TU Eindhoven, 2WAG0

# Problems in Measure theory

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*“What I don’t like about measure theory is that you have to say “almost everywhere” almost everywhere”*

– Kurt Friedrichs

**Disclaimer:**

This is a broad selection of exercises for the course *Measure, integration and probability theory*. They are meant to accompany the lecture notes and give you the opportunity to exercise. If you wish to have your solution checked, send it in LaTeX, and we will correct and polish it together, so that it can be featured in this notes in the “Solutions” part.

These collection of exercises are still in progress and they might contain small typos. If you see any or if you think that the statement of the problems is not yet crystal clear, feel free to drop a line. The most efficient way is to send an email to me, a.chiarini@tue.nl. All comments and suggestions will be greatly appreciated.

# Contents

# Part I.

# Problems

# 1. Warming up

In this section we will review some of the basic set operations which will be much needed in the sequel.

**Problem 1.1.** Let $A$, $B$ and $C$ be sets, show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

**Problem 1.2.** Let $A$, $B$ be sets, show that $A \cap (A \cup B) = A$.

**Problem 1.3.** Let $A, B \subseteq \Omega$. We define the symmetric difference to be

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Show that $A \cup B$ is the disjoint union of $A \Delta B$ and $A \cap B$.

**Problem 1.4.** Let $A, B \subseteq \Omega$, show that

$$\Omega \setminus (A \cup B) = (\Omega \setminus A) \cap (\Omega \setminus B).$$

**Problem 1.5.** (De Morgan's law) Let $I$ be any index set and let $\{A_i\}_{i \in I} \subseteq 2^{\Omega}$ be a family subsets of $\Omega$. Show that

$$\Omega \setminus \left( \bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} \Omega \setminus A_i.$$

**Problem 1.6.** Let $f : \Omega \to E$ be some function. Recall that for any $D \subseteq \Omega$ the *image* of $D$ under $f$ is the set

$$f(D) = \{f(x) : \, x \in D\},$$

Let $A, B \subseteq \Omega$. Show that

- $f(A \cap B) \subseteq f(A) \cap f(B)$,
- $f(A \cup B) = f(A) \cup f(B)$.

Find an example where $f(A \cap B) \neq f(A) \cap f(B)$. Is it true that $f(\Omega \setminus A) = E \setminus f(A)$?

**Problem 1.7.** Let $f : \Omega \to E$ be some function. Recall that for any $F \subseteq E$ the *inverse image* of $F$ under $f$ is the set

$$f^{-1}(F) = \{x : \, f(x) \in F\}.$$

Let $H, K \subseteq E$. Show that, taking the inverse image commutes with the set operations:

- $f^{-1}(H \cap K) = f^{-1}(H) \cap f^{-1}(K)$,

- $f^{-1}(H \cup K) = f^{-1}(H) \cup f^{-1}(K)$,
- $f^{-1}(E \setminus H) = \Omega \setminus f^{-1}(H)$.

**Problem 1.8.** Let $f : \Omega \to E$ be some function.

- Let $A \subseteq \Omega$. Is it true that $f^{-1}(f(A)) = A$? Provide a proof or a counterexample.
- Let $H \subseteq E$. Is it true that $f(f^{-1}(H)) = H$? Provide a proof or a counterexample.

**Problem 1.9.** Recall that given a set $\Omega$, $2^\Omega$ denotes the set of all subsets of $\Omega$. Suppose $\Omega = \{0, 1\}$, list all the elements of $2^\Omega$. What is $|2^\Omega|$, where $|\cdot|$ denotes the number of elements of a set? Suppose that $|\Omega| < \infty$, what is $|2^\Omega|$ in this case?

