

TU EINDHOVEN, 2WAG0

Problems in Measure theory



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“What I don’t like about measure theory is that you have to say “almost everywhere” almost everywhere”

– Kurt Friedrichs

Disclaimer:

This is a broad selection of exercises for the course *Measure, integration and probability theory*. They are meant to accompany the lecture notes and give you the opportunity to exercise. If you wish to have your solution checked, send it in \LaTeX , and we will correct and polish it together, so that it can be featured in this notes in the “Solutions” part.

These collection of exercises are still in progress and they might contain small typos. If you see any or if you think that the statement of the problems is not yet crystal clear, feel free to drop a line. The most efficient way is to send an email to me, a.chiarini@tue.nl. All comments and suggestions will be greatly appreciated.

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Part I.

Problems

1. Warming up

In this section we will review some of the basic set operations which will be much needed in the sequel.

Problem 1.1. Let A, B and C be sets, show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Problem 1.2. Let A, B be sets, show that $A \cap (A \cup B) = A$.

Problem 1.3. Let $A, B \subseteq \Omega$. We define the symmetric difference to be

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Show that $A \cup B$ is the disjoint union of $A \Delta B$ and $A \cap B$.

Problem 1.4. Let $A, B \subseteq \Omega$, show that

$$\Omega \setminus (A \cup B) = (\Omega \setminus A) \cap (\Omega \setminus B).$$

Problem 1.5. (De Morgan's law) Let I be any index set and let $\{A_i\}_{i \in I} \subseteq 2^\Omega$ be a family subsets of Ω . Show that

$$\Omega \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} \Omega \setminus A_i.$$

Problem 1.6. Let $f : \Omega \rightarrow E$ be some function. Recall that for any $D \subseteq \Omega$ the *image* of D under f is the set

$$f(D) = \{f(x) : x \in D\},$$

Let $A, B \subseteq \Omega$. Show that

$$\blacktriangleright f(A \cap B) \subseteq f(A) \cap f(B),$$

$$\blacktriangleright f(A \cup B) = f(A) \cup f(B).$$

Find an example where $f(A \cap B) \neq f(A) \cap f(B)$. Is it true that $f(\Omega \setminus A) = E \setminus f(A)$?

Problem 1.7. Let $f : \Omega \rightarrow E$ be some function. Recall that for any $F \subseteq E$ the *inverse image* of F under f is the set

$$f^{-1}(F) = \{x : f(x) \in F\}.$$

Let $H, K \subseteq E$. Show that, taking the inverse image commutes with the set operations:

$$\blacktriangleright f^{-1}(H \cap K) = f^{-1}(H) \cap f^{-1}(K),$$

- $f^{-1}(H \cup K) = f^{-1}(H) \cup f^{-1}(K)$,
- $f^{-1}(E \setminus H) = \Omega \setminus f^{-1}(H)$.

Problem 1.8. Let $f : \Omega \rightarrow E$ be some function.

- Let $A \subseteq \Omega$. Is it true that $f^{-1}(f(A)) = A$? Provide a proof or a counterexample.
- Let $H \subseteq E$. Is it true that $f(f^{-1}(H)) = H$? Provide a proof or a counterexample.

Problem 1.9. Recall that given a set Ω , 2^Ω denotes the set of all subsets of Ω . Suppose $\Omega = \{0, 1\}$, list all the elements of 2^Ω . What is $|2^\Omega|$, where $|\cdot|$ denotes the number of elements of a set? Suppose that $|\Omega| < \infty$, what is $|2^\Omega|$ in this case?

Problem 1.10. Let $\{a_i\}_{i \in I} \subseteq [0, \infty]$, where I is an arbitrary (index) set. Recall that their sum is defined by

$$\sum_{i \in I} a_i = \sup \left\{ \sum_{i \in K} a_i : K \subseteq I, K \text{ finite} \right\}.$$

Now, suppose that $I = \mathbb{N}$. Show that the above definition agrees with the standard one, that is

$$\sum_{i \in I} a_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i.$$

Show that the value of the series does not depend on the ordering of the elements in the sequence. That is, if $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ is a bijection, then

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\sigma(i)}.$$

Problem 1.11. Let $\{a_i\}_{i \in I} \subseteq [0, \infty)$, where I is an arbitrary (index) set. Suppose that

$$\sum_{i \in I} a_i < \infty.$$

Show that the set $J_n = \{i \in I : a_i > 1/n\}$ is finite. Conclude that the set of $i \in I$ such that $a_i > 0$ is at most countable.

