CRAIOVA UNIVERSITY
FACULTY OF AUTOMATION, COMPUTERS AND ELECTRONICS

DEPARTMENT OF COMPUTERS AND INFORMATION TECHNOLOGY

DIPLOMA PROJECT

Ștefan Adrian-Cătălin

SCIENTIFIC COORDINATOR

Conf. Univ. Dr. Ing. Eugen Ganea

&

Assoc. Prof. Dr. Boureanu Maria-Magdalena

July 2024

CRAIOVA

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Numerical methods for linear algebra

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DECLARATION OF ORIGINALITY

The undersigned Ștefan Adrian-Cătălin, a student at the Computers and Information Technology specialization at the Faculty of Automation, Computers and Electronics of the University of Craiova, hereby certify that I have read the following and that I assume, in this context, the originality of my bachelor's project:

- with the title Numerical methods for linear algebra,
- coordinated by Conf. Univ. Dr. Ing. Eugen Ganea & Assoc. Prof. Dr. Boureanu Maria-Magdalena,
- presented in the July 2024 session.

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....
Head of
Department,
Prof. dr. ing.
Nicolae Enescu

THE DIPLOMA PROJECT

Name and surname of the student:	Ștefan Adrian-Cătălin
Theme Statement:	Numerical methods for linear algebra
Start dates:	How do linear systems help us in everyday life.
Project content:	The paper contains several methods for solving linear systems, and for each we have explained the method, the advantages of using this method, a solved example and the pseudocode algorithm.
Compulsory graphic material:	
Consultations:	*Weekly*
Scientific leader (title, name and surname, signature):	Conf. Univ. Dr. Ing. Eugen Ganea & Assoc. Prof. Dr. Boureanu Maria-Magdalena
Theme release date:	15.10.2023
Estimated deadline for handing over the project:	01.06.2024
Date of handing in the project by the student and his signature:	

REPORT OF THE SCIENTIFIC MANAGER

Name and surname of the candidate:	Ștefan Adrian-Cătălin
Specialization:	Department of Computers and Information Technology
Project title:	Numerical methods for linear algebra
The location where the documentation practice was carried out (check one or more of the options on the right):	In the faculty ☐ In production ☐ In research ☐ Other location: [*detail*]

Following the analysis of the candidate's work, the following were found:

Level of documentation		Insufficient ☐	Satisfactorily ☐	Good ☐	Very good ☐
Project type		Research ☐	Projection ☐	Practical realization ☐	Other [*detail*]
The mathematical apparatus used		Simple ☐	Middle ☐	Complex ☐	Absent ☐
Utility		Research contract ☐	Internal research ☐	Equipment ☐	Other [*detail*]
Writing the paper		Insufficient ☐	Satisfactorily ☐	Good ☐	Very good ☐
The graphic part, drawings		Insufficiency ☐	Satisfying ☐	Good ☐	Very good ☐
Practical realization	Author's contribution	Insufficiency ☐	Satisfying ☐	Big ☐	Very big ☐
	Theme complexity	Simple ☐	Middle ☐	Big ☐	Complex ☐
	Requirements analysis	Insufficient ☐	Satisfactorily ☐	Good ☐	Very good ☐
	Architecture	Simple ☐	Middle ☐	Big ☐	Complex ☐

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In conclusion, it is proposed:

ADMISSION OF THE PROJECT	REJECTION OF THE PROJECT
☐	☐

Date,

Signature of the scientific supervisor,

Contents

Chapter 1

Introduction

Mathematics has always been my favorite subject, and its applications in informatics and the IT industry have always been a point of interest for me. And one of the aspects that attracted me was **solving linear systems**.

What are **linear systems** good for?
In mathematics, a set of many equations involving the same variables is referred to as a system of equations. It is clear from the term “system” itself that the equations are regarded as communal rather than individual. System solving, whether through numerical or other means, is crucial in fields including engineering, chemistry, physics and computer science. Allow me to elaborate on the well-known query that the majority of individuals who are not very interested in mathematics or who do not work in these fields have, which is “what good is this to me?”. I would like to challenge the widespread belief that mathematics is overly abstract in the sections that follow. Yes, when we combine all the parts, we learn that, without even realizing it, we encounter equation systems on a daily basis.
An example of a system of equations in mathematics would be the following and it would seem something trivial:

$$\begin{cases} 3x + 4y = 11 \\ 2x + 5y = 12 \end{cases}$$

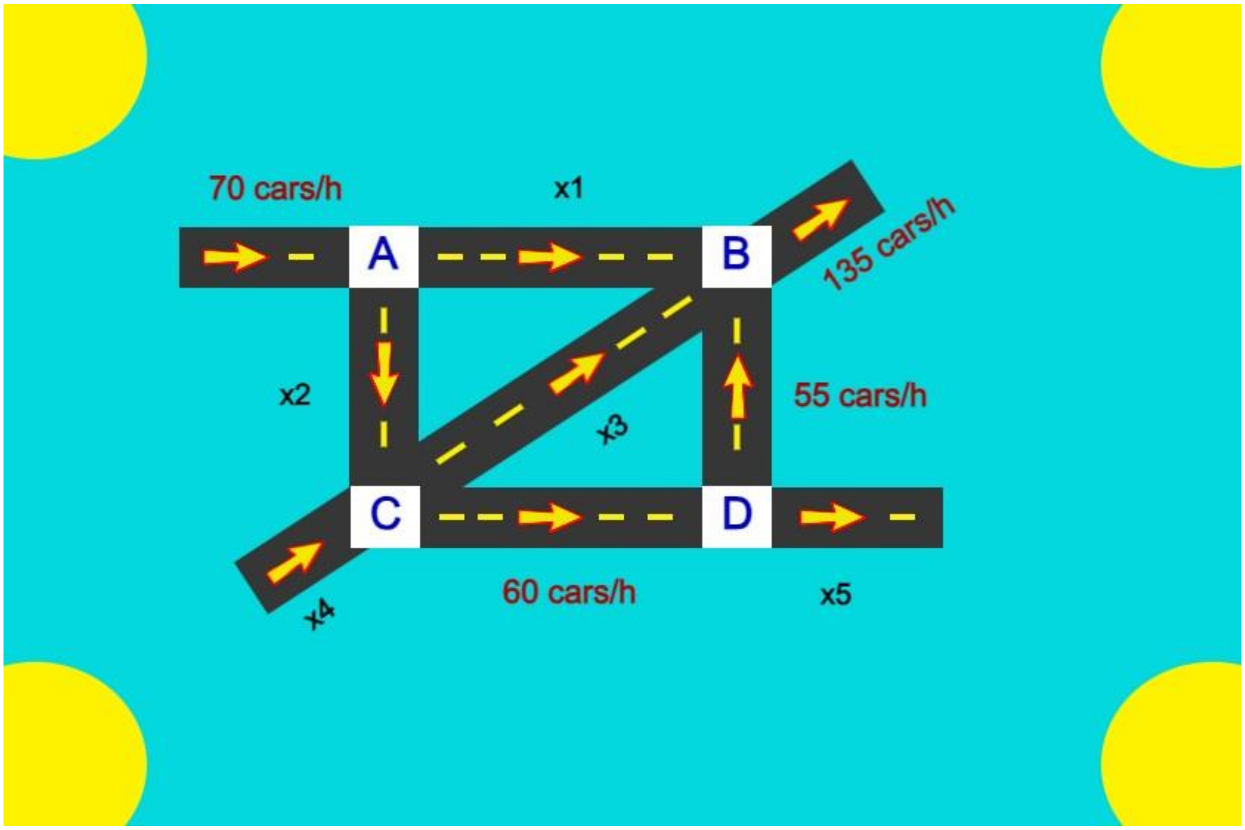
$$\begin{cases} 3x + 4y = 11 & | * 5 \\ 2x + 5y = 12 & | * 4 \end{cases}$$

$$\begin{cases} 15x + 20y = 55 \\ 8x + 20y = 48 \end{cases}$$

$$\overline{}(-)$$

$7x = 7$
$x = 1$
$4y = 11 - 3$
$y = 2.$

If we look at this system, we do not understand how it connects to reality. Indeed, it does not seem to be interesting to us from this point of view. But if we imagine two kids that just bought from a general store pencils and candies, things change. Let us assume that the first kid bought 3 pencils and 4 candies and paid 11 euros, while the second kid bought 2 pencils and 5 candies and paid 12 euros. At school, they meet a colleague who asks: “How much does a candy cost?” The first two kids realize that they do not know, they simply paid what the seller asked. How can they find out how much a candy costs? Well, in the above system, if we denote by x the price of a pencil and by y the price of a candy, we just solved this situation. And the mathematics involved here does not seem meaningless anymore.
Another day to day example was inspired by [1]. In the flow of the traffic we also use systems without even noticing it. The most frequently utilized streets must be identified by engineers and architects when creating a traffic light circuit.



$$\begin{cases} 70 + x4 = 135 + x5 \\ 70 = x1 + x2 \\ x1 + x3 + 55 = 135 \\ x2 + x4 = 60 + x3 \\ 60 = 55 + x5 \end{cases}$$

$$\begin{cases} x4 - x5 = 65 \\ x1 + x2 = 70 \\ x1 + x3 = 80 \\ x2 - x3 + x4 = 60 \\ x5 = 5 \end{cases}$$

In this way, the solutions to several issues in our daily lives are provided by linear systems.
Other real-world situations that are solved with the aid of systems of equations include: calculating flight and marine routes, creating a business plan, calculating the difference between bank interest rates and IFNs in the event of a loan, railroad management, identifying the ideal job in relation to the required criteria etc see [1].

In **linear systems**, the mathematical solution obtained by hand is done for systems with 3-4 unknowns, but when we talk about larger systems, which have over 10, 20 or 100 unknowns, solving them by hand is particularly laborious and unfeasible. Here comes the role of the computer and numerical methods.
From a numerical point of view, there are two types of methods, **direct methods** and **iterative methods**.

Through a linear system with m linear equations and n unknown variables, with real coefficients, we consider the following set of equalities

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $a_{ij}, b_i \in R$, for all $i \in \{1, 2, ..., m\}$ and all $j \in \{1, 2, ..., n\}$. Furthermore, the system mentioned above is referred to as a homogeneous system of linear equations if $b_1 = b_2 = ... = b_m = 0$.
When a polynomial function of the first degree is used in an equations, it is referred to as linear.

We will consider a solution of the system from above, that is, a vector $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \cdots \\ \tilde{x}_n \end{pmatrix} \in \mathbf{R}^n$ such that in the moment we substitute each x_i by $\tilde{x}_i$, for all $i \in \{1, 2, \dots , n\}$, all the equalities in the system hold simultaneously. Depending on whether the system described above has at least one solution, we distinguish two types of systems: incompatible systems, which are systems with no solution, and compatible systems, which are systems which admit at least one solution. A system that admits a unique solution is called a determined compatible system, and a system that admits infinitely many solutions is called an undetermined compatible system.
To every system like the one above we can associate the following matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The free terms on the right-hand side of the linear equations will be kept in a vector

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}.$$

The system from above can be written in a compact form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, 2, \dots, m.$$

Finally, the matrix form of the system is the following $Ax = b$, where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}.$$

So, we can denote the following matrix

$$\bar{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right),$$

the extended matrix associated to the above system.

There are two types of methods that can be used to solve a system of linear equations via numerical methods: direct methods (they are generally used for systems with less than 100 unknown variables and equations; these allow us to arrive at the solution of the system after a finite number of steps) and iterative methods. As examples of direct methods, we will refer to three variants of Gauss's method, presented in Chapters 2, 3 and 4. Since these methods allow, in addition to solving the systems, to also calculate the determinant of the associated matrix, Chapter 5 presents another method for calculating the determinants, namely Chio's method of pivotal condensation. This method is more efficient from a numerical point of view. Chapters 6 and 7 present two iterative methods for calculating linear systems, namely Jacobi and Seidel-Gauss. The iterative methods are more useful than the direct methods when we deal with larger systems, which have more than 100-200 unknown variables.

We will comment on the advantages and disadvantages of each method and we are going to illustrate these by mathematically solving the exercises (by hand). The mathematical calculus is important because it represents the source of the algorithm. Moreover, the diverse situations encountered when we solve mathematically a linear system lead to the improvement of the corresponding pseudocode algorithm.

Each chapter includes a description of the method under discussion, examples of exercises solved by hand, versions of the pseudocode algorithms and comments. For the theoretical aspects we refer to [2], [6] and [7], and for the pseudocode algorithms we refer to [3] and [5]. However, some algorithms were modified and improved by me. Also, some of the mathematical examples were constructed by me.

Chapter 2

Gaussian elimination method

In 1823, Gauss established a general method that is known as Gauss's method, which, as said above, represents a direct method of solving linear systems. Based on a successive elimination schema of the system's unknowns, this method involves basic matrix manipulations, such as the permutation of two rows (horizontal lines) or columns (vertical lines), one-row multiplications by scalars, and the sums of this among themselves, etc.

We consider the linear system $A \cdot x = b$, where $A \in M_n(\mathbf{R})$ is the matrix associated to the system from above and $b \in \mathbf{R}^n$ is the vector containing the free terms of the system. Our objective is to find, if possible, $x \in \mathbf{R}^n$, where x represents the unique solution of the system.

Now we take the augmented matrix $(A \mid b) = (a_{ij})$ with $1 \le i \le n$ and $1 \le j \le n+1$, where $a_{i,n+1} = b_i$, $1 \le i \le n$.

The Gauss method consists of processing the augmented matrix $(A \mid b)$ such that, in $n-1$ steps the matrix A becomes upper-triangular:

$$\left(\begin{array}{ccccc|c} a_{11}^{(n)} & a_{12}^{(n)} & \cdots & a_{1,n-1}^{(n)} & a_{1,n}^{(n)} & a_{1,n+1}^{(n)} \\ 0 & a_{22}^{(n)} & \cdots & a_{2,n-1}^{(n)} & a_{2,n}^{(n)} & a_{2,n+1}^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1}^{(n)} & a_{n-1,n}^{(n)} & a_{n-1,n+1}^{(n)} \\ 0 & 0 & \cdots & 0 & a_{n,n}^{(n)} & a_{n,n+1}^{(n)} \end{array}\right) = A^{(n)}, where\ A^{(1)} = (A|b).$$

If $a_{kk}^{(k)} \neq 0, 1 \le k \le n-1$, where the element $a_{kk}^{(k)}$ is called **pivot**, in order to arrive at matrix from above we apply the following algorithm. For $k = 1, 2, \ldots, n-1$, we copy the first k rows (lines); on column "k", under pivot, the elements will be null (zero); the remaining elements, situated below the row "k" and at the right hand side of the column "k", will be determined using the so-called "rectangle rule", described below:

$$\begin{array}{lccccl} & \vdots & & \vdots & & \\ & a_{kk}^{(k)} & & a_{kj}^{(k)} & & \\ row\ k \cdots & \vdots & \cdots\cdots\cdots & \vdots & \cdots & \\ row\ i \cdots & a_{ik}^{(k)} & \cdots\cdots\cdots & a_{ij}^{(k)} & \cdots & \\ & \vdots & & \vdots & & \\ & column\ k & & column\ j & & \end{array} \Rightarrow a_{ij}^{(k+1)} = \frac{a_{kk}^{(k)} a_{ij}^{(k)} - a_{ik}^{(k)} a_{kj}^{(k)}}{a_{kk}^{(k)}}.$$

Therefore, for $1 \leq k \leq n-1$, we will use the following formula:

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)} & 1 \leq i \leq k, \quad i \leq j \leq n+1 \\ 0 & 1 \leq j \leq k, \quad j+1 \leq i \leq n \\ a_{ij}^{(k)} - \dfrac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \cdot a_{kj}^{(k)} & k+1 \leq i \leq n, \quad k+1 \leq j \leq n+1 \end{cases}.$$

After arriving at the upper-triangular matrix, we notice that we have actually arrived at an upper-triangular system equivalent to the first one from above:

$$\begin{cases} a_{11}^{(n)} x_1 + a_{12}^{(n)} x_2 + \cdots + a_{1n}^{(n)} x_n = a_{1,n+1}^{(n)} \\ a_{22}^{(n)} x_1 + \cdots + a_{2n}^{(n)} x_n = a_{2,n+1}^{(n)} \\ \cdots\cdots\cdots \\ a_{ii}^{(n)} x_1 + \cdots + a_{in}^{(n)} x_n = a_{i,n+1}^{(n)} \\ \cdots\cdots\cdots \\ a_{nn}^{(n)} x_n = a_{n.n+1}^{(n)}. \end{cases}$$

This system can be solved by using the back substitution method, that is, by applying the following formula:

$$x_n = a_{n,n+1}^{(n)} \,/\, a_{nn}^{(n)}, \ if\ a_{nn}^{(n)} \neq 0,$$

and, for i = n – 1, n – 2, ..., 1,

$$x_i = \left(a_{i,n+1}^{(n)} - \sum_{j=i+1}^{n} a_{ij}^{(n)} \cdot x_j \right) / \ a_{ii}^{(n)}.$$

Pseudocode Algorithm for Gaussian method

// Here we read n, the dimension of matrix A and the augmented matrix (A | b)

1. read n, a_{ij}, $1 \leq i \leq n, 1 \leq j \leq n+1$
2. for k = 1, 2, ..., n-1
   - 2.1. if $a_{kk} \neq 0$ then

     //Here we apply the formula from Gauss method (the rectangle rule),
     - 2.1.1. for i = k + 1, k + 2, ..., n
       - 2.1.1.1. for j = k + 1, k + 2, ..., n + 1
         - 2.1.1.1.1. $a_{ij} \Leftarrow a_{ij} - a_{ik} \cdot a_{kj} / a_{kk}$
3. if $a_{nn} = 0$ then
   - 3.1. write 'The system does not have unique solution'
   - 3.2. exit