**Problem 1.10.** Let $\{a_i\}_{i\in I} \subseteq [0, \infty]$, where $I$ is an arbitrary (index) set. Recall that their sum is defined by

$$\sum_{i\in I} a_i = \sup\Big\{\sum_{i\in K} a_i : \ K \subseteq I, \ K \text{ finite}\Big\}.$$

Now, suppose that $I = \mathbb{N}$. Show that the above definition agrees with the standard one, that is

$$\sum_{i\in I} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Show that the value of the series does not depend on the ordering of the elements in the sequence. That is, if $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, then

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\sigma(i)}.$$

**Problem 1.11.** Let $\{a_i\}_{i\in I} \subseteq [0, \infty)$, where $I$ is an arbitrary (index) set. Suppose that

$$\sum_{i\in I} a_i < \infty.$$

Show that the set $J_n = \{i \in I : \ a_i > 1/n\}$ is finite. Conclude that the set of $i \in I$ such that $a_i > 0$ is at most countable.

**Problem 1.12.** Let $\{a_i\}_{i\in I} \subseteq (0, \infty)$ be a family of *positive* real numbers, where $I$ is an (index) set with uncountably many elements. Show that

$$\sum_{i\in I} a_i = \infty.$$

**Problem 1.13.** (*) Let $\Omega$ be a non-empty set and $p_\omega \in [0, 1]$, $\omega \in \Omega$ be real numbers such that

$$\sum_{\omega\in\Omega} p_\omega = 1.$$

Define the set function $\mathbb{P} : 2^\Omega \to [0, 1]$ by

$$\mathbb{P}(A) = \sum_{\omega\in A} p_\omega.$$

Show that $\mathbb{P}$ is a measure on $2^\Omega$.

**Problem 1.14.** (*) Let $A \subset \mathbb{R}$ be an open set. Show that $A$ is the union of at most countable many intervals. (*Hint:* define for all $x \in A$ the interval $I_x = \bigcup_{I \text{ interval}: x \in I \subseteq A} I$ to be the largest interval contained in $A$ containing $x$)

**Problem 1.15.** Let $I$ and $J$ be two index sets and $a_{i,j}$, $i \in I$ and $j \in J$ be non-negative real numbers. Show that

$$\sum_{i \in I} \sum_{j \in J} a_{i,j} = \sum_{j \in J} \sum_{i \in I} a_{i,j}.$$

# 2. Measurable sets and $\sigma$-algebras

**Problem 2.1.** Show that there is no $\sigma$-algebra with an odd number of elements.

**Problem 2.2.** Let $(\Omega, \mathscr{F})$ be a measurable space and $A, B \in \mathscr{F}$. Show, starting from the definition of $\sigma$-algebra, that $A \cup B$, $A \cap B$, $A \setminus B$, and $A \Delta B$ all belong to $\mathscr{F}$.

**Problem 2.3.** Let $(\Omega, \mathscr{F})$ be a measurable space and $A_1, A_2, \ldots$ be a sequence of sets in $\mathscr{F}$. Define the following sets

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m.$$

Show that:

- $(\limsup_{n\to\infty} A_n)^c = \liminf_{n\to\infty} A_n^c$,
- $\liminf_{n\to\infty} A_n \in \mathscr{F}$ and $\limsup_{n\to\infty} A_n \in \mathscr{F}$,
- $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.
- $\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}$.
- $\liminf_{n\to\infty} A_n = \{\omega \in \Omega : \exists m \in \mathbb{N} \text{ such that } \omega \in A_n \text{ for all } n \geq m\}$.

**Problem 2.4.** Let $\Omega$, $E$ be non-empty, $\mathscr{G}$ a $\sigma$-algebra on $E$ and $f : \Omega \to E$. Show that

$$\mathscr{F} = \{f^{-1}(B) : B \in \mathscr{G}\},$$

is a $\sigma$-algebra on $\Omega$.

**Problem 2.5.** Let $(\Omega, \mathscr{F})$ be a measurable space and $(A_n)_{n\in\mathbb{N}}$ a collection of sets in $\mathscr{F}$. Show that:

- There are $(E_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ mutually disjoint such that $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} E_n$.
- There are $(F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ such that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} F_n$.

**Problem 2.6.** Let $\Omega$ be a non-empty set and $\mathscr{F}$ a non-empty collection of subsets of $\Omega$ which is closed under taking complements and finite unions (such a collection is called an *algebra*). Show that $\mathscr{F}$ is a $\sigma$-algebra if and only if it is closed under countable increasing unions (i.e., if $\{A_n\} \subseteq \mathscr{F}$ and $A_1 \subseteq A_2 \subseteq \ldots$, then $\cup_{n\in\mathbb{N}} A_n \in \mathscr{F}$).