Problem 1.12. Let $\{a_i\}_{i \in I} \subseteq (0, \infty)$ be a family of *positive* real numbers, where I is an (index) set with uncountably many elements. Show that

$$\sum_{i \in I} a_i = \infty.$$

Problem 1.13. (*) Let Ω be a non-empty set and $p_\omega \in [0, 1]$, $\omega \in \Omega$ be real numbers such that

$$\sum_{\omega \in \Omega} p_\omega = 1.$$

Define the set function $\mathbb{P} : 2^\Omega \rightarrow [0, 1]$ by

$$\mathbb{P}(A) = \sum_{\omega \in A} p_\omega.$$

Show that \mathbb{P} is a measure on 2^Ω .

Problem 1.14. (*) Let $A \subset \mathbb{R}$ be an open set. Show that A is the union of at most countable many intervals. (*Hint:* define for all $x \in A$ the interval $I_x = \bigcup_{I \text{ interval}: x \in I \subseteq A} I$ to be the largest interval contained in A containing x)

Problem 1.15. Let I and J be two index sets and $a_{i,j}$, $i \in I$ and $j \in J$ be non-negative real numbers. Show that

$$\sum_{i \in I} \sum_{j \in J} a_{i,j} = \sum_{j \in J} \sum_{i \in I} a_{i,j}.$$

2. Measurable sets and σ -algebras

Problem 2.1. Show that there is no σ -algebra with an odd number of elements.

Problem 2.2. Let (Ω, \mathcal{F}) be a measurable space and $A, B \in \mathcal{F}$. Show, starting from the definition of σ -algebra, that $A \cup B$, $A \cap B$, $A \setminus B$, and $A \Delta B$ all belong to \mathcal{F} .

Problem 2.3. Let (Ω, \mathcal{F}) be a measurable space and A_1, A_2, \dots be a sequence of sets in \mathcal{F} . Define the following sets

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m, \quad \liminf_{n \rightarrow \infty} A_n = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} A_m.$$

Show that:

- $(\limsup_{n \rightarrow \infty} A_n)^c = \liminf_{n \rightarrow \infty} A_n^c$,
- $\liminf_{n \rightarrow \infty} A_n \in \mathcal{F}$ and $\limsup_{n \rightarrow \infty} A_n \in \mathcal{F}$,
- $\liminf_{n \rightarrow \infty} A_n \subseteq \limsup_{n \rightarrow \infty} A_n$.
- $\liminf_{n \rightarrow \infty} A_n = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}$.
- $\limsup_{n \rightarrow \infty} A_n = \{\omega \in \Omega : \exists m \in \mathbb{N} \text{ such that } \omega \in A_n \text{ for all } n \geq m\}$.

Problem 2.4. Let Ω, E be non-empty, \mathcal{G} a σ -algebra on E and $f : \Omega \rightarrow E$. Show that

$$\mathcal{F} = \{f^{-1}(B) : B \in \mathcal{G}\},$$

is a σ -algebra on Ω .

Problem 2.5. Let (Ω, \mathcal{F}) be a measurable space and $(A_n)_{n \in \mathbb{N}}$ a collection of sets in \mathcal{F} . Show that:

- There are $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ mutually disjoint such that $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} E_n$.
- There are $(F_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ such that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} F_n$.

Problem 2.6. Let Ω be a non-empty set and \mathcal{F} a non-empty collection of subsets of Ω which is closed under taking complements and finite unions (such a collection is called an *algebra*). Show that \mathcal{F} is a σ -algebra if and only if it is closed under countable increasing unions (i.e., if $\{A_n\} \subseteq \mathcal{F}$ and $A_1 \subseteq A_2 \subseteq \dots$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}$).

Problem 2.7. (Restriction of σ -algebra) Let \mathcal{F} be a σ -algebra of subsets of Ω . Suppose that $A \subseteq \Omega$ is non-empty. Show that

$$\mathcal{F}_A = \{B \cap A : B \in \mathcal{F}\}$$

is a σ -algebra on A .