//Here we determine x_n

4. $a_{n,n+1} \Leftarrow a_{n,n+1}/a_{nn}$

//Here we determine x_i, $n - 1 \geq i \geq 1$

5. for i = n – 1, n – 2, ..., 1

5.1. S ⇦ 0
5.2. for j = i +1, i + 2, ..., n
5.2.1. S ⇦ S + $a_{ij} \cdot a_{j,n+1}$
5.3. $a_{i,n+1}$ ⇦ ($a_{i,n+1}$ – S)/a_{ii}
6. write 'x_i =', $a_{i,n+1}$, $1 \leq i \leq n$.
//Here we calculate the determinant
7. det = 1
8. For i = 1,2, ..., n
8.1. det = det * a_{ii}
9. write "det A=" det

Remark 2.1: We notice that we can have the case where a_{kk} = 0. In this situation, we will have to find another line. For better numerical stability, even in the situation where the pivot is zero, it will be chosen according to a certain rule, depending on this chosen rule we distinguish between two methods, Gauss's method with partial pivoting and the one with total pivoting.
From solving mathematical examples by hand, by understanding them and by meeting particular situations, we learn how to improve algorithms.
As a mathematical example, we present the following example, see [3]

$$\begin{cases} 2x1 + 2x2 + 3x3 + x4 = 6 \\ 3x1 + 3x2 + 2x3 + x4 = 2 \\ x1 + x4 = 0 \\ x1 + x2 + x3 = 2. \end{cases}$$

The extended matrix corresponding to the system is

$$A^{(1)} = (A|b) = \left(\begin{array}{cccc|c} 2 & 2 & 3 & 1 & 6 \\ 3 & 3 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{array}\right).$$

To expose the point of view from the previous observation, we will solve this example following the Gauss method with partial pivoting to obtain an a_{kk} = 0 and visualize the necessary steps.
So we search the pivot to be put in the position of a_{11} on the first column, more exactly we will choose the element with the largest absolute value from the first column of the matrix, in this case is a_{21}=3, so we will interchange row 1 with row 2 and we obtain

$$A^{(1)} \overline{\overline{L1 \leftrightarrow L2}} \left(\begin{array}{cccc|c} 3 & 3 & 2 & 1 & 2 \\ 2 & 2 & 3 & 1 & 6 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{array}\right).$$

Now we proceed with the Gaussian elimination method as usual, with $a^{(1)}_{11}$=3 as pivot:

$$a_{22}^{(2)} = \frac{3 \cdot 2 - 2 \cdot 3}{3} = 0,$$
$$a_{23}^{(2)} = \frac{3 \cdot 3 - 2 \cdot 2}{3} = \frac{5}{3},$$
$$a_{24}^{(2)} = \frac{3 \cdot 1 - 2 \cdot 1}{3} = \frac{1}{3},$$
$$a_{25}^{(2)} = \frac{3 \cdot 6 - 2 \cdot 2}{3} = \frac{14}{3},$$

$$a_{32}^{(2)} = \frac{3 \cdot 0 - 1 \cdot 3}{3} = -1,$$

$$a_{33}^{(2)} = \frac{3 \cdot 0 - 1 \cdot 2}{3} = -\frac{2}{3},$$

$$a_{34}^{(2)} = \frac{3 \cdot 1 - 1 \cdot 1}{3} = \frac{2}{3},$$

$$a_{35}^{(2)} = \frac{3 \cdot 0 - 1 \cdot 2}{3} = -\frac{2}{3},$$

$$a_{42}^{(2)} = \frac{3 \cdot 1 - 1 \cdot 3}{3} = 0,$$

$$a_{43}^{(2)} = \frac{3 \cdot 1 - 1 \cdot 2}{3} = \frac{1}{3},$$

$$a_{44}^{(2)} = \frac{3 \cdot 0 - 1 \cdot 1}{3} = \frac{1}{3},$$

$$a_{45}^{(2)} = \frac{3 \cdot 2 - 1 \cdot 2}{3} = \frac{4}{3}.$$

So we obtain the matrix

$$A^{(2)} = \left(\begin{array}{cccc|c} 3 & 3 & 2 & 1 & 2 \\ 0 & 0 & 5/3 & 1/3 & 14/3 \\ 0 & -1 & -2/3 & 2/3 & -2/3 \\ 0 & 0 & 1/3 & -1/3 & 4/3 \end{array}\right).$$

Now we notice the case where $a_{kk} = 0$, so as we did before, we choose the absolute value from the column, which is $a_{32} = -1$, so we interchange row 2 with row 3 and we get

$$A^{(2)} \overline{\overline{L2 \leftrightarrow L3}} \left(\begin{array}{cccc|c} 3 & 3 & 2 & 1 & 2 \\ 0 & -1 & -2/3 & 2/3 & -2/3 \\ 0 & 0 & 5/3 & 1/3 & 14/3 \\ 0 & 0 & 1/3 & -1/3 & 4/3 \end{array}\right).$$

Now we apply again the rectangular rule with $a^{(2)}_{22}=-1$ as pivot

$$a_{33}^{(3)} = \frac{-1 \cdot \frac{5}{3} - 0 \cdot -\frac{2}{3}}{-1} = \frac{5}{3},$$

$$a_{34}^{(3)} = \frac{-1 \cdot \frac{1}{3} - 0 \cdot \frac{2}{3}}{-1} = \frac{1}{3},$$

$$a_{35}^{(3)} = \frac{-1 \cdot \frac{14}{3} - 0 \cdot -\frac{2}{3}}{-1} = \frac{14}{3},$$

$$a_{43}^{(3)} = \frac{-1 \cdot \frac{1}{3} - 0 \cdot -\frac{2}{3}}{-1} = \frac{1}{3},$$

$$a_{44}^{(3)} = \frac{-1 \cdot -\frac{1}{3} - 0 \cdot \frac{2}{3}}{-1} = -\frac{1}{3},$$

$$a_{45}^{(3)} = \frac{-1 \cdot \frac{4}{3} - 0 \cdot -\frac{2}{3}}{-1} = \frac{4}{3}.$$

Remark 2.2: Notice that $A^{(3)} = A^{(2)}$.

We repeat the steps before, we choose $a_{33}=\frac{5}{3}$ to be pivot and we keep rows as they were, and we start solving using the "rectangle rule":

$$a_{44}^{(4)} = \frac{\frac{5}{3} \cdot (-\frac{1}{3}) - \frac{1}{3} \cdot \frac{1}{3}}{\frac{5}{3}} = -\frac{2}{5},$$

$$a_{45}^{(4)} = \frac{\frac{5}{3} \cdot \frac{4}{3} - \frac{1}{3} \cdot \frac{14}{3}}{\frac{5}{3}} = \frac{2}{5}.$$

We obtain

$$A^{(4)} = \left(\begin{array}{cccc|c} 3 & 3 & 2 & 1 & 2 \\ 0 & -1 & -2/3 & 2/3 & -2/3 \\ 0 & 0 & 5/3 & 1/3 & 14/3 \\ 0 & 0 & 0 & -2/5 & -2/5 \end{array}\right).$$

The corresponding system is

$$\begin{cases} 3x1 + 3x2 + 2x3 + x4 = 2 \\ -x2 - \frac{2}{3}x3 + \frac{2}{3}x4 = -\frac{2}{3} \\ \frac{5}{3}x3 + \frac{1}{3}x4 = \frac{14}{3} \\ -\frac{2}{5}x4 = \frac{2}{5}. \end{cases}$$

So the solution is

$$\begin{cases} x4 = -1 \\ x3 = 3 \\ x2 = -2 \\ x1 = 1, \end{cases}$$

and, therefore

$$\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -1 \end{pmatrix}.$$

One of our constant concerns is to improve as much as possible the algorithms. For example, as pointed out in **Remark 2.2**, in the exercise above, we noticed that every time we have 0 under the pivot, the corresponding line does not change. Therefore we can improve the algorithm so that when under the pivot is 0 all the time, the line will remain unchanged. This way we can skip a step in the Gaussian method, and the improved algorithm is the following:

Improved version of the pseudocode algorithm for Gaussian method

// Here we read n, the dimension of matrix A and the augmented matrix (A | b)

1. read n, a_{ij}, $1 \le i \le n, 1 \le j \le n+1$

2. for $k = 1, 2, \ldots, n-1$

    2.1. if $a_{kk} \neq 0$ then

    //Here we apply the formula from Gauss method (the rectangle rule),

    2.1.1. for $i = k + 1, k + 2, \ldots, n$

        2.1.1.1. for $j = k + 1, k + 2, \ldots, n + 1$

            2.1.1.1.1. $a_{ij} \Leftarrow a_{ij} - a_{ik} \cdot a_{kj} / a_{kk}$

    2.2. else $a_{kk} = 0$ then

        2.2.1. $l = k$

        2.2.2. do l++ while $a_{lk} = 0$ and $l <= n$

        2.2.3. if $l > n$ then

            2.2.3.1. write 'The system does not have unique solution'

        2.2.4. for $j=k, k + 1, \ldots, n + 1$

            2.2.4.1. swap a_{kj} with a_{lj}

        2.2.5. np++

3. if $a_{nn} = 0$ then

    3.1. write 'The system does not have unique solution'

    3.2. exit

//Here we determine x_n

4. $a_{n,n+1} \Leftarrow a_{n,n+1}/a_{nn}$

//Here we determine $x_i, n - 1 \geq i \geq 1$

5. For $i = n - 1, n - 2, \ldots, 1$

    5.1. $S \Leftarrow 0$

    5.2. for $j = i + 1, i + 2, \ldots, n$

        5.2.1. $S \Leftarrow S + a_{ij} \cdot a_{j,n+1}$

    5.3. $a_{i,n+1} \Leftarrow (a_{i,n+1} - S)/a_{ii}$

6. write 'x_i =', $a_{i,n+1}$, $1 \leq i \leq n$.

//Here we calculate the determinant

7. $det = -1^{(np)}$

8. For $i = 1, 2, \ldots, n$

    8.1. $det = det * a_{ii}$

9. write "det A=" det

Moreover, in the previous pseudocode algorithm, we have included the calculus of the matrix A since from a mathematical point of view the determinant of a triangular matrix is equal to the product of the elements from the main diagonal. We have taken into consideration the fact that when passing from the initial form of A to its triangular form, we may interchange some lines, meaning that we have to change the sign of the determinant each time we make an interchange. In addition, for better numerical stability, it is important to ensure that the pivot is as far away as possible from 0. Therefore, the following variants of this method were developed:

- Gauss's method with partial pivoting
- Gauss's method with total pivoting

Chapter 3

Gauss's method with partial pivoting

In this method, even if the pivot is different from 0, at each step we choose as the pivot the element with the largest absolute value from the respective column. As a mathematical example, we present the following example

$$\begin{cases} 2x1+3x2+x3-4x4=5 \\ -4x1-2x2+3x3+7x4=3 \\ x1+x2-x3+x4=6 \\ 3x1+2x2-x3-3x4=9. \end{cases}$$

The extended matrix corresponding to the system is

$$A^{(1)}=(A|b)=\left(\begin{array}{cccc|c} 2 & 3 & 1 & -4 & 5 \\ -4 & -2 & 3 & 7 & 3 \\ 1 & 1 & -1 & 1 & 6 \\ 3 & 2 & -1 & -3 & 9 \end{array}\right).$$

So to respect the partial pivoting method, we choose as the pivot to be put in the position of a_{11} on the first column the element with the largest absolute value from the first column of the matrix, in this case is a_{21}=-4, so we will interchange row 1 with row 2 and we obtain

$$A^{(1)}\overline{\overline{L1\leftrightarrow L2}}\left(\begin{array}{cccc|c} -4 & -2 & 3 & 7 & 3 \\ 2 & 3 & 1 & -4 & 5 \\ 1 & 1 & -1 & 1 & 6 \\ 3 & 2 & -1 & -3 & 9 \end{array}\right).$$

Now we start solving using the "rectangle rule", with $a^{(1)}_{11}$=-4 as pivot:

$$a_{22}^{(2)}=\frac{-4\cdot 3-2\cdot -2}{-4}=2,$$

$$a_{23}^{(2)}=\frac{-4\cdot 1-2\cdot 3}{-4}=\frac{5}{2},$$

$$a_{24}^{(2)}=\frac{-4\cdot -4-2\cdot 7}{-4}=-\frac{1}{2},$$

$$a_{25}^{(2)}=\frac{-4\cdot 5-2\cdot 3}{-4}=\frac{13}{2},$$

$$a_{32}^{(2)}=\frac{-4\cdot 1-1\cdot -2}{-4}=\frac{1}{2},$$

$$a_{33}^{(2)}=\frac{-4\cdot -1-1\cdot 3}{-4}=-\frac{1}{4},$$

$$a_{34}^{(2)}=\frac{-4\cdot 1-1\cdot 7}{-4}=\frac{11}{4},$$

$$a_{35}^{(2)}=\frac{-4\cdot 6-1\cdot 3}{-4}=\frac{27}{4},$$

$$a_{42}^{(2)}=\frac{-4\cdot 2-3\cdot -2}{-4}=\frac{1}{2},$$

$$a_{43}^{(2)}=\frac{-4\cdot -1-3\cdot 3}{-4}=\frac{5}{4},$$

$$a_{44}^{(2)}=\frac{-4\cdot -3-3\cdot 7}{-4}=\frac{9}{4},$$

$$a_{45}^{(2)} = \frac{-4 \cdot 9 - 3 \cdot 3}{-4} = \frac{45}{4}.$$

So we obtain the matrix

$$A^{(2)} = \left(\begin{array}{cccc|c} -4 & -2 & 3 & 7 & 3 \\ 0 & 2 & 5/2 & -1/2 & 13/2 \\ 0 & 1/2 & -1/4 & 11/4 & 27/4 \\ 0 & 1/2 & 5/4 & 9/4 & 45/4 \end{array}\right).$$

Now, we have to choose again from the second column, the element with the largest absolute value as pivot, so in our case this element is a_{22}=2, in this case we don't have to interchange any line so we have the same matrix as before

$$A^{(2)} = \left(\begin{array}{cccc|c} -4 & -2 & 3 & 7 & 3 \\ 0 & 2 & 5/2 & -1/2 & 13/2 \\ 0 & 1/2 & -1/4 & 11/4 & 27/4 \\ 0 & 1/2 & 5/4 & 9/4 & 45/4 \end{array}\right).$$

Now we apply again the rectangular rule with $a^{(2)}_{22}$=2 as pivot

$$a_{33}^{(3)} = \frac{2 \cdot -\frac{1}{4} - \frac{1}{2} \cdot \frac{5}{2}}{2} = -\frac{7}{8},$$

$$a_{34}^{(3)} = \frac{2 \cdot \frac{11}{4} - \frac{1}{2} \cdot -\frac{1}{2}}{2} = \frac{23}{8},$$

$$a_{35}^{(3)} = \frac{2 \cdot \frac{27}{4} - \frac{1}{2} \cdot \frac{13}{2}}{2} = \frac{41}{8},$$

$$a_{43}^{(3)} = \frac{2 \cdot \frac{5}{4} - \frac{1}{2} \cdot \frac{5}{2}}{2} = \frac{5}{8},$$

$$a_{44}^{(3)} = \frac{2 \cdot \frac{9}{4} - \frac{1}{2} \cdot -\frac{1}{2}}{2} = \frac{19}{8},$$

$$a_{45}^{(3)} = \frac{2 \cdot \frac{45}{4} - \frac{1}{2} \cdot \frac{13}{2}}{2} = \frac{77}{8}.$$

So we obtain the matrix

$$A^{(3)} = \left(\begin{array}{cccc|c} -4 & -2 & 3 & 7 & 3 \\ 0 & 2 & 5/2 & -1/2 & 13/2 \\ 0 & 0 & -7/8 & 23/8 & 41/8 \\ 0 & 0 & 5/8 & 19/8 & 77/8 \end{array}\right).$$

We repeat the steps from before, again we will choose the element with the largest absolute value as pivot, in our case $a_{33}= -\frac{7}{8}$, so we will not have to interchange any line, in this case the matrix is the same as before

$$A^{(3)} = \left(\begin{array}{cccc|c} -4 & -2 & 3 & 7 & 3 \\ 0 & 2 & 5/2 & -1/2 & 13/2 \\ 0 & 0 & -7/8 & 23/8 & 41/8 \\ 0 & 0 & 5/8 & 19/8 & 77/8 \end{array}\right).$$

With $a_{33} = -\frac{7}{8}$ as pivot, we start solving using the “rectangle rule”:

$$a_{44}^{(4)} = \frac{-\frac{7}{8}\cdot\frac{19}{8} - \frac{5}{8}\cdot\frac{23}{8}}{-\frac{7}{8}} = \frac{31}{7},$$

$$a_{45}^{(4)} = \frac{-\frac{7}{8}\cdot\frac{77}{8} - \frac{5}{8}\cdot\frac{41}{8}}{-\frac{7}{8}} = \frac{93}{7}.$$

We obtain

$$A^{(4)} = \left(\begin{array}{cccc|c} -4 & -2 & 3 & 7 & 3 \\ 0 & 2 & 5/2 & -1/2 & 13/2 \\ 0 & 0 & -7/8 & 23/8 & 41/8 \\ 0 & 0 & 0 & 31/7 & 93/7 \end{array}\right).$$

The corresponding system is

$$\begin{cases} -4x1 - 2x2 + 3x3 + 7x4 = 3 \\ 2x2 + \frac{5}{2}x3 - \frac{1}{2}x4 = \frac{13}{2} \\ -\frac{7}{8}x3 + \frac{23}{8}x4 = \frac{41}{8} \\ \frac{31}{7}x4 = \frac{93}{7}. \end{cases}$$

Then the solution is

$$\begin{cases} x4 = 3 \\ x3 = 4 \\ x2 = -1 \\ x1 = 8, \end{cases}$$

and, therefore

$$\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 4 \\ 3 \end{pmatrix}.$$

Pseudocode Algorithm for the Gauss method with partial pivoting

```
// Here we read n, the dimension of matrix A and the augmented matrix (A | b)
1. read n, a_ij, 1 ≤ i ≤ n, 1 ≤ j ≤ n + 1
// We find the element with the largest absolute value to be selected as pivot
2. for k = 1,2,...,n – 1
        2.1. piv ⇦ |a_kk|
        2.2. lin ⇦ k
        2.3. for i = k + 1, k + 2, ..., n
                2.3.1. if piv < |a_ik| then
                        2.3.1.1. piv ⇦ |a_ik|
                        2.3.1.2. lin ⇦ i
        2.4. if piv = 0 then
                2.4.1. write “The system does not have a unique solution”
```

2.4.2. exit

2.5. if lin $\neq$ k then

2.5.1. for j = k, k+1, ..., n+1

2.5.1.1. swap a_{kj} with $a_{lin,j}$

2.5.2. np++

//Here we apply the formula from Gauss method (the rectangle rule),

2.6 for i = k+1, k+2, ..., n

2.6.1. for j = k+1, k+2, ..., n+1

2.6.1.1. $a_{ij} \Leftarrow a_{ij} - a_{ik} \cdot a_{kj} / a_{kk}$

3. if $a_{nn} = 0$ then

3.1. write "The system does not have a unique solution"

3.2. exit

//Here we determine x_n

4. $a_{n,n+1} \Leftarrow a_{n,n+1} / a_{nn}$

//Here we determine $x_i, n - 1 \geq i \geq 1$

5. for i = n-1, n-2, ..., 1

5.1. S $\Leftarrow$ 0

5.2. for j = i+1, i+2, ..., n

5.2.1. $S \Leftarrow S + a_{ij} \cdot a_{j,n+1}$

5.3. $a_{i,n+1} \Leftarrow (a_{i,n+1} - S) / a_{ii}$

6. write "x_i =", $a_{i,n+1}$, $1 \leq i \leq n$.

