1) $Q = [\rho, \rho u, \rho e_t]^T \qquad F = [\rho u, \rho u^2 + p, \rho u H]^T$

$$e_t = e + u^2/2 \qquad H = h + u^2/2 = \underbrace{e + u^2/2}_{e_t} + \frac{p}{\rho}$$

Start by manipulating F, assuming a calorically perfect gas $\underbrace{\text{eq. of state}}$:

$$e = p/[(\gamma-1)\rho]$$

$$F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u\left(e_t + \frac{p}{\rho}\right) \end{bmatrix} \qquad \left.\begin{aligned} \rho u &= m \\ \rho e_t &= \epsilon \end{aligned}\right\} \text{by definition}$$

$$\boxed{F_1 = m}$$

Try to manipulate F_2 so it contains only γ, ϵ, m, and ρ:

$$\rho u^2 + p = \underbrace{\rho u^2 + e(\gamma-1)\rho}_{\text{from EOS}} = \rho u^2 + \left[e_t - \frac{u^2}{2}\right](\gamma-1)\rho$$

$$= \rho u^2 + \underbrace{\rho e_t(\gamma-1)}_{(\gamma-1)\epsilon} - \frac{\rho u^2}{2}(\gamma-1) = (\gamma-1)\epsilon + \rho u^2\left[1 - \frac{(\gamma-1)}{2}\right]$$

$$= (\gamma-1)\epsilon + \rho u^2\left[1 - \frac{\gamma}{2} + \frac{1}{2}\right] = (\gamma-1)\epsilon + \rho u^2\left[\frac{3}{2} - \frac{\gamma}{2}\right]$$

recognize that $m^2 = \rho^2 u^2$, so $\rho u^2 = \frac{m^2}{\rho}$

$$\boxed{F_2 = (\gamma-1)\epsilon + (3-\gamma)m^2/2\rho}$$

Now, manipulate F_3 so that it is only in terms of γ, ϵ, m, and ρ:

→

$$\rho u\left(e_t + \frac{p}{\rho}\right) = \rho u\left(e_t + \underbrace{\frac{(\gamma-1)\rho e}{\rho}}_{\text{from EOS}}\right) = \rho u\left(e_t + (\gamma-1)e\right)$$

$$= \rho u\left[e_t + (\gamma-1)\left(e_t - \frac{u^2}{2}\right)\right] = \rho u\left[\underbrace{e_t + (\gamma-1)e_t}_{e_t + \gamma e_t - \cancel{e_t}} - (\gamma-1)u^2/2\right]$$

$$= \rho u\left[\gamma e_t - (\gamma-1)u^2/2\right] = \gamma \underbrace{\rho e_t}_{\epsilon} u - (\gamma-1)\rho u^3/2$$

recognize that $u = \frac{m}{\rho}$ and $u^3 = \frac{m^3}{\rho^3}$

$$F_2 = \frac{\gamma m \epsilon}{\rho} - (\gamma-1)\frac{\cancel{\rho} m^3}{2\rho^{\cancel{3}\,2}}$$

$$\boxed{F_2 = \frac{\gamma m \epsilon}{\rho} - (\gamma-1)m^3/2\rho^2}$$

So, our Q and F vectors are now:

$$F = \begin{bmatrix} m \\ (\gamma-1)\epsilon + (3-\gamma)m^2/2\rho \\ \gamma m\epsilon/\rho - (\gamma-1)m^3/2\rho^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} \rho \\ m \\ \epsilon \end{bmatrix}$$ now, calculate $\underset{\sim}{A} = \frac{dF}{dQ}$

$$\underset{\sim}{A} = \frac{dF}{dQ} = \begin{bmatrix} \frac{\partial F_1}{\partial Q_1} & \frac{\partial F_1}{\partial Q_2} & \frac{\partial F_1}{\partial Q_3} \\ \frac{\partial F_2}{\partial Q_1} & \frac{\partial F_2}{\partial Q_2} & \frac{\partial F_2}{\partial Q_3} \\ \frac{\partial F_3}{\partial Q_1} & \frac{\partial F_3}{\partial Q_2} & \frac{\partial F_3}{\partial Q_3} \end{bmatrix}$$