**Problem 2.7.** (Restriction of $\sigma$-algebra) Let $\mathscr{F}$ be a $\sigma$-algebra of subsets of $\Omega$. Suppose that $A \subseteq \Omega$ is non-empty. Show that

$$\mathscr{F}_A = \{B \cap A : B \in \mathscr{F}\}$$

is a $\sigma$-algebra on $A$.

**Problem 2.8.** (Extension of $\sigma$-algebra) Let $(\Omega, \mathscr{F})$ be a measurable space, and let $K$ be some non-empty set such that $\Omega \cap K = \varnothing$. Define $\overline{\Omega} = \Omega \cup K$ and $\overline{\mathscr{F}} = \sigma(\mathscr{F} \cup K)$ be a $\sigma$-algebra on $\overline{\Omega}$. Show that $\overline{\mathscr{F}} = \{A \subseteq \overline{\Omega} : A \cap \Omega \in \mathscr{F}\}$.

**Problem 2.9.** Let $\Omega$ be a infinite non-empty set.

- Define the collection of sets $\mathscr{F} = \{A \subseteq \Omega : A \text{ is countable or } \Omega \setminus A \text{ is countable}\}$. Is $\mathscr{F}$ a $\sigma$-algebra? Prove or disprove.
- Define the collection of sets $\mathscr{F} = \{A \subseteq \Omega : A \text{ is finite or } \Omega \setminus A \text{ is finite}\}$. Is $\mathscr{F}$ a $\sigma$-algebra? Prove or disprove.

**Problem 2.10.** Let $\mathscr{F}$ and $\mathscr{G}$ be $\sigma$-algebras on $\Omega$. Show that $\mathscr{F} \cap \mathscr{G}$ is a $\sigma$-algebra. Prove or disprove whether $\mathscr{F} \cup \mathscr{G}$ is in general a $\sigma$-algebra.

**Problem 2.11.** Let $\mathscr{F}_n$, $n \in \mathbb{N}$ be $\sigma$-algebras on $\Omega$ such that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ for all $n \in \mathbb{N}$ (such a sequence $\{\mathscr{F}_n\}$ is called a *filtration*).

- Show that $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ is an algebra.
- Is $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ a $\sigma$-algebra? Consider $\Omega = \mathbb{N}$ and $\mathscr{F}_n = \sigma(\{A : A \subseteq \mathbb{N} \cap \{1, \ldots n\}\}))$.

**Problem 2.12.** Let $\mathscr{E} \subseteq \mathscr{A}$ be two collections of sets. Show that $\sigma(\mathscr{E}) \subseteq \sigma(\mathscr{A})$.

**Problem 2.13.** (Product sigma algebra) Let $\Omega_1$ and $\Omega_2$ be two non-empty sets, and let $\mathscr{F}_1$ and $\mathscr{F}_2$ be $\sigma$-algebras on $\Omega_1$ and $\Omega_2$ respectively. Consider the *product $\sigma$-algebra* on $\Omega_1 \times \Omega_2$

$$\mathscr{F}_1 \otimes \mathscr{F}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}.$$

Suppose that $\mathscr{F}_1$ is generated by $\mathscr{A}_1$ and $\mathscr{F}_2$ is generated by $\mathscr{A}_2$. Show that $\mathscr{F}_1 \otimes \mathscr{F}_2$ is generated by $A_1 \times A_2$ with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$.

**Problem 2.14.** Show that the Borel $\sigma$-algebra $\mathscr{B}_{\mathbb{R}^2}$ equals $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$.