//Here we calculate the determinant

7. det = $-1^{(np)}$

8. For i = 1,2, ..., n

8.1. det = det * a_{ii}

9. write "det A=" det

We will solve the following system from exercise 4 of laboratory 2, see [3]

$$\begin{cases} 12x1 + 6x2 + 4x3 + x4 = -22 \\ 24x1 + 10x2 + 4x3 + x4 = -54 \\ -2x1 + x4 = 6 \\ 8x1 + 4x2 + 2x3 + x4 = -16. \end{cases}$$

The extended matrix corresponding to the system is

$$A^{(1)} = (A|b) = \left(\begin{array}{cccc|c} 12 & 6 & 4 & 1 & -22 \\ 24 & 10 & 4 & 1 & -54 \\ -2 & 0 & 0 & 1 & 6 \\ 8 & 4 & 2 & 1 & -16 \end{array}\right).$$

We will solve this example following the steps of the Gauss method with partial pivoting, we will choose the element with the largest absolute value from the first column to be our pivot, in this case is a_{21}=24, thus we will interchange row 1 with row 2 and we obtain

$$A^{(1)} \overline{\overline{L1 \leftrightarrow L2}} \left(\begin{array}{cccc|c} 24 & 10 & 4 & 1 & -54 \\ 12 & 6 & 4 & 1 & -22 \\ -2 & 0 & 0 & 1 & 6 \\ 8 & 4 & 2 & 1 & -16 \end{array}\right).$$

Now we start solving using the "rectangle rule", with $a^{(1)}_{11}$=24 as pivot:

$$a_{22}^{(2)} = \frac{24 \cdot 6 - 12 \cdot 10}{24} = 1,$$
$$a_{23}^{(2)} = \frac{24 \cdot 4 - 12 \cdot 4}{24} = 2,$$
$$a_{24}^{(2)} = \frac{24 \cdot 1 - 12 \cdot 1}{24} = \frac{1}{2},$$
$$a_{25}^{(2)} = \frac{24 \cdot -22 - 12 \cdot -54}{24} = 5,$$
$$a_{32}^{(2)} = \frac{24 \cdot 0 - (-2) \cdot 10}{24} = \frac{5}{6},$$
$$a_{33}^{(2)} = \frac{24 \cdot 0 - (-2) \cdot 4}{24} = \frac{1}{3},$$
$$a_{34}^{(2)} = \frac{24 \cdot 1 - (-2) \cdot 1}{24} = \frac{13}{12},$$
$$a_{35}^{(2)} = \frac{24 \cdot 6 - (-2) \cdot -54}{24} = \frac{3}{2},$$
$$a_{42}^{(2)} = \frac{24 \cdot 4 - 8 \cdot 10}{24} = \frac{2}{3},$$
$$a_{43}^{(2)} = \frac{24 \cdot 2 - 8 \cdot 4}{24} = \frac{2}{3},$$
$$a_{44}^{(2)} = \frac{24 \cdot 1 - 8 \cdot 1}{24} = \frac{2}{3},$$
$$a_{45}^{(2)} = \frac{24 \cdot -16 - 8 \cdot -54}{24} = 2.$$

So we obtain the matrix

$$A^{(2)} = \left(\begin{array}{cccc|c} 24 & 10 & 4 & 1 & -54 \\ 0 & 1 & 2 & 1/2 & 5 \\ 0 & 5/6 & 1/3 & 13/12 & 3/2 \\ 0 & 2/3 & 2/3 & 2/3 & 2 \end{array}\right).$$

Now, we have to choose again from the second column, the element with the largest absolute value as pivot, so in our case this element is a_{22}=1, in this case we don't have to interchange any line so we have the same matrix as before

$$A^{(2)} = \left(\begin{array}{cccc|c} 24 & 10 & 4 & 1 & -54 \\ 0 & 1 & 2 & 1/2 & 5 \\ 0 & 5/6 & 1/3 & 13/12 & 3/2 \\ 0 & 2/3 & 2/3 & 2/3 & 2 \end{array}\right).$$

Now we apply again the rectangular rule with $a^{(2)}_{22}$=1 as pivot:

$$a_{33}^{(3)} = \frac{1 \cdot \frac{1}{3} - \frac{5}{6} \cdot 2}{1} = -\frac{4}{3},$$

$$a_{34}^{(3)} = \frac{1 \cdot \frac{13}{12} - \frac{5}{6} \cdot \frac{1}{2}}{1} = \frac{2}{3},$$

$$a_{35}^{(3)} = \frac{1 \cdot \frac{3}{2} - \frac{5}{6} \cdot 5}{1} = -\frac{16}{6},$$

$$a_{43}^{(3)} = \frac{1 \cdot \frac{2}{3} - \frac{2}{3} \cdot 2}{1} = -\frac{2}{3},$$

$$a_{44}^{(3)} = \frac{1 \cdot \frac{2}{3} - \frac{2}{3} \cdot \frac{1}{2}}{1} = \frac{1}{3},$$

$$a_{45}^{(3)} = \frac{1 \cdot 2 - \frac{2}{3} \cdot 5}{1} = -\frac{4}{3}.$$

So we obtain the matrix

$$A^{(3)} = \left(\begin{array}{cccc|c} 24 & 10 & 4 & 1 & -54 \\ 0 & 1 & 2 & 1/2 & 5 \\ 0 & 0 & -4/3 & 2/3 & -16/6 \\ 0 & 0 & -2/3 & 1/3 & -4/3 \end{array}\right).$$

We repeat the steps from before, again we will choose the element with the largest absolute value as pivot, in our case $a_{33} = -\frac{4}{3}$, so we will not have to interchange any line, in this case the matrix is the same as before

$$A^{(3)} = \left(\begin{array}{cccc|c} 24 & 10 & 4 & 1 & -54 \\ 0 & 1 & 2 & 1/2 & 5 \\ 0 & 0 & -4/3 & 2/3 & -16/6 \\ 0 & 0 & -2/3 & 1/3 & -4/3 \end{array}\right).$$

With $a_{33} = -\frac{4}{3}$ as pivot, we start solving using the "rectangle rule":

$$a_{44}^{(4)} = \frac{-\frac{4}{3} \cdot \frac{1}{3} - (-\frac{2}{3}) \cdot \frac{2}{3}}{-\frac{4}{3}} = 0,$$

$$a_{45}^{(4)} = \frac{-\frac{4}{3} \cdot -\frac{4}{3} - \left(-\frac{2}{3}\right) \cdot -\frac{16}{6}}{-\frac{4}{3}} = 0.$$

We obtain

$$A^{(4)} = \left(\begin{array}{cccc|c} 24 & 10 & 4 & 1 & -54 \\ 0 & 1 & 2 & 1/2 & 5 \\ 0 & 0 & -4/3 & 2/3 & -16/6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

In this case we notice that $a_{nn}=0$, therefore the system does not have a unique solution.
This is the solution done by hand, but if we solve the system with the help of the previous algorithm the matrix will undergo the following changes:

- at step 3 we have $A^{(3)} = \left(\begin{array}{cccc|c} 24 & 10 & 4 & 1 & -54 \\ 0 & 1 & 2 & 1/2 & 5 \\ 0 & 0 & -4/3 & 2/3 & -16/6 \\ 0 & 0 & -2/3 & 1/3 & -4/3 \end{array}\right)$

➢ at step 4 we have $A^{(4)} = \left(\begin{array}{cccc|c} 24 & 10 & 4 & 1 & -54 \\ 0 & 1 & 2 & 1/2 & 5 \\ 0 & 0 & -4/3 & 2/3 & -16/6 \\ 0 & 0 & 0 & -2.38e & 0 \end{array}\right)$

Therefore, based on the previous pseudocode algorithm, the computer will give us that the system has the following unique solution, {x1=-3, x2=1, x3=2, x4=0}, which is false. This happens due to the truncations performed when we solve the system. Indeed, it may happen that a_{nn} would give us 0 when is calculated by hand, but after the calculation performed by the computer to give us something different from 0 like 0.00001. In this case, although normally the system does not have a unique solution, the computer will consider that it has a unique solution and will give us this solution. To avoid this kind of situation we can improve the algorithm, replacing the "if a_{nn}=0" test with "if $|a_{nn}|$ < 0.00001". We make this change in all similar situations and reach the next version of the code.

Improved version of the pseudocode algorithm

// Here we read n, the dimension of matrix A and the augmented matrix (A | b)

1. read n, a_{ij}, $1 \le i \le n, 1 \le j \le n+1$

// We find the element with the largest absolute value to be selected as pivot

2. for k = 1,2,...,n – 1
    - 2.1. piv ⇦ $|a_{kk}|$
    - 2.2. lin ⇦ k
    - 2.3. for i = k + 1, k + 2, ..., n
        - 2.3.1. if piv < $|a_{ik}|$ then
            - 2.3.1.1. piv ⇦ $|a_{ik}|$
            - 2.3.1.2. lin ⇦ i
    - 2.4. if |piv| < 0.00001 then
        - 2.4.1. write “The system does not have unique solution”
        - 2.4.2. exit
    - 2.5. if lin ≠ k then
        - 2.5.1. for j = k, k+1, ..., n+1
            - 2.5.1.1. swap a_{kj} with $a_{lin,j}$
            - 2.5.2. np++

    //Here we apply the formula from Gauss method (the rectangle rule),

    - 2.6 for i = k+1, k+2, ..., n
        - 2.6.1. for j = k+1, k+2, ..., n+1
            - 2.6.1.1. a_{ij} ⇦ $a_{ij} - a_{ik} \cdot a_{kj} / a_{kk}$

3. if $|a_{nn}|$ < 0.00001 then
    - 3.1. write “The system does not have unique solution”
    - 3.2. exit

//Here we determine x_n

4. $a_{n,n+1}$ ⇦ $a_{n,n+1} / a_{nn}$

//Here we determine x_i, n – 1 ≥ i ≥ 1

5. for i = n-1, n-2, ..., 1
    - 5.1. S ⇦ 0
    - 5.2. for j = i+1, i+2, ..., n
        - 5.2.1. S ⇦ S + $a_{ij} \cdot a_{j,n+1}$
    - 5.3. $a_{i,n+1}$ ⇦ $(a_{i,n+1} - S) / a_{ii}$

6. write “x_i =”, $a_{i,n+1}$, $1 \le i \le n$.
//Here we calculate the determinant
7. det = $-1^{(np)}$
8. For i = 1,2, …, n
    8.1. det = det * a_{ii}
9. write “det A=” det

Using this improved algorithm on the previous example we arrive at the correct form of the upper triangular matrix.

Remark 3.1: The situation illustrated above is one of the reasons why in this method we always look for the modulus of the pivot to be as far away as possible from 0. A variant of the method with even better numerical stability is Gauss's method with total pivoting.

Chapter 4

Gauss's method with total pivoting

This variant of Gauss’s method follows the main idea of the other two variants, only that this time, when we choose the pivot, we will look for the element with the largest absolute value, both on the rows and on the columns situated bellow the position of the pivot

$$\left|piv^{(k)}\right| = \max_{k\le i,j\le n}\left|a_{ij}^{(k)}\right|.$$

As opposed to the permutation of lines, permutation of columns will change the order of the unknowns, which fact must be taken into account at the end when we give the solution. That is why it is necessary to note which column changes occur. As a mathematical example, we present the following example, see [3]

$$\begin{cases} x1 + 3x2 - 2x3 - 4x4 = -2 \\ 2x1 + 6x2 - 7x3 - 10x4 = -6 \\ -x1 - x2 + 5x3 + 9x4 = 9 \\ -3x1 - 5x2 + 15x4 = 13. \end{cases}$$

The extended matrix corresponding to the system is

$$A^{(1)} = (A|b) = \left(\begin{array}{cccc|c} 1 & 3 & -2 & -4 & -2 \\ 2 & 6 & -7 & -10 & -6 \\ -1 & -1 & 5 & 9 & 9 \\ -3 & -5 & 0 & 15 & 13 \end{array}\right).$$

Now we are looking for the element that will be chosen as a pivot, thus we are looking for the element with the largest absolute value, $a^{(1)}_{ij}$ where $1\le i, j \le 4$. So the pivot will be a_{44}=15, in this case we will interchange row 1 with row 4, and we can observe that this change has no impact on the solution of the system, we also interchange column 1 with column 4, but this will mean that we also interchange x1 with x4, so we will keep that in mind at the end. We obtain the following matrix

$$A^{(1)}\overline{\overline{L1 \leftrightarrow L4}}\left(\begin{array}{cccc|c} -3 & -5 & 0 & 15 & 13 \\ 2 & 6 & -7 & -10 & -6 \\ -1 & -1 & 5 & 9 & 9 \\ 1 & 3 & -2 & -4 & -2 \end{array}\right)\overline{\overline{C1 \leftrightarrow C4}}\left(\begin{array}{cccc|c} 15 & -5 & 0 & -3 & 13 \\ -10 & 6 & -7 & 2 & -6 \\ 9 & -1 & 5 & -1 & 9 \\ -4 & 3 & -2 & 1 & -2 \end{array}\right).$$

Now we start solving using the “rectangle rule”, with $a^{(1)}_{11}$=15 as pivot:

$$a_{22}^{(2)} = \frac{15 \cdot 6 - (-10) \cdot -5}{15} = \frac{8}{3},$$

$$a_{23}^{(2)} = \frac{15 \cdot -7 - (-10) \cdot 0}{15} = -7,$$

$$a_{24}^{(2)} = \frac{15 \cdot 2 - (-10) \cdot -3}{15} = 0,$$

$$a_{25}^{(2)} = \frac{15 \cdot -6 - (-10) \cdot 13}{15} = \frac{8}{3},$$

$$a_{32}^{(2)} = \frac{15 \cdot -1 - 9 \cdot -5}{15} = 2,$$

$$a_{33}^{(2)} = \frac{15 \cdot 5 - 9 \cdot 0}{15} = 5,$$

$$a_{34}^{(2)} = \frac{15 \cdot -1 - 9 \cdot -3}{15} = \frac{4}{5},$$

$$a_{35}^{(2)} = \frac{15 \cdot 9 - 9 \cdot 13}{15} = \frac{6}{5},$$

$$a_{42}^{(2)} = \frac{15 \cdot 3 - (-4) \cdot -5}{15} = \frac{5}{3},$$

$$a_{43}^{(2)} = \frac{15 \cdot -2 - (-4) \cdot 0}{15} = -2,$$

$$a_{44}^{(2)} = \frac{15 \cdot 1 - (-4) \cdot -3}{15} = \frac{1}{5},$$

$$a_{45}^{(2)} = \frac{15 \cdot -2 - (-4) \cdot 13}{15} = \frac{22}{15}.$$

So we obtain the matrix

$$A^{(2)} = \left(\begin{array}{cccc|c} 15 & -5 & 0 & -3 & 13 \\ 0 & 8/3 & -7 & 0 & 8/3 \\ 0 & 2 & 5 & 4/5 & 6/5 \\ 0 & 5/3 & -2 & 1/5 & 22/15 \end{array}\right).$$

