Decision procedures for a non-Fregean logic SCI

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Test: ążźćłóęń

1 Sentential logic and the Fregean axiom

2 SCI

2.1 Basic notions

Definition 2.1 (Vocabulary of SCI)**.** The vocabulary of SCI consists of symbols from the following pairwise disjoint sets:

- $\mathtt{V} = \{p, q, r, ...\}$ – a countable infinite set of propositional variables,
- $\{\neg, \rightarrow, \equiv\}$ – the set consisting of the unary operator of negation ($\neg$) and binary operators of implication ($\rightarrow$) and identity ($\equiv$),
- $\{(,)\}$ – the set of auxiliary symbols.

Definition 2.2 (Formulas of SCI)**.** The set of formulas of SCI is defined with the following grammar:

$$\mathtt{FOR} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \rightarrow \varphi) \mid (\varphi \equiv \varphi)$$

where $p \in \mathtt{V}$ is a propositional variable.

The propositional variables will also be called atomic formulas.

From now on whenever we write $p, q, r, ...$ we will mean the atomic formulas. We will omit brackets when it will lead to no misunderstanding. We will write $\varphi \not\equiv \psi$ as a shorthand for $\neg(\varphi \equiv \psi)$.

The set of identities $\mathtt{ID}$ is a set of formulas $\varphi \equiv \psi$ where $\varphi, \psi \in \mathtt{FOR}$. Formulas $\varphi \equiv \varphi$ are the trivial identities.

Definition 2.3 (Subformulas)**.** For a formula $\varphi \in \mathtt{FOR}$ let us define the set of subformulas of φ as:

$$\mathtt{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in \mathtt{V}, \\ \{\varphi\} \cup \mathtt{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \mathtt{SUB}(\psi) \cup \mathtt{SUB}(\vartheta), & \text{if } \varphi = \psi \rightarrow \vartheta, \text{or } \varphi = \psi \equiv \vartheta \end{cases}$$

By $\varphi(\psi/\vartheta)$ we will denote the formula φ with all occurrences of its subformula ψ substituted with ϑ.

Definition 2.4 (Simple formulas)**.** The formula φ is called a simple formula if it has one of the following form:

$$p,\ \neg p,\ p \equiv q,\ p \not\equiv q,\ p \equiv \neg q,\ p \equiv (q \rightarrow r),\ p \equiv (q \equiv r)$$

for $p, q, r \in \mathtt{V}$.

Definition 2.5 (Size of a formula)**.** Given a formula φ, let us define its size $s(\varphi)$:

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta, \text{ or } \varphi = \psi \rightarrow \vartheta. \end{cases}$$

$\mathtt{V}$ is a countable set. Let us take any full ordering of it and mark it as $<$. Let us then extend it by saying that for each $p \in \mathtt{V}$: $p < \neg < (<) < \rightarrow < \equiv$. Now, let us define a ordering of formulas $\prec$ to be a lexicographical ordering with $<$.

If we consider formulas that contain only the negation and implication operators, they form a classical Propositional Calculus. For simplicity, in SCI we'll consider every tautology of the classical Propositional Calculus to be an axiom.

Definition 2.6 (Axiomatization of SCI)**.** SCI is axiomatized with the following axioms:

- Any tautology of the classical Propositional Calculus
- $(Ax1)$ $\varphi \equiv \varphi$,
- $(Ax2)$ $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$,
- $(Ax3)$ $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$,
- $(Ax4)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \theta)))$,
- $(Ax5)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$.

The only inference rule is the *modus ponens* rule:

$$\mathsf{MP} : \frac{\varphi,\ \varphi \rightarrow \psi}{\psi}$$

Definition 2.7 (A thesis of SCI)**.** A formula φ is *a thesis of* SCI if there exists a finite sequence of formulas $\varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula φ_i is either an axiom of SCI, or it is inferred from formulas φ_j, φ_k $(j, k < i)$ via the *modus ponens* rule. If φ is a thesis of SCI we will denote it by $\vdash \varphi$, say that φ is *provable in* SCI and call the sequence $\varphi_1, ..., \varphi_n$ the *proof of* φ.

