TU Eindhoven, 2WAG0

Problems in Measure theory

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"What I don't like about measure theory is that you have to say "almost everywhere" almost everywhere"

– Kurt Friedrichs

Disclaimer:

This is a broad selection of exercises for the course *Measure, integration and probability theory*. They are meant to accompany the lecture notes and give you the opportunity to exercise. If you wish to have your solution checked, send it in LaTeX, and we will correct and polish it together, so that it can be featured in this notes in the "Solutions" part.

These collection of exercises are still in progress and they might contain small typos. If you see any or if you think that the statement of the problems is not yet crystal clear, feel free to drop a line. The most efficient way is to send an email to me, a.chiarini@tue.nl. All comments and suggestions will be greatly appreciated.

Contents

Part I.

Problems

1. Warming up

In this section we will review some of the basic set operations which will be much needed in the sequel.

Problem 1.1. Let A, B and C be sets, show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Problem 1.2. Let A, B be sets, show that $A \cap (A \cup B) = A$.

Problem 1.3. Let $A, B \subseteq \Omega$. We define the symmetric difference to be

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Show that $A \cup B$ is the disjoint union of $A \Delta B$ and $A \cap B$.

Problem 1.4. Let $A, B \subseteq \Omega$, show that

$$\Omega \setminus (A \cup B) = (\Omega \setminus A) \cap (\Omega \setminus B).$$

Problem 1.5. (De Morgan's law) Let I be any index set and let $\{A_i\}_{i \in I} \subseteq 2^{\Omega}$ be a family subsets of Ω. Show that

$$\Omega \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} \Omega \setminus A_i.$$

Problem 1.6. Let $f : \Omega \to E$ be some function. Recall that for any $D \subseteq \Omega$ the *image* of D under f is the set

$$f(D) = \{f(x) : \, x \in D\},$$

Let $A, B \subseteq \Omega$. Show that

- $f(A \cap B) \subseteq f(A) \cap f(B)$,
- $f(A \cup B) = f(A) \cup f(B)$.

Find an example where $f(A \cap B) \neq f(A) \cap f(B)$. Is it true that $f(\Omega \setminus A) = E \setminus f(A)$?

Problem 1.7. Let $f : \Omega \to E$ be some function. Recall that for any $F \subseteq E$ the *inverse image* of F under f is the set

$$f^{-1}(F) = \{x : \, f(x) \in F\}.$$

Let $H, K \subseteq E$. Show that, taking the inverse image commutes with the set operations:

- $f^{-1}(H \cap K) = f^{-1}(H) \cap f^{-1}(K)$,

- $f^{-1}(H \cup K) = f^{-1}(H) \cup f^{-1}(K)$,
- $f^{-1}(E \setminus H) = \Omega \setminus f^{-1}(H)$.

Problem 1.8. Let $f : \Omega \to E$ be some function.

- Let $A \subseteq \Omega$. Is it true that $f^{-1}(f(A)) = A$? Provide a proof or a counterexample.
- Let $H \subseteq E$. Is it true that $f(f^{-1}(H)) = H$? Provide a proof or a counterexample.

Problem 1.9. Recall that given a set Ω, 2^Ω denotes the set of all subsets of Ω. Suppose $\Omega = \{0, 1\}$, list all the elements of 2^Ω. What is $|2^\Omega|$, where $|\cdot|$ denotes the number of elements of a set? Suppose that $|\Omega| < \infty$, what is $|2^\Omega|$ in this case?

Problem 1.10. Let $\{a_i\}_{i\in I} \subseteq [0, \infty]$, where I is an arbitrary (index) set. Recall that their sum is defined by

$$\sum_{i\in I} a_i = \sup \Big\{ \sum_{i\in K} a_i : \ K \subseteq I, \ K \text{ finite} \Big\}.$$

Now, suppose that $I = \mathbb{N}$. Show that the above definition agrees with the standard one, that is

$$\sum_{i\in I} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Show that the value of the series does not depend on the ordering of the elements in the sequence. That is, if $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, then

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\sigma(i)}.$$

Problem 1.11. Let $\{a_i\}_{i\in I} \subseteq [0, \infty)$, where I is an arbitrary (index) set. Suppose that

$$\sum_{i\in I} a_i < \infty.$$

Show that the set $J_n = \{i \in I : \ a_i > 1/n\}$ is finite. Conclude that the set of $i \in I$ such that $a_i > 0$ is at most countable.

Problem 1.12. Let $\{a_i\}_{i\in I} \subseteq (0, \infty)$ be a family of *positive* real numbers, where I is an (index) set with uncountably many elements. Show that

$$\sum_{i\in I} a_i = \infty.$$

Problem 1.13. (*) Let Ω be a non-empty set and $p_\omega \in [0, 1]$, $\omega \in \Omega$ be real numbers such that

$$\sum_{\omega\in\Omega} p_\omega = 1.$$

Define the set function $\mathbb{P} : 2^\Omega \to [0, 1]$ by

$$\mathbb{P}(A) = \sum_{\omega\in A} p_\omega.$$

Show that $\mathbb{P}$ is a measure on 2^Ω.

Problem 1.14. (*) Let $A \subset \mathbb{R}$ be an open set. Show that A is the union of at most countable many intervals. (*Hint:* define for all $x \in A$ the interval $I_x = \bigcup_{I \text{ interval}: x \in I \subseteq A} I$ to be the largest interval contained in A containing x)

Problem 1.15. Let I and J be two index sets and $a_{i,j}$, $i \in I$ and $j \in J$ be non-negative real numbers. Show that

$$\sum_{i \in I} \sum_{j \in J} a_{i,j} = \sum_{j \in J} \sum_{i \in I} a_{i,j}.$$

2. Measurable sets and σ-algebras

Problem 2.1. Show that there is no σ-algebra with an odd number of elements.

Problem 2.2. Let $(\Omega, \mathscr{F})$ be a measurable space and $A, B \in \mathscr{F}$. Show, starting from the definition of σ-algebra, that $A \cup B$, $A \cap B$, $A \setminus B$, and $A \Delta B$ all belong to $\mathscr{F}$.

Problem 2.3. Let $(\Omega, \mathscr{F})$ be a measurable space and $A_1, A_2, \ldots$ be a sequence of sets in $\mathscr{F}$. Define the following sets

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m.$$

Show that:

- $(\limsup_{n\to\infty} A_n)^c = \liminf_{n\to\infty} A_n^c$,
- $\liminf_{n\to\infty} A_n \in \mathscr{F}$ and $\limsup_{n\to\infty} A_n \in \mathscr{F}$,
- $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.
- $\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}$.
- $\liminf_{n\to\infty} A_n = \{\omega \in \Omega : \exists m \in \mathbb{N} \text{ such that } \omega \in A_n \text{ for all } n \geq m\}$.