Now we search the element with the largest absolute value, $a^{(2)}_{ij}$ where $2 \leq i, j \leq 4$. In this case the pivot will be a_{23}=-7, so the row will remain 2, and we interchange column 2 with column 3, keeping in mind that we also interchange x2 with x3. We obtain

$$A^{(2)} \xrightarrow{C2 \leftrightarrow C3} \left(\begin{array}{cccc|c} 15 & 0 & -5 & -3 & 13 \\ 0 & -7 & 8/3 & 0 & 8/3 \\ 0 & 5 & 2 & 4/5 & 6/5 \\ 0 & -2 & 5/3 & 1/5 & 22/15 \end{array}\right).$$

Now we start solving using the “rectangle rule”, with $a^{(2)}_{22}$=-7 as pivot:

$$a_{33}^{(3)} = \frac{-7 \cdot 2 - (-5) \cdot \frac{8}{3}}{-7} = \frac{574}{3},$$

$$a_{34}^{(3)} = \frac{-7 \cdot \frac{4}{5} - (-5) \cdot 0}{-7} = \frac{196}{5},$$

$$a_{35}^{(3)} = \frac{-7 \cdot \frac{6}{5} - (-5) \cdot \frac{8}{3}}{-7} = \frac{2282}{15},$$

$$a_{43}^{(3)} = \frac{-7 \cdot \frac{5}{3} - (-2) \cdot \frac{8}{3}}{-7} = \frac{133}{3},$$

$$a_{44}^{(3)} = \frac{-7 \cdot \frac{1}{5} - (-2) \cdot 0}{-7} = \frac{49}{5},$$

$$a_{45}^{(3)} = \frac{-7 \cdot \frac{22}{15} - (-2) \cdot \frac{8}{3}}{-7} = \frac{518}{15}.$$

So we obtain the matrix

$$A^{(3)} = \left(\begin{array}{cccc|c} 15 & 0 & -5 & -3 & 13 \\ 0 & -7 & 8/3 & 0 & 8/3 \\ 0 & 0 & 574/3 & 196/5 & 2282/15 \\ 0 & 0 & 133/3 & 49/5 & 518/15 \end{array}\right).$$

We search the element with the largest absolute value, $a^{(3)}_{ij}$ where $3 \leq i, j \leq 4$. In this case the pivot will be $a_{33}=\frac{574}{3}$, so we do not need to apply any changes to the matrix, then we start solving using the "rectangle rule", with $a^{(3)}_{33}=\frac{574}{3}$ as pivot:

$$a_{44}^{(4)} = \frac{\frac{574}{3} \cdot \frac{49}{5} - \frac{133}{3} \cdot \frac{196}{5}}{\frac{574}{3}} = \frac{1029}{1435},$$

$$a_{45}^{(4)} = \frac{\frac{574}{3} \cdot \frac{518}{15} - \frac{133}{3} \cdot \frac{2282}{15}}{\frac{574}{3}} = -\frac{3087}{4305}.$$

We obtain

$$A^{(4)} = \left(\begin{array}{cccc|c} 15 & 0 & -5 & -3 & 13 \\ 0 & -7 & 8/3 & 0 & 8/3 \\ 0 & 0 & 574/3 & 196/5 & 2282/15 \\ 0 & 0 & 0 & 1029/1435 & -3087/4305 \end{array}\right).$$

The corresponding system is

$$\begin{cases} 15x1 - 5x3 - 3x4 = 13 \\ -7x2 + \frac{8}{3}x3 = \frac{8}{3} \\ \frac{574}{3}x3 + \frac{196}{5}x4 = \frac{2282}{15} \\ \frac{1029}{1435}x4 = -\frac{3087}{4305}. \end{cases}$$

So by back substitution method, the intermediate solution is

$$\begin{cases} x4 = -1 \\ x3 = 1 \\ x2 = 0 \\ x1 = 1 \end{cases} => \begin{cases} x1 = 1 \\ x2 = 0 \\ x3 = 1 \\ x4 = -1, \end{cases}$$

and we will interchange in the following order keeping in mind the previous permutations

$$\begin{cases} component\ 2 \ \leftrightarrow component\ 3\ (since\ C2 \leftrightarrow C3) \\ component\ 1 \ \leftrightarrow component\ 4\ (since\ C1 \leftrightarrow C4). \end{cases}$$

so we have

$$\begin{cases} x1 = 1 \\ x2 = 0 \\ x3 = 1 \\ x4 = -1 \end{cases} \xrightarrow{x2 \leftrightarrow x3} \begin{cases} x1 = 1 \\ x2 = 1 \\ x3 = 0 \\ x4 = -1 \end{cases} \xrightarrow{x1 \leftrightarrow x4} \begin{cases} x1 = -1 \\ x2 = 1 \\ x3 = 0 \\ x4 = 1. \end{cases}$$

So the solution of the system is

$$\begin{cases} x1 = -1 \\ x2 = 1 \\ x3 = 0 \\ x4 = 1. \end{cases}$$

Pseudocode Algorithm for the Gauss method with total pivoting

// Here we read n, the dimension of matrix A and the augmented matrix (A | b)

1. read n, a_{ij}, $1 \le i \le n, 1 \le j \le n+1$,
2. npc ⇦ 0

// We find the element with the largest absolute value to be selected as pivot

3. for k = 1, 2, ..., n – 1
   - 3.1. piv ⇦ $|a_{kk}|$
   - 3.2. lin ⇦ k
   - 3.3. col ⇦ k
   - 3.4. for j = k, k+1, ..., n
     - 3.4.1. for i = k, k+1, ..., n
       - 3.4.1.1. if piv < $|a_{ij}|$ then
         - 3.4.1.1.1. piv ⇦ $|a_{ij}|$
         - 3.4.1.1.2. lin ⇦ i
         - 3.4.1.1.3. col ⇦ j
   - 3.5. piv ≤ 0.00001 then
     - 3.5.1. write "The system does not have a unique solution"
     - 3.5.2. exit
   - 3.6. if lin ≠ k then
     - 3.6.1 for j=k, k+1, ..., n+1
       - 3.6.1.1. swap a_{kj} with $a_{lin,j}$
   - 3.7. if col ≠ k then
     - 3.7.1. npc ⇦ npc + 1
     - 3.7.2. $c_{npc,1}$ ⇦ k
     - 3.7.3. $c_{npc,2}$ ⇦ col
     - 3.7.4. for i = 1, 2, ...,n
       - 3.7.4.1. swap a_{ik} with $a_{i,col}$
     - 3.7.5. np++

   //Here we apply the formula from Gauss method (the rectangle rule),

   - 3.8. for i=k+1, k+2, ..., n
     - 3.8.1. if $a_{ik} \neq 0$

3.8.1.2. for j=k+1, k+2, ..., n+1
3.8.1.2.1. if $a_{kj} \neq 0$
3.8.1.2.1.1. $a_{ij} \Leftarrow a_{ij} - a_{ik} \cdot a_{kj} / a_{kk}$

4. if $|a_{nn}| \leq 0.00001$ then
4.1. write "The system does not have a unique solution"
4.2. exit

//Here we determine x_n

5. $a_{n,n+1} \Leftarrow a_{n,n+1} / a_{nn}$

//Here we determine $x_i, n - 1 \geq i \geq 1$

6. for i=n-1, n-2, ..., 1
6.1. $S \Leftarrow 0$
6.2. for j=i+1, i+2, ..., n
6.2.1. $S \Leftarrow S + a_{ij} \cdot a_{j,n+1}$
6.3. $a_{i,n+1} \Leftarrow (a_{i,n+1} - S) / a_{ii}$

7. if npc $\neq 0$ then
7.1. for i=npc, npc-1, ..., 1
7.1.1. swap $a_{c[i,1],n+1}$ with $a_{c[i,2],n+1}$

8. write "$x_i =$", $a_{i,n+1}$, $1 \leq i \leq n$.

//Here we calculate the determinant

9. det = $-1^{(np)}$
10. For i = 1,2, ..., n
10.1. det = det * a_{ii}
11. write "det A=" det

As we have seen so far, Gauss's method also helps us to calculate the determinant of the matrix, but this calculation is done in $O(n^3)$ steps, where n is the dimension of the matrix A. A more suitable method for calculating the determinant of a matrix is Chio's method, which has a lower algorithm cost.

Chapter 5

Chio pivotal condensation method

Chio's method is used to find the value of a determinant of the n x n type, which can also be called a determinant of order n. The general idea is to transform the determinant into $(n-1)^2$ minors of order 2. Then we will repeat this procedure until we reach a determinant of order 2.

Based on what was said, we will apply the following formula

$$\det(A) = \frac{1}{a_{11}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \cdots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix},$$

where $a_{11} \neq 0$, and we will apply this formula repetitively until we obtain a determinant of order 2.

Remark 5.1: If we have $a_{11} = 0$ and we can find an $a_{i1} \neq 0$, where $2 \leq i \leq n$, then we will change row 1 with i, and we remember to change the sign of the determinant.

Remark 5.2: If we have $a_{i1} = 0$ for all $1 \leq i \leq n$, then the determinant will be 0.

As a mathematical example, we present the following example:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 2 & 1 \\ 6 & 3 & 2 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 6 \\ 1 & 1 & -2 & 3 & 0 & 10 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 2 & 1 & -2 & -1 & 1 & 10 \end{pmatrix},$$

we will apply the formula presented previously

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 & 1 & 2 & 1 \\ 6 & 3 & 2 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 6 \\ 1 & 1 & -2 & 3 & 0 & 10 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 2 & 1 & -2 & -1 & 1 & 10 \end{vmatrix}$$

$$= \frac{1}{2^{6-2}} \begin{vmatrix} \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 6 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 6 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 6 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 6 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 10 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 10 \end{vmatrix} \end{vmatrix}$$

$$= \frac{1}{16} \begin{vmatrix} 0 & 4 & -8 & -12 & -4 \\ 3 & 2 & -1 & -2 & 11 \\ 1 & -4 & 5 & -2 & 19 \\ -1 & 0 & -3 & -2 & 1 \\ 0 & -4 & -4 & -2 & 18 \end{vmatrix}.$$

By performing the calculus of several determinants of order 2, we have arrived at calculus of a determinant of order 5 instead of the initial determinant of order 6. At this point we observe that the first element a_{11} is zero, so we are in the case of remark 5.1. In this case we will interchange the first two rows of the determinant. So we have,

$$\det(A) = \frac{-1}{16} \begin{vmatrix} 3 & 2 & -1 & -2 & 11 \\ 0 & 4 & -8 & -12 & -4 \\ 1 & -4 & 5 & -2 & 19 \\ -1 & 0 & -3 & -2 & 1 \\ 0 & -4 & -4 & -2 & 18 \end{vmatrix}.$$

We apply again the formula this time for a determinant of order 5 and we obtain:

$$\det(A) = \frac{-1}{16 \cdot 3^{5-2}} \begin{vmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 0 & -8 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 0 & -12 \end{vmatrix} & \begin{vmatrix} 3 & 11 \\ 0 & -4 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 11 \\ 1 & 19 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -1 & -3 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 11 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 11 \\ 0 & 18 \end{vmatrix} \end{vmatrix}$$

$$= \frac{-1}{432}\begin{vmatrix} 12 & -24 & -36 & -12 \\ -14 & 16 & -4 & 46 \\ 2 & -10 & -8 & 14 \\ -12 & -12 & -6 & 54 \end{vmatrix}.$$

We continue now with a determinant of order 4:

$$\det(A) = \frac{-1}{432 \cdot 12^{4-2}} \begin{vmatrix} \begin{vmatrix} 12 & -24 \\ -14 & 16 \end{vmatrix} & \begin{vmatrix} 12 & -36 \\ -14 & -4 \end{vmatrix} & \begin{vmatrix} 12 & -12 \\ -14 & 46 \end{vmatrix} \\ \begin{vmatrix} 12 & -24 \\ 2 & -10 \end{vmatrix} & \begin{vmatrix} 12 & -36 \\ 2 & -8 \end{vmatrix} & \begin{vmatrix} 12 & -12 \\ 2 & 14 \end{vmatrix} \\ \begin{vmatrix} 12 & -24 \\ -12 & -12 \end{vmatrix} & \begin{vmatrix} 12 & -36 \\ -12 & -6 \end{vmatrix} & \begin{vmatrix} 12 & -12 \\ -12 & 54 \end{vmatrix} \end{vmatrix}$$

$$= \frac{-1}{62208}\begin{vmatrix} -144 & -552 & 384 \\ -72 & -24 & 192 \\ -432 & -504 & 504 \end{vmatrix}.$$

Now we have a determinant of order 3:

$$\det(A) = \frac{-1}{62208 \cdot (-144)^{3-2}} \begin{vmatrix} \begin{vmatrix} -144 & -552 \\ -72 & -24 \end{vmatrix} & \begin{vmatrix} -144 & 384 \\ -72 & 192 \end{vmatrix} \\ \begin{vmatrix} -144 & -552 \\ -432 & -504 \end{vmatrix} & \begin{vmatrix} -144 & 384 \\ -432 & 504 \end{vmatrix} \end{vmatrix}$$

$$= \frac{1}{8957952}\begin{vmatrix} -36288 & 0 \\ -165888 & 93312 \end{vmatrix}$$

$$= \frac{-3386105856}{8957952}$$

$$= -378.$$

Pseudocode Algorithm for Chio pivotal condensation method

// Here we read n, the dimension of matrix A and the matrix A

1. read n, a_{ij}, $1 \le i, j \le n$
2. det ⇦ 1
3. repeat
    - 3.1. if a_{11}=0 then
        - 3.1.1. i ⇦ 2
        - 3.1.2. while (i≤ n) and (a_{i1}=0)
            - 3.1.2.1. i ⇦ i + 1
        - 3.1.3. if i > n then
            - 3.1.3.1. write "det(A)=0"
            - 3.1.3.2. exit
        - 3.1.4. for j = 1,2,...,n
            - 3.1.4.1. swap a_{1j} with a_{ij}
        - 3.1.5. det ⇦ −det
    - 3.2. for i = 1,2,...,n-2
        - 3.2.1. det ⇦ det · a_{11}
    - 3.3. for i = 2,3,...,n
        - 3.3.1. for j = 2,3,...,n
            - 3.3.1.1. a_{ij} ⇦ $a_{ij} \cdot a_{11} - a_{i1} \cdot a_{1j}$
    - 3.4. n ⇦ n-1

3.5. for i = 1,2,…,n

3.5.1. for j = 1,2,…,n

3.5.1.1. $a_{ij} \Leftarrow a_{i+1,j+1}$

until (n=1)

4. det ⇦ a_{11}/det

5. Write “det(A)=”, det.

Remark 5.3: Among other things, by calculating the determinant of a matrix, we determine if the matrix is invertible, which happens when the determinant is different from 0. In addition, when the determinant is nonzero, if we talk about solving linear systems, we deduce that the system has a unique solution.

Chapter 6

Jacobi’s method for solving linear systems

This is the first iterative method for solving linear systems that we present in this thesis. As their name suggests, the iterative approaches are predicated on iterations. When we talk about an iteration, we mean the repeating of a specific calculus technique since the outcome from one step is applied to the subsequent step, and so forth. Iterative methods are generally used for very large systems that have over 100 equations and unknowns, or even over 200 equations and unknowns. The mathematical principle on which it is based the method presented here is the Contraction Principle also known as Banach Theorem of fixed point. For more details concerning the definition of a contraction and for the statement of this principle we refer for instance to [2] and [4]. Starting from this we can construct a sequence (Sn)n that converges to the solution of the system. When we consider that an element Sn of the sequence approximates the solution with sufficient accuracy we will stop. When the sequence (Sn)n constructed by Jacobi's method converges, we say that Jacobi's method converges.

We consider the linear system A · x = b, where A ∈ M_n (**R**) is the matrix associated to the linear system and b ∈ $\mathbf{R}^n$ is the vector containing the free terms of the system. Our objective is to find, if possible, x ∈ $\mathbf{R}^n$, where x represents the unique solution of the system.

We take $x^{(0)}$ ∈ $\mathbf{R}^n$ to be the initial approximation of the solution of the system, for example a good choice is the null vector. Based on $x^{(0)}$ we obtain $x^{(1)}$, based on $x^{(1)}$ we obtain $x^{(2)}$ and so on. We obtain that by applying the formula:

$$x_i^{(k+1)} = \left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(k)} \right) where\ a_{ii}, 1 \leq i \leq n, k \geq 0,$$

until

$$dist\left(x^{(k+1)} - x^{(k)}\right) = \max_{1\leq i\leq n}\left|x_i^{(k+1)} - x_i^{(k)}\right| \leq \varepsilon, (6.1)$$

where ε is the error that we consider acceptable when we try to approximate the exact solution of the system. Then $x \approx x^{(k+1)}$.

Remark 6.1: Relation 6.1 represents a stopping condition that arises from the inequality

$$dist\left(x^{(k+1)} - x^{(k)}\right) \leq \varepsilon,$$

where

$$dist\left(x^{(k+1)} - x^{(k)}\right) = \max_{1 \le i \le n} \left| x_i^{(k+1)} - x_i^{(k)} \right|$$

represents a metric in space $\mathbf{R}^n$. Therefore, we can have other stopping criteria if we use other metrics, for example the Euclidean metric, where we have dist : $\mathbf{R}^n \times \mathbf{R}^n \Longrightarrow \mathbf{R}$,

$$dist(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2},$$

where $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$.
See again [2].

A criterion which ensures the convergence of Jacobi Method is the following.