$\longrightarrow$

$$\boxed{\frac{\partial F_1}{\partial Q_1} = \frac{\partial m}{\partial \rho} = 0} \qquad \boxed{\frac{\partial F_1}{\partial Q_2} = \frac{\partial m}{\partial m} = 1}$$

$$\boxed{\frac{\partial F_1}{\partial Q_3} = \frac{\partial m}{\partial E} = 0} \qquad \frac{\partial F_2}{\partial Q_1} = \frac{\partial}{\partial \rho}\left(\cancel{(\gamma-1)E}^{\,0} + (3-\gamma)m^2/2\rho\right)$$

$$\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2} \rightarrow \boxed{\frac{\partial F_2}{\partial Q_1} = (\gamma-3)m^2/2\rho^2}$$

$$\frac{\partial F_2}{\partial Q_2} = \frac{\partial}{\partial m}\left[\underset{0}{\cancel{(\gamma-1)E}} + (3-\gamma)\underset{2m}{m^2}/2\rho\right] \rightarrow \boxed{\frac{\partial F_2}{\partial Q_2} = (3-\gamma)m/\rho}$$

$$\frac{\partial F_2}{\partial Q_3} = \frac{\partial}{\partial E}\left[(\gamma-1)E + (3-\gamma)m^2/2\rho\right] \rightarrow \boxed{\frac{\partial F_2}{\partial Q_3} = \gamma-1}$$

$$\frac{\partial F_3}{\partial Q_1} = \frac{\partial}{\partial \rho}\left[\frac{\gamma m E}{\rho} - \frac{(\gamma-1)m^3}{2\rho^2}\right] = -\frac{\gamma m E}{\rho^2} + \frac{(\gamma-1)m^3}{\rho^3}$$

$$\frac{\partial}{\partial \rho}\left(-\rho^{-2}\right) = 2\rho^{-3}$$

$$\boxed{\frac{\partial F_3}{\partial Q_1} = \frac{(\gamma-1)m^3}{\rho^3} - \frac{\gamma m E}{\rho^2}} \qquad \frac{\partial F_3}{\partial Q_2} = \frac{\partial}{\partial m}\left[\frac{\gamma m E}{\rho} - \frac{(\gamma-1)m^3}{2\rho^2}\right]$$

$$\boxed{\frac{\partial F_3}{\partial Q_2} = \frac{\gamma E}{\rho} - \frac{3(\gamma-1)m^2}{2\rho^2}} \qquad \frac{\partial F_3}{\partial Q_3} = \frac{\partial}{\partial E}\left[\frac{\gamma m E}{\rho} - \frac{(\gamma-1)m^3}{2\rho^2}\right]$$

$$\boxed{\frac{\partial F_3}{\partial Q_3} = \frac{\gamma m}{\rho}}$$

Now, we can assemble the $\underset{\sim}{A}$ matrix

$\longrightarrow$

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(\gamma-3)m^2}{2\rho^2} & (3-\gamma)\frac{m}{\rho} & \gamma-1 \\ \frac{(\gamma-1)m^3}{\rho^3} - \frac{\gamma m \epsilon}{\rho^2} & \frac{\gamma\epsilon}{\rho} - \frac{3(\gamma-1)m^2}{2\rho^2} & \frac{\gamma m}{\rho} \end{bmatrix}$$