Let us now give some semantic definitions:

Definition 2.8 (SCI-model)**.** A model of SCI (or an SCI-model) model is a structure $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ where:

- $U \neq \emptyset$ is an *universe* of M,
- $\emptyset \neq D \subsetneq U$ is a *set of designated values*,
- $\tilde{\neg}$ is an unary operation on U, such that for all $a \in U$: $\tilde{\neg}a \in D$ if and only if $a \notin D$,
- $\tilde{\rightarrow}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$,
- $\tilde{\equiv}$ is a binary operation on U, such that for all $a, b \in U$: $a \tilde{\equiv} b \in D$ if and only if $a = b$ (that is, if and only if a and b denote the same element of the universe U).

If an universe U is finite, we'll call a given SCI-model a *finite* SCI*-model*.

Definition 2.9 (Valuation)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a valuation in $\mathcal{M}$ is a function $V : \mathtt{FOR} \longrightarrow U$ that assigns a value $V(p) \in U$ for all propositional variables p, and such that for all $\varphi, \psi \in \mathtt{FOR}$:

- $V(\neg\varphi) = \tilde{\neg}V(\varphi)$
- $V(\varphi \rightarrow \psi) = V(\varphi)\tilde{\rightarrow}V(\psi)$
- $V(\varphi \equiv \psi) = V(\varphi)\tilde{\equiv}V(\psi)$

If $V(\varphi) = a$ we will call a the *denotation of* φ.

Definition 2.10 (Satisfaction of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ and a valuation V in $\mathcal{M}$, a formula φ is *satisfied in* $\mathcal{M}$ *by* V if and only if $V(\varphi) \in D$. If a formula φ is satisfied in $\mathcal{M}$ by a valuation V, then will denote it by $M, V \models \varphi$.

Definition 2.11 (Truth of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a formula is *true in* $\mathcal{M}$ if and only if it is satisfied in $\mathcal{M}$ by all valuations of $\mathcal{M}$. If a formula φ is true in $\mathcal{M}$, then we will denote it by $\mathcal{M} \models \varphi$.

Definition 2.12 (Validity of a formula)**.** A formula is *valid in* SCI (or SCI-*valid*) if and only if it is true in all models. If a formula φ is SCI-valid we will denote it by $\models \varphi$.

Let us see some examples:
TODO

Theorem 1 (Soundness of SCI)**.** *For every* SCI*-formula φ, if φ is provable in* SCI*, then φ is valid in* SCI*.*

Proof. The proof will have two main parts:

(a) Every axiom of SCI is valid in SCI.

(b) By applying the *modus ponens* rule to valid formulas φ and $\varphi \to \psi$ the inferred formula ψ is also valid.

Let us show (a) first:

- First, we need to prove that every tautology of the classical Propositional Calculus (PC) is valid in SCI.

  Let φ be a tautology of PC. Let us take any SCI-model $\mathcal{M}$ and any SCI-valuation $\mathcal{M}, V$ in it. We want to show that $\mathcal{M}, V \models \varphi$.

  Given $\mathcal{M}, V$, we will construct V' that will be a PC valuation of φ. We can do this, because φ doesn't contain operator $\equiv$.

  For every $\psi \in \mathtt{SUB}(\varphi)$, let us define:

$$V'(\psi) = \begin{cases} 1, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \in D, \\ 0, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \notin D, \\ 1 - V'(\chi), & \text{if } \psi = \neg\chi, \\ max(1 - V'(\chi), V'(\theta)), & \text{if } \psi = \chi \to \theta. \end{cases}$$

  It is easy to see that for every $\psi \in \mathtt{SUB}(\varphi)$, $V'(\psi) = 1$ if and only if $V(\psi) \in D$.

  The V' function is constructed in the same way the valuation function in the PC is constructed, therefore since φ is a tautology of PC, we have that $V'(\varphi) = 1$. So, we have that $V(\varphi) \in D$, whis is what we wanted to show.

- Second, we want to show that axioms $(Ax1) - (Ax5)$ are valid in SCI.

  - $(Ax1)$ $\varphi \equiv \varphi$

    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.9, we have that $V(\varphi \equiv \varphi) = V(\varphi) \tilde{\equiv} V(\varphi)$. Based on definition of $\tilde{\equiv}$ from 2.8, we have that $V(\varphi) \tilde{\equiv} V(\varphi)$ is in D if and only if $V(\varphi) = V(\varphi)$, which is trivially the case. So, we have that $\mathcal{M}, V \models \varphi \equiv \varphi$.