Criterion 1: Les us consider $A \in M(R)$ the matrix associated with the linear system. Jacobi's method converges if A is strictly diagonal dominant on rows or columns. We say that A is strictly diagonal dominant on rows if and only if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|, i = \overline{1, n} \quad (6.2)$$

We say that A is strictly diagonal dominant on columns if and only if

$$|a_{jj}| > \sum_{\substack{i=1 \\ i \ne j}}^{n} |a_{ij}|, j = \overline{1, n} \quad (6.3)$$

Notice that only one of the above conditions needs to be met ((6.2) or (6.3)).

In what follows we are going to provide three examples of systems to illustrate various situations that we may encounter.

Example 1. Solve the following system within the error 10^{-2}:

$$\begin{cases} 10x1 - 2x2 + 2x3 = -28 \\ x1 + 5x2 - 2x3 + x4 = -4 \\ x1 + x2 - 10x3 + 2x4 = -26 \\ x1 - x2 - 20x4 = 36. \end{cases}$$

We have

$$A = \begin{pmatrix} 10 & -2 & 2 & 0 \\ 1 & 5 & -2 & 1 \\ 1 & 1 & -10 & 2 \\ 1 & -1 & 0 & -20 \end{pmatrix} \; and \; b = \begin{pmatrix} -28 \\ -4 \\ -26 \\ 36 \end{pmatrix}.$$

Let us check if the matrix A is strictly diagonal dominant on rows

$$\left.\begin{matrix} |a_{11}| = 10 \\ |a_{12}| + |a_{13}| + |a_{14}| = |-2| + |2| + |0| = 4 \end{matrix}\right\} \Longrightarrow |a_{11}| > |a_{12}| + |a_{13}| + |a_{14}|$$

$$\left.\begin{matrix} |a_{22}| = 5 \\ |a_{21}| + |a_{23}| + |a_{24}| = |1| + |-2| + |1| = 4 \end{matrix}\right\} \Longrightarrow |a_{22}| > |a_{21}| + |a_{23}| + |a_{24}|$$

$$\left.\begin{matrix} |a_{33}| = 10 \\ |a_{31}| + |a_{32}| + |a_{34}| = |1| + |1| + |2| = 4 \end{matrix}\right\} \Longrightarrow |a_{33}| > |a_{31}| + |a_{32}| + |a_{34}|$$

$$\left.\begin{matrix} |a_{44}| = 20 \\ |a_{41}| + |a_{42}| + |a_{43}| = |1| + |-1| + |0| = 2 \end{matrix}\right\} \Longrightarrow |a_{44}| > |a_{41}| + |a_{42}| + |a_{43}|.$$

So, the matrix A is strictly diagonal dominant on rows.
And because of that, we can be sure that the Jacobi's method enables us to obtain an approximation of the solutions with the desired precision.
Remark 6.2: If it happens that a matrix A is neither diagonally dominant on the lines nor strictly diagonally dominant on the columns, it does not mean that the Jacobi method does not converge. As we said, Criterion 1 represents a sufficient condition, it is not a necessary condition.

We write the initial system in an equivalent form

$$\begin{cases} x1 = (-28 + 2x2 - 2x3)/10 \\ x2 = (-4 - x1 + 2x3 - x4)/5 \\ x3 = (-26 - x1 - x2 - 2x4)/(-10) \\ x4 = (36 - x1 + x2)/(-20). \end{cases}$$

We choose as initial approximation $x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, and we consider the recurrence

$$\begin{cases} x_1^{(k+1)} = (-28 + 2x_2^{(k)} - 2x_3^{(k)})/10 \\ x_2^{(k+1)} = (-4 - x_1^{(k)} + 2x_3^{(k)} - x_4^{(k)})/5 \\ x_3^{(k+1)} = (-26 - x_1^{(k)} - x_2^{(k)} - 2x_4^{(k)})/(-10) \\ x_4^{(k+1)} = (36 - x_1^{(k)} + x_2^{(k)})/(-20). \end{cases}$$

For k = 0 we obtain

$$\begin{cases} x_1^{(1)} = (-28 + 2x_2^{(0)} - 2x_3^{(0)})/10 = \ -28/10 \\ x_2^{(1)} = -4 - x_1^{(0)} + 2x_3^{(0)} - x_4^{(0)}/5 = -8/10 \\ x_3^{(1)} = (-26 - x_1^{(0)} - x_2^{(0)} - 2x_4^{(0)})/(-10) = 26/10 \\ x_4^{(1)} = (36 - x_1^{(0)} + x_2^{(0)})/(-20) = 18/10. \end{cases}$$

We check the stop condition

$$d\left(x^{(1)} - x^{(0)}\right) = \max_{1\le i\le 4}\left|x_i^{(1)} - x_i^{(0)}\right|$$

$$= max\left\{\left|x_1^{(1)} - x_1^{(0)}\right|, \left|x_2^{(1)} - x_2^{(0)}\right|, \left|x_3^{(1)} - x_3^{(0)}\right|, \left|x_4^{(1)} - x_4^{(0)}\right|\right\}$$

$$= max\left\{\frac{28}{10}, \frac{8}{10}, \frac{26}{10}, \frac{18}{10}\right\} = 2.8 > \varepsilon = 0.01.$$

For k = 1 we obtain

$$\begin{cases} x_1^{(2)} = (-28 + 2x_2^{(1)} - 2x_3^{(1)})/10 = \ -348/100 \\ x_2^{(2)} = -4 - x_1^{(1)} + 2x_3^{(1)} - x_4^{(1)}/5 = -92/100 \\ x_3^{(2)} = (-26 - x_1^{(1)} - x_2^{(1)} - 2x_4^{(1)})/(-10) = 260/100 \\ x_4^{(2)} = (36 - x_1^{(1)} + x_2^{(1)})/(-20) = -190/100. \end{cases}$$

We check the stop condition

$$d\left(x^{(2)}-x^{(1)}\right)=\max_{1\le i\le 4}\left|x_i^{(2)}-x_i^{(1)}\right|$$

$$=max\left\{\left|x_1^{(2)}-x_1^{(1)}\right|,\left|x_2^{(2)}-x_2^{(1)}\right|,\left|x_3^{(2)}-x_3^{(1)}\right|,\left|x_4^{(2)}-x_4^{(1)}\right|\right\}$$

$$=max\left\{\frac{68}{100},\frac{12}{100},0,\frac{370}{100}\right\}=3.7>\varepsilon=0.01.$$

For k = 2 we obtain

$$\begin{cases} x_1^{(3)}=(-28+2x_2^{(2)}-2x_3^{(2)})/10=\ -3504/1000\\ x_2^{(3)}=-4-x_1^{(2)}+2x_3^{(2)}-x_4^{(2)}/5=1316/1000\\ x_3^{(3)}=(-26-x_1^{(2)}-x_2^{(2)}-2x_4^{(2)})/(-10)=1780/1000\\ x_4^{(3)}=(36-x_1^{(2)}+x_2^{(2)})/(-20)=-1928/1000.\end{cases}$$

We check the stop condition

$$d\left(x^{(3)}-x^{(2)}\right)=\max_{1\le i\le 4}\left|x_i^{(3)}-x_i^{(2)}\right|$$

$$=max\left\{\left|x_1^{(3)}-x_1^{(2)}\right|,\left|x_2^{(3)}-x_2^{(2)}\right|,\left|x_3^{(3)}-x_3^{(2)}\right|,\left|x_4^{(3)}-x_4^{(2)}\right|\right\}$$

$$=max\left\{\frac{24}{1000},\frac{2236}{1000},\frac{720}{1000},\frac{28}{1000}\right\}=2.236>\varepsilon=0.01.$$

For k = 3 we obtain

$$\begin{cases} x_1^{(4)}=(-28+2x_2^{(3)}-2x_3^{(3)})/10=\ -28928/10000\\ x_2^{(4)}=-4-x_1^{(3)}+2x_3^{(3)}-x_4^{(3)}/5=9984/10000\\ x_3^{(4)}=(-26-x_1^{(3)}-x_2^{(3)}-2x_4^{(3)})/(-10)=19956/10000\\ x_4^{(4)}=(36-x_1^{(3)}+x_2^{(3)})/(-20)=-20410/10000.\end{cases}$$

We check the stop condition

$$d\left(x^{(4)}-x^{(3)}\right)=\max_{1\le i\le 4}\left|x_i^{(4)}-x_i^{(3)}\right|$$

$$=max\left\{\left|x_1^{(4)}-x_1^{(3)}\right|,\left|x_2^{(4)}-x_2^{(3)}\right|,\left|x_3^{(4)}-x_3^{(3)}\right|,\left|x_4^{(4)}-x_4^{(3)}\right|\right\}$$

$$=max\left\{\frac{6112}{10000},\frac{3176}{10000},\frac{2156}{10000},\frac{10130}{10000}\right\}=1.013>\varepsilon=0.01.$$

For k = 4 we obtain

$$\begin{cases} x_1^{(5)}=(-28+2x_2^{(4)}-2x_3^{(4)})/10=\ -299944/100000\\ x_2^{(5)}=-4-x_1^{(4)}+2x_3^{(4)}-x_4^{(4)}/5=98500/100000\\ x_3^{(5)}=(-26-x_1^{(4)}-x_2^{(4)}-2x_4^{(4)})/(-10)=200236/100000\\ x_4^{(5)}=(36-x_1^{(4)}+x_2^{(4)})/(-20)=-199456/100000.\end{cases}$$

We check the stop condition

$$d\left(x^{(5)}-x^{(4)}\right)=\max_{1\le i\le 4}\left|x_i^{(5)}-x_i^{(4)}\right|$$

$$=max\left\{\left|x_1^{(5)}-x_1^{(4)}\right|,\left|x_2^{(5)}-x_2^{(4)}\right|,\left|x_3^{(5)}-x_3^{(4)}\right|,\left|x_4^{(5)}-x_4^{(4)}\right|\right\}$$

$$= max\left\{\frac{10664}{100000}, \frac{1350}{100000}, \frac{686}{100000}, \frac{4644}{100000}\right\} = 0.10664 > \varepsilon = 0.01.$$

For k = 5 we obtain

$$\begin{cases} x_1^{(6)} = (-28 + 2x_2^{(5)} - 2x_3^{(5)})/10 = -3003472/1000000 \\ x_2^{(6)} = -4 - x_1^{(5)} + 2x_3^{(5)} - x_4^{(5)}/5 = 999744/1000000 \\ x_3^{(6)} = (-26 - x_1^{(5)} - x_2^{(5)} - 2x_4^{(5)})/(-10) = 1999644/1000000 \\ x_4^{(6)} = (36 - x_1^{(5)} + x_2^{(5)})/(-20) = -1999222/1000000. \end{cases}$$

We check the stop condition

$$d\left(x^{(6)} - x^{(5)}\right) = \max_{1 \le i \le 4}\left|x_i^{(6)} - x_i^{(5)}\right|$$

$$= max\left\{\left|x_1^{(6)} - x_1^{(5)}\right|, \left|x_2^{(6)} - x_2^{(5)}\right|, \left|x_3^{(6)} - x_3^{(5)}\right|, \left|x_4^{(6)} - x_4^{(5)}\right|\right\}$$

$$= max\left\{\frac{4032}{1000000}, \frac{14744}{1000000}, \frac{2760}{1000000}, \frac{4562}{1000000}\right\} = 0.014744 > \varepsilon = 0.01.$$

For k = 6 we obtain

$$\begin{cases} x_1^{(7)} = (-28 + 2x_2^{(6)} - 2x_3^{(6)})/10 = -29999800/10000000 \\ x_2^{(7)} = -4 - x_1^{(6)} + 2x_3^{(6)} - x_4^{(6)}/5 = 10003964/10000000 \\ x_3^{(7)} = (-26 - x_1^{(6)} - x_2^{(6)} - 2x_4^{(6)})/(-10) = 19997828/10000000 \\ x_4^{(7)} = (36 - x_1^{(6)} + x_2^{(6)})/(-20) = -20001608/10000000. \end{cases}$$

We check the stop condition

$$d\left(x^{(7)} - x^{(6)}\right) = \max_{1 \le i \le 4}\left|x_i^{(7)} - x_i^{(6)}\right|$$

$$= max\left\{\left|x_1^{(7)} - x_1^{(6)}\right|, \left|x_2^{(7)} - x_2^{(6)}\right|, \left|x_3^{(7)} - x_3^{(6)}\right|, \left|x_4^{(7)} - x_4^{(6)}\right|\right\}$$

$$= max\left\{\frac{34920}{10000000}, \frac{6524}{10000000}, \frac{1388}{10000000}, \frac{9388}{10000000}\right\} = 0.003492 < \varepsilon = 0.01.$$

We obtain the solution with precision given by $\varepsilon = 0.01$ at the step 7

$$\begin{cases} x_1^{(7)} = -2.99998 \\ x_2^{(7)} = 1.0003964 \\ x_3^{(7)} = 1.9997828 \\ x_4^{(7)} = -2.0001608. \end{cases}$$

Notice that the exact solution of the system is $\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \\ -2 \end{pmatrix}$.

The above example will be solved in the next chapter by the Seidel-Gauss method to show the fact that when both methods converge, the Seidel-Gauss method converges faster. Thus we ask ourselves what is the interest in solving by the Jacobi method. The answer to this question is given by the fact that there are situations in which the Jacobi method converges and the Seidel-Gauss method does not. One such example is the following system that was solved in Chapter 3 of [2], see example 11 which we discuss below.

Example 2. Solve the following system within the error 10^{-4}

$$\begin{cases} x1 - x2 + x3 = 0 \\ -2x1 + x2 + x3 = -4 \\ -4x1 + 2x2 + x3 = -7. \end{cases}$$

We will try to solve the above system using Jacobi's method.

We have

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & 1 \\ -4 & 2 & 1 \end{pmatrix} \ and\ b = \begin{pmatrix} 0 \\ -4 \\ -7 \end{pmatrix}.$$

We check if the matrix A is diagonal dominant on rows or columns. Then we have

$$\left.\begin{matrix} |a_{11}| = 1 \\ |a_{12}| + |a_{13}| = |-1| + |21| = 4 \end{matrix}\right\} \Rightarrow |a_{11}| < |a_{12}| + |a_{13}|$$

$$\left.\begin{matrix} |a_{11}| = 1 \\ |a_{21}| + |a_{31}| = |-2| + |-4| = 6 \end{matrix}\right\} \Rightarrow |a_{11}| < |a_{21}| + |a_{31}|.$$

Therefore, the matrix A is not diagonal dominant on rows or columns, but it is still possible for the two iterative methods to converge even in this situation. That being said we proceed with the methods and we see what happens.

We write the initial system in an equivalent form

$$\begin{cases} x1 = x2 - x3 \\ x2 = -4 + 2x1 - x3 \\ x3 = -7 + 4x1 - 2x2. \end{cases}$$

We choose as initial approximation $x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and we consider the recurrence

$$\begin{cases} x_1^{(k+1)} = x_2^{(k)} - x_3^{(k)} \\ x_2^{(k+1)} = -4 + 2x_1^{(k)} - x_3^{(k)} \\ x_3^{(k+1)} = -7 + 4x_1^{(k)} - 2x_2^{(k)}. \end{cases}$$

For k = 0 we obtain

$$\begin{cases} x_1^{(1)} = x_2^{(0)} - x_3^{(0)} = 0 \\ x_2^{(1)} = -4 + 2x_1^{(0)} - x_3^{(0)} = -4 \\ x_3^{(1)} = -7 + 4x_1^{(0)} - 2x_2^{(0)} = -7. \end{cases}$$

We check the stop condition

$$d\left(x^{(1)} - x^{(0)}\right) = \max_{1 \le i \le 3} \left|x_i^{(1)} - x_i^{(0)}\right|$$

$$= max\left\{\left|x_1^{(1)} - x_1^{(0)}\right|, \left|x_2^{(1)} - x_2^{(0)}\right|, \left|x_3^{(1)} - x_3^{(0)}\right|\right\}$$

$$= max\{0, 4, 7\} = 7 > \varepsilon = 0.0001.$$

For k = 1 we obtain

$$\begin{cases} x_1^{(2)} = x_2^{(1)} - x_3^{(1)} = -4 + 7 = 3 \\ x_2^{(2)} = -4 + 2x_1^{(1)} - x_3^{(1)} = -4 + 0 + 7 = 3 \\ x_3^{(2)} = -7 + 4x_1^{(1)} - 2x_2^{(1)} = -7 + 0 + 8 = 1. \end{cases}$$

We check the stop condition

$$d\left(x^{(2)} - x^{(1)}\right) = \max_{1 \le i \le 3} \left|x_i^{(2)} - x_i^{(1)}\right|$$

$$= max\left\{\left|x_1^{(2)} - x_1^{(1)}\right|, \left|x_2^{(2)} - x_2^{(1)}\right|, \left|x_3^{(2)} - x_3^{(1)}\right|\right\}$$

$$= max\{3, 7, 8\} = 8 > \varepsilon = 0.0001.$$

For k = 2 we obtain

$$\begin{cases} x_1^{(3)} = x_2^{(2)} - x_3^{(2)} = 3 - 1 = 2 \\ x_2^{(3)} = -4 + 2x_1^{(2)} - x_3^{(2)} = -4 + 6 - 1 = 1 \\ x_3^{(3)} = -7 + 4x_1^{(2)} - 2x_2^{(2)} = -7 + 12 - 6 = -1. \end{cases}$$

We check the stop condition

$$d\left(x^{(3)} - x^{(2)}\right) = \max_{1 \le i \le 3} \left|x_i^{(3)} - x_i^{(2)}\right|$$

$$= max\left\{\left|x_1^{(3)} - x_1^{(2)}\right|, \left|x_2^{(3)} - x_2^{(2)}\right|, \left|x_3^{(3)} - x_3^{(2)}\right|\right\}$$

$$= max\{1, 2, 2\} = 2 > \varepsilon = 0.0001.$$

For k = 3 we obtain

$$\begin{cases} x_1^{(4)} = x_2^{(3)} - x_3^{(3)} = 1 + 1 = 2 \\ x_2^{(4)} = -4 + 2x_1^{(3)} - x_3^{(3)} = -4 + 4 + 1 = 1 \\ x_3^{(4)} = -7 + 4x_1^{(3)} - 2x_2^{(3)} = -7 + 8 - 2 = -1. \end{cases}$$

We check the stop condition

$$d\left(x^{(4)} - x^{(3)}\right) = \max_{1 \le i \le 3} \left|x_i^{(4)} - x_i^{(3)}\right|$$

$$= max\left\{\left|x_1^{(4)} - x_1^{(3)}\right|, \left|x_2^{(4)} - x_2^{(3)}\right|, \left|x_3^{(4)} - x_3^{(3)}\right|\right\}$$

$$= max\{0, 0, 0\} = 0 < \varepsilon = 0.0001.$$

So, the solution of the system within the error $\varepsilon = 0.0001\ is \begin{pmatrix} x_1^{(4)} \\ x_2^{(4)} \\ x_3^{(4)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

In fact, if we substitute this solution into the initial system, we notice that this is not just an approximation of the solution, instead we have arrived at the exact solution of the system.