Problem 2.4. Let Ω, E be non-empty, $\mathscr{G}$ a σ-algebra on E and $f : \Omega \to E$. Show that

$$\mathscr{F} = \{f^{-1}(B) : B \in \mathscr{G}\},$$

is a σ-algebra on Ω.

Problem 2.5. Let $(\Omega, \mathscr{F})$ be a measurable space and $(A_n)_{n\in\mathbb{N}}$ a collection of sets in $\mathscr{F}$. Show that:

- There are $(E_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ mutually disjoint such that $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} E_n$.
- There are $(F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ such that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} F_n$.

Problem 2.6. Let Ω be a non-empty set and $\mathscr{F}$ a non-empty collection of subsets of Ω which is closed under taking complements and finite unions (such a collection is called an *algebra*). Show that $\mathscr{F}$ is a σ-algebra if and only if it is closed under countable increasing unions (i.e., if $\{A_n\} \subseteq \mathscr{F}$ and $A_1 \subseteq A_2 \subseteq \ldots$, then $\cup_{n\in\mathbb{N}} A_n \in \mathscr{F}$).

Problem 2.7. (Restriction of σ-algebra) Let $\mathscr{F}$ be a σ-algebra of subsets of Ω. Suppose that $A \subseteq \Omega$ is non-empty. Show that

$$\mathscr{F}_A = \{B \cap A : B \in \mathscr{F}\}$$

is a σ-algebra on A.

Problem 2.8. (Extension of σ-algebra) Let $(\Omega, \mathscr{F})$ be a measurable space, and let K be some non-empty set such that $\Omega \cap K = \varnothing$. Define $\overline{\Omega} = \Omega \cup K$ and $\overline{\mathscr{F}} = \sigma(\mathscr{F} \cup K)$ be a σ-algebra on $\overline{\Omega}$. Show that $\overline{\mathscr{F}} = \{A \subseteq \overline{\Omega} : A \cap \Omega \in \mathscr{F}\}$.

Problem 2.9. Let Ω be a infinite non-empty set.

- Define the collection of sets $\mathscr{F} = \{A \subseteq \Omega : A \text{ is countable or } \Omega \setminus A \text{ is countable}\}$. Is $\mathscr{F}$ a σ-algebra? Prove or disprove.
- Define the collection of sets $\mathscr{F} = \{A \subseteq \Omega : A \text{ is finite or } \Omega \setminus A \text{ is finite}\}$. Is $\mathscr{F}$ a σ-algebra? Prove or disprove.

Problem 2.10. Let $\mathscr{F}$ and $\mathscr{G}$ be σ-algebras on Ω. Show that $\mathscr{F} \cap \mathscr{G}$ is a σ-algebra. Prove or disprove whether $\mathscr{F} \cup \mathscr{G}$ is in general a σ-algebra.

Problem 2.11. Let $\mathscr{F}_n$, $n \in \mathbb{N}$ be σ-algebras on Ω such that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ for all $n \in \mathbb{N}$ (such a sequence $\{\mathscr{F}_n\}$ is called a *filtration*).

- Show that $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ is an algebra.
- Is $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ a σ-algebra? Consider $\Omega = \mathbb{N}$ and $\mathscr{F}_n = \sigma(\{A : A \subseteq \mathbb{N} \cap \{1, \ldots n\}\})$.

Problem 2.12. Let $\mathscr{E} \subseteq \mathscr{A}$ be two collections of sets. Show that $\sigma(\mathscr{E}) \subseteq \sigma(\mathscr{A})$.

Problem 2.13. (Product sigma algebra) Let Ω_1 and Ω_2 be two non-empty sets, and let $\mathscr{F}_1$ and $\mathscr{F}_2$ be σ-algebras on Ω_1 and Ω_2 respectively. Consider the *product σ-algebra* on $\Omega_1 \times \Omega_2$

$$\mathscr{F}_1 \otimes \mathscr{F}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}).$$

Suppose that $\mathscr{F}_1$ is generated by $\mathscr{A}_1$ and $\mathscr{F}_2$ is generated by $\mathscr{A}_2$. Show that $\mathscr{F}_1 \otimes \mathscr{F}_2$ is generated by $A_1 \times A_2$ with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$.

Problem 2.14. Show that the Borel σ-algebra $\mathscr{B}_{\mathbb{R}^2}$ equals $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$.

Problem 2.15. Show that the Borel σ-algebra on $\mathbb{R}$ is generated by each of the following:

i. the open intervals: $\mathscr{A}_1 = \{(a, b) : a < b\}$,

ii. the closed intervals: $\mathscr{A}_2 = \{[a, b] : a < b\}$,

iii. the half open intervals $\mathscr{A}_3 = \{[a, b) : a < b\}$ or $\mathscr{A}_4 = \{(a, b] : a < b\}$,

iv. the open rays: $\mathscr{A}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathscr{A}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$,

v. the closed rays: $\mathscr{A}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathscr{A}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

Problem 2.16. Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. Also recall that

$$\mathscr{B}_{\overline{\mathscr{B}}} = \{A \subseteq \overline{\mathscr{B}} : A \cap \mathbb{R} \in \mathscr{B}_{\mathbb{R}}\}$$

is a σ-algebra on $\overline{\mathbb{R}}$. Show that $\mathscr{B}_{\overline{\mathscr{B}}}$ is generated by the family of closed rays $\mathscr{A} = \{[-\infty, a] : a \in \mathbb{R}\}$.

Problem 2.17. Let $\mathscr{F}$ be an infinite σ-algebra.

- ▶ Show that $\mathscr{F}$ contains an infinite sequence of disjoint sets.
- ▶ (*) Show that $\mathrm{Card}(\mathscr{F}) \geq \mathrm{Card}([0, 1])$.
  (*Hint:* think about binary representation of numbers in $[0, 1]$).

Problem 2.18. Show that Λ is a λ-system on Ω if and only if

I. $\Omega \in \Lambda$,

II. if $A, B \in \Lambda$ and $A \subseteq B$, then $B \setminus A \in \Lambda$,

III. if $A_1, A_2, \ldots$ is a sequence of subsets in Λ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\bigcup_{n \in \mathbb{N}} A_n \in \Lambda.$$

Problem 2.19. Let Λ be a λ-system. Show that $\varnothing \in \Lambda$.

Problem 2.20. Let $\mathscr{A}$ be both a λ-system and a π-system. Show that $\mathscr{A}$ is a σ-algebra.