  - $(Ax2)$ $(\varphi \equiv \psi) \to (\varphi \to \psi)$

    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.9, we have that $V((\varphi \equiv \psi) \to (\varphi \to \psi)) = V(\varphi \equiv \psi) \tilde{\to} V(\varphi \to \psi)$. There are two cases.

    1° $V(\varphi \equiv \psi) \notin D$. Then, by definition of $\tilde{\to}$ in 2.8, we have that $V(\varphi \equiv \psi) \tilde{\to} V(\varphi \to \psi) \in D$, so $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

    2° $V(\varphi \equiv \psi) \in D$. By definition 2.9 it means, that $V(\varphi) \tilde{\equiv} V(\psi)$, which by definition 2.8 means that $V(\varphi) = V(\psi)$.
    By definition 2.9 $V(\varphi \to \psi)$ is equal to $V(\varphi) \tilde{\to} V(\psi)$. By definition 2.8 $V(\varphi) \tilde{\to} V(\psi)$ is in D if and only if $V(\varphi) \notin D$ or $V(\psi) \in D$, but since $V(\varphi) = V(\psi)$ we have that $V(\varphi) \tilde{\to} V(\psi)$ is in D if and only if $V(\varphi) \notin D$ or $V(\varphi) \in D$, which is trivially the case.

    So, we have that $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

  - Validity of axioms $(Ax3), (Ax4)$ and $(Ax5)$ can be shown in a similar way. (TODO: a może rozpisać dla jasności?)

To show (b), let take any valid formulas φ and $\varphi \to \psi$ and any SCI model and valuation $\mathcal{M}, V$. From validity of $\varphi \to \psi$, we know that $V(\varphi \to \psi) \in D$. From definition 2.9 we have that $D \ni V(\varphi \to \psi) = V(\varphi)\tilde{\to}V(\psi)$. From definition 2.8, since we know that $V(\varphi)\tilde{\to}V(\psi)$ we know that $V(\varphi) \notin D$, or $V(\psi) \in D$. But we have that φ is valid, so $V(\varphi) \in D$, so it must be that $V(\psi) \in D$, so $\mathcal{M}, V \models \psi$. The same holds for any other SCI model and valuation, so $\models \psi$, which is what we wanted to show.

Now, looking at definition 2.7, for a given provable φ, let us take its proof $\varphi_1, ..., \varphi_n = \varphi$. Every subsequent formula in this proof is either an SCI axiom and thus, by (a), valid, or is inferred by the *modus ponens* rule from valid formulas and thus, by (b) valid. So, φ is valid. □

Theorem 2 (Completeness of SCI). *For every* SCI*-formula* φ*, if* φ *is valid in* SCI*, then* φ *is provable in* SCI.

Proof. To show that every valid SCI-formula is SCI-provable is, by contraposition, to show that every non-SCI-provable formula is non-SCI-valid.

To do this, we will construct a SCI-model and a valuation in which every SCI-provable formula is satisfied (this model will be similar in style to the Herbrandt model). Then, it will be shown that for every formula that is not SCI-provable, this model and this valuation will be a witness of its non-SCI-validity, thus proving completeness of SCI

Wydaje mi się, że pomieszałem coś w notatkach z tego dowodu. Na wykładzie co prawda dla każdego niesprzecznego zbioru formuł X konstruowaliśmy model, w którym były prawdziwe wszystkie formuły $X \vdash \varphi$, ale dla samego dowodu pełności i tak stawialiśmy na koniec (chyba?) $X = \emptyset$, wiec uprościłem.

Próbowałem to porównać z [Suszko, Bloom 1972], ale tam ten dowód jest bardzo lakonicznie opisany.

A może bierze się niedowodliwą formułę i dla niej konkretnie buduje model (lub wartościowanie), które ją inwaliduje?

First, let us define a relation R on SCI-formulas:

$$\varphi R \psi \text{ if and only if} \vdash \varphi \equiv \psi$$

Note that for clarity we're using infix notation for relations. We want to show that R is an equivalence relation.

(a) R is reflexive. It is trivially true from axiom $(Ax1)$.