Finally, express each entry in $\underset{\sim}{A}$ in terms of u, γ, and a $\left(=\sqrt{\frac{\gamma p}{\rho}}\right)$

$$\boxed{\underset{\sim}{A}_{21} = \frac{(\gamma-3)m^2}{2\rho^2} = (\gamma-3)u^2/2} \quad \boxed{\underset{\sim}{A}_{22} = (3-\gamma)\frac{m}{\rho} = (3-\gamma)u}$$

$$A_{31} = \frac{(\gamma-1)m^3}{\rho^3} - \frac{\gamma m \epsilon}{\rho^2} = (\gamma-1)u^3 - \frac{\gamma u \epsilon}{\rho} = (\gamma-1)u^3 - \frac{\gamma u \rho e_+}{\rho}$$

$$= (\gamma-1)u^3 - \gamma u e_+ = (\gamma-1)u^3 - \gamma u(e + u^2/2)$$

$$e = \frac{p}{(\gamma-1)\rho} \text{ from EOS}$$

$$= (\gamma-1)u^3 - \underbrace{\frac{\gamma p u}{\rho(\gamma-1)}}_{\frac{\gamma p}{\rho} = a^2} - \frac{\gamma u^3}{2}$$

$$\boxed{A_{31} = (\gamma-1)u^3 - \frac{u a^2}{\gamma-1} - \frac{1}{2}\gamma u^3}$$

$$A_{32} = \frac{\gamma\epsilon}{\rho} - \frac{3(\gamma-1)m^2}{2\rho^2} = \frac{\gamma\rho(e + u^2/2)}{\rho} - \frac{3(\gamma-1)u^2}{2}$$

$$\frac{m^2}{\rho^2} \to u^2$$

$$= \frac{\gamma u^2}{2} + \underbrace{\frac{\gamma p}{\rho(\gamma-1)}}_{\frac{\gamma p}{\rho} = a^2} - \frac{3(\gamma-1)u^2}{2} = \left[\gamma - 3(\gamma-1)\right]\frac{u^2}{2} + \frac{a^2}{\gamma-1}$$

$\rightarrow$

$$A_{32} = \frac{(\gamma - 3\gamma + 3)}{2} u^2 + \frac{a^2}{\gamma - 1}$$

$$\boxed{A_{32} = \left(\frac{3}{2} - \gamma\right) u^2 + \frac{a^2}{\gamma - 1}} \qquad \boxed{A_{33} = \frac{\gamma m}{\rho} = \gamma u}$$

Finally, assemble $\underset{\sim}{A}$ matrix in terms of u, γ, a:

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(\gamma - 3) u^2}{2} & (3 - \gamma) u & \gamma - 1 \\ (\gamma - 1) u^3 - \frac{u a^2}{\gamma - 1} - \frac{1}{2} \gamma u^3 & \left(\frac{3}{2} - \gamma\right) u^2 + \frac{a^2}{\gamma - 1} & \gamma u \end{bmatrix}$$

2) $F \equiv \underset{\sim}{A} Q = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma-3)u^2/2 & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - \frac{ua^2}{\gamma-1} - \frac{\gamma u^3}{2} & \left(\frac{3}{2}-\gamma\right)u^2 + \frac{a^2}{\gamma-1} & \gamma u \end{bmatrix} \begin{bmatrix} \rho \\ m \\ \epsilon \end{bmatrix}$

$$F = \begin{bmatrix} 0\rho + 1m + 0\epsilon \\ (\gamma-3)\rho u^2/2 + (3-\gamma)mu + (\gamma-1)\epsilon \\ \rho(\gamma-1)u^3 - \frac{\rho u a^2}{\gamma-1} - \frac{\gamma\rho u^3}{2} + \left(\frac{3}{2}-\gamma\right)mu^2 + \frac{ma^2}{\gamma-1} + \gamma u\epsilon \end{bmatrix}$$

$u = \frac{m}{\rho}$ $\quad u = \frac{m}{\rho}$

$\boxed{F_1 = m}$ $\quad F_2 = (\gamma-3)\rho\left(\frac{m}{\rho}\right)^2/2 + (3-\gamma)m\cdot\left(\frac{m}{\rho}\right) + (\gamma-1)\epsilon$

$F_2 = (\gamma-3)m^2/2\rho + (3-\gamma)m^2/\rho + (\gamma-1)\epsilon = (\gamma-1)\epsilon + \left[\frac{1}{2}(\gamma-3) + (3-\gamma)\right]m^2/\rho$