We end the chapter by showing that there is also a situation in which Jacobi's method does not converge. We present example 9 from chapter 3 from [2].

Example 3. Solve the following system within the error 10^{-2}

$$\begin{cases} x1 - \dfrac{3}{4}x2 + \dfrac{1}{4}x3 = 1 \\ -\dfrac{3}{4}x1 + x2 - \dfrac{1}{2}x3 = -\dfrac{3}{2} \\ \dfrac{1}{4}x1 - \dfrac{1}{2}x2 + x3 = \dfrac{3}{2}. \end{cases}$$

We will try to solve the above system using Jacobi's method.
We have

$$A = \begin{pmatrix} 1 & -3/4 & 1/4 \\ -3/4 & 1 & -1/2 \\ 1/4 & -1/2 & 1 \end{pmatrix} \; and \; b = \begin{pmatrix} 1 \\ -3/2 \\ 3/2 \end{pmatrix}.$$

We check if the matrix A is diagonal dominant on rows or columns. Then we have

$$\left.\begin{array}{c} |a_{22}| = 1 \\ |a_{21}| + |a_{23}| = |-3/4| + |-1/2| = 1.25 \end{array}\right\} \Rightarrow |a_{22}| < |a_{21}| + |a_{23}|$$

$$\left.\begin{array}{c} |a_{22}| = 1 \\ |a_{12}| + |a_{32}| = |-3/4| + |-1/2| = 1.25 \end{array}\right\} \Rightarrow |a_{11}| < |a_{12}| + |a_{32}|.$$

So, the matrix A is not diagonal dominant on rows or columns, but it is still possible for the two iterative methods to converge even in this situation. That being said we proceed with the methods and we see what happens.
We write the initial system in an equivalent form

$$\begin{cases} x1 = 1 + \dfrac{3}{4}x2 - \dfrac{1}{4}x3 \\ x2 = -\dfrac{3}{2} + \dfrac{3}{4}x1 + \dfrac{1}{2}x3 \\ x3 = \dfrac{3}{2} - \dfrac{1}{4}x1 + \dfrac{1}{2}x2. \end{cases}$$

We choose as initial approximation $x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and we consider the recurrence

$$\begin{cases} x_1^{(k+1)} = 1 + \dfrac{3}{4}x_2^{(k)} - \dfrac{1}{4}x_3^{(k)} \\ x_2^{(k+1)} = -\dfrac{3}{2} + \dfrac{3}{4}x_1^{(k)} + \dfrac{1}{2}x_3^{(k)} \\ x_3^{(k+1)} = \dfrac{3}{2} - \dfrac{1}{4}x_1^{(k)} + \dfrac{1}{2}x_2^{(k)}. \end{cases}$$

For k = 0 we obtain

$$\begin{cases} x_1^{(1)} = 1 + \dfrac{3}{4}x_2^{(0)} - \dfrac{1}{4}x_3^{(0)} = 1 \\ x_2^{(1)} = -\dfrac{3}{2} + \dfrac{3}{4}x_1^{(0)} + \dfrac{1}{2}x_3^{(0)} = -\dfrac{3}{2} \\ x_3^{(1)} = \dfrac{3}{2} - \dfrac{1}{4}x_1^{(0)} + \dfrac{1}{2}x_2^{(0)} = \dfrac{3}{2}. \end{cases}$$

We check the stop condition

$$d\left(x^{(1)} - x^{(0)}\right) = \max_{1 \le i \le 3} \left| x_i^{(1)} - x_i^{(0)} \right|$$

$$= max \left\{ \left| x_1^{(1)} - x_1^{(0)} \right|, \left| x_2^{(1)} - x_2^{(0)} \right|, \left| x_3^{(1)} - x_3^{(0)} \right| \right\}$$

$$= max \left\{ 1, \frac{3}{2}, \frac{3}{2} \right\} = 1.5 > \varepsilon = 0.01.$$

For k = 1 we obtain

$$\begin{cases} x_1^{(2)} = 1 + \frac{3}{4} x_2^{(1)} - \frac{1}{4} x_3^{(1)} = 1 - \frac{9}{8} - \frac{3}{8} = -\frac{1}{2} \\ x_2^{(2)} = -\frac{3}{2} + \frac{3}{4} x_1^{(1)} + \frac{1}{2} x_3^{(1)} = -\frac{3}{2} + \frac{3}{4} + \frac{3}{4} = 0 \\ x_3^{(2)} = \frac{3}{2} - \frac{1}{4} x_1^{(1)} + \frac{1}{2} x_2^{(1)} = \frac{3}{2} - \frac{1}{4} - \frac{3}{4} = \frac{1}{2}. \end{cases}$$

We check the stop condition

$$d\left(x^{(2)} - x^{(1)}\right) = \max_{1 \le i \le 3} \left| x_i^{(2)} - x_i^{(1)} \right|$$

$$= max \left\{ \left| x_1^{(2)} - x_1^{(1)} \right|, \left| x_2^{(2)} - x_2^{(1)} \right|, \left| x_3^{(2)} - x_3^{(1)} \right| \right\}$$

$$= max \left\{ \frac{3}{2}, \frac{3}{2}, 1 \right\} = 1.5 > \varepsilon = 0.01.$$

For k = 2 we obtain

$$\begin{cases} x_1^{(3)} = 1 + \frac{3}{4} x_2^{(2)} - \frac{1}{4} x_3^{(2)} = 1 - \frac{1}{8} = \frac{7}{8} \\ x_2^{(3)} = -\frac{3}{2} + \frac{3}{4} x_1^{(2)} + \frac{1}{2} x_3^{(2)} = -\frac{3}{2} - \frac{3}{8} + \frac{1}{4} = -\frac{13}{8} \\ x_3^{(3)} = \frac{3}{2} - \frac{1}{4} x_1^{(2)} + \frac{1}{2} x_2^{(2)} = \frac{3}{2} + \frac{1}{8} = \frac{13}{8}. \end{cases}$$

We check the stop condition

$$d\left(x^{(3)} - x^{(2)}\right) = \max_{1 \le i \le 3} \left| x_i^{(3)} - x_i^{(2)} \right|$$

$$= max \left\{ \left| x_1^{(3)} - x_1^{(2)} \right|, \left| x_2^{(3)} - x_2^{(2)} \right|, \left| x_3^{(3)} - x_3^{(2)} \right| \right\}$$

$$= max \left\{ \frac{3}{8}, \frac{13}{8}, \frac{17}{8} \right\} = 2.125 > \varepsilon = 0.01.$$

We can observe that the distance between two consecutive terms of the sequence that we construct via Jacobi's method becomes larger and larger. That being said, we conclude that the Jacobi's method does not converge for this system.

Pseudocode Algorithm for Jacobi's method for solving linear systems

// Here we read n, the dimension of matrix A, the matrix A, the matrix B, the error value, the maximum number of iterations and the initial solution of vector X

1. read n, a_{ij}, $1 \le i, j \le n, b_i, 1 \le i \le n, \varepsilon, itmax, x_i, 1 \le i \le n$
2. it ⇦ 0
3. okl ⇦ 0

4. okc ⇦ 0

// Here we check the Criterion 1 of the method to see if it converge,

5. for i = 1,2,…,n
    5.1. S ⇦ 0
    5.2. for j = 1,2,…,n
        5.2.1. if $i \neq j$
            5.2.1.1. S ⇦ S + $|a_{ij}|$
    5.3. if $|a_{ii}| \leq S$
        5.3.1. okl ⇦ 1
        5.3.2. break
6. for i = 1,2,…,n
    6.1. S ⇦ 0
    6.2. for j = 1,2,…,n
        6.2.1. if $i \neq j$
            6.2.1.1. S ⇦ S + $|a_{ij}|$
    6.3. if $|a_{ii}| \leq S$
        6.3.1. okc ⇦ 1
        6.3.2. break
7. if okl = 0
    7.1. write "A is strictly diagonal on lines."
8. else
    8.1. write "A is not strictly diagonal on lines."
9. if okc = 0
    9.1. write "A is strictly diagonal on columns."
10. else
    10.1. write "A is not strictly diagonal on columns."

// Here we determine the approximate solution,

11. repeat
    11.1. max ⇦ 0
    11.2. for i = 1,2,…,n
        11.2.1. S ⇦ 0
        11.2.2. for j = 1,2,…,n
            11.2.2.1. if $j \neq i$ then
                11.2.2.1.1. S ⇦ S + $a_{ij} \cdot x_j$
        11.2.3. y_i ⇦ $(b_i - S)/a_{ii}$
        11.2.4. if max < $|y_i - x_i|$ then
            11.2.4.1. max ⇦ $|y_i - x_i|$
    11.3. for i = 1,2,…,n
        11.3.1. x_i ⇦ y_i
    11.4. it ⇦ it + 1

until (max $\leq \varepsilon$) or (it > itmax)

12. if it > itmax then
    12.1. write "We cannot obtain the solution in", itmax, "iterations, with precision", ε
    12.2. exit
13. write "The approximation of the solution found in ", it, "iterations within error", ε, "is", x_i, $1 \leq i \leq n$

Remark 6.3: We can encounter situations in which even if the criterion of convergence on lines or columns is not met, the method can converge.

Chapter 7

Seidel-Gauss method for solving linear systems

The second iterative method that we will discuss in this thesis is the Seidel-Gauss method. Similarly, for Jacobi's method, we consider the linear system A · x = b, where A ∈ M_n (**R**) is the matrix associated to the linear system and b ∈ $\mathbf{R}^n$ is the vector containing the free terms of the system. Our objective is to find, if possible, x ∈ $\mathbf{R}^n$, where x represents the unique solution of the system. To solve the system by Seidel-Gauss's method, we take $x^{(0)}$ ∈ $\mathbf{R}^n$ to be the initial approximation of the solution of the system, for example a good choice is the null vector. Based on $x^{(0)}$we obtain $x^{(1)}$, based on $x^{(1)}$ we obtain $x^{(2)}$and so on. We obtain that by applying the formula:

$$x_i^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) where\ a_{ii}, 1 \le i \le n, k \ge 0,$$

until

$$\max_{1 \le i \le n} \left| x_i^{(k+1)} - x_i^{(k)} \right| \le \varepsilon,$$

where ε is the error that we consider acceptable when we try to approximate the exact solution of the system. Then $x \approx x^{(k+1)}$.

Although this method resembles to Jacobi's method for solving linear systems, there are a few differences that we are going to emphasize. As in the case of Jacobi's method, a sufficient condition for the convergence of this method is based on Criterion 1 presented in **Chapter 6** which says that a sufficient condition to obtain the solution of system within the error ε is that matrix A should be strictly diagonally dominant on rows or columns. A second criterion that gives us sufficient conditions for Seidel-Gauss method to converge (but it is not working for Jacobi's method) is the following.

Criterion 2: Seidel-Gauss method converges if A is symmetric and positive-definite.

We recall that A is symmetric if and only if A = A^t, meaning that,

$$\left(a_{ij} = a_{ji} \right)_{(\forall) i,j = \overline{1,n}.}$$

A is positive-definite if and only if all the principal diagonal minors are positive.

Remark 7.1: Criterion 1 and Criterion 2 represent sufficient conditions for the convergence of the Seidel-Gauss method, but they are not necessary. Therefore, it can happen that the Seidel-Gauss method converges even if these two criteria are not met, see [2] for more details.

In what follows, we will try to solve the three examples from the previous chapter using the Seidel-Gauss method.

Example 1. Solve the following system within the error 10^{-2}

$$\begin{cases} 10x1 - 2x2 + 2x3 = -28 \\ x1 + 5x2 - 2x3 + x4 = -4 \\ x1 + x2 - 10x3 + 2x4 = -26 \\ x1 - x2 - 20x4 = 36. \end{cases}$$

We have

$$A = \begin{pmatrix} 10 & -2 & 2 & 0 \\ 1 & 5 & -2 & 1 \\ 1 & 1 & -10 & 2 \\ 1 & -1 & 0 & -20 \end{pmatrix} \; and \; b = \begin{pmatrix} -28 \\ -4 \\ -26 \\ 36 \end{pmatrix}.$$

We already saw that matrix A is strictly diagonal dominant on rows, thus we can be sure that the Seidel-Gauss method enables us to obtain an approximation of the solutions with the desired precision.
We write the initial system in an equivalent form

$$\begin{cases} x1 = (-28 + 2x2 - 2x3)/10 \\ x2 = (-4 - x1 + 2x3 - x4)/5 \\ x3 = (-26 - x1 - x2 - 2x4)/(-10) \\ x4 = (36 - x1 + x2)/(-20). \end{cases}$$

We choose as initial approximation $x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, and we consider the recurrence

$$\begin{cases} x_1^{(k+1)} = (-28 + 2x_2^{(k)} - 2x_3^{(k)})/10 \\ x_2^{(k+1)} = (-4 - x_1^{(k+1)} + 2x_3^{(k)} - x_4^{(k)})/5 \\ x_3^{(k+1)} = (-26 - x_1^{(k+1)} - x_2^{(k+1)} - 2x_4^{(k)})/(-10) \\ x_4^{(k+1)} = (36 - x_1^{(k+1)} + x_2^{(k+1)})/(-20). \end{cases}$$

For k = 0 we obtain

$$\begin{cases} x_1^{(1)} = (-28 + 2x_2^{(0)} - 2x_3^{(0)})/10 = -2800/1000 \\ x_2^{(1)} = -4 - x_1^{(1)} + 2x_3^{(0)} - x_4^{(0)}/5 = -240/1000 \\ x_3^{(1)} = (-26 - x_1^{(1)} - x_2^{(1)} - 2x_4^{(0)})/(-10) = 2296/1000 \\ x_4^{(1)} = (36 - x_1^{(1)} + x_2^{(1)})/(-20) = -1928/1000. \end{cases}$$

We check the stop condition

$$d(x^{(1)} - x^{(0)}) = \max_{1 \le i \le 4} \left| x_i^{(1)} - x_i^{(0)} \right|$$

$$= max \left\{ \left| x_1^{(1)} - x_1^{(0)} \right|, \left| x_2^{(1)} - x_2^{(0)} \right|, \left| x_3^{(1)} - x_3^{(0)} \right|, \left| x_4^{(1)} - x_4^{(0)} \right| \right\}$$

$$= max \left\{ \frac{2800}{1000}, \frac{240}{1000}, \frac{2296}{1000}, \frac{1928}{1000} \right\} = 2.8 > \varepsilon = 0.01.$$

For k = 1 we obtain

$$\begin{cases} x_1^{(2)} = (-28 + 2x_2^{(1)} - 2x_3^{(1)})/10 = -3211200/1000000 \\ x_2^{(2)} = -4 - x_1^{(2)} + 2x_3^{(1)} - x_4^{(1)}/5 = 1146240/1000000 \\ x_3^{(2)} = (-26 - x_1^{(2)} - x_2^{(2)} - 2x_4^{(1)})/(-10) = 2007904/1000000 \\ x_4^{(2)} = (36 - x_1^{(2)} + x_2^{(2)})/(-20) = -2017872/1000000. \end{cases}$$

We check the stop condition

$$d(x^{(2)} - x^{(1)}) = \max_{1 \le i \le 4} \left| x_i^{(2)} - x_i^{(1)} \right|$$

$$= max\left\{\left|x_1^{(2)} - x_1^{(1)}\right|, \left|x_2^{(2)} - x_2^{(1)}\right|, \left|x_3^{(2)} - x_3^{(1)}\right|, \left|x_4^{(2)} - x_4^{(1)}\right|\right\}$$

$$= max\left\{\frac{411200}{1000000}, \frac{1386240}{1000000}, \frac{288096}{1000000}, \frac{89872}{1000000}\right\} = 1.38624 > \varepsilon = 0.01.$$

For k = 2 we obtain

$$\begin{cases} x_1^{(3)} = (-28 + 2x_2^{(2)} - 2x_3^{(2)})/10 = -2972332800/1000000000 \\ x_2^{(3)} = -4 - x_1^{(3)} + 2x_3^{(2)} - x_4^{(2)}/5 = 1001202560/1000000000 \\ x_3^{(3)} = (-26 - x_1^{(3)} - x_2^{(3)} - 2x_4^{(2)})/(-10) = 1999312576/1000000000 \\ x_4^{(3)} = (36 - x_1^{(3)} + x_2^{(3)})/(-20) = -1998676768/1000000000. \end{cases}$$