(b) R is symmetrical, that is for all $\varphi, \psi \in \texttt{FOR}$, $\varphi R \psi$ if and only if $\psi R \varphi$.

As we have shown in example TODO, we can prove that for all φ, ψ, $\vdash (\varphi \equiv \psi) \to (\psi \equiv \varphi)$. If $\varphi R \psi$, then $\vdash \varphi \equiv \psi$, so by *modus ponens* we can infer $\vdash \psi \equiv \varphi$, so $\psi R \varphi$. In the same way we can show that if $\psi R \varphi$ then $\varphi R \psi$, so R is symmetrical.

(c) R is transitive, that is for all $\varphi, \psi, \vartheta \in \texttt{FOR}$, if $\varphi R \psi$ and $\psi R \vartheta$ then $\varphi R \vartheta$.

As we have shown in example TODO, we can prove that for all $\varphi, \psi, \vartheta \in \texttt{FOR}$, $\vdash (\varphi \equiv \psi) \to ((\psi \equiv \vartheta) \to (\varphi \equiv \vartheta))$. Let us assume that $\varphi R \psi$ and $\psi R \vartheta$. This gives us that $\vdash \varphi \equiv \psi$ and $\vdash \psi \equiv \vartheta$. Applying *modus ponens* twice we get $\vdash \varphi \equiv \vartheta$, so $\varphi R \vartheta$.

So, R is an equivalence relation. Let us now show that it is a congruence, that is:

(a) For all $\varphi, \psi \in \texttt{FOR}$, if $\varphi R \psi$, then $\neg\varphi R \neg\psi$. This is trivially true from axiom $(Ax3)$.

(b) For all $\varphi, \psi, \vartheta, \chi \in \texttt{FOR}$, if $\varphi R \psi$ and $\vartheta R \chi$, then $\varphi \to \vartheta R \psi \to \chi$. This is easily proven from $(Ax4)$.

(c) For all $\varphi, \psi, \vartheta, \chi \in \texttt{FOR}$, if $\varphi R \psi$ and $\vartheta R \chi$, then $\varphi \equiv \vartheta R \psi \equiv \chi$. This is easily proven from $(Ax5)$.

Now that we know that R is a congruent equivalence relation, we can define a structure: $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\to}, \tilde{\equiv})$, such that:

- $U = \{|\varphi|_R : \varphi \in \texttt{FOR}\}$ (by $|\varphi|_R$ we mean an equivalence class of R of which φ is a member),
- $D = \{|\varphi|_R : \varphi \in \texttt{FOR}, \vdash \varphi\}$
- $\tilde{\neg}$ is an unary function on equivalence classes of R, such that $\tilde{\neg}|\varphi|_R = |\neg\varphi|_R$
- $\tilde{\to}$ is a binary function on equivalence classes of R, such that $|\varphi|_R \tilde{\to} |\psi|_R = |\varphi \to \psi|_R$
- $\tilde{\equiv}$ is a binary function on equivalence classes of R, such that $|\varphi|_R \tilde{\equiv} |\psi|_R = |\varphi \equiv \psi|_R$

Let us add some comments:

- For any SCI-formulas φ, ψ, such that $\varphi \equiv \psi$, $\vdash \varphi$ if and only if $\vdash \psi$. This follows from $(Ax2)$. Because of that, $|\varphi|_R = |\psi|_R$ and so D is well defined.
- From congruence of R, we know that for any $\varphi, \psi \in \texttt{FOR}$ if $|\varphi|_R = |\psi|_R$, then $|\neg\varphi|_R = |\neg\psi|_R$, so $\tilde{\neg}$ is a well defined function.
- Similarly, form the congruence of R we can see that $\tilde{\rightarrow}$ and $\tilde{\equiv}$ are well defined functions.

Now we will want to show that $\mathcal{M}$ is an SCI-model. From the definition 2.8 we want to show that:

- $U \neq \emptyset$ – this is trivial,
- $\emptyset \neq D$ – this is true, because for example $\vdash p \equiv p$
- $D \subsetneq U$ – $D \subseteq U$ is trivial from the definition of D. $D \neq U$, because there are formulas which are not valid (e.g. $\neg(p \equiv p)$), which from theorem 1 are not provable.
- For all $a \in U : \tilde{\neg}a \in D$ if and only if $a \notin D$

  Czy to nie jest fałsz? Weźmy $a = |p \equiv q|$.