$F_2 = (\gamma-1)\epsilon + \left[\frac{\gamma}{2} - \frac{3}{2} + 3 - \gamma\right]m^2/\rho = (\gamma-1)\epsilon + \left[\frac{3}{2} - \frac{\gamma}{2}\right]m^2/\rho$

$\boxed{F_2 = (\gamma-1)\epsilon + (3-\gamma)m^2/2\rho}$

$F_3 = \rho(\gamma-1)\frac{m^3}{\rho^3} - \frac{ma^2}{\gamma-1} - \gamma\rho m^3/2\rho^3 + \left(\frac{3}{2}-\gamma\right)m\frac{m^2}{\rho^2} + \frac{ma^2}{\gamma-1} + \frac{\gamma m\epsilon}{\rho}$

$u^3 = \left(\frac{m}{\rho}\right)^3$ $\quad$ these cancel

$F_3 = (\gamma-1)m^3/\rho^2 - \gamma m^3/2\rho^2 + \left(\frac{3}{2}-\gamma\right)m^3/\rho^2 + \frac{\gamma m\epsilon}{\rho}$

$F_3 = \left[(\gamma-1) - \frac{\gamma}{2} + \left(\frac{3}{2}-\gamma\right)\right]m^3/\rho^2 + \frac{\gamma m\epsilon}{\rho}$

$F_3 = \left[\frac{1}{2} - \frac{\gamma}{2}\right]m^3/\rho^2 + \frac{\gamma m\epsilon}{\rho} = (1-\gamma)m^3/\rho^2 + \frac{\gamma m\epsilon}{\rho}$

$\rightarrow$

$$\boxed{F_3 = \frac{\gamma m \epsilon}{\rho} - \frac{(\gamma-1)m^3}{\rho^2}}$$

Now, reassemble F vector:

$$F = \underset{\sim}{A}Q = \begin{bmatrix} m \\ (\gamma-1)\epsilon + (3-\gamma)m^2/2\rho \\ \gamma m\epsilon/\rho - (\gamma-1)m^3/\rho^2 \end{bmatrix}$$

which is exactly eq. (3.16) from the typeset class notes

3) $F_1^{\pm} = \frac{\rho}{2\gamma}\left\{\lambda_1^{\pm} + \lambda_2^{\pm} + 2(\gamma-1)\lambda_3^{\pm}\right\}$

manipulate each entry in $F^{\pm}$ so that it is in terms of ρ, a, M, and γ

$$\lambda_1^{\pm} = \frac{(u-a) \pm |u-a|}{2}$$

$$\lambda_2^{\pm} = \frac{(u+a) \pm |u+a|}{2} \qquad \lambda_3^{\pm} = \frac{u \pm |u|}{2}$$

$$\lambda_1^{\pm} = a\frac{[(M-1) \pm |M-1|]}{2} \qquad \lambda_2^{\pm} = a\frac{[(M+1) \pm |M+1|]}{2}$$

$$\lambda_3^{\pm} = a\frac{[M \pm |M|]}{2}$$

These substitutions will be used for $F_2^{\pm}$ and $F_3^{\pm}$ as well.

$$F_1^{\pm} = \frac{\rho a}{2\gamma}\left\{\frac{(M-1) \pm |M-1|}{2} + \frac{(M+1) \pm |M+1|}{2} + \cancel{2}(\gamma-1)\frac{(M \pm |M|)}{\cancel{2}}\right\}$$

$$= \frac{\rho a}{2\gamma}\left\{\underbrace{\frac{M-1}{2} + \frac{M+1}{2}}_{=M} \pm \underbrace{\left(\frac{|M+1|}{2} + \frac{|M-1|}{2}\right)} + \gamma M \pm \gamma|M| - M \mp |M|\right\}$$

if $-1 \le M \le 1$, $\frac{|M+1|}{2} + \frac{|M-1|}{2} = 1$

This substitution will appear again later!

