with $\varphi = q \equiv r$
   $\psi = (p \equiv q) \equiv (p \equiv r)$
   $\vartheta = (p \equiv q) \to (p \equiv r)$

5. $((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))) \to ((q \equiv r) \to ((p \equiv q) \to (p \equiv r)))$ MP$(3,4)$
6. $p \equiv p$ $(Ax1)$
7. $(p \equiv p) \to ((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r)))$ $(Ax5)$
8. $(q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))$ MP$(6,7)$
9. $(q \equiv r) \to ((p \equiv q) \to (p \equiv r))$ MP$(8,5)$

In 1974 Michaels presented a differet proof system for SCI, in the style of Gentzen's sequent calculus [2]. It was refined by Wasilewska and presented in 1976 as a decision procedure [3]. Although the procedure presented there ws proved to terminate for every input formula, it was too complicated and, if implemented, would have too big time comlexity to be practical. In 2018 Chlebowski further modified and discussed the sequent calculus system, but his result was not a decision procedure.

In 2007 Golińska-Pilarek presented a proof system in the style of Rasiowa-Sikorski dual tableau [5] it was expanded and refined in [6] and [7], but it was not yet a terminating decision procedure.

An important difference between both the above proof systems and the Hilbert style system showed before is that the former start with the formula we want to prove or disprove and at each step offer a finite amount of possible rules to apply. Even though they were not a decision procedures, they could be helpful when manually proving some formulas. Finally, they were important steps at arriving at the full decision procedure of SCI.

The a full decision procedure $\mathsf{T}_{\mathsf{SCI}}$ was presented in 2021 by Golińska-Pilarek, Huuskonen and Zawidzki [8]. Also in 2021 the dual tableau system mentioned above was modified by the same authors to become a terminating decision procedure $\mathsf{DT}_{\mathsf{SCI}}$ [9].

Let us now take a brief look at both $\mathsf{T}_{\mathsf{SCI}}$ and $\mathsf{DT}_{\mathsf{SCI}}$, starting with the latter as it is easier to present.

3.2 $\mathsf{DT}_{\mathsf{SCI}}$

TODO

$$(\neg^+)\quad \dfrac{w^+ : \neg\varphi}{v^- : \varphi} \qquad\qquad (\neg^-)\quad \dfrac{w^- : \neg\varphi}{v^+ : \varphi}$$

$$(\to^+)\quad \dfrac{w^+ : \varphi \to \psi}{\begin{array}{c|c|c} v^- : \varphi & v^- : \varphi & v^+ : \varphi \\ u^- : \psi & u^+ : \psi & u^+ : \psi \end{array}} \qquad\qquad (\to^-)\quad \dfrac{w^- : \varphi \to \psi}{\begin{array}{c} v^+ : \varphi \\ u^- : \psi \end{array}}$$

$$(\equiv^+)\quad \dfrac{w^+ : \varphi \equiv \psi}{\begin{array}{c|c} v^+ : \varphi & v^- : \varphi \\ u^+ : \psi & u^- : \psi \\ v^+ = u^+ & v^- = u^- \end{array}} \qquad (\equiv^-)\quad \dfrac{w^- : \varphi \equiv \psi}{\begin{array}{c|c|c|c} v^+ : \varphi & & & v^- : \varphi \\ u^+ : \psi & v^+ : \varphi & v^- : \varphi & u^- : \psi \\ v^+ \neq u^+ & u^- : \psi & u^+ : \psi & v^- \neq u^- \end{array}}$$

$$(\equiv^\neg)\quad \dfrac{\begin{array}{c} \varphi \approx \psi \\ u : \neg\varphi \\ y : \neg\psi \end{array}}{u = y} \qquad (\equiv^\to)\quad \dfrac{\begin{array}{c} \varphi \approx \psi \\ \chi \approx \vartheta \\ x : \varphi \to \chi \\ z : \psi \to \vartheta \end{array}}{x = z} \qquad (\equiv^\equiv)\quad \dfrac{\begin{array}{c} \varphi \approx \psi \\ \chi \approx \vartheta \\ x : \varphi \equiv \chi \\ z : \psi \equiv \vartheta \end{array}}{x = z} \qquad (\mathtt{F})\quad \dfrac{\begin{array}{c} w : \varphi \\ v : \varphi \end{array}}{w = v}$$

$$(\mathtt{sym})\quad \dfrac{w = v}{v = w} \qquad (\mathtt{tran})\quad \dfrac{\begin{array}{c} w = v \\ v = u \end{array}}{w = u} \qquad (\bot_1)\quad \dfrac{\begin{array}{c} w = v \\ w \neq v \end{array}}{\bot} \qquad (\bot_2)\quad \dfrac{w^+ = v^-}{\bot}$$

Figure 1: Deduction rules of the $\mathsf{T}_{\mathsf{SCI}}$ system.

3.3 $\mathsf{T}_{\mathsf{SCI}}$

4 $\mathsf{T}^*_{\mathsf{SCI}}$

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