Problem 2.8. (Extension of σ -algebra) Let (Ω, \mathcal{F}) be a measurable space, and let K be some non-empty set such that $\Omega \cap K = \emptyset$. Define $\bar{\Omega} = \Omega \cup K$ and $\bar{\mathcal{F}} = \sigma(\mathcal{F} \cup K)$ be a σ -algebra on $\bar{\Omega}$. Show that $\bar{\mathcal{F}} = \{A \subseteq \bar{\Omega} : A \cap \Omega \in \mathcal{F}\}$.

Problem 2.9. Let Ω be a infinite non-empty set.

- Define the collection of sets $\mathcal{F} = \{A \subseteq \Omega : A \text{ is countable or } \Omega \setminus A \text{ is countable}\}$. Is \mathcal{F} a σ -algebra? Prove or disprove.
- Define the collection of sets $\mathcal{F} = \{A \subseteq \Omega : A \text{ is finite or } \Omega \setminus A \text{ is finite}\}$. Is \mathcal{F} a σ -algebra? Prove or disprove.

Problem 2.10. Let \mathcal{F} and \mathcal{G} be σ -algebras on Ω . Show that $\mathcal{F} \cap \mathcal{G}$ is a σ -algebra. Prove or disprove whether $\mathcal{F} \cup \mathcal{G}$ is in general a σ -algebra.

Problem 2.11. Let $\mathcal{F}_n, n \in \mathbb{N}$ be σ -algebras on Ω such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$ (such a sequence $\{\mathcal{F}_n\}$ is called a *filtration*).

- Show that $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is an algebra.
- Is $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ a σ -algebra? Consider $\Omega = \mathbb{N}$ and $\mathcal{F}_n = \sigma(\{A : A \subseteq \mathbb{N} \cap \{1, \dots, n\}\})$.

Problem 2.12. Let $\mathcal{E} \subseteq \mathcal{A}$ be two collections of sets. Show that $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{A})$.

Problem 2.13. (Product sigma algebra) Let Ω_1 and Ω_2 be two non-empty sets, and let \mathcal{F}_1 and \mathcal{F}_2 be σ -algebras on Ω_1 and Ω_2 respectively. Consider the *product σ -algebra* on $\Omega_1 \times \Omega_2$

$$\mathcal{F}_1 \otimes \mathcal{F}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}).$$

Suppose that \mathcal{F}_1 is generated by \mathcal{A}_1 and \mathcal{F}_2 is generated by \mathcal{A}_2 . Show that $\mathcal{F}_1 \otimes \mathcal{F}_2$ is generated by $A_1 \times A_2$ with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Problem 2.14. Show that the Borel σ -algebra $\mathcal{B}_{\mathbb{R}^2}$ equals $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

Problem 2.15. Show that the Borel σ -algebra on \mathbb{R} is generated by each of the following:

- i. the open intervals: $\mathcal{A}_1 = \{(a, b) : a < b\}$,
- ii. the closed intervals: $\mathcal{A}_2 = \{[a, b] : a < b\}$,
- iii. the half open intervals $\mathcal{A}_3 = \{[a, b) : a < b\}$ or $\mathcal{A}_4 = \{(a, b] : a < b\}$,
- iv. the open rays: $\mathcal{A}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{A}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$,

v. the closed rays: $\mathcal{A}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{A}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

Problem 2.16. Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. Also recall that

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{A \subseteq \overline{\mathbb{R}} : A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$$

is a σ -algebra on $\overline{\mathbb{R}}$. Show that $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the family of closed rays $\mathcal{A} = \{[-\infty, a] : a \in \mathbb{R}\}$.

Problem 2.17. Let \mathcal{F} be an infinite σ -algebra.

- Show that \mathcal{F} contains an infinite sequence of disjoint sets.
- (*) Show that $\text{Card}(\mathcal{F}) \geq \text{Card}([0, 1])$.
(Hint: think about binary representation of numbers in $[0, 1]$).

Problem 2.18. Show that Λ is a λ -system on Ω if and only if

- I. $\Omega \in \Lambda$,
- II. if $A, B \in \Lambda$ and $A \subseteq B$, then $B \setminus A \in \Lambda$,
- III. if A_1, A_2, \dots is a sequence of subsets in Λ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\bigcup_{n \in \mathbb{N}} A_n \in \Lambda.$$

Problem 2.19. Let Λ be a λ -system. Show that $\emptyset \in \Lambda$.