3. Measures

Problem 3.1. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that for all $E, F \in \mathscr{F}$ such that $E \subseteq F$, one has $\mu(E) \leq \mu(F)$.

Problem 3.2. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that for all $E, F \in \mathscr{F}$ then

$$\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F).$$

Conclude that $\mu(E \cup F) \leq \mu(E) + \mu(F)$.

Problem 3.3. Let $(\Omega, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, \infty]$ be a set function. Assume that $\mu(\varnothing) = 0$, μ is finitely additive and continuous from below. Show that μ is a measure.

Problem 3.4. Consider a non-empty uncountable set Ω and the σ-algebra

$$\mathscr{F} = \{A \subseteq \Omega : \ A \text{ is countable or } \Omega \setminus A \text{ is countable}\}.$$

We define the set function $\mu : \mathscr{F} \to [0, \infty]$ by $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ if $\Omega \setminus E$ is countable. Show that μ is a measure on $(\Omega, \mathscr{F})$.

Problem 3.5. Let Ω be an infinite set and $\mathscr{F} = 2^\Omega$. Define μ on $\mathscr{F}$ by $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is not finite. Show that μ is finitely additive but not a measure.

Problem 3.6. Let $(\Omega, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, 1]$ be an additive set function, that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathscr{F}$ such that $A \cap B = \varnothing$. Show that $\mu(\varnothing) = 0$.

Problem 3.7. (Inclusion-exclusion) Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Let $A_1, A_2, A_3 \in \mathscr{F}$. Show that

$$\begin{aligned} \mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) &+ \mu(A_2) + \mu(A_3) \\ &- \mu(A_1 \cap A_2) - \mu(A_2 \cap A_3) - \mu(A_3 \cap A_1) + \mu(A_1 \cap A_2 \cap A_3). \end{aligned}$$

Let $A_1, \dots, A_n \in \mathscr{F}$. Show that

$$\mu\Big(\bigcup_{i=1}^n A_i\Big) = \sum_{j=1}^n (-1)^{j-1} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = j}} \mu\Big(\bigcap_{i \in I} A_i\Big).$$

Problem 3.8. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space.

- If $E, F \in \mathscr{F}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

► Define $\rho(E,F) = \mu(E\Delta F)$ for all $E, F \in \mathscr{F}$. Show that $\rho(E,F) \leq \rho(E,G) + \rho(G,F)$ for all $E, F, G \in \mathscr{F}$.

Problem 3.9. If $\mu_1, \ldots, \mu_n$ are measures on a measurable space $(\Omega, \mathscr{F})$, and $a_1, \ldots, a_n$ are non-negative real numbers, then $\mu := \sum_{i=1}^n a_i \mu_i$ is a measure on $\mathscr{F}$. Moreover μ is σ-finite if μ_i is σ-finite for all $i = 1, \ldots, n$.

Problem 3.10. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $\mathscr{G} \subseteq \mathscr{F}$ be a σ-algebra. Show that $(\Omega, \mathscr{G}, \mu)$ is a measure space.

Problem 3.11. Let μ be a finite measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty, x]), \quad \forall x \in \mathbb{R}.$$

Show that F is non-decreasing, right-continuous, and

$$\lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to\infty} F(x) = \mu(\mathbb{R}).$$

Problem 3.12. Find a measure space $(\Omega, \mathscr{F}, \mu)$ and a decreasing sequence $B_1 \supseteq B_2 \supseteq \ldots \in \mathscr{F}$ such that $\lim_{n\to\infty} \mu(B_n) > \mu(\cap_{n\in\mathbb{N}} B_n)$.

Problem 3.13. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $A \in \mathscr{F}$. Show that the set function $\mu_A : \mathscr{F} \to [0, \infty]$ defined by $\mu_A(B) := \mu(B \cap A)$ is a measure on $(\Omega, \mathscr{F})$.

Problem 3.14. Let $(\Omega, \mathscr{F})$ be a measure space $\mu : \mathscr{F} \to [0, \infty]$ be a set function which is finitely additive and such that $\mu(\varnothing) = 0$. Show that μ is a measure on $(\Omega, \mathscr{F})$ if and only if μ is continuous from below.

Problem 3.15. Let $(\Omega, \mathscr{F})$ be a measure space $\mu : \mathscr{F} \to [0, \infty]$ be a set function which is finitely additive and such that $\mu(\Omega) < \infty$. Show that μ is a measure on $(\Omega, \mathscr{F})$ if and only if μ is continuous from above.

Problem 3.16. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$. Recall that

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m.$$

Show that

$$\limsup_{n\to\infty} \mu(A_n) \leq \mu\Big(\limsup_{n\to\infty} A_n\Big), \qquad \liminf_{n\to\infty} \mu(A_n) \geq \mu\Big(\liminf_{n\to\infty} A_n\Big).$$

Problem 3.17. Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space. Let $\mathscr{E} \subseteq \mathscr{F}$ be a π-system such that there exists $E_1, E_2, \ldots \in \mathscr{E}$ such that $E_n \uparrow \Omega$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Show that μ is uniquely determined by its values on $\mathscr{E}$.

Problem 3.18. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function. Let ν_F be the unique measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ associated to F. Show that $\nu_F(\{x\}) = F(x) - F(x-)$ where we define

$$F(x-) := \lim_{y\uparrow x} F(y).$$

Conclude that it if F is continuous, then $\nu_F(\mathbb{Q}) = 0$.

Problem 3.19. Let λ be the unique measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ such that $\lambda((a,b]) = b - a$ for all $a < b$. Show that λ is translation invariant, that is $\lambda(A + x) = \lambda(A)$ for all $A \in \mathscr{B}_{\mathbb{R}}$ and all $x \in \mathbb{R}$, where we write $A + x := \{a + x : a \in A\}$ for the translation of A by x. (*Hint:* a solution can be obtained with the $\pi - \lambda$ theorem.)

Problem 3.20. Let λ be the unique measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ such that $\lambda((a,b]) = b-a$ for all $a < b$. Show that $\lambda(\tau A) = |\tau|\lambda(A)$ for all $A \in \mathscr{B}_{\mathbb{R}}$ and all $\tau \neq 0$, where we write $\tau A := \{\tau a : a \in A\}$ for the dilation of A by τ.

Problem 3.21. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function. Let ν_F be the unique measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ such that $\nu_F((a,b]) = F(b) - F(a)$ for all $a < b$. Show that ν_F is σ-finite.