$$F_1^{\pm} = \frac{\rho a}{2\gamma}\left\{M \pm 1 + \gamma M \pm \gamma|M| - M \mp |M|\right\}$$

eliminate

→

$$F_1^{\pm} = \frac{\rho a}{2}\left\{ \pm\frac{1}{\gamma} \underbrace{+ M \pm |M|}_{\text{good}} \mp \frac{1}{\gamma}|M| \right\}$$

$$\boxed{F_1^{\pm} = \frac{\rho a}{2}\left\{ M \pm |M| \pm \left(1 - |M|\right)\frac{1}{\gamma} \right\}}$$

$$F_2^{\pm} = \frac{\rho}{2\gamma}\left\{ (u-a)\lambda_1^{\pm} + (u+a)\lambda_2^{\pm} + 2(\gamma-1)u\lambda_3^{\pm} \right\}$$

$$= \frac{\rho a}{2\gamma}\left\{ (M-1)\lambda_1^{\pm} + (M+1)\lambda_2^{\pm} + 2(\gamma-1)M\lambda_3^{\pm} \right\}$$

substitute in $\lambda_1^{\pm}, \lambda_2^{\pm}, \lambda_3^{\pm}$ definitions as before

$$= \frac{\rho a^2}{2\gamma}\left\{ (M-1)a\frac{[(M-1) \pm |M-1|]}{2} + (M+1)a\frac{[(M+1) \pm |M-1|]}{2} \right.$$

$$\left. + 2(\gamma-1)Ma\frac{[M \pm |M|]}{2} \right\}$$

$$= \frac{\rho a^2}{2\gamma}\left\{ \frac{(M-1)^2 \pm (M-1)|M-1|}{2} + \frac{(M+1)^2 \pm (M+1)|M+1|}{2} \right.$$

$$\left. + (\gamma-1)\left[M^2 \pm M|M|\right] \right\}$$

$$= \frac{\rho a^2}{2\gamma}\left\{ \frac{M^2 - 2M + 1}{2} + \frac{M^2 + 2M + 1}{2} \pm \left[\frac{(M+1)|M+1|}{2} + \frac{(M-1)|M-1|}{2}\right] \right.$$

$$\left. + \gamma M^2 \pm \gamma M|M| - M^2 \mp M|M| \right\}$$

$$= \frac{\rho a^2}{2\gamma}\left\{ M^2 + 1 \pm \left[M\left(\frac{|M+1|}{2} + \frac{|M-1|}{2}\right) + \frac{|M+1|}{2} - \frac{|M-1|}{2} \right] \right.$$

$$\left. + \gamma M^2 \pm \gamma M|M| - M^2 \mp M|M| \right\}$$

$\rightarrow$

recall that $\frac{|M+1|}{2} + \frac{|M-1|}{2} = 1$ for $-1 \le M \le 1$

also recognize that $\frac{|M+1|}{2} - \frac{|M-1|}{2} = M$ for $-1 \le M \le 1$

make both of these substitutions:

$$F_2^{\pm} = \frac{\rho a^2}{2\gamma}\left\{1 \pm [M(1) + M] \pm \gamma M|M| \mp M|M| + \gamma M^2\right\}$$

$$= \frac{\rho a^2}{2}\left\{M^2 + \frac{1}{\gamma} \pm \frac{2M}{\gamma} \pm M|M| \mp \frac{M|M|}{\gamma}\right\}$$

$$= \frac{\rho a^2}{2}\left\{M^2 + \frac{1}{\gamma} \pm \left(\frac{2M}{\gamma} + M|M| - \frac{M|M|}{\gamma}\right)\right\}$$

recognize $M|M| = \frac{\gamma}{\gamma} M|M|$, so $\frac{\gamma}{\gamma}M|M| - \frac{M|M|}{\gamma} = M|M|\left(\frac{\gamma}{\gamma} - \frac{1}{\gamma}\right)$