  Chcemy pokazać: $\tilde{\neg}|p \equiv q| \in D$ iff $|p \equiv q| \notin D$.

  Prawa strona: Prawdziwa, bo $\nvdash |p \equiv q|$

  Lewa strona: $\tilde{\neg}|p \equiv q| \overset{\text{def.}\tilde{\neg}}{\Longleftrightarrow} |\neg(p \equiv q)|$. Ale: $\nvdash \neg(p \equiv q)$, więc $|\neg(p \equiv q)| \notin D$, więc lewa nieprawdziwa.
- For all $a, b \in U : a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$.

  TODO.
- For all $a, b \in U : a \tilde{\equiv} b \in D$ if and only if $a = b$.

  TODO.

So, $\mathcal{M}$ is a SCI-model. Now, let us define a valuation V for all $\varphi \in \texttt{FOR}$:

$$V(\varphi) = |\varphi|_R$$

We need to show that V is indeed a valuation:

- TODO (chyba proste wszystko)

Now, for the final proof of completeness let us take a non-provable formula φ. Since $\nvdash \varphi$ we have that $|\varphi|_R \notin D$, so $V(\varphi) \notin D$. So, we have found a model and a valuation $\mathcal{M}, V$, such that $\mathcal{M}, V \not\models \varphi$, so φ is not valid.

W notatkach mam: “

- Mamy dane X, φ: $X \nvdash \varphi$
- budujemy model $\mathcal{M}^X, V^X$, t.że $X \vdash \psi \Rightarrow |\psi|_{R_X} \in D^X$
- Ale oczywiście skoro $X \nvdash \varphi$ to $|\varphi|_{R_X} \notin D^X$, więc $X \not\models \varphi$

”

Więc X nic tutaj nie pomaga, stawiając $X = \emptyset$ dostaję to co jest u góry.

□

Definition 2.13 (Decidability)**.** A logic is *decidable* if there exists an effective method, that is, an algorithm that will always terminate, to determine whether any formula of this logic is a theorem.

We will want to show that SCI is decidable. To do this, first let us show the following:

Theorem 3 (Finite model property of SCI)**.** *For any $\varphi \in \texttt{FOR}$, if there exists $\mathcal{M}, V$, such that $\mathcal{M}, V \models \varphi$, then there exists a finite* SCI*-model $\mathcal{M}', V'$, such that $\mathcal{M}', V' \models \varphi$.*

Proof. We are given $\mathcal{M}, V$ and $\varphi \in FOR$, such that $\mathcal{M}, v \models \varphi$. Let us define $\mathcal{M}' = (U', D', \tilde{\neg}', \tilde{\rightarrow}')$ and V':

- $U' = \{v(\psi) : \psi \in \texttt{SUB}(\varphi)\} \cup \{1, 0\}$ – let us create $1, 0$, such that $1, 0 \notin U$,

- $D' = \{V(\psi) : \psi \in \mathtt{SUB}(\varphi), V(\psi) \in D\} \cup \{1\}$
- $\tilde{\neg}' a = \begin{cases} \tilde{\neg} a, & \text{if exists } \psi \text{ such that } a = V(\psi) \text{ and } \neg\psi \in \mathtt{SUB}(\varphi) \\ 0, & \text{otherwise, if } a \in D' \\ 1, & \text{otherwise, if } a \notin D' \end{cases}$
- $a \tilde{\rightarrow}' b = \begin{cases} a \tilde{\rightarrow} b, & \text{if exist } \psi, \vartheta \text{ such that } a = V(\psi), b = V(\vartheta) \text{ and } \psi \rightarrow \vartheta \in \mathtt{SUB}(\varphi) \\ 1, & \text{otherwise, if } a \notin D' \text{ or } b \in D' \\ 0, & \text{otherwise, if } a \in D' \text{ and } b \notin D' \end{cases}$
- $a \tilde{\equiv}' b = TODO \begin{cases} a \tilde{\equiv} b, & \text{if } a = v(\psi), b = v(\vartheta) \text{ and } \psi \equiv \vartheta \in \mathtt{SUB}(\varphi) \\ 1, & \text{otherwise, if } a = b, \\ 0, & \text{otherwise.} \end{cases}$
- $V'(\psi) = \begin{cases} V(\psi), & \text{if } \psi \in \mathtt{SUB}(\varphi) \\ 1, & \text{if } \psi \notin \mathtt{SUB}(\varphi) \text{ and } V(\psi) \in D \\ 0, & \text{if } \psi \notin \mathtt{SUB}(\varphi) \text{ and } V(\psi) \notin D \end{cases}$