Problem 3.22. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. Let $(A_i)_{i \in \mathbb{N}} \subseteq \mathscr{F}$ be a sequence such that $\mathbb{P}(A_i) = 1$ for all $i \in \mathbb{N}$. Show that $\mathbb{P}(\cap_{i \in \mathbb{N}} A_i) = 1$.

Problem 3.23. Let μ be a finite measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Show that the set $\{x \in \mathbb{R} : \mu(\{x\}) > 0\}$ is at most countable.

4. Null sets, completion and independence

Problem 4.1. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Suppose that $A, N \in \mathscr{F}$ and $\mu(N) = 0$. Show that $\mu(A \cup N) = \mu(A)$.

Problem 4.2. Suppose that $(\Omega, \mathscr{F}, \mu)$ is a measure space and let N be a null set. Show that for all $M \subseteq N$, M is a null set.

Problem 4.3. Let $(N_n)_{n\in\mathbb{N}}$ be null sets in a measure space $(\Omega, \mathscr{F}, \mu)$. Show that $\cup_{n\in\mathbb{N}} N_n$ is a null set.

Problem 4.4. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of function. Show that $f_n = 0$ almost everywhere for all $n \in \mathbb{N}$ if and only if almost everywhere $f_n = 0$ for all $n \in \mathbb{N}$. (Careful with the quantifiers!)

Problem 4.5. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability, we say that $A \in \mathscr{F}$ happens almost surely if $\Omega \setminus A$ is a null set for $\mathbb{P}$.

- Show that $A \in \mathscr{F}$ happens almost surely if and only if $\mathbb{P}(A) = 1$.
- Assume now that $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ is such that A_n happens almost surely for all $n \in \mathbb{N}$. Show that $\mathbb{P}(\cap_{n\in\mathbb{N}} A_n) = 1$.

Problem 4.6. Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set and let V be the Vitali set. Show that if $E \subseteq V$, then E is a null-set.

Problem 4.7. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $(\Omega, \overline{\mathscr{F}}, \overline{\mu})$ its completion. Show that $\overline{A} \in \overline{\mathscr{F}}$ if and only if there is $A \in \mathscr{F}$ such that $A \Delta \overline{A}$ is a null set.

Problem 4.8. (*) Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $\mathscr{L}(E) > 0$. Show that there exists $N \subseteq E$ not Lebesgue measurable. (Hint: assume first $E \subseteq (0, 1)$ and look at $V \cap E$ where V is the Vitali set.)

Problem 4.9. (The Cantor set) The Lebesgue null sets include not only the countable sets but also many sets having the cardinality of the continuum. The Cantor set C is the set of all $x \in [0, 1]$ that have a base-3 expansion

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \quad \text{with } a_j \in \{0, 2\} \text{ for all } j \in \mathbb{N}.$$

Thus C is obtained from $[0, 1]$ by removing the open middle third $(1/3, 2/3)$, then removing the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining intervals and so forth. Show that

- C is compact and with zero Lebesgue measure.

- $\mathrm{Card}(C) = \mathrm{Card}([0,1])$. Hint: consider the so called Cantor function, for $x \in C$, $x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$, define
$$f(x) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}, \quad b_j = a_j/2.$$
- (*) Show that C has empty interior and is totally disconnected (that is for all $x < y \in C$ there is $z \in (x, y)$ such that $z \notin C$). Moreover C has no isolated points.

Problem 4.10. Show that for any Lebesgue measurable set $E \subseteq \mathbb{R}$ and any real number $\lambda \in \mathbb{R}$, $\mathscr{L}(E + \lambda) = \mathscr{L}(E)$ and $\mathscr{L}(\lambda E) = |\lambda|\mathscr{L}(E)$.

Problem 4.11. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. We say that two σ-algebras $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{F}$ are independent if
$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \quad \forall A_1 \in \mathscr{A}_1, \forall A_2 \in \mathscr{A}_2.$$
Suppose that $\mathscr{E}_1, \mathscr{E}_2$ are π-systems generating $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. Show that $\mathscr{A}_1$ and $\mathscr{A}_2$ are independent if and only if
$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \quad \forall A_1 \in \mathscr{E}_1, \forall A_2 \in \mathscr{E}_2.$$

Problem 4.12. Are the following true of false?

- If A is an open subset of $[0,1]$, then $\mathscr{L}^1(A) = \mathscr{L}^1(\overline{A})$, where $\overline{A}$ is the closure of the set.
- If A is a subset of $[0,1]$ such that $\mathscr{L}^1(\mathrm{int}(A)) = \mathscr{L}^1(\overline{A})$, then A is measurable. Here $\mathrm{int}(A)$ denotes the interior of the set A.

Problem 4.13. Show that if $A \subset [0,1]$ and $\mathscr{L}^1(A) > 0$, then there are x and y in A such that $|x - y|$ is an irrational number.

5. Measurable functions

Problem 5.1. Let $(\Omega, \mathscr{F})$ be a measure space. Let $A \subseteq \Omega$ and consider the indicator function $\mathbb{1}_A : \Omega \to \mathbb{R}$, defined by

$$\mathbb{1}_A(\omega) = 1, \text{ if } \omega \in A, \quad \mathbb{1}_A(\omega) = 0, \text{ if } \omega \notin A.$$

Show that $\mathbb{1}_A$ is $(\mathscr{F}, \mathscr{B})$-measurable if and only if $A \in \mathscr{F}$.

Problem 5.2. Let $(\Omega, \mathscr{F})$ be a measure space and $A, B \in \mathscr{F}$. Show that

$$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B}, \qquad \mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B.$$

Problem 5.3. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and $X, Y : \Omega \to \mathbb{R}$ be random variables. Recall that X and Y are called independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B), \qquad \forall A \in \sigma(X), \forall B \in \sigma(Y).$$

Show that X and Y are independent if and only if

$$\mathbb{P}(X \geq s, Y \geq t) = \mathbb{P}(X \geq s)\mathbb{P}(Y \geq t), \qquad \forall s, t \in \mathbb{R}.$$

(Hint: you might need the $\pi - \lambda$ Theorem.)

Problem 5.4. Let $(\Omega, \mathscr{F})$ be a measurable space and $f, g : \Omega \to \mathbb{R}$ be $\mathscr{F}$-measurable functions. Show that $\max(f, g)$, $\min(f, g)$ and $|f|$ are measurable.

Problem 5.5. Let $(\Omega, \mathscr{F})$ be a measurable space and $E_1, \ldots, E_n \in \mathscr{F}$ be measurable sets. Show that if $a_1, \ldots, a_n$ are real numbers, then

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}$$

is $\mathscr{F}$-measurable.