$= M|M|\frac{\gamma-1}{\gamma}$ (a similar substitution is made later)

$$\boxed{F_2^{\pm} = \frac{\rho a^2}{2}\left\{M^2 + \frac{1}{\gamma} \pm M\left(\frac{2}{\gamma} + |M|\frac{\gamma-1}{\gamma}\right)\right\}}$$

$$F_3^{\pm} = \frac{\rho}{2\gamma}\left\{\underbrace{\frac{1}{2}(u-a)^2\lambda_1^{\pm} + \frac{1}{2}(u+a)^2\lambda_2^{\pm}}_{1} + \underbrace{(\gamma-1)u^2\lambda_3^{\pm}}_{2} + \underbrace{\frac{(3-\gamma)a^2(\lambda_1^{\pm}+\lambda_2^{\pm})}{2(\gamma-1)}}_{3}\right\}$$

Due to the size of this, it is convenient to group terms and treat each group independently

$\longrightarrow$

$$①: \frac{\rho}{2\gamma}\left\{\frac{1}{2}(u-a)^2\lambda_1^{\pm} + \frac{1}{2}(u+a)^2\lambda_2^{\pm}\right\}$$

$$= \frac{\rho}{2\gamma}\left\{\frac{1}{2}a^2(M-1)^2 a\frac{[(M-1)\pm|M-1|]}{2} + \frac{1}{2}a^2(M+1)^2 a\frac{[(M+1)\pm|M+1|]}{2}\right\}$$

$$= \frac{\rho a^3}{2\gamma}\left\{\frac{1}{2}\frac{(M-1)^3\pm(M-1)^2|M-1|}{2} + \frac{1}{2}\frac{(M+1)^3\pm(M+1)^2|M+1|}{2}\right\}$$

expand $(M-1)^3$, $(M+1)^3$, $(M-1)^2$, $(M+1)^2$

$$= \frac{\rho a^3}{2\gamma}\left\{\frac{1}{2}\left[\frac{M^3-3M^2+3M-1}{2} + \frac{M^3+3M^2+3M+1}{2} \pm (M^2\boxed{-2M}+1)\frac{|M-1|}{2} \pm (M^2+\boxed{2M}+1)\frac{|M+1|}{2}\right]\right\}$$

eliminate some terms, separate the 2M

$$= \frac{\rho a^3}{2\gamma}\left\{\frac{1}{2}\left[M^3+3M \pm\left((M^2+1)\frac{|M-1|}{2} - 2M\frac{|M-1|}{2} + (M^2+1)\frac{|M+1|}{2} + 2M\frac{|M+1|}{2}\right)\right]\right\}$$

collect terms on $|M-1|$ and $|M+1|$

$$= \frac{\rho a^3}{2\gamma}\left\{\frac{1}{2}\left[M^3+3M \pm\left((M^2+1)\underbrace{\left(\frac{|M+1|}{2}+\frac{|M-1|}{2}\right)}_{=1} + 2M\underbrace{\left(\frac{|M+1|}{2}-\frac{|M-1|}{2}\right)}_{=M}\right)\right]\right\}$$

make the same substitutions as previously

$$= \frac{\rho a^3}{2\gamma}\left\{\frac{1}{2}\left[M^3+3M\pm\left(M^2+1+2M^2\right)\right]\right\}$$

$$① = \frac{\rho a^3}{2\gamma}\left\{\frac{1}{2}\left[M^3+3M\pm(3M^2+1)\right]\right\}$$

$$① = \frac{\rho a^3}{2}\left\{\frac{M^3}{2\gamma} + \frac{3M}{2\gamma} \pm\left(\frac{3M^2}{2\gamma}+\frac{1}{2\gamma}\right)\right\}$$

now work on term ②

→

②: $\frac{\rho}{2\gamma}\left\{(\gamma-1)u^2\lambda_3^{\pm}\right\}$

$$= \frac{\rho}{2\gamma}\left\{(\gamma-1)M^2a^2\frac{a[M\pm|M|]}{2}\right\} = \frac{\rho a^3}{2\gamma}\left\{(\gamma-1)\frac{M^3\pm M^2|M|}{2}\right\}$$

