# Decision procedures for a non-Fregean logic SCI

Adrian Siwiec

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Test: ążźćłóęń

## 1 Sentential logic and the Fregean axiom

## 2 SCI

### 2.1 Basic notions

**Definition 2.1** (Vovabulary of SCI)**.** The vocabulary of SCI consists of symbols from the following pairwise disjoint sets:

- $\mathtt{V} = \{p, q, r, ...\}$ – a countable infinite set of propositional variables,
- $\{\neg, \rightarrow, \equiv\}$ – the set consisting of the unary operator of negation ($\neg$) and binary operators of implication ($\rightarrow$) and identity ($\equiv$),
- $\{(,)\}$ – the set of auxiliary symbols.

**Definition 2.2** (Formulas of SCI)**.** The set of formulas of SCI is defined with the following grammar:

$$\mathtt{FOR} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \rightarrow \varphi) \mid (\varphi \equiv \varphi)$$

where $p \in \mathtt{V}$ is a propositional variable.

The propositional variables will also be called atomic formulas.

From now on whenever we write $p, q, r, ...$ we will mean the atomic formulas. We will omit brackets when it will lead to no misunderstanding. We will write $\varphi \not\equiv \psi$ as a shorthand for $\neg(\varphi \equiv \psi)$.

The set of identities $\mathtt{ID}$ is a set of formulas $\varphi \equiv \psi$ where $\varphi, \psi \in \mathtt{FOR}$. Formulas $\varphi \equiv \varphi$ are the trivial identities.

**Definition 2.3** (Subformulas)**.** For a formula $\varphi \in \mathtt{FOR}$ let us define the set of subformulas of $\varphi$ as:

$$\mathtt{SUB}(\varphi) = \begin{cases} \{p\}, & \text{if } \varphi = p \in \mathtt{V}, \\ \{\varphi\} \cup \mathtt{SUB}(\psi), & \text{if } \varphi = \neg\psi, \\ \{\varphi\} \cup \mathtt{SUB}(\psi) \cup \mathtt{SUB}(\vartheta), & \text{if } \varphi = \psi \rightarrow \vartheta, \text{or } \varphi = \psi \equiv \vartheta \end{cases}$$

By $\varphi(\psi/\vartheta)$ we will denote the formula $\varphi$ with all occurrences of its subformula $\psi$ substituted with $\vartheta$.

**Definition 2.4** (Simple formulas)**.** The formula $\varphi$ is called a simple formula if it has one of the following form:

$$p,\ \neg p,\ p \equiv q,\ p \not\equiv q,\ p \equiv \neg q,\ p \equiv (q \rightarrow r),\ p \equiv (q \equiv r)$$

for $p, q, r \in \mathtt{V}$.

**Definition 2.5** (Size of a formula)**.** Given a formula $\varphi$, let us define its size $s(\varphi)$:

$$s(\varphi) = \begin{cases} 1, & \text{if } \varphi = p, \\ s(\psi) + 1, & \text{if } \varphi = \neg\psi, \\ s(\psi) + s(\vartheta) + 1, & \text{if } \varphi = \psi \equiv \vartheta, \text{ or } \varphi = \psi \rightarrow \vartheta. \end{cases}$$

$\mathtt{V}$ is a countable set. Let us take any full ordering of it and mark it as $<$. Let us then extend it by saying that for each $p \in \mathtt{V}$: $p < \neg < ( < ) < \rightarrow < \equiv$. Now, let us define a ordering of formulas $\prec$ to be a lexicographical ordering with $<$.

If we consider formulas that contain only the negation and implication operators, they form a classical Propositional Calculus. For simplicity, in SCI we'll consider every tautology of the classical Propositional Calculus to be an axiom.

**Definition 2.6** (Axiomatization of SCI)**.** SCI is axiomatized with the following axioms:

- Any tautology of the classical Propositional Calculus
- $(Ax1)$ $\varphi \equiv \varphi$,
- $(Ax2)$ $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$,
- $(Ax3)$ $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$,
- $(Ax4)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \theta)))$,
- $(Ax5)$ $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \theta) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \theta)))$.

The only inference rule is the *modus ponens* rule:

$$\mathsf{MP} : \frac{\varphi,\ \varphi \rightarrow \psi}{\psi}$$

**Definition 2.7** (A thesis of SCI)**.** A formula $\varphi$ is *a thesis of* SCI if there exists a finite sequence of formulas $\varphi_1, ..., \varphi_n$ $(n \geq 1)$, such that $\varphi = \varphi_n$, and for all $i \in \{1, ..., n\}$ the formula $\varphi_i$ is either an axiom of SCI, or it is inferred from formulas $\varphi_j$, $\varphi_k$ $(j, k < i)$ via the *modus ponens* rule. If $\varphi$ is a thesis of SCI we will denote it by $\vdash \varphi$, say that $\varphi$ is *provable in* SCI and call the sequence $\varphi_1, ..., \varphi_n$ the *proof of* $\varphi$.