**Problem 2.15.** Show that the Borel $\sigma$-algebra on $\mathbb{R}$ is generated by each of the following:

i. the open intervals: $\mathscr{A}_1 = \{(a, b) : a < b\}$,

ii. the closed intervals: $\mathscr{A}_2 = \{[a, b] : a < b\}$,

iii. the half open intervals $\mathscr{A}_3 = \{[a, b) : a < b\}$ or $\mathscr{A}_4 = \{(a, b] : a < b\}$,

iv. the open rays: $\mathscr{A}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathscr{A}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$,

v. the closed rays: $\mathscr{A}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathscr{A}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

**Problem 2.16.** Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. Also recall that

$$\mathscr{B}_{\overline{\mathscr{B}}} = \{A \subseteq \overline{\mathscr{B}} : A \cap \mathbb{R} \in \mathscr{B}_{\mathbb{R}}\}$$

is a $\sigma$-algebra on $\overline{\mathbb{R}}$. Show that $\mathscr{B}_{\overline{\mathscr{B}}}$ is generated by the family of closed rays $\mathscr{A} = \{[-\infty, a] : a \in \mathbb{R}\}$.

**Problem 2.17.** Let $\mathscr{F}$ be an infinite $\sigma$-algebra.

- Show that $\mathscr{F}$ contains an infinite sequence of disjoint sets.
- (*) Show that $\mathrm{Card}(\mathscr{F}) \geq \mathrm{Card}([0, 1])$.
  (*Hint:* think about binary representation of numbers in $[0, 1]$).

**Problem 2.18.** Show that $\Lambda$ is a $\lambda$-system on $\Omega$ if and only if

I. $\Omega \in \Lambda$,

II. if $A, B \in \Lambda$ and $A \subseteq B$, then $B \setminus A \in \Lambda$,

III. if $A_1, A_2, \ldots$ is a sequence of subsets in $\Lambda$ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\bigcup_{n \in \mathbb{N}} A_n \in \Lambda.$$

**Problem 2.19.** Let $\Lambda$ be a $\lambda$-system. Show that $\varnothing \in \Lambda$.

**Problem 2.20.** Let $\mathscr{A}$ be both a $\lambda$-system and a $\pi$-system. Show that $\mathscr{A}$ is a $\sigma$-algebra.

# 3. Measures

**Problem 3.1.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that for all $E, F \in \mathscr{F}$ such that $E \subseteq F$, one has $\mu(E) \leq \mu(F)$.

**Problem 3.2.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that for all $E, F \in \mathscr{F}$ then

$$\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F).$$

Conclude that $\mu(E \cup F) \leq \mu(E) + \mu(F)$.

**Problem 3.3.** Let $(\Omega, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, \infty]$ be a set function. Assume that $\mu(\varnothing) = 0$, $\mu$ is finitely additive and continuous from below. Show that $\mu$ is a measure.

**Problem 3.4.** Consider a non-empty uncountable set $\Omega$ and the $\sigma$-algebra

$$\mathscr{F} = \{A \subseteq \Omega : \ A \text{ is countable or } \Omega \setminus A \text{ is countable}\}.$$

We define the set function $\mu : \mathscr{F} \to [0, \infty]$ by $\mu(E) = 0$ if $E$ is countable, and $\mu(E) = 1$ if $\Omega \setminus E$ is countable. Show that $\mu$ is a measure on $(\Omega, \mathscr{F})$.

**Problem 3.5.** Let $\Omega$ be an infinite set and $\mathscr{F} = 2^{\Omega}$. Define $\mu$ on $\mathscr{F}$ by $\mu(E) = 0$ if $E$ is finite and $\mu(E) = \infty$ if $E$ is not finite. Show that $\mu$ is finitely additive but not a measure.

**Problem 3.6.** Let $(\Omega, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, 1]$ be an additive set function, that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathscr{F}$ such that $A \cap B = \varnothing$. Show that $\mu(\varnothing) = 0$.

**Problem 3.7.** (Inclusion-exclusion) Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Let $A_1, A_2, A_3 \in \mathscr{F}$. Show that

$$\begin{aligned} \mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) + \mu(A_2) + \mu(A_3) \\ - \mu(A_1 \cap A_2) - \mu(A_2 \cap A_3) - \mu(A_3 \cap A_1) + \mu(A_1 \cap A_2 \cap A_3). \end{aligned}$$

Let $A_1, \ldots, A_n \in \mathscr{F}$. Show that

$$\mu\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = j}} \mu\Big(\bigcap_{i \in I} A_i\Big).$$

**Problem 3.8.** Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space.

- If $E, F \in \mathscr{F}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

- ► Define $\rho(E,F) = \mu(E\Delta F)$ for all $E,F \in \mathscr{F}$. Show that $\rho(E,F) \leq \rho(E,G) + \rho(G,F)$ for all $E,F,G \in \mathscr{F}$.