$$= \frac{\rho a^3}{2\gamma}\left\{\frac{\gamma M^3}{2}-\frac{M^3}{2}\pm\frac{\gamma M^2|M|}{2}\mp\frac{M^2|M|}{2}\right\}$$

$$= \frac{\rho a^3}{2\gamma}\left\{\frac{\gamma M^3}{2}-\frac{M^3}{2}\pm\left(\frac{\gamma M^2|M|}{2}-\frac{M^2|M|}{2}\right)\right\}$$

$$= \frac{\rho a^3}{2}\left\{\frac{M^3}{2}-\frac{M^3}{2\gamma}\pm\left(\underbrace{\frac{M^2|M|}{2}-\frac{M^2|M|}{2\gamma}}\right)\right\}$$

$$\frac{M^2|M|}{2} = \frac{\gamma M^2|M|}{2\gamma}$$

$$= \frac{\rho a^3}{2}\left\{\frac{M^3}{2}-\frac{M^3}{2\gamma}\pm\left(\frac{\gamma M^2|M|}{2\gamma}-\frac{M^2|M|}{2\gamma}\right)\right\}$$

$$② = \frac{\rho a^3}{2}\left\{\frac{M^3}{2}-\frac{M^3}{2\gamma}\pm\frac{M^2|M|}{2}\left(\frac{\gamma-1}{\gamma}\right)\right\}$$

now, work on term ③

③: $\frac{\rho}{2\gamma}\left\{\frac{(3-\gamma)a^2}{2(\gamma-1)}\left(\underbrace{\lambda_1^{\pm}+\lambda_2^{\pm}}\right)\right\}$

$$\frac{a[(M-1)\pm|M-1|]}{2}+\frac{a[(M+1)\pm|M+1|]}{2} = a\left[\underbrace{\frac{(M-1)}{2}+\frac{(M+1)}{2}}_{=M}\right.$$

$$\left.\pm\underbrace{\left(\frac{|M+1|}{2}+\frac{|M-1|}{2}\right)}_{=1 \text{ from before}}\right] = a[M\pm 1]$$

$\longrightarrow$

$$③ = \frac{\rho}{2\gamma}\left\{\frac{(3-\gamma)a^3}{2(\gamma-1)}\left[M \pm 1\right]\right\} = \frac{\rho a^3}{2}\left\{\frac{3-\gamma}{2\gamma(\gamma-1)}M \pm \frac{3-\gamma}{2\gamma(\gamma-1)}\right\}$$

now, assemble $F_3^{\pm} = ① + ② + ③$

$$F_3^{\pm} = \frac{\rho a^3}{2}\left\{\frac{M^3}{2\gamma} + \frac{3M}{2\gamma} \pm \left(\frac{3M^2}{2\gamma} + \frac{1}{2\gamma}\right) + \frac{M^3}{2} \cancel{- \frac{M^3}{2\gamma}} \pm \frac{M^2|M|}{2}\left(\frac{\gamma-1}{\gamma}\right)\right.$$

$$\left. + \frac{(3-\gamma)M}{2\gamma(\gamma-1)} \pm \frac{3-\gamma}{2\gamma(\gamma-1)}\right\}$$

eliminate circled terms

collect terms on M^3, $M^2|M|$

$$F_3^{\pm} = \frac{\rho a^3}{2}\left\{\frac{M^3}{2} \pm \frac{M^2|M|}{2}\left(\frac{\gamma-1}{\gamma}\right) + \frac{3M}{2\gamma} \pm \left(\frac{3M^2}{2\gamma} + \frac{1}{2\gamma}\right)\right.$$

$$\left. + \frac{(3-\gamma)M}{2\gamma(\gamma-1)} \pm \frac{3-\gamma}{2\gamma(\gamma-1)}\right\}$$

$$F_3^{\pm} = \frac{\rho a^3}{2}\left\{\underbrace{\frac{M^2}{2}\left(M \pm |M|\frac{\gamma-1}{\gamma}\right)}_{\text{good!}} + \underbrace{\frac{3M(\gamma-1)}{2\gamma(\gamma-1)}}_{\cdot\frac{\gamma-1}{\gamma-1}} \pm \left(\frac{3M^2}{2\gamma} + \boxed{\underbrace{\frac{\gamma-1}{2\gamma(\gamma-1)}}_{\cdot\frac{\gamma-1}{\gamma-1}}}\right)\right.$$