We need to show that $\mathcal{M}'$ is an SCI-model, V' is a valuation and that $V'(\varphi) = V(\varphi)$:

- $\emptyset \neq U' \supsetneq D' \neq \emptyset$ – trivially from $0, 1 \in U', 1 \in D'$.
- $\tilde{\neg}' a \in D'$ if and only if $a \notin D'$ – there are cases:

  1° There exists ψ, such that $a = V(\psi)$ and $\neg\psi \in \mathtt{SUB}(\varphi)$. Then from the definition of $\tilde{\neg}'$ we have $\tilde{\neg}' a = V'(\neg\psi)$, from the definition of V' we have that $V'(\neg\psi) = V(\neg\psi)$ and since V is a valuation we have that $V(\neg\psi) = \tilde{\neg} V(\psi)$. But we assumed that $a = V(\psi)$, so we have that $\tilde{\neg}' a = \tilde{\neg} a$. Then since $\mathcal{M}$ is a model, we know that $\tilde{\neg} a \in D$ if and only if $a \notin D$, and since $\neg\psi \in \mathtt{SUB}(\varphi)$, we have $\tilde{\neg}' a \in D'$ if and only if $a \notin D'$.

  2° Otherwise, and $a \in D'$. Then $\tilde{\neg}' a = 0 \notin D'$.

  3° Otherwise, and $a \notin D'$. Then $\tilde{\neg}' a = 1 \in D'$.

- For all $\psi \in \mathtt{FOR}$, $V'(\neg\psi) = \tilde{\neg}' V'(\psi)$ – there are cases:

  1° $\neg\psi \in \mathtt{SUB}(\varphi)$. Then from the definition of $\tilde{\neg}'$ we have that $\tilde{\neg}' V'(\psi) = V'(\neg\psi)$.

  2° $\neg\psi \notin \mathtt{SUB}(\varphi)$.

  Kontrprzykład:

  - weźmy ψ: $\neg\psi \notin \mathtt{SUB}(\varphi)$
  - Z definicji V': $V'(\neg\psi) = 1$ lub 0, w zeleżności czy $V(\psi) \in D$.
  - Niech $V'(\psi) = b$
  - Niech istnieje χ, takie że: $\neg\chi \in \mathtt{SUB}(\varphi)$, $V'(\chi) = b$, $V'(\neg\chi) = c \notin \{1, 0\}$
  - wtedy, z definicji $\tilde{\neg}' V'(\psi) = \tilde{\neg}' b = c \notin \{1, 0\}$. Sprzeczność.

  Konkretny kontrprzykład:

  - $\varphi = \neg(p \equiv \neg p)$
  - $U = \{a, b\}$
  - $D = \{b\}$
  - $V(p) = V(p \equiv \neg p) = a$
  - $V(\neg p) = V(\neg(p \equiv \neg p)) = b$
  - Konstruujemy $\mathcal{M}', V'$ jak wyżej
  - $\psi := \neg(p \equiv \neg p)$
  - $V'(\neg\psi) = 0$, bo $\neg\psi \notin \mathtt{SUB}(\varphi)$
  - $V'(\psi) = a$
  - $\tilde{\neg}' a = \tilde{\neg} a = b$, bo istnieje $\chi = p$ t.że $a = V(\chi)$ i $\neg\chi \in \mathtt{SUB}(\varphi)$
  - więc $\tilde{\neg}' V'(\psi) = b \neq 0$ – sprzeczność.

  Trzeba chyba skomplikować definicję $V'(\psi)$ rozbijając ψ. Np. jeśli $\exists \chi : \neg\chi = \psi, \chi \in \mathtt{SUB}(\varphi)$ to $V'(\psi) = \tilde{\neg}' V'(\chi)$. Wtedy V' przestaje mieć tą ładną właściwość, że $\psi \notin \mathtt{SUB}(\varphi) \Rightarrow V'(\psi) \in \{1, 0\}$, np. w przykładzie u góry $V'(\neg\neg(p \equiv \neg p))$ będzie równe a. Ale nie ma to wpływu na liczność modelu i wydaje mi się że będzie się wszystko zgadzać.