Let us now give some semantic definitions:

**Definition 2.8** (SCI-model)**.** A model of SCI (or an SCI-model) model is a structure $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ where:

- $U \neq \emptyset$ is an *universe* of $M$,
- $\emptyset \neq D \subsetneq U$ is a *set of designated values*,
- $\tilde{\neg}$ is an unary operation on $U$, such that for all $a \in U$: $\tilde{\neg}a \in D$ if and only if $a \notin D$,
- $\tilde{\rightarrow}$ is a binary operation on $U$, such that for all $a, b \in U$: $a \tilde{\rightarrow} b \in D$ if and only if $a \notin D$ or $b \in D$,
- $\tilde{\equiv}$ is a binary operation on $U$, such that for all $a, b \in U$: $a \tilde{\equiv} b \in D$ if and only if $a = b$ (that is, if and only if $a$ and $b$ denote the same element of the universe $U$).

**Definition 2.9** (Valuation)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a valuation in $\mathcal{M}$ is a function $V : \mathtt{FOR} \longrightarrow U$ that assigns a value $V(p) \in U$ for all propositional variables $p$, and such that for all $\varphi, \psi \in \mathtt{FOR}$:

- $V(\neg\varphi) = \tilde{\neg}V(\varphi)$
- $V(\varphi \rightarrow \psi) = V(\varphi)\tilde{\rightarrow}V(\psi)$
- $V(\varphi \equiv \psi) = V(\varphi)\tilde{\equiv}V(\psi)$

If $V(\varphi) = a$ we will call $a$ the *denotation of* $\varphi$.

**Definition 2.10** (Satisfaction of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$ and a valuation $V$ in $\mathcal{M}$, a formula $\varphi$ is *satisfied in* $\mathcal{M}$ *by* $V$ if and only if $V(\varphi) \in D$. If a formula $\varphi$ is satisfied in $\mathcal{M}$ by a valuation $V$, then will denote it by $M, V \models \varphi$.

**Definition 2.11** (Truth of a formula)**.** Given an SCI-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\rightarrow}, \tilde{\equiv})$, a formula is *true in* $\mathcal{M}$ if and only if it is satisfied in $\mathcal{M}$ by all valuations of $\mathcal{M}$. If a formula $\varphi$ is true in $\mathcal{M}$, then we will denote it by $\mathcal{M} \models \varphi$.

**Definition 2.12** (Validity of a formula). A formula is *valid in* SCI (or SCI-*valid*) if and only if it is true in all models. If a formula $\varphi$ is SCI-valid we will denote it by $\models \varphi$.

Let us see some examples:
TODO

**Theorem 1** (Soundness and completeness of SCI). *For every* SCI*-formula* $\varphi$*, the following conditions are equivalent:*

- $\varphi$ *is provable in* SCI*,*
- $\varphi$ *is valid in* SCI*.*

Let us show an outline of the proof.

*Proof.* First, let us show that every provable SCI-formula is valid in SCI. To show that, we want to prove that:

(a) Every axiom of SCI is valid in SCI.

(b) By applying the *modus ponens* rule to valid formulas $\varphi$ and $\varphi \to \psi$ the inferred formula $\psi$ is also valid.

Let us show (a) first:

- First, we need to prove that every tautology of the classical Propositional Calculus (PC) is valid in SCI.

  Let $\varphi$ be a tautology of PC. Let us take any SCI-model $\mathcal{M}$ and any SCI-valuation $\mathcal{M}, V$ in it. We want to show that $\mathcal{M}, V \models \varphi$.

  Given $\mathcal{M}, V$, we will construct $V'$ that will be a PC valuation of $\varphi$. We can do this, because $\varphi$ doesn't contain operator $\equiv$.

  For every $\psi \in \mathtt{SUB}(\varphi)$, let us define:

$$V'(\psi) = \begin{cases} 1, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \in D, \\ 0, & \text{if } \psi = p \in \mathtt{V} \text{ and } V(p) \notin D, \\ 1 - V'(\chi), & \text{if } \psi = \neg\chi, \\ max(1 - V'(\chi), V'(\theta)), & \text{if } \psi = \chi \to \theta. \end{cases}$$

  It is easy to see that for every $\psi \in \mathtt{SUB}(\varphi)$, $V'(\psi) = 1$ if and only if $V(\psi) \in D$.