**Problem 3.9.** If $\mu_1,\dots,\mu_n$ are measures on a measurable space $(\Omega,\mathscr{F})$, and $a_1,\dots,a_n$ are non-negative real numbers, then $\mu := \sum_{i=1}^n a_i\mu_i$ is a measure on $\mathscr{F}$. Moreover $\mu$ is $\sigma$-finite if $\mu_i$ is $\sigma$-finite for all $i = 1,\dots,n$.

**Problem 3.10.** Let $(\Omega,\mathscr{F},\mu)$ be a measure space and $\mathscr{G} \subseteq \mathscr{F}$ be a $\sigma$-algebra. Show that $(\Omega,\mathscr{G},\mu)$ is a measure space.

**Problem 3.11.** Let $\mu$ be a finite measure on $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty,x]), \quad \forall x \in \mathbb{R}.$$

Show that $F$ is non-decreasing, right-continuous, and

$$\lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to\infty} F(x) = \mu(\mathbb{R}).$$

**Problem 3.12.** Find a measure space $(\Omega,\mathscr{F},\mu)$ and a decreasing sequence $B_1 \supseteq B_2 \supseteq \ldots \in \mathscr{F}$ such that $\lim_{n\to\infty}\mu(B_n) > \mu(\cap_{n\in\mathbb{N}}B_n)$.

**Problem 3.13.** Let $(\Omega,\mathscr{F},\mu)$ be a measure space and $A \in \mathscr{F}$. Show that the set function $\mu_A : \mathscr{F} \to [0,\infty]$ defined by $\mu_A(B) := \mu(B\cap A)$ is a measure on $(\Omega,\mathscr{F})$.

**Problem 3.14.** Let $(\Omega,\mathscr{F})$ be a measure space $\mu : \mathscr{F} \to [0,\infty]$ be a set function which is finitely additive and such that $\mu(\varnothing) = 0$. Show that $\mu$ is a measure on $(\Omega,\mathscr{F})$ if and only if $\mu$ is continuous from below.

**Problem 3.15.** Let $(\Omega,\mathscr{F})$ be a measure space $\mu : \mathscr{F} \to [0,\infty]$ be a set function which is finitely additive and such that $\mu(\Omega) < \infty$. Show that $\mu$ is a measure on $(\Omega,\mathscr{F})$ if and only if $\mu$ is continuous from above.

**Problem 3.16.** Let $(\Omega,\mathscr{F},\mu)$ be a finite measure space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$. Recall that

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}}\bigcup_{m\geq n} A_m, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n} A_m.$$

Show that

$$\limsup_{n\to\infty}\mu(A_n) \leq \mu\Big(\limsup_{n\to\infty} A_n\Big), \qquad \liminf_{n\to\infty}\mu(A_n) \geq \mu\Big(\liminf_{n\to\infty} A_n\Big).$$

**Problem 3.17.** Let $(\Omega,\mathscr{F},\mu)$ be a $\sigma$-finite measure space. Let $\mathscr{E} \subseteq \mathscr{F}$ be a $\pi$-system such that there exists $E_1,E_2,\ldots \in \mathscr{E}$ such that $E_n \uparrow \Omega$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Show that $\mu$ is uniquely determined by its values on $\mathscr{E}$.

**Problem 3.18.** Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function. Let $\nu_F$ be the unique measure on $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$ associated to $F$. Show that $\nu_F(\{x\}) = F(x) - F(x-)$ where we define

$$F(x-) := \lim_{y\uparrow x} F(y).$$

Conclude that it if $F$ is continuous, then $\nu_F(\mathbb{Q}) = 0$.

**Problem 3.19.** Let $\lambda$ be the unique measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ such that $\lambda((a,b]) = b - a$ for all $a < b$. Show that $\lambda$ is translation invariant, that is $\lambda(A + x) = \lambda(A)$ for all $A \in \mathscr{B}_{\mathbb{R}}$ and all $x \in \mathbb{R}$, where we write $A + x := \{a + x : a \in A\}$ for the translation of $A$ by $x$. (*Hint:* a solution can be obtained with the $\pi - \lambda$ theorem.)