Problem 5.6. Let $(\Omega, \mathscr{F})$ be a measurable space, $f : \Omega \to \overline{\mathbb{R}}$, and $Y = f^{-1}(\mathbb{R})$. Show that f is measurable if and only if $Y \in \mathscr{F}$, $f^{-1}(\{-\infty\}) \in \mathscr{F}$, $f^{-1}(\{\infty\}) \in \mathscr{F}$, and f is measurable when restricted to Y.

Problem 5.7. Let $(\Omega, \mathscr{F})$ be a measurable space and f_i, $i \in \mathbb{N}$ be measurable functions from $(\Omega, \mathscr{F})$ to $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Show that the set

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) = 0\}$$

is measurable, that is, it belongs to $\mathscr{F}$. (Hint: recall $\lim_{n\to\infty} f_n(\omega) = 0$ means that for all $m > 0$ there exists $N > 0$ such that $|f_n(\omega)| \leq 1/m$ for all $n \geq N$. Can you now write the set as countable union and intersections of measurable sets.)

Problem 5.8. Let $(\Omega, \mathscr{F})$ be a measurable space and $f, g : \Omega \to \mathbb{R}$ be two measurable functions. Show that $\{\omega \in \mathscr{F} : f(\omega) = g(\omega)\}$ is a measurable set.

Problem 5.9. Let $(\Omega, \mathscr{F})$ be a measurable space and (f_n) be a sequence of $\mathscr{F}$-measurable functions. Show that the set

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) \text{ exists}\}$$

is measurable.

Problem 5.10. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Show that f is Lebesgue measurable if and only if there exists a Borel measurable function g such that $f \equiv g$ almost everywhere.

Problem 5.11. Show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel-measurable.

Problem 5.12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an homeomorphism, that is f is continuous with continuous inverse. Show that f maps Borel measurable sets to Borel measurable sets.

Problem 5.13. Let $(\Omega, \mathscr{F}, \mu)$ be a complete measure space. Show that

- if f is measurable and $f \equiv g$ μ-a.e. then g is measurable.
- if $f_n, n \in \mathbb{N}$ are measurable and $f_n \to f$ μ-a.e. then f is measurable.

Problem 5.14. Let $(\Omega, \mathscr{F})$ be a measurable space and $f : \Omega \to \overline{\mathbb{R}}$ be a function. Define $f^+(\omega) = \max(f(\omega), 0)$ and $f^-(\omega) = -\min(f(\omega), 0)$. Show that f is measurable if and only if f^+ and f^- are measurable.

Problem 5.15. Let $(\mathscr{X}, \mathscr{F})$ and $(\mathscr{Y}, \mathscr{G})$ be two measurable spaces. Let $E \in \mathscr{F} \otimes \mathscr{G}$, and for $y \in \mathscr{Y}$ define the y-section of E to be

$$E_y = \{x \in \mathscr{X} : (x, y) \in E\},$$

Similarly define for $x \in \mathscr{X}$ the x-section of E to be

$$E_x = \{y \in \mathscr{Y} : (x, y) \in E\}.$$

Show that E_y is measurable for all $y \in \mathscr{Y}$.

Problem 5.16. Let $(\Omega, \mathscr{F})$ and $(E, \mathscr{G})$ be measurable spaces, and let $f : \Omega \to E$ be a function. Suppose that $A \in \mathscr{F}$, we say that f is measurable on A if $f^{-1}(B) \cap A \in \mathscr{F}$ for all $B \in \mathscr{G}$. Show that f is measurable on $A \in \mathscr{F}$ if and only if $f|_A$ is $(\mathscr{F}_A, \mathscr{G})$-measurable, where $\mathscr{F}_A = \{A \cap B : B \in \mathscr{F}\}$. Show that if f is $(\mathscr{F}, \mathscr{G})$-measurable, then f is measurable on A for all $A \in \mathscr{F}$.

Problem 5.17. Let $(\Omega, \mathscr{F})$ be a measurable space, $f : \Omega \to \overline{\mathbb{R}}$ and $\mathscr{Y} = f^{-1}(\mathbb{R})$. Then f is measurable if and only if $f^{-1}(\{-\infty\}) \in \mathscr{F}$, $f^{-1}(\{\infty\}) \in \mathscr{F}$, and f is measurable when restricted on $\mathscr{Y}$.

Problem 5.18. If a function $Y_A : A \to E$ is $(\mathscr{F}_A, \mathscr{G})$-measurable, and $p \in E$, then the extension Y defined by

$$Y(\omega) := \begin{cases} Y_A(\omega), & \omega \in A, \\ p, & \omega \notin A, \end{cases}$$

is $(\mathscr{F}, \mathscr{G})$-measurable.

Problem 5.19. Does there exist a non-measurable function $f \geq 0$ such that $\sqrt{f}$ is measurable?

Problem 5.20. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous almost everywhere, then f is Lebesgue-measurable.

Problem 5.21. Is the following true of false? If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is measurable.

Problem 5.22. (Convergence in measure/probability) Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $X_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$ be random variables. We say that X_n converges in probability to a random variable X, if for all $\epsilon > 0$

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0, \qquad \text{as } n \to \infty.$$

Show that if X_n converges to a random variable X in probability, then there exists a subsequence of $(X_n)_{n\in\mathbb{N}}$ which converges almost surely to X. (*Hint: use Borell-Cantelli Lemma.)*

Part II.

Solutions

6. Solutions: Warming up

Solution to Problem 1.1 *(by Kempen, S.F.M.)*
a) To be proven: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
" $\subseteq$ " Let $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$, which means $x \in A$ and ($x \in B$ or $x \in C$).

- If $x \in A$ and $x \in B$, then $x \in A \cap B$ so also $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in A$ and $x \in C$, then $x \in A \cap C$ so also $x \in (A \cap B) \cup (A \cap C)$.

" $\supseteq$ " Let $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap B$ or $x \in A \cap C$.

- If $x \in A \cap B$, then $x \in A$ and $x \in B$ so $x \in B \cup C$ so also $x \in A \cap (B \cup C)$.
- If $x \in A \cap C$, then $x \in A$ and $x \in C$ so $x \in B \cup C$ so also $x \in A \cap (B \cup C)$.

b) To be proven: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
" $\subseteq$ " Let $x \in A \cup (B \cap C)$ then $x \in A$ or $x \in B \cap C$, which means $x \in A$ or ($x \in B$ and $x \in C$).

- If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ so also $x \in (A \cup B) \cap (A \cup C)$.
- If $x \in B$ and $x \in C$, then $x \in A \cup B$ and $x \in A \cup C$ so also $x \in (A \cup B) \cap (A \cup C)$.