We check the stop condition

$$d\left(x^{(3)} - x^{(2)}\right) = \max_{1 \le i \le 4}\left|x_i^{(3)} - x_i^{(2)}\right|$$

$$= max\left\{\left|x_1^{(3)} - x_1^{(2)}\right|, \left|x_2^{(3)} - x_2^{(2)}\right|, \left|x_3^{(3)} - x_3^{(2)}\right|, \left|x_4^{(3)} - x_4^{(2)}\right|\right\}$$

$$= max\left\{\frac{238867200}{1000000000}, \frac{145037440}{1000000000}, \frac{8591424}{1000000000}, \frac{19195232}{1000000000}\right\} = 0.2388672 > \varepsilon = 0.01.$$

For k = 3 we obtain

$$\begin{cases} x_1^{(4)} = (-28 + 2x_2^{(3)} - 2x_3^{(3)})/10 = -2999622003200/1000000000000 \\ x_2^{(4)} = -4 - x_1^{(4)} + 2x_3^{(3)} - x_4^{(3)}/5 = 999384784640/1000000000000 \\ x_3^{(4)} = (-26 - x_1^{(4)} - x_2^{(4)} - 2x_4^{(3)})/(-10) = 2000240924544/1000000000000 \\ x_4^{(4)} = (36 - x_1^{(4)} + x_2^{(4)})/(-20) = -1999950339392/1000000000000. \end{cases}$$

We check the stop condition

$$d\left(x^{(4)} - x^{(3)}\right) = \max_{1 \le i \le 4}\left|x_i^{(4)} - x_i^{(3)}\right|$$

$$= max\left\{\left|x_1^{(4)} - x_1^{(3)}\right|, \left|x_2^{(4)} - x_2^{(3)}\right|, \left|x_3^{(4)} - x_3^{(3)}\right|, \left|x_4^{(4)} - x_4^{(3)}\right|\right\}$$

$$= max\left\{\frac{27289203200}{1000000000000}, \frac{1817775360}{1000000000000}, \frac{928348544}{1000000000000}, \frac{1273571392}{1000000000000}\right\}$$

$$= 0.0272892032 > \varepsilon = 0.01.$$

For k = 4 we obtain

$$\begin{cases} x_1^{(5)} = (-28 + 2x_2^{(4)} - 2x_3^{(4)})/10 = -3000171227980800/1000000000000000 \\ x_2^{(5)} = -4 - x_1^{(5)} + 2x_3^{(4)} - x_4^{(4)}/5 = 1000120683292160/1000000000000000 \\ x_3^{(5)} = (-26 - x_1^{(5)} - x_2^{(5)} - 2x_4^{(4)})/(-10) = 2000004877652736/1000000000000000 \\ x_4^{(5)} = (36 - x_1^{(5)} + x_2^{(5)})/(-20) = -2000014595563648/1000000000000000. \end{cases}$$

We check the stop condition

$$d\left(x^{(5)} - x^{(4)}\right) = \max_{1 \le i \le 4}\left|x_i^{(5)} - x_i^{(4)}\right|$$

$$= max\left\{\left|x_1^{(5)} - x_1^{(4)}\right|, \left|x_2^{(5)} - x_2^{(4)}\right|, \left|x_3^{(5)} - x_3^{(4)}\right|, \left|x_4^{(5)} - x_4^{(4)}\right|\right\}$$

$$= max\left\{\frac{549224780800}{1000000000000000}, \frac{735898652160}{1000000000000000}, \frac{236046891264}{1000000000000000}, \frac{64256171648}{1000000000000000}\right\}$$

$$= 0.00073589865216 < \varepsilon = 0.01.$$

We obtain the solution with precision given by $\varepsilon = 0.01$ at the step 5

$$\begin{cases} x_1^{(5)} = -3.0001712279808 \\ x_2^{(5)} = 1.00012068329216 \\ x_3^{(5)} = 2.000004877652736 \\ x_4^{(5)} = -2.000014595563648. \end{cases}$$

We recall that the exact solution of the system is $\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \\ -2 \end{pmatrix}$.

As we can see, with the Seidel-Gauss method we reached an approximate solution in 5 steps, while with Jacobi's method we needed 7 steps. As we said, when both methods converge, the Seidel-Gauss method is faster. But we can have cases like Example 2 from **Chapter 6** which cannot be solved with the Seidel-Gauss method even though it can be solved with Jacobi's method.

Example 2. Solve the following system within the error 10^{-4}

$$\begin{cases} x1 - x2 + x3 = 0 \\ -2x1 + x2 + x3 = -4 \\ -4x1 + 2x2 + x3 = -7. \end{cases}$$

An equivalent form of the system is

$$\begin{cases} x1 = x2 - x3 \\ x2 = -4 + 2x1 - x3 \\ x3 = -7 + 4x1 - 2x2. \end{cases}$$

We choose as initial approximation $x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and we consider the recurrence

$$\begin{cases} x_1^{(k+1)} = x_2^{(k)} - x_3^{(k)} \\ x_2^{(k+1)} = -4 + 2x_1^{(k+1)} - x_3^{(k)} \\ x_3^{(k+1)} = -7 + 4x_1^{(k+1)} - 2x_2^{(k+1)}. \end{cases}$$

For k = 0 we obtain

$$\begin{cases} x_1^{(1)} = x_2^{(0)} - x_3^{(0)} = 0 \\ x_2^{(1)} = -4 + 2x_1^{(1)} - x_3^{(0)} = -4 \\ x_3^{(1)} = -7 + 4x_1^{(1)} - 2x_2^{(1)} = 1. \end{cases}$$

We check the stop condition

$$d\left(x^{(1)} - x^{(0)}\right) = \max_{1 \le i \le 3}\left|x_i^{(1)} - x_i^{(0)}\right|$$

$$= max\left\{\left|x_1^{(1)} - x_1^{(0)}\right|, \left|x_2^{(1)} - x_2^{(0)}\right|, \left|x_3^{(1)} - x_3^{(0)}\right|\right\}$$

$$= max\{0, 4, 1\} = 4 > \varepsilon = 0.0001.$$

For k = 1 we obtain

$$\begin{cases} x_1^{(2)} = x_2^{(1)} - x_3^{(1)} = -5 \\ x_2^{(2)} = -4 + 2x_1^{(2)} - x_3^{(1)} = -15 \\ x_3^{(2)} = -7 + 4x_1^{(2)} - 2x_2^{(2)} = 3. \end{cases}$$

We check the stop condition

$$d\left(x^{(2)} - x^{(1)}\right) = \max_{1 \le i \le 3} \left| x_i^{(2)} - x_i^{(1)} \right|$$

$$= max\left\{ \left| x_1^{(2)} - x_1^{(1)} \right|, \left| x_2^{(2)} - x_2^{(1)} \right|, \left| x_3^{(2)} - x_3^{(1)} \right| \right\}$$

$$= max\{5, 11, 2\} = 11 > \varepsilon = 0.0001.$$

For k = 2 we obtain

$$\begin{cases} x_1^{(3)} = x_2^{(2)} - x_3^{(2)} = -18 \\ x_2^{(3)} = -4 + 2x_1^{(3)} - x_3^{(2)} = -43 \\ x_3^{(3)} = -7 + 4x_1^{(3)} - 2x_2^{(3)} = 7. \end{cases}$$

We check the stop condition

$$d\left(x^{(3)} - x^{(2)}\right) = \max_{1 \le i \le 3} \left| x_i^{(3)} - x_i^{(2)} \right|$$

$$= max\left\{ \left| x_1^{(3)} - x_1^{(2)} \right|, \left| x_2^{(3)} - x_2^{(2)} \right|, \left| x_3^{(3)} - x_3^{(2)} \right| \right\}$$

$$= max\{13, 28, 4\} = 28 > \varepsilon = 0.0001.$$

For k = 3 we obtain

$$\begin{cases} x_1^{(4)} = x_2^{(3)} - x_3^{(3)} = -50 \\ x_2^{(4)} = -4 + 2x_1^{(4)} - x_3^{(3)} = -111 \\ x_3^{(4)} = -7 + 4x_1^{(4)} - 2x_2^{(4)} = 15. \end{cases}$$

We check the stop condition

$$d\left(x^{(4)} - x^{(3)}\right) = \max_{1 \le i \le 3} \left| x_i^{(4)} - x_i^{(3)} \right|$$

$$= max\left\{ \left| x_1^{(4)} - x_1^{(3)} \right|, \left| x_2^{(4)} - x_2^{(3)} \right|, \left| x_3^{(4)} - x_3^{(3)} \right| \right\}$$

$$= max\{68, 68, 8\} = 68 > \varepsilon = 0.0001.$$

For k = 4 we obtain

$$\begin{cases} x_1^{(5)} = x_2^{(4)} - x_3^{(4)} = -125 \\ x_2^{(5)} = -4 + 2x_1^{(5)} - x_3^{(4)} = -271 \\ x_3^{(5)} = -7 + 4x_1^{(5)} - 2x_2^{(5)} = 31. \end{cases}$$

Going further to iterate over and over again with the help of a computer, we notice that the Seidel-Gauss method does not converge because the distance between two consecutive terms of the sequence grows larger and larger, instead of getting smaller. Thus we have seen in the above example a system for which the Seidel-Gauss method does not converge, while Jacobi's method converges quite rapidly, allowing us to obtain

the exact solution of the system at the fourth step.
On the other hand, there are opposite situations where the Seidel-Gauss method converges and the Jacobi method does not converges, and here we can refer to Example 3 from the previous chapter where we notice that Criterion 2 is fulfilled.

Example 3. Solve the following system within the error 10^{-4}

$$\begin{cases} x1-\frac{3}{4}x2+\frac{1}{4}x3=1 \\ -\frac{3}{4}x1+x2-\frac{1}{2}x3=-\frac{3}{2} \\ \frac{1}{4}x1-\frac{1}{2}x2+x3=\frac{3}{2}. \end{cases}$$

An equivalent form of this system is

$$\begin{cases} x1=1+\frac{3}{4}x2-\frac{1}{4}x3 \\ x2=-\frac{3}{2}+\frac{3}{4}x1+\frac{1}{2}x3 \\ x3=\frac{3}{2}-\frac{1}{4}x1+\frac{1}{2}x2. \end{cases}$$

We choose as initial approximation $x^{(0)}=\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and we consider the recurrence

$$\begin{cases} x_1^{(k+1)}=1+\frac{3}{4}x_2^{(k)}-\frac{1}{4}x_3^{(k)} \\ x_2^{(k+1)}=-\frac{3}{2}+\frac{3}{4}x_1^{(k+!)}+\frac{1}{2}x_3^{(k)} \\ x_3^{(k+1)}=\frac{3}{2}-\frac{1}{4}x_1^{(k+1)}+\frac{1}{2}x_2^{(k+1)}. \end{cases}$$

For k = 0 we obtain

$$\begin{cases} x_1^{(1)}=1+\frac{3}{4}x_2^{(0)}-\frac{1}{4}x_3^{(0)}=1 \\ x_2^{(1)}=-\frac{3}{2}+\frac{3}{4}x_1^{(1)}+\frac{1}{2}x_3^{(0)}=-\frac{3}{4} \\ x_3^{(1)}=\frac{3}{2}-\frac{1}{4}x_1^{(1)}+\frac{1}{2}x_2^{(1)}=\frac{7}{8}. \end{cases}$$

We check the stop condition

$$d\left(x^{(1)}-x^{(0)}\right)=\max_{1\le i\le 3}\left|x_i^{(1)}-x_i^{(0)}\right|$$

$$=max\left\{\left|x_1^{(1)}-x_1^{(0)}\right|,\left|x_2^{(1)}-x_2^{(0)}\right|,\left|x_3^{(1)}-x_3^{(0)}\right|\right\}$$

$$=max\left\{1,\frac{3}{4},\frac{7}{8}\right\}=1>\varepsilon=0.01.$$

For k = 1 we obtain

$$\begin{cases} x_1^{(2)} = 1 + \frac{3}{4}x_2^{(1)} - \frac{1}{4}x_3^{(1)} = 1 - \frac{9}{16} - \frac{7}{32} = \frac{7}{32} \\ x_2^{(2)} = -\frac{3}{2} + \frac{3}{4}x_1^{(2)} + \frac{1}{2}x_3^{(1)} = -\frac{3}{2} + \frac{21}{128} + \frac{7}{16} = -\frac{115}{128} \\ x_3^{(2)} = \frac{3}{2} - \frac{1}{4}x_1^{(2)} + \frac{1}{2}x_2^{(2)} = \frac{3}{2} - \frac{7}{128} - \frac{115}{256} = \frac{255}{256}. \end{cases}$$

We check the stop condition

$$d\left(x^{(2)} - x^{(1)}\right) = \max_{1 \le i \le 3}\left|x_i^{(2)} - x_i^{(1)}\right|$$

$$= max\left\{\left|x_1^{(2)} - x_1^{(1)}\right|, \left|x_2^{(2)} - x_2^{(1)}\right|, \left|x_3^{(2)} - x_3^{(1)}\right|\right\}$$

$$= max\left\{\frac{25}{32}, \frac{31}{128}, \frac{31}{256}\right\} = 0.12109375 > \varepsilon = 0.01.$$

Continuing to make these iterations, we reach step 6 and we will obtain the following approximate solution

$$\begin{cases} x_1^{(6)} = 0.01489 \\ x_2^{(6)} = -0.98644 \\ x_3^{(6)} = 1.003052. \end{cases}$$

So, we notice that with the Seidel-Gauss method we managed to reach an approximate solution on an example on which Jacobi's method has failed.

Pseudocode Algorithm for Seidel-Gauss method for solving linear systems

// Here we read n, the dimension of matrix A, the matrix A, the matrix B, the error value, the maximum number of iterations and the initial solution of vector X

1. read n, a_{ij}, $1 \le i, j \le n, b_i, 1 \le i \le n, \varepsilon, itmax, x_i, 1 \le i \le n$
2. it ⇦ 0
3. okl ⇦ 0
4. okc ⇦ 0
5. oks ⇦ 0

// Here we check the Criterion 1 and Criterion 2 of the method to see if it converge,

6. for i = 1,2,…,n
    - 6.1. S ⇦ 0
    - 6.2. for j = 1,2,…,n
        - 6.2.1. if i ≠ j
            - 6.2.1.1. S ⇦ S + $|a_{ij}|$
    - 6.3. if $|a_{ii}| \le$ S
        - 6.3.1. okl ⇦ 1
        - 6.3.2. break
7. for i = 1,2,…,n
    - 7.1. S ⇦ 0
    - 7.2. for j = 1,2,…,n
        - 7.2.1. if i ≠ j
            - 7.2.1.1. S ⇦ S + $|a_{ij}|$
    - 7.3. if $|a_{ii}| \le$ S
        - 7.3.1. okc ⇦ 1
        - 7.3.2. break

```
8. for i = 1,2,…,n
        8.1. for j = 1,2,…,n
                8.1.1. if aij ≠ aji
                        8.1.1.1. oks ⇦ 1
                        8.1.1.2. break
9. if okl = 0
        9.1. write "A is strictly diagonal on lines."
10. else
        10.1. write "A is not strictly diagonal on lines."
11. if okc = 0
        11.1. write "A is strictly diagonal on columns."
12. else
        12.1. write "A is not strictly diagonal on columns."
13. if oks = 0
        13.1. write "A is symmetric and positive-definite"
14. else
        14.1. write "A is not symmetric and positive-definite"
// Here we determine approximate solution,
15. repeat
        15.1. max ⇦ 0
        15.2. for i = 1,2,…,n
                15.2.1. S ⇦ 0
                15.2.2. for j = 1,2,…,i-1
                        15.2.2.1. S ⇦ S + aij · yj
                15.2.3. for j = i+1,i+2,…,n
                        15.2.3.1.  S ⇦ S + aij · xj
                15.2.4. yi ⇦ (bi – S)/aii
                15.2.5. if max < |yi – xi| then
                        15.2.5.1. max ⇦ |yi – xi|
        15.3. for i = 1,2,…,n
                15.3.1. xi ⇦ yi
        15.4. it ⇦ it + 1
until (max ≤ ε) or ( it > itmax)
16. if it > itmax then
        16.1. write "We cannot obtain the solution in", itmax, "iterations, with precision", ε
        16.2. exit
17. write "The approximation of the solution found in ", it, "iterations within error", ε, "is", xi, 1 ≤ i ≤ n
```

To occupy less memory space, we can improve the algorithm by using instead of the vector $y = (y_1, y_2, \dots, y_n)$ a scalar variable y as it was done in the pseudocode algorithm from Chapter 10 of [5]. That being said our algorithm can be rewritten as below.