**Problem 3.20.** Let $\lambda$ be the unique measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ such that $\lambda((a,b]) = b - a$ for all $a < b$. Show that $\lambda(\tau A) = |\tau|\lambda(A)$ for all $A \in \mathscr{B}_{\mathbb{R}}$ and all $\tau \neq 0$, where we write $\tau A := \{\tau a : a \in A\}$ for the dilation of $A$ by $\tau$.

**Problem 3.21.** Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function. Let $\nu_F$ be the unique measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ such that $\nu_F((a,b]) = F(b) - F(a)$ for all $a < b$. Show that $\nu_F$ is $\sigma$-finite.

**Problem 3.22.** Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. Let $(A_i)_{i \in \mathbb{N}} \subseteq \mathscr{F}$ be a sequence such that $\mathbb{P}(A_i) = 1$ for all $i \in \mathbb{N}$. Show that $\mathbb{P}(\cap_{i \in \mathbb{N}} A_i) = 1$.

**Problem 3.23.** Let $\mu$ be a finite measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Show that the set $\{x \in \mathbb{R} : \mu(\{x\}) > 0\}$ is at most countable.

# 4. Null sets, completion and independence

**Problem 4.1.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Suppose that $A, N \in \mathscr{F}$ and $\mu(N) = 0$. Show that $\mu(A \cup N) = \mu(A)$.

**Problem 4.2.** Suppose that $(\Omega, \mathscr{F}, \mu)$ is a measure space and let $N$ be a null set. Show that for all $M \subseteq N$, $M$ is a null set.

**Problem 4.3.** Let $(N_n)_{n\in\mathbb{N}}$ be null sets in a measure space $(\Omega, \mathscr{F}, \mu)$. Show that $\cup_{n\in\mathbb{N}} N_n$ is a null set.

**Problem 4.4.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of function. Show that $f_n = 0$ almost everywhere for all $n \in \mathbb{N}$ if and only if almost everywhere $f_n = 0$ for all $n \in \mathbb{N}$. (Careful with the quantifiers!)

**Problem 4.5.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability, we say that $A \in \mathscr{F}$ happens almost surely if $\Omega \setminus A$ is a null set for $\mathbb{P}$.

- Show that $A \in \mathscr{F}$ happens almost surely if and only if $\mathbb{P}(A) = 1$.
- Assume now that $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ is such that $A_n$ happens almost surely for all $n \in \mathbb{N}$. Show that $\mathbb{P}(\cap_{n\in\mathbb{N}} A_n) = 1$.

**Problem 4.6.** Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set and let $V$ be the Vitali set. Show that if $E \subseteq V$, then $E$ is a null-set.

**Problem 4.7.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $(\Omega, \overline{\mathscr{F}}, \overline{\mu})$ its completion. Show that $\overline{A} \in \overline{\mathscr{F}}$ if and only if there is $A \in \mathscr{F}$ such that $A \Delta \overline{A}$ is a null set.

**Problem 4.8.** (*) Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $\mathscr{L}(E) > 0$. Show that there exists $N \subseteq E$ not Lebesgue measurable. (Hint: assume first $E \subseteq (0, 1)$ and look at $V \cap E$ where $V$ is the Vitali set.)

**Problem 4.9.** (The Cantor set) The Lebesgue null sets include not only the countable sets but also many sets having the cardinality of the continuum. The Cantor set $C$ is the set of all $x \in [0, 1]$ that have a base-3 expansion

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \quad \text{with } a_j \in \{0, 2\} \text{ for all } j \in \mathbb{N}.$$

Thus $C$ is obtained from $[0, 1]$ by removing the open middle third $(1/3, 2/3)$, then removing the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining intervals and so forth. Show that

- $C$ is compact and with zero Lebesgue measure.