  The $V'$ function is constructed in the same way the valuation function in the PC is constructed, therefore since $\varphi$ is a tautology of PC, we have that $V'(\varphi) = 1$. So, we have that $V(\varphi) \in D$, whis is what we wanted to show.

- Second, we want to show that axioms $(Ax1) - (Ax5)$ are valid in SCI.

  – $(Ax1)$ $\varphi \equiv \varphi$
    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.9, we have that $V(\varphi \equiv \varphi) = V(\varphi) \tilde{\equiv} V(\varphi)$. Based on definition of $\tilde{\equiv}$ from 2.8, we have that $V(\varphi) \tilde{\equiv} V(\varphi)$ is in $D$ if and only if $V(\varphi) = V(\varphi)$, which is trivially the case. So, we have that $\mathcal{M}, V \models \varphi \equiv \varphi$.

  – $(Ax2)$ $(\varphi \equiv \psi) \to (\varphi \to \psi)$
    Let us take any SCI model and valuation $\mathcal{M}, V$. Based on definition 2.9, we have that $V((\varphi \equiv \psi) \to (\varphi \to \psi)) = V(\varphi \equiv \psi) \tilde{\to} V(\varphi \to \psi)$. There are two cases.

    1° $V(\varphi \equiv \psi) \notin D$. Then, by definition of $\tilde{\to}$ in 2.8, we have that $V(\varphi \equiv \psi) \tilde{\to} V(\varphi \to \psi) \in D$, so $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

    2° $V(\varphi \equiv \psi) \in D$. By definition 2.9 it means, that $V(\varphi) \tilde{\equiv} V(\psi)$, which by definition 2.8 means that $V(\varphi) = V(\psi)$.
    By definition 2.9 $V(\varphi \to \psi)$ is equal to $V(\varphi) \tilde{\to} V(\psi)$. By definition 2.8 $V(\varphi) \tilde{\to} V(\psi)$ is in $D$ if and only if $V(\varphi) \notin D$ or $V(\psi) \in D$, but since $V(\varphi) = V(\psi)$ we have that $V(\varphi) \tilde{\to} V(\psi)$ is in $D$ if and only if $V(\varphi) \notin D$ or $V(\varphi) \in D$, which is trivially the case.

    So, we have that $\mathcal{M}, V \models (\varphi \equiv \psi) \to (\varphi \to \psi)$.

  – Validity of axioms $(Ax3), (Ax4)$ and $(Ax5)$ can be shown in a similar way. (TODO: a może rozpisać dla jasności?)

To show (b), let take any valid formulas $\varphi$ and $\varphi \to \psi$ and any $\mathsf{SCI}$ model and valuation $\mathcal{M}, V$. From validity of $\varphi \to \psi$, we know that $V(\varphi \to \psi) \in D$. From definition 2.9 we have that $D \ni V(\varphi \to \psi) = V(\varphi)\tilde{\to}V(\psi)$. From definition 2.8, since we know that $V(\varphi)\tilde{\to}V(\psi)$ we know that $V(\varphi) \notin D$, or $V(\psi) \in D$. But we have that $\varphi$ is valid, so $V(\varphi) \in D$, so it must be that $V(\psi) \in D$, so $\mathcal{M}, V \models \psi$. The same holds for any other $\mathsf{SCI}$ model and valuation, so $\models \psi$, which is what we wanted to show.

Looking at definition 2.7, for a given provable $\varphi$, let us take its proof $\varphi_1, ..., \varphi_n = \varphi$. Every subsequent formula in this proof is either an $\mathsf{SCI}$ axiom and thus, by (a), valid, or is inferred by the *modus ponens* rule from valid formulas and thus, by (b) valid. So, $\varphi$ is valid.

Now, let us show that any valid $\mathsf{SCI}$-formula is also provable.

TODO □

The proof of the correctness and fullness theorem can be found in [1] (Theorem 1.9).

TODO: podać ideę dowodu

**Definition 2.13** (Decidability)**.** A logic is *decidable* if there exists an effective method to determine whether a given formula of this logic is a theorem.

TODO: definicja rozstrzygalności

**Theorem 2** (Decidability of $\mathsf{SCI}$)**.** $\mathsf{SCI}$ *is decidable.*

The proof of the decidability theorem can be found in [1] (Corollary 2.4).

TODO: dowód.

# 3 Deduction in SCI

## 3.1 Deduction systems

TODO: Let us give an example of a provable formula with its full proof:

# References


[1] Stephen L. Bloom, Roman Suszko (1972) *Investigations into the Sentential Calculus with Identity*, Notre Dame Journal of Formal Logic, vol. XIII, no 3.