" $\supseteq$ " Let $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup B$ and $x \in A \cup C$, which means $(x \in A$ or $x \in B)$ and ($x \in A$ or $x \in C$).

- If $x \in A$ then definitely $x \in A \cup (B \cap C)$.
- If $x \notin A$ then $x \in B$ and $x \in C$ which means $x \in B \cap C$ so also $x \in A \cup (B \cap C)$.

😄

Solution to Problem 1.2 *(by Kempen, S.F.M.)* To be proven: $A \cap (A \cup B) = A$.
" $\subseteq$ " Let $x \in A \cap (A \cup B)$, then $x \in A$ so we are done.
" $\supseteq$ " Let $x \in A$, then also ($x \in A$ or $x \in B$) is true, therefore $x \in A \cup B$. So $x \in A \cap (A \cup B)$. 😄

Solution to Problem 1.5 *(by Kempen, S.F.M.)* To be proven: $\Omega \setminus \left(\bigcup_{i \in I} A_i\right) = \bigcap_{i \in I} \Omega \setminus A_i$.
" $\subseteq$ " Let $x \in \Omega \setminus \left(\bigcup_{i \in I} A_i\right)$ then $x \in \Omega$ and $x \notin \bigcup_{i \in I} A_i$, so for all $i \in I$ holds $x \notin A_i$. Then for all $i \in I$ we have $x \in \Omega \setminus A_i$. Since this is true for any $i \in I$, we can write $x \in \bigcap_{i \in I} \Omega \setminus A_i$.
" $\supseteq$ " Let $x \in \bigcap_{i \in I} \Omega \setminus A_i$ then for all $i \in I$ we have $x \in \Omega \setminus A_i$, so $x \in \Omega$ and $x \notin A_i$. Since this holds for all $i \in I$, we can write $x \notin \bigcup_{i \in I} A_i$ and therefore $x \in \Omega \setminus \left(\bigcup_{i \in I} A_i\right)$. 😄

Solution to Problem 1.8 *(by Kempen, S.F.M.)*
a) The statement $f^{-1}(f(A)) = A$ is not true since f is not assumed to be injective. As a counterexample, take $\Omega = \{0,1\}, E = \{0\}, f(\{0\}) = f(\{1\}) = \{0\}, A = \{0\}$ then $f(A) = \{0\}$ and $f^{-1}(f(A)) = f^{-1}(\{0\}) = \{0,1\} \neq A$.
b) The statement $f(f^{-1}(H)) = H$ is not true since f is not assumed to be surjective. As a counterexample, take $\Omega = \{1\}$, $E = \{1,2\}$, $f(\{1\}) = \{1\}$, $H = \{1,2\}$ then $f^{-1}(H) = f^{-1}(\{1,2\}) = \{1\}$ and $f(f^{-1}(H)) = f(\{1\}) = \{1\} \neq H$. 😄

Solution to Problem 1.10 *(by Beurskens, T.P.J.)* Let $n \in \mathbb{N}$, and define $K = \{1,\dots,n\}$. By definition of the supremum, we then have

$$\sum_{i=1}^{n} a_i = \sum_{i\in K} a_i \leq \sup\left(\sum_{i\in K} a_i : K \subseteq I, K \text{ finite}\right) = \sum_{i\in I} a_i.$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i \leq \sum_{i\in I} a_i.$$

Next, let $K \subseteq \mathbb{N}$ be finite, so that $K \subset \{1,\dots,n\}$ for some $n \in \mathbb{N}$. Note that $n \geq \sup K$. We get

$$\sum_{i\in K} a_i \leq \sum_{i\in\{1,\dots,n\}} a_i = \sum_{i=1}^{n} a_i \leq \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Since this holds for arbitrary finite K, it holds for all finite K. Thus we get

$$\sum_{i\in I} a_i = \sup\left(\sum_{i\in K} a_i : K \subseteq I, K \text{ finite}\right) \leq \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Using both inequalities, we see that indeed $\sum_{i\in I} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.

We are left with showing that the sum $\sum_{i=1}^{\infty} a_i$ does not depend on the ordering of the elements in the sequence (a_i). This follows immediately from the fact that for any index set I

$$\sum_{i\in I} a_i = \sup\Big\{\sum_{i\in K} a_i \,:\, K \subseteq I,\ K \text{ finite}\Big\}$$

is completely blind to any ordering of I, in fact I is possibly not even ordered. To be more precise, if $\sigma : I \to I$ is a bijection, then

$$\begin{aligned}\sum_{i\in I} a_{\sigma(i)} &= \sup\Big\{\sum_{i\in K} a_{\sigma(i)} \,:\, K \subseteq I,\ K \text{ finite}\Big\}\\ &= \sup\Big\{\sum_{i\in\sigma^{-1}(K)} a_i \,:\, \sigma^{-1}(K) \subseteq I,\ \sigma^{-1}(K) \text{ finite}\Big\} = \sum_{i\in I} a_i,\end{aligned}$$

where we used that $K \subseteq I$ is finite if and only if $\sigma^{-1}(K)$ is finite. So, from this observation and the first part of the problem, one has that for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$

$$\sum_{i=1}^{\infty} a_i = \sum_{i\in\mathbb{N}} a_i = \sum_{i\in\mathbb{N}} a_{\sigma(i)} = \sum_{i=1}^{\infty} a_{\sigma(i)},$$

where the summation in the middle is with respect the new notion. 😃

Solution to Problem 1.12 *(by Bakker, A.)* Proof by contradiction. Suppose $\sum_{i \in I} a_i < \infty$, then by Problem 1.11 we have that the set I contains at most a countable number of elements i with a_i positive. This, together with the fact that $a_i > 0$ for all $i \in I$, contradicts that there are uncountable many elements in I. Hence $\sum_{i \in I} a_i = \infty$. 😃