- $a \tilde{\rightarrow}' b \in D'$ if and only if $a \notin D'$ or $b \in D'$ – there are cases:

  1° TODO There exists ψ and ϑ such that $a = v(\psi), b = v(\vartheta)$ and $\psi \rightarrow \vartheta \in \texttt{SUB}(\varphi)$. Then from the definition of $\tilde{\rightarrow}'$ we have that $a \tilde{\rightarrow}' b \in D' \iff a \tilde{\rightarrow} b \in D$, from the definition of $\tilde{\rightarrow}$ we have that $a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$, which from the definition of D' is if and only if $a \notin D'$ or $b \in D'$.

  2° TODO There are no such ψ and ϑ and $a \notin D$. Then, from the definition of $\tilde{\rightarrow}'$ we have that $a \tilde{\rightarrow}' b = 1 \in D'$ and from the definition of D' we have that $a \notin D'$.

  3° TODO There are no such ψ and ϑ and $b \in D$. Then again, from the definition of $\tilde{\rightarrow}'$ we have that $a \tilde{\rightarrow}' b = 1 \in D'$ and from the definition of D' we have that $b \in D'$.

□

Theorem 4 (Decidability of SCI)**.** SCI *is decidable.*

Proof. Let us take any SCI-formula φ, we want to know if φ is a SCI-theorem. From fullness and completeness of SCI we know that φ is a theorem if and only if $\neg\varphi$ has no SCI-model and valuation $\mathcal{M}, V$, such that $\mathcal{M}, V \models \neg\varphi$. But from theorem 3 we know that if there is $\mathcal{M}, V$, such that $\mathcal{M}, V \models \neg\varphi$, then there is a finite model $\mathcal{M}'$ and V' such that $\mathcal{M}', V' \models \neg\varphi$. Furthermore, from the proof of theorem 3 we can see that $|U'| \leq |\texttt{SUB}(\varphi)|$ ($|X|$ here meaning the size of a set X).

For any finite model $\mathcal{M}'$ we can construct an isomorphic model $\mathcal{M}''$ with $U'' = \{1, 2, ..., |U'|\}$ by setting any bijection $f : U' \rightarrow U''$ and then setting the rest of the model and V'' accordingly. So, we know that if there is $\mathcal{M}, V$ such that $\mathcal{M}, V \models \neg\varphi$, then and only then there is a $\mathcal{M}'', V''$, such that $\mathcal{M}'', V'' \models \neg\varphi$ and $U'' = \{1, 2, ..., |\texttt{SUB}(\neg\varphi)|\}$, and there is a finite amount of such finite models.

We can have a procedure to set D'' to be any of $2^{|\texttt{SUB}(\neg\varphi)|}$ subsets of U'', then for any $a \in U''$ to set $\tilde{\neg} a$ to be any element of U'', then for any $a, b \in U''$ to set $a \tilde{\rightarrow} b$ to be any element of U'', then for any $a, b \in U''$ to set $a \tilde{\equiv} b$ to be any element of U'' and then to set every variable of $\neg\varphi$ to be any element of U'' and finally to check, whether such constructed $\mathcal{M}'', V''$ is a SCI-model and a correct valuation. If it happens to be the case, then we have found a SCI-model invalidating φ. If after checking every such combination we have not found any SCI-model invalidating φ, we know that φ is valid, and so that it is a theorem.

If we set $n = s(\varphi)$, then this procedure will take $O(n^{2n^2})$ operations, so it is very far from practical, but it is nevetheless finite, so we have a finite procedure to decide whether φ is a SCI-theorem.

□

3 Deduction in SCI

3.1 Deduction systems

The axiomatization of SCI we presented in definition 2.6 is an example of a Hilbert-style deductive system. Such systems consist of a set of axioms and inference rules (often *modus ponens* is the only inference rule) that can be used to deduce theses, as explained in definition 2.7.