- $\mathrm{Card}(C) = \mathrm{Card}([0,1])$. Hint: consider the so called Cantor function, for $x \in C$, $x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$, define
$$f(x) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}, \quad b_j = a_j/2.$$
- (*) Show that $C$ has empty interior and is totally disconnected (that is for all $x < y \in C$ there is $z \in (x,y)$ such that $z \notin C$). Moreover $C$ has no isolated points.

**Problem 4.10.** Show that for any Lebesgue measurable set $E \subseteq \mathbb{R}$ and any real number $\lambda \in \mathbb{R}$, $\mathscr{L}(E+\lambda) = \mathscr{L}(E)$ and $\mathscr{L}(\lambda E) = |\lambda| \mathscr{L}(E)$.

**Problem 4.11.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. We say that two $\sigma$-algebras $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{F}$ are independent if
$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \quad \forall A_1 \in \mathscr{A}_1, \forall A_2 \in \mathscr{A}_2.$$
Suppose that $\mathscr{E}_1, \mathscr{E}_2$ are $\pi$-systems generating $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. Show that $\mathscr{A}_1$ and $\mathscr{A}_2$ are independent if and only if
$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \quad \forall A_1 \in \mathscr{E}_1, \forall A_2 \in \mathscr{E}_2.$$

# Part II.

# Solutions

# 5. Solutions: Warming up

**Solution to Problem 1.1** *(by Kempen, S.F.M.)*
a) To be proven: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
" $\subseteq$ " Let $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$, which means $x \in A$ and ($x \in B$ or $x \in C$).

- If $x \in A$ and $x \in B$, then $x \in A \cap B$ so also $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in A$ and $x \in C$, then $x \in A \cap C$ so also $x \in (A \cap B) \cup (A \cap C)$.

" $\supseteq$ " Let $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap B$ or $x \in A \cap C$.

- If $x \in A \cap B$, then $x \in A$ and $x \in B$ so $x \in B \cup C$ so also $x \in A \cap (B \cup C)$.
- If $x \in A \cap C$, then $x \in A$ and $x \in C$ so $x \in B \cup C$ so also $x \in A \cap (B \cup C)$.

b) To be proven: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
" $\subseteq$ " Let $x \in A \cup (B \cap C)$ then $x \in A$ or $x \in B \cap C$, which means $x \in A$ or ($x \in B$ and $x \in C$).

- If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ so also $x \in (A \cup B) \cap (A \cup C)$.
- If $x \in B$ and $x \in C$, then $x \in A \cup B$ and $x \in A \cup C$ so also $x \in (A \cup B) \cap (A \cup C)$.

" $\supseteq$ " Let $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup B$ and $x \in A \cup C$, which means ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$).

- If $x \in A$ then definitely $x \in A \cup (B \cap C)$.
- If $x \notin A$ then $x \in B$ and $x \in C$ which means $x \in B \cap C$ so also $x \in A \cup (B \cap C)$.

😄

**Solution to Problem 1.2** *(by Kempen, S.F.M.)* To be proven: $A \cap (A \cup B) = A$.
" $\subseteq$ " Let $x \in A \cap (A \cup B)$, then $x \in A$ so we are done.
" $\supseteq$ " Let $x \in A$, then also ($x \in A$ or $x \in B$) is true, therefore $x \in A \cup B$. So $x \in A \cap (A \cup B)$. 😄

**Solution to Problem 1.5** *(by Kempen, S.F.M.)* To be proven: $\Omega \setminus \left(\bigcup_{i \in I} A_i\right) = \bigcap_{i \in I} \Omega \setminus A_i$.
" $\subseteq$ " Let $x \in \Omega \setminus \left(\bigcup_{i \in I} A_i\right)$ then $x \in \Omega$ and $x \notin \bigcup_{i \in I} A_i$, so for all $i \in I$ holds $x \notin A_i$. Then for all $i \in I$ we have $x \in \Omega \setminus A_i$. Since this is true for any $i \in I$, we can write $x \in \bigcap_{i \in I} \Omega \setminus A_i$.
" $\supseteq$ " Let $x \in \bigcap_{i \in I} \Omega \setminus A_i$ then for all $i \in I$ we have $x \in \Omega \setminus A_i$, so $x \in \Omega$ and $x \notin A_i$. Since this holds for all $i \in I$, we can write $x \notin \bigcup_{i \in I} A_i$ and therefore $x \in \Omega \setminus \left(\bigcup_{i \in I} A_i\right)$. 😄