Improved version of the pseudocode algorithm for Seidel-Gauss method for solving linear systems, inspired by [5]

```
// Here we read n, the dimension of matrix A, the matrix A, the matrix B, the error value, the maximum
number of iterations and the initial solution of vector X
```

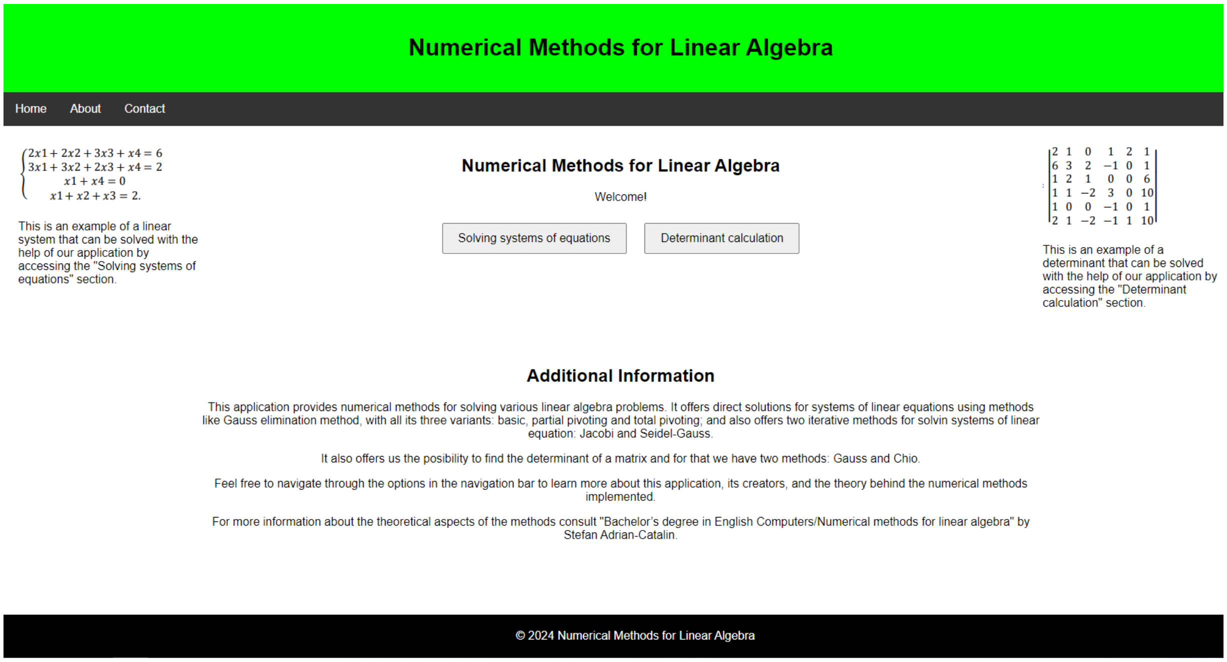
```
1. read n, aij, 1 ≤ i,j ≤ n, b_i, 1 ≤ i ≤ n, ε, itmax, x_i, 1 ≤ i ≤ n
2. it ⇦ 0
3. okl ⇦ 0
4. okc ⇦ 0
5. oks ⇦ 0
// Here we check the Criterion 1 and Criterion 2 of the method to see if it converge,
6. for i = 1,2,…,n
        6.1. S ⇦ 0
        6.2. for j = 1,2,…,n
                6.2.1. if i ≠ j
                        6.2.1.1. S ⇦ S + |aij|
        6.3. if |aii| ≤ S
                6.3.1. okl ⇦ 1
                6.3.2. break
7. for i = 1,2,…,n
        7.1. S ⇦ 0
        7.2. for j = 1,2,…,n
                7.2.1. if i ≠ j
                        7.2.1.1. S ⇦ S + |aij|
        7.3. if |aii| ≤ S
                7.3.1. okc ⇦ 1
                7.3.2. break
8. for i = 1,2,…,n
        8.1. for j = 1,2,…,n
                8.1.1. if aij ≠ aji
                        8.1.1.1. oks ⇦ 1
                        8.1.1.2. break
9. if okl = 0
        9.1. write "A is strictly diagonal on lines."
10. else
        10.1. write "A is not strictly diagonal on lines."
11. if okc = 0
        11.1. write "A is strictly diagonal on columns."
12. else
        12.1. write "A is not strictly diagonal on columns."
13. if oks = 0
        13.1. write "A is symmetric and positive-definite"
14. else
        14.1. write "A is not symmetric and positive-definite"
// Here we determine approximate solution,
15. repeat
        15.1. max ⇦ 0
        15.2. for i = 1,2,…,n
                15.2.1. S ⇦ 0
                15.2.2. for j = 1,2,…,n
                        15.2.2.1. if j ≠ i
                                15.2.2.1. S ⇦ S + aij · xj
```

15.2.3. y ⇦ $(b_i - S)/a_{ii}$
15.2.4. if max < $|y - x_i|$ then
15.2.4.1. max ⇦ $|y - x_i|$
15.2.5. x_i ⇦ y
15.3. it ⇦ it + 1
until (max $\le \varepsilon$) or (it > itmax)
16. if it > itmax then
16.1. write “We cannot obtain the solution in”, itmax, “iterations, with precision”, ε
16.2. exit
17. write “The approximation of the solution found in “, it, “iterations within error”, ε, “is”, x_i, $1 \le i \le n$.

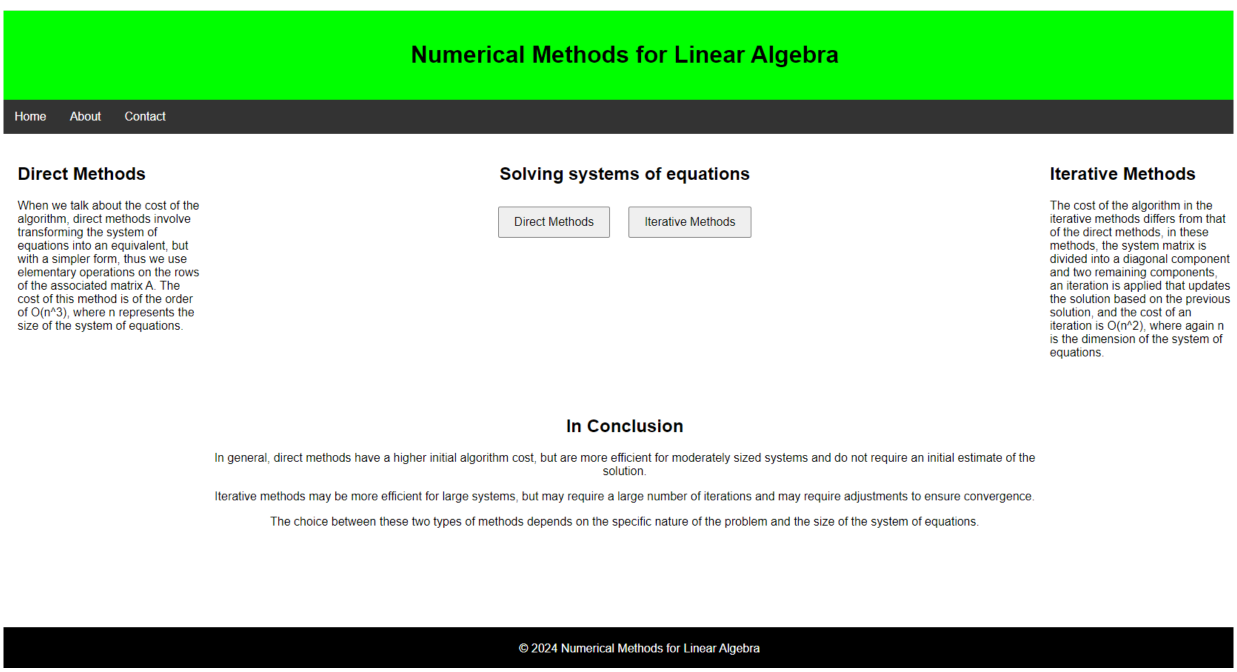
Chapter 8

The online application

In order to put into practice and have a physical representation of the calculation methods of the systems and determinants presented previously, we created an online application that contains all these methods and where a user can enter the data of a system or determinant and obtain with the method on which the user chooses from those presented above, the solution of the system or the result of the determinant. Thus, in the following we will present to you the way in which we created this online application as well as the way in which a user can use this application. To create this application, I used the following programming languages: C++, HTML, CSS and JavaScript. In the first phase, based on the pseudocodes presented previously, we built the C++ codes for each method presented and used the Visual Studio 2022 environment to test each code and see if they work and provide the same results that we obtained by hand calculation presented in the work. Afterwards, we used Visual Studio Code to create the HTML pages that make up the online application, and in which we included the HTML, CSS and JavaScript codes for the part of calculation methods and for other functions of the application. In order to integrate the C++ codes in the HTML pages, we converted the C++ code to JavaScript code, thus being able to add the methods in the online application. After we completed the conversion and the introduction of the JavaScript code in the application, we tested again, this time through the online application, if we obtain the same res ults as we obtained by hand calculation. Having said that, the role of the application is to help solve the systems, but not in the cases that we presented in the paper, and in the cases where we have a lot of unknowns and by hand calculation becomes ineffective, so to speak for these systems we can use the online application that will give us the solution much faster and more efficiently than trying to solve it by hand. Having said that, we will move on to the part where we present you how to use our application. When you open the online application, you will be on the Home page of the application.



The Home page contains information that is helpful for the user regarding what he can do on the application, also two buttons appear on the page and depending on what he wants to do, he can choose either to solve a system of equations, or to calculate the determinant of a matrices. Also in the navigation bar of pages, the user has the option to go to the About page where he will find more detailed information about the systems and their role, as well as information about how to enter the data and use the application, information that will also be found in this paper. He can also use the Contact page in case he has to report a bug or a problem he encountered on the application. If the user chooses to solve a system and presses the related button, he will be redirected to a new page where he will have a choice between the two types of methods: direct and iterative.



Here the user will have presented the advantages and disadvantages of the two types of methods in terms of the cost of the algorithm and the efficiency depending on the size of the system that he wants to solve.

Solving systems of equations

Direct methods:

Basic Gaussian method | Gauss's with partial pivoting | Gauss's with total pivoting

If the user chooses the direct methods, he will have a choice between the three available methods: basic Gauss, with partial pivoting and with total pivoting, and for each method, when the mouse is placed on the button of the respective method, on a short description of the method will appear on the page. Now after the user chooses a method, he will have to enter the data of the system he wants to solve, and to better understand how to do this we will present again how we transform a system into a matrix. As an example we will use the system from **Chapter 7,** example 1:

$$\begin{cases} 10x1 - 2x2 + 2x3 = -28 \\ x1 + 5x2 - 2x3 + x4 = -4 \\ x1 + x2 - 10x3 + 2x4 = -26 \\ x1 - x2 - 20x4 = 36. \end{cases}$$

As you can see this system has four unknowns, so the size of the system is four. The matrix for this system will be represented by the coefficients of the four unknowns on all four rows to which the column with the free terms will be added, so the matrix will look like this,

$$A = \left(\begin{array}{cccc|c} 10 & -2 & 2 & 0 & -28 \\ 1 & 5 & -2 & 1 & -4 \\ 1 & 1 & -10 & 2 & -26 \\ 1 & -1 & 0 & -20 & 36 \end{array}\right)$$

which can also be represented as

$$A = \left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right).$$

Having said that, we will apply the same principle in our application, so when you have to enter a system at the beginning you will be asked what is the size of the system, after you enter the value several boxes similar to the previously presented matrices will open and there you will the user must enter the data.

Matrix Size: 4 Submit

a11	a12	a13	a14	a15
a21	a22	a23	a24	a25
a31	a32	a33	a34	a35
a41	a42	a43	a44	a45

Calculate

After the user enters the data correctly, he will press the Calculate button, after which the application will calculate the system solution through the JavaScript code and offer it to the user, also at the end he has the option to find out what the cost of the algorithm was. That was for the direct methods, if the user chooses the iterative methods, he has to do the following steps.

Solving systems of equations

Iterative Methods:

Jacobi's method | Seidel-Gauss method

The two available methods will be presented, similar to the direct ones, and the user will have to choose the method with which he wants to solve the system, Jacobi or Seidel-Gauss. The data entry method is somewhat similar to the direct methods only that this time we will transform the system into two matrices, matrix A which will contain the coefficients of the unknowns and matrix B which will contain the column of free terms as for example,

$$\begin{cases} 10x1 - 2x2 + 2x3 = -28 \\ x1 + 5x2 - 2x3 + x4 = -4 \\ x1 + x2 - 10x3 + 2x4 = -26 \\ x1 - x2 - 20x4 = 36. \end{cases}$$

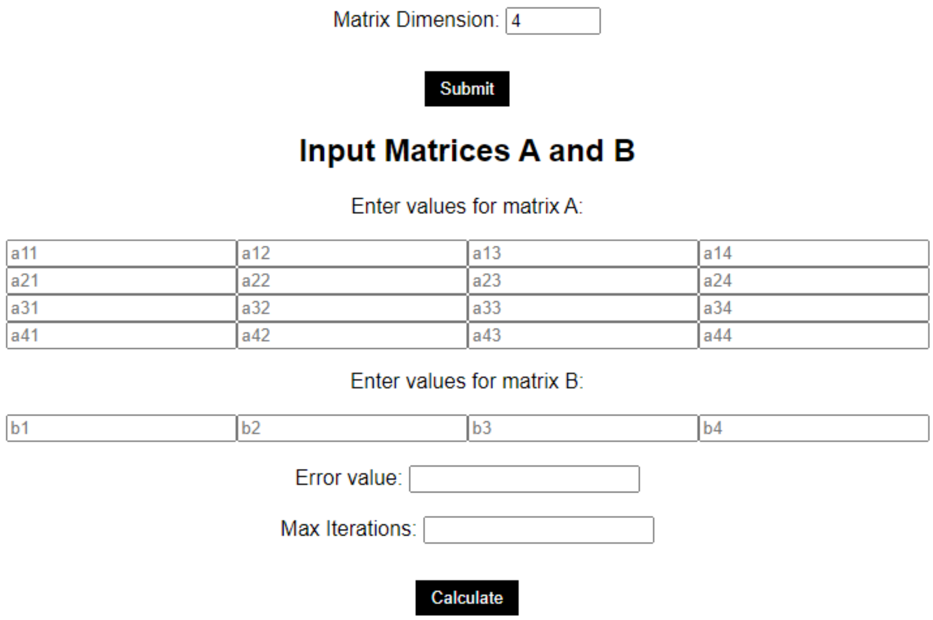
the system presented previously will be transformed into

$$A = \begin{pmatrix} 10 & -2 & 2 & 0 \\ 1 & 5 & -2 & 1 \\ 1 & 1 & -10 & 2 \\ 1 & -1 & 0 & -20 \end{pmatrix} \; and \; b = \begin{pmatrix} -28 \\ -4 \\ -26 \\ 36 \end{pmatrix}$$

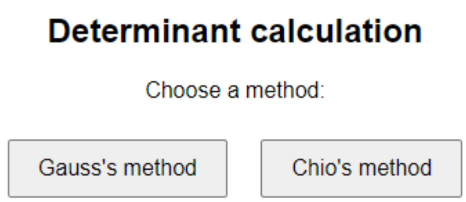
which can also be represented as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \; and \; b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

Thus, when the user chooses one of the iterative methods, he will have to give the size of the system, after which he will have to enter the corresponding data in the two matrices A and B, and in addition to this information, because we are talking about iterative methods, where we are looking for an approximation of the solution, the user will have to enter the error value with which he wants to approximate the solution and the maximum number of iterations in which he wants to find the solution.



After entering the data correctly, the user will press the Calculate button and the application will offer the system solution, if the method converges to the system entered by the user, and if it manages to find the solution in the maximum number of iterations entered by the user. Now if the user does not want to solve a system and chooses to calculate a determinant, he will be redirected to a page where the two methods of calculating the available determinants, Gauss and Chio, will be presented, with the advantages and disadvantages of each method.



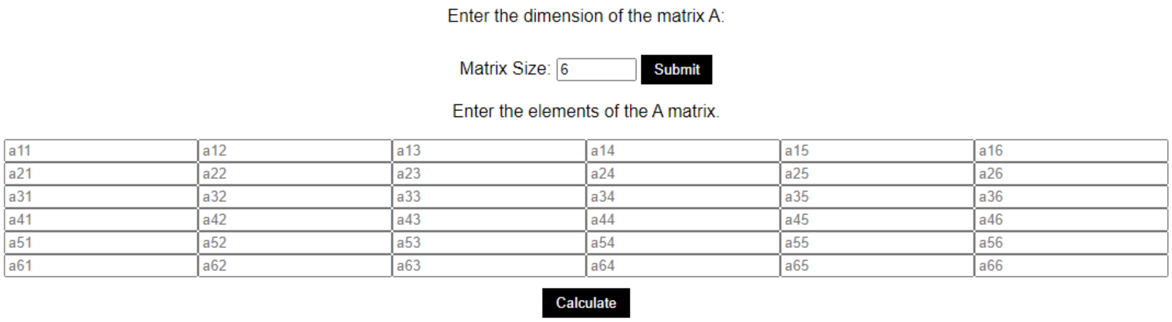
After choosing one of the two methods, the user will have to enter the data of the determinant and to exemplify in an easier way how to do this we will present the following determinant from **Chapter 5**,

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 2 & 1 \\ 6 & 3 & 2 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 6 \\ 1 & 1 & -2 & 3 & 0 & 10 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 2 & 1 & -2 & -1 & 1 & 10 \end{pmatrix},$$

which can also be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}.$$

Thus, when the user chooses one of the methods, he will be asked the size of the matrix A that represents the determinant, after which he will have to enter the data correctly and then he will be able to press the Calculate button and get the desired result.



At the end, he will also have the option of being shown the cost of the algorithm for his determinant. For each of the methods there is this option to show the cost of the algorithm, so besides using the application to find out the solution of a system that has a larger size and it is inefficient to do it by hand, or a determinant, the user can use our application to see what would be the cost of the algorithm for the same system or determinant using different methods where different methods can be applied on the same system, because we can have cases where one method works for a system and another does not. Through this application we also wanted to combine mathematics and the computer to demonstrate how useful technology is today, in our case we chose to solve systems of equations, and such an application can be extremely useful both for a student who needs help when he wants to solve a system, but also for engineers when we talk about the use of systems of equations in more complex activities.

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