7. Solutions: Measurable sets and σ-algebras

Solution to Problem 2.2 *(by Castella, A.)* By the definition of a σ-algebra, all countable unions of sets in $\mathscr{F}$ are also in $\mathscr{F}$. Thus let create a sequence $(A_n)_{n\in\mathbb{N}}$ such that $A_1 = A$ and $A_i = B$ for all $i > 1$. Clearly

$$\bigcup_{i=1}^{\infty} A_i = A \cup B \in \mathscr{F}.$$

Since A and B are arbitrary, this must hold for all pairs of sets in $\mathscr{F}$. From the fact that $\mathscr{F}$ is a σ-algebra we find that $A^c, B^c \in \mathscr{F}$. As we have proven that unions of pairs are in $\mathscr{F}$, we find that

$$(A^c \cup B^c)^c = A \cap B \in \mathscr{F}.$$

This again holds for all possible pairs of sets in $\mathscr{F}$. Now let us note that

$$A \setminus B = A \cap B^c.$$

It is clear from the properties that we have already proven and the complementation property of σ-algebras, we can conclude that

$$A \setminus B \in \mathscr{F}.$$

The final set, $A\Delta B$ follows by definition. The set is defined by $(A \cup B) \setminus (A \cap B)$. It is clear that this is a composition of the properties we have already proven. Since the sets A and B were arbitrary, we can conclude that

$$A\Delta B \in \mathscr{F}.$$

☺

Solution to Problem 2.4 *(by Castella, A.)* A σ-algebra $\mathscr{G}$ is defined as a set of subsets such that

- $E \in \mathscr{G}$,
- for all infinite sequences $\{A_i\}$ in $\mathscr{G}$ the union $\bigcup_{i=1}^{\infty} A_i$ also belongs to the set $\mathscr{G}$,
- for all sets A in $\mathscr{G}$, the set A^c belongs to $\mathscr{G}$.

Therefore, we begin by verifying that $\Omega \in \mathscr{F}$. From the definition of f it directly follows that

$$f^{-1}(E) = \Omega.$$

Since $\mathscr{G}$ is a σ-algebra, we find that $E \in \mathscr{G}$. Therefore, we conclude that $\Omega \in \mathscr{F}$.

We now verify the second condition. Let us take an arbitrary infinite sequence $\{A_i\}$ in $\mathscr{F}$. By the definition of $\mathscr{F}$ we know that for all A_i, there exists $B_i \in \mathscr{G}$ such that $A_i = f^{-1}(B_i)$. In order to prove the condition, we notice that

$$\bigcup_{i=1}^{\infty} B_i \in \mathscr{G}.$$

Before we continue with the proof, we need to prove an intermediary. Let us take some family of sets $\{C_\alpha\}_{\alpha \in \mathscr{I}}$ where $\mathscr{I}$ is an index set. We will prove that for some function $g : \Omega \to E$ we have

$$g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right) = \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha).$$

We begin by showing that $g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right) \subset \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha)$. Let us take an arbitrary $a \in g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right)$. Then we find that there exists some $b \in \bigcup_{\alpha \in \mathscr{I}} C_\alpha$ such that $g(a) = b$. This implies that there exists an $\alpha \in \mathscr{I}$ such that $b \in C_\alpha$. From this we find that

$$a \in g^{-1}(C_\alpha) \subset \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha).$$

Since a was chosen arbitrarily, we can conclude that

$$g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right) \subset \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha).$$

Now we show that the converse is also true. Let us take an arbitrary $a' \in \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha)$. Then we find that there must exist some $\alpha' \in \mathscr{I}$ such that $a' \in g^{-1}(C_{\alpha'})$. We can now choose some $b' \in C_{\alpha'}$ such that $g(a') = b'$. It is clear that $b' \in \bigcup_{\alpha \in \mathscr{I}} C_\alpha$ as well. Thus we know that

$$a' \in g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right).$$

Since again, our choice of a' was arbitrary, we can conclude that

$$\bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha) \subset g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right).$$

These two inequalities clearly imply that

$$g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right) = \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha).$$

Using the result that we have just proven we can continue with the proof. For the infinite sequence $\{A_i\}$ we find that

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) = \bigcup_{i=1}^{\infty} A_i.$$

By the definition of our set $\mathscr{F}$ we now find that

$$\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}.$$

Therefore, we conclude that the second condition holds for $\mathscr{F}$ as well.

We now prove the third and final condition. Let us take $A \in \mathscr{F}$. By the definition of $\mathscr{F}$ we find that there exists $B \in \mathscr{G}$ such that

$$A = f^{-1}(B).$$

By the definition of a σ-algebra we know that $B^c \in \mathscr{G}$. Additionally, we know that

$$f^{-1}(B^c) = f^{-1}(B)^c = A^c.$$

By using these two facts and the definition of the set $\mathscr{F}$ again, we find that

$$A^c \in \mathscr{F}.$$

This proves that the third condition holds as well.

Since all of the required conditions hold, we can come to the final conclusion that $\mathscr{F}$ is indeed a σ-algebra. 😄

Solution to Problem 2.5 *(by Castella, A.)*

- Before we begin with the proof, we will prove the intermediary that if $A, B \in \mathscr{F}$, then $A \setminus B \in \mathscr{F}$. We begin by noting that

  $$A \setminus B = A \cap (B^c).$$

  From this it becomes very clear that the statement is true. We know by the definition of a σ-algebra that $B \in \mathscr{F}$ implies $B^c \in \mathscr{F}$. Additionally, we know that intersections of infinite sequences also belong to the same σ-algebra. We take the sequence $A_1 = A$, $A_i = B^c$ for $i \in \mathbb{N} \setminus \{1\}$. With this we find that $A \cap B^c \in \mathscr{F}$ and therefore

  $$A \setminus B \in \mathscr{F}.$$

  We now proceed to showing that for all infinite sequences $(A_n)_{n\in\mathbb{N}}$, there exits a mutually disjoint sequence who's union is equal to $\cup_{n\in\mathbb{N}} A_n$. We define the sequence $(E_n)_{n\in\mathbb{N}}$ such that

  $$E_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i.$$

  By the intermediary and since infinite unions, and by the same argument as in the intermediary, also finite unions are contained in the σ-algebra, it is easy to see that

  $$E_n \in \mathscr{F}$$

  for all $n \in \mathbb{N}$. It is clear from the definition of the sequence that it is mutually disjoint and that its union is equal to the union of $(A_n)_{n\in\mathbb{N}}$.

- We take the same arbitrary sequence $(A_n)_{n\in\mathbb{N}}$ as in the previous item. We define the sequence $(F_n)_{n\in\mathbb{N}}$ by
$$F_n = A_n \cup \left(\bigcup_{i=1}^{n-1} A_i\right) = \bigcup_{i=1}^{n} A_i.$$
We first note that it is clear from the definition that this is an increasing sequence, as each element is the union of A_n and all of its predecessors. As mentioned in the previous item, infinite unions are contained in the σ-algebra as well as finite unions. Therefore we know that
$$F_n \in \mathscr{F}$$
for all $n \in \mathbb{N}$. From the definition of a union we also know that
$$\bigcup_{i=1}^{\infty}\left(\bigcup_{j=1}^{i} A_j\right) = \bigcup_{i=1}^{\infty} A_n.$$
Thus we have now proven that the sequence $(F_n)_{n\in\mathbb{N}}$ is such that $F_n \subset F_{n+1}$ and $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} F_n$.