Problem 2.20. Let \mathcal{A} be both a λ -system and a π -system. Show that \mathcal{A} is a σ -algebra.

Part II.

Solutions

3. Solutions: Warming up

Solution to Problem 1.1 (by Kempen, S.F.M.)

a) To be proven: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

“ \subseteq ” Let $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$, which means $x \in A$ and ($x \in B$ or $x \in C$).

► If $x \in A$ and $x \in B$, then $x \in A \cap B$ so also $x \in (A \cap B) \cup (A \cap C)$.

► If $x \in A$ and $x \in C$, then $x \in A \cap C$ so also $x \in (A \cap B) \cup (A \cap C)$.

“ \supseteq ” Let $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap B$ or $x \in A \cap C$.

► If $x \in A \cap B$, then $x \in A$ and $x \in B$ so $x \in B \cup C$ so also $x \in A \cap (B \cup C)$.

► If $x \in A \cap C$, then $x \in A$ and $x \in C$ so $x \in B \cup C$ so also $x \in A \cap (B \cup C)$.

b) To be proven: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

“ \subseteq ” Let $x \in A \cup (B \cap C)$ then $x \in A$ or $x \in B \cap C$, which means $x \in A$ or ($x \in B$ and $x \in C$).

► If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ so also $x \in (A \cup B) \cap (A \cup C)$.

► If $x \in B$ and $x \in C$, then $x \in A \cup B$ and $x \in A \cup C$ so also $x \in (A \cup B) \cap (A \cup C)$.

“ \supseteq ” Let $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup B$ and $x \in A \cup C$, which means ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$).

► If $x \in A$ then definitely $x \in A \cup (B \cap C)$.

► If $x \notin A$ then $x \in B$ and $x \in C$ which means $x \in B \cap C$ so also $x \in A \cup (B \cap C)$.

☺

Solution to Problem 1.2 (by Kempen, S.F.M.) To be proven: $A \cap (A \cup B) = A$.

“ \subseteq ” Let $x \in A \cap (A \cup B)$, then $x \in A$ so we are done.

“ \supseteq ” Let $x \in A$, then also ($x \in A$ or $x \in B$) is true, therefore $x \in A \cup B$. So $x \in A \cap (A \cup B)$. ☺

Solution to Problem 1.5 (by Kempen, S.F.M.) To be proven: $\Omega \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} \Omega \setminus A_i$.

“ \subseteq ” Let $x \in \Omega \setminus (\bigcup_{i \in I} A_i)$ then $x \in \Omega$ and $x \notin \bigcup_{i \in I} A_i$, so for all $i \in I$ holds $x \notin A_i$. Then for all $i \in I$ we have $x \in \Omega \setminus A_i$. Since this is true for any $i \in I$, we can write $x \in \bigcap_{i \in I} \Omega \setminus A_i$.

“ \supseteq ” Let $x \in \bigcap_{i \in I} \Omega \setminus A_i$ then for all $i \in I$ we have $x \in \Omega \setminus A_i$, so $x \in \Omega$ and $x \notin A_i$. Since this holds for all $i \in I$, we can write $x \notin \bigcup_{i \in I} A_i$ and therefore $x \in \Omega \setminus (\bigcup_{i \in I} A_i)$. ☺

Solution to Problem 1.8 (by Kempen, S.F.M.)

a) The statement $f^{-1}(f(A)) = A$ is not true since f is not assumed to be injective. As a counterexample, take $\Omega = \{0, 1\}, E = \{0\}, f(\{0\}) = f(\{1\}) = \{0\}, A = \{0\}$ then $f(A) = \{0\}$ and $f^{-1}(f(A)) = f^{-1}(\{0\}) = \{0, 1\} \neq A$.

b) The statement $f(f^{-1}(H)) = H$ is not true since f is not assumed to be surjective. As a counterexample, take $\Omega = \{1\}, E = \{1, 2\}, f(\{1\}) = \{1\}, H = \{1, 2\}$ then $f^{-1}(H) = f^{-1}(\{1, 2\}) = \{1\}$ and $f(f^{-1}(H)) = f(\{1\}) = \{1\} \neq H$. ☺