Hilbert-style systems are historically first and most fundamental deductive systems. For every thesis of such systems, there is a sequence of formulas all the way back to the axioms, from which a resulting thesis is inferred. But trying to prove some specific formula can be very challenging as it requires creative usages of the axioms so that the given formula is inferred at the end. Those proof systems give no clue how to do that.

The same is true for the deduction system presented in definitions 2.6 and 2.7. What is more, from theorem 4 we know that SCI is decidable, but so far we have no practical decision procedure to determine whether a given formula is a tautology.

Let us show some examples to illustrate these points:

Example 3.1. Let us show a full proof of a formula: $(p \equiv q) \rightarrow (q \equiv p)$

1. $p \equiv p$ $\quad (Ax1)$
2. $(p \equiv q) \rightarrow ((p \equiv p) \rightarrow ((p \equiv p) \equiv (q \equiv p)))$ $\quad (Ax5)$
3. $((p \equiv p) \equiv (q \equiv p)) \rightarrow ((p \equiv p) \rightarrow (q \equiv p))$ $\quad (Ax2)$
4. $(p \equiv p) \rightarrow (((p \equiv q) \rightarrow ((p \equiv p) \rightarrow ((p \equiv p) \equiv (q \equiv p)))) \rightarrow ((((p \equiv p) \equiv (q \equiv p)) \rightarrow ((p \equiv p) \rightarrow (q \equiv p))) \rightarrow ((p \equiv q) \rightarrow (q \equiv p))))$

PC axiom $\varphi \to ((\psi \to (\varphi \to \vartheta)) \to ((\vartheta \to (\varphi \to \chi)) \to (\psi \to \chi)))$
with $\varphi = p \equiv p$
$\psi = p \equiv q$
$\vartheta = q \equiv p$
$\chi = (p \equiv p) \equiv (q \equiv p)$.

5. $((p \equiv q) \to ((p \equiv p) \to ((p \equiv p) \equiv (q \equiv p)))) \to ((((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))) \to ((p \equiv q) \to (q \equiv p)))$ MP(1, 4)

6. $(((p \equiv p) \equiv (q \equiv p)) \to ((p \equiv p) \to (q \equiv p))) \to ((p \equiv q) \to (q \equiv p))$ MP(2, 5)

7. $(p \equiv q) \to (q \equiv p)$ MP(3, 6)

As can be seen, althouth the finished proof isn't very long, it would be very difficult to come up with the precise axioms to use.

Example 3.2. Let us take a formula $p \equiv q$. We could be deducing many theses from our axioms, but at no point will we deduce it, nor $\neg(p \equiv q)$, because those formulas aren't tautologies. But we cannot arrive at this conclusion using just our Hilbert-style system.

Example 3.3. Let us show a full proof of a formula $(q \equiv r) \to ((p \equiv q) \to (p \equiv r))$.

1. $(((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))) \to ((q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))))$

   PC axiom $\varphi \to (\psi \to \varphi)$
   with $\varphi = ((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))$
   $\psi = q \equiv r$

2. $((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r))$ $(Ax2)$

3. $(q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r)))$ MP(2, 1)

4. $((q \equiv r) \to (((p \equiv q) \equiv (p \equiv r)) \to ((p \equiv q) \to (p \equiv r)))) \to (((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))) \to ((q \equiv r) \to ((p \equiv q) \to (p \equiv r))))$

   PC axiom $(\varphi \to (\psi \to \vartheta)) \to ((\varphi \to \psi) \to (\varphi \to \vartheta))$
   with $\varphi = q \equiv r$
   $\psi = (p \equiv q) \equiv (p \equiv r)$
   $\vartheta = (p \equiv q) \to (p \equiv r)$

5. $((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))) \to ((q \equiv r) \to ((p \equiv q) \to (p \equiv r)))$ MP(3, 4)

6. $p \equiv p$ $(Ax1)$

7. $(p \equiv p) \to ((q \equiv r) \to ((p \equiv q) \equiv (p \equiv r)))$ $(Ax5)$

8. $(q \equiv r) \to ((p \equiv q) \equiv (p \equiv r))$ MP(6, 7)

9. $(q \equiv r) \to ((p \equiv q) \to (p \equiv r))$ MP(8, 5)

References

[1] Stephen L. Bloom, Roman Suszko (1972) *Investigations into the Sentential Calculus with Identity*, Notre Dame Journal of Formal Logic, vol. XIII, no 3.