**Solution to Problem 1.8** *(by Kempen, S.F.M.)*
a) The statement $f^{-1}(f(A)) = A$ is not true since $f$ is not assumed to be injective. As a counterexample, take $\Omega = \{0,1\}, E = \{0\}, f(\{0\}) = f(\{1\}) = \{0\}, A = \{0\}$ then $f(A) = \{0\}$ and $f^{-1}(f(A)) = f^{-1}(\{0\}) = \{0,1\} \neq A$.
b) The statement $f(f^{-1}(H)) = H$ is not true since f is not assumed to be surjective. As a counterexample, take $\Omega = \{1\}$, $E = \{1,2\}$, $f(\{1\}) = \{1\}$, $H = \{1,2\}$ then $f^{-1}(H) = f^{-1}(\{1,2\}) = \{1\}$ and $f(f^{-1}(H)) = f(\{1\}) = \{1\} \neq H$. 😄

**Solution to Problem 1.10** *(by Beurskens, T.P.J.)* Let $n \in \mathbb{N}$, and define $K = \{1,\dots,n\}$. By definition of the supremum, we then have

$$\sum_{i=1}^{n} a_i = \sum_{i\in K} a_i \le \sup\left(\sum_{i\in K} a_i : K \subseteq I, K \text{ finite}\right) = \sum_{i\in I} a_i.$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i \le \sum_{i\in I} a_i.$$

Next, let $K \subseteq \mathbb{N}$ be finite, so that $K \subset \{1,\dots,n\}$ for some $n \in \mathbb{N}$. Note that $n \geq \sup K$. We get

$$\sum_{i\in K} a_i \le \sum_{i\in\{1,\dots,n\}} a_i = \sum_{i=1}^{n} a_i \le \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Since this holds for arbitrary finite $K$, it holds for all finite $K$. Thus we get

$$\sum_{i\in I} a_i = \sup\left(\sum_{i\in K} a_i : K \subseteq I, K \text{ finite}\right) \le \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Using both inequalities, we see that indeed $\sum_{i\in I} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.

We are left with showing that the sum $\sum_{i=1}^{\infty} a_i$ does not depend on the ordering of the elements in the sequence $(a_i)$. This follows immediately from the fact that for any index set $I$

$$\sum_{i\in I} a_i = \sup\Big\{\sum_{i\in K} a_i \,:\, K \subseteq I,\ K \text{ finite}\Big\}$$

is completely blind to any ordering of $I$, in fact $I$ is possibly not even ordered. To be more precise, if $\sigma : I \to I$ is a bijection, then

$$\begin{aligned}\sum_{i\in I} a_{\sigma(i)} &= \sup\Big\{\sum_{i\in K} a_{\sigma(i)} \,:\, K \subseteq I,\ K \text{ finite}\Big\}\\ &= \sup\Big\{\sum_{i\in\sigma^{-1}(K)} a_i \,:\, \sigma^{-1}(K) \subseteq I,\ \sigma^{-1}(K) \text{ finite}\Big\} = \sum_{i\in I} a_i,\end{aligned}$$

where we used that $K \subseteq I$ is finite if and only if $\sigma^{-1}(K)$ is finite. So, from this observation and the first part of the problem, one has that for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$

$$\sum_{i=1}^{\infty} a_i = \sum_{i\in\mathbb{N}} a_i = \sum_{i\in\mathbb{N}} a_{\sigma(i)} = \sum_{i=1}^{\infty} a_{\sigma(i)},$$

where the summation in the middle is with respect the new notion.

**Solution to Problem 1.12** *(by Bakker, A.)* Proof by contradiction. Suppose $\sum_{i \in I} a_i < \infty$, then by Problem 1.11 we have that the set $I$ contains at most a countable number of elements $i$ with $a_i$ positive. This, together with the fact that $a_i > 0$ for all $i \in I$, contradicts that there are uncountable many elements in $I$. Hence $\sum_{i \in I} a_i = \infty$.