😄

Solution to Problem 2.13 *(by Beerens, L.)* Let Ω_1 and Ω_2 be two non-empty sets, and let $\mathscr{F}_1$ and $\mathscr{F}_2$ be σ-algebras on Ω_1 and Ω_2 respectively. We consider the product σ-algebra on $\Omega_1 \times \Omega_2$ given by
$$\mathscr{F}_1 \otimes \mathscr{F}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}).$$
Suppose that $\mathscr{F}_1$ is generated by $\mathscr{A}_1$ and $\mathscr{F}_2$ is generated by $\mathscr{A}_2$. Let $\mathscr{F} := \mathscr{F}_1 \otimes \mathscr{F}_2$ and
$$\mathscr{S} := \sigma(\{A_1 \times A_2 : A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2\}).$$
Let
$$A \in \{A_1 \times A_2 : A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2\}.$$
Then $A = A_1 \times A_2$ for some $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. Since $\mathscr{A}_1 \subset \mathscr{F}_1$ and $\mathscr{A}_2 \subset \mathscr{F}_2$, we find that
$$A \in \{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}.$$
Therefore,
$$\{A_1 \times A_2 : A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2\} \subset \{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\},$$
from which it follows that $\mathscr{S} \subset \mathscr{F}$ (By Problem 2.12).

To prove the converse, suppose that $F \in \{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}$. Then there exist $A_1 \in \mathscr{F}_1$ and $A_2 \in \mathscr{F}_2$ such that $F = A_1 \times A_2 = (A_1 \times \Omega_2) \cap (\Omega_1 \times A_2)$. By definition for all $A \in \mathscr{A}_1$ we have
$$A \times \Omega_2 \in \mathscr{S}.$$

Thus we find that

$$\sigma(\mathscr{A}_1) \times \{\Omega_2\} = \sigma(\mathscr{A}_1 \times \{\Omega_2\}) \subset \mathscr{S}.$$

Since $A_1 \in \mathscr{F}_1 = \sigma(\mathscr{A}_1)$, it now follows that $A_1 \times \Omega_2 \in \mathscr{S}$. Analogously, $\Omega_1 \times A_2 \in \mathscr{S}$. Since $\mathscr{S}$ is a σ-algebra, it follows that $F = (A_1 \times \Omega_2) \cap (\Omega_1 \times A_2) \in \mathscr{S}$. Thus, $\mathscr{F} \subset \mathscr{S}$, from which we can conclude that $\mathscr{F} = \mathscr{S}$, as was to be shown. 😄

Solution to Problem 2.14 *(by Castella, A.)* In order to prove equality of the sets, we will prove that they are both subsets of one another.

- As we know, the set $\mathscr{B}_{\mathbb{R}^2}$ is generated by the π-system of open rectangles. Let us assume that $\mathscr{A}$ is the set of open rectangles in $\mathbb{R}^2$. Let us take some arbitrary $A \in \mathscr{A}$, then there exists $a, b, c, d \in \mathbb{R}$ such that $a < b$, $c < d$, and $A = (a, b) \times (c, d)$. Since the Borel set $\mathscr{B}_{\mathbb{R}}$ is generated by the set of open intervals in $\mathbb{R}$, we find that $(a, b), (c, d) \in \mathscr{B}_{\mathbb{R}}$. From this we immediately find that $(a, b) \times (c, d) \in \mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}}$. Since $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$ is the σ-algebra generated by the cross product $\mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}}$ we find that

$$\mathscr{B}_{\mathbb{R}^2} \subset \mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}.$$

- We now prove the converse. Let us begin by defining the projections π_1 and π_2 as

$$\pi_1(x, y) = x$$

and

$$\pi_2(x, y) = y.$$

We will show that these two functions are measurable with respect to $\mathscr{B}_{\mathbb{R}^2}$ and $\mathscr{B}_{\mathbb{R}}$. Let us take an arbitrary $t \in \mathbb{R}$. We will show that the set

$$A = \{(x \times y) \in \mathbb{R}^2 : \pi_1(x, y) < t\}$$

is a Borel set. We find that $\pi_1(x, y) < t$ if and only if $x < t$. Thus we find that

$$A = (-\infty, t) \times \mathbb{R}.$$

Let us define the set B_n as

$$B_n = (-n, t) \times (-n, n)$$

Clearly, for all $n \in \mathbb{N}$, the set B_n is an open rectangle and therefore $B_n \in \mathscr{B}_{\mathbb{R}^2}$. We also find that

$$\bigcup_{i=1}^{\infty} B_n = (-\infty, t) \times \mathbb{R} = A.$$

Since $\mathscr{B}_{\mathbb{R}^2}$ is a σ-algebra, it contains all countable unions of its sets. Therefore, we find that

$$\bigcup_{i=1}^{\infty} B_n = A \in \mathscr{B}_{\mathbb{R}^2}.$$

Thus the set A is indeed a Borel measurable set. Since t was chosen arbitrarily, this holds for all $t \in \mathbb{R}$. Thus π_1 is a measurable function. The proof is analogous for π_2. Since both of these functions are measurable, we know that the preimage of a set in $\mathscr{B}_{\mathbb{R}}$ is in $\mathscr{B}_{\mathbb{R}^2}$. Therefore, for all $A \in \mathscr{B}_{\mathbb{R}}$, we find that

$$A \times \mathbb{R} \in \mathscr{B}_{\mathbb{R}^2},$$

by the measurability of π_1 and

$$\mathbb{R} \times A \in \mathscr{B}_{\mathbb{R}^2},$$

by the measurability of π_2. Let us take arbitrary $A, B \in \mathscr{B}_{\mathbb{R}}$. By our previous result and since $\mathscr{B}_{\mathbb{R}^2}$ is a σ-algebra and thus contains countable intersections, we know that

$$(A \times \mathbb{R}) \cap (\mathbb{R} \times B) = A \times B \in \mathscr{B}_{\mathbb{R}^2}.$$

By the fact that A and B were chosen arbitrarily we find that

$$\mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}} \subset \mathscr{B}_{\mathbb{R}^2}.$$

Since $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$ is the σ-algebra generated by the Cartesian product of $\mathscr{B}_{\mathbb{R}}$ with itself, we can conclude that

$$\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}} \subset \mathscr{B}_{\mathbb{R}^2}.$$

Combining both of the results, we arrive at the final conclusion that

$$\mathscr{B}_{\mathbb{R}^2} = \mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}.$$

😄