$$\left. + \frac{(3-\gamma)M}{2\gamma(\gamma-1)} \boxed{\pm \frac{3-\gamma}{2\gamma(\gamma-1)}}\right\}$$

add terms circled

$$F_3^{\pm} = \frac{\rho a^3}{2}\left\{\frac{M^2}{2}\left(M \pm |M|\frac{\gamma-1}{\gamma}\right) + \underbrace{\frac{3M\gamma - 3M + 3M - M\gamma}{2\gamma(\gamma-1)}}_{= \frac{2\gamma M}{2\gamma(\gamma-1)}} \pm \left(\frac{3M^2}{2\gamma} + \right.\right.$$

$$\left.\left. \underbrace{\frac{3-\gamma+\gamma-1}{2\gamma(\gamma-1)}}_{= \frac{2}{2\gamma(\gamma-1)}}\right)\right\} \longrightarrow$$

Finally, we can make some simplifications

$$F_3^{\pm} = \frac{\rho a^3}{2}\left[\frac{M^2}{2}\left(M \pm |M|\frac{\gamma-1}{\gamma}\right) + \frac{M}{\gamma-1} \pm \left(\frac{3M^2}{2\gamma} + \frac{1}{\gamma(\gamma-1)}\right)\right\}$$

bring this inside, it becomes $\pm \frac{M}{\gamma-1}$

$$\boxed{F_3^{\pm} = \frac{\rho a^3}{2}\left\{\frac{M^2}{2}\left(M \pm |M|\frac{\gamma-1}{\gamma}\right) \pm \left(\frac{3M^2}{2\gamma} \pm \frac{M}{\gamma-1} + \frac{1}{\gamma(\gamma-1)}\right)\right\}}$$

Now we can assemble $F^{\pm}$!

$$\boxed{F^{\pm} = \begin{bmatrix} \frac{\rho a}{2}\left\{M \pm |M| \pm (1-|M|)\frac{1}{\gamma}\right\} \\ \frac{\rho a^2}{2}\left\{M^2 + \frac{1}{\gamma} \pm M\left(\frac{2}{\gamma} + |M|\frac{\gamma-1}{\gamma}\right)\right\} \\ \frac{\rho a^3}{2}\left\{\frac{M^2}{2}\left(M \pm |M|\frac{\gamma-1}{\gamma}\right) \pm \left(\frac{3M^2}{2\gamma} \pm \frac{M}{\gamma-1} + \frac{1}{\gamma(\gamma-1)}\right)\right\} \end{bmatrix}}$$

4) Starting from

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho e_t \end{bmatrix} \qquad F = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho u H \end{bmatrix} \qquad Z = \sqrt{\rho} \begin{bmatrix} 1 \\ u \\ H \end{bmatrix}$$

Write out each Roe parameter vector term:

$$Z_1 = \sqrt{\rho} \qquad Z_2 = \sqrt{\rho}\, u \qquad Z_3 = \sqrt{\rho}\, H = \sqrt{\rho}\left(e + \frac{P}{\rho} + \frac{u^2}{2}\right)$$

Useful to know e, P in terms of Z_1, Z_2, Z_3, γ

$$Z_3 = \sqrt{\rho}\left(e + \underbrace{\frac{P}{\rho}}_{\substack{=(\gamma-1)e \\ \text{from EOS}}} + u^2/2\right) = \underbrace{\sqrt{\rho}}_{=Z_1}\left(e + (\gamma-1)e + u^2/2\right)$$

$$Z_3 = Z_1\left(\gamma e + u^2/2\right) \qquad \underbrace{\rho}_{Z_1^2} Z_3 = Z_1\left(\rho\gamma e + \underbrace{\rho u^2/2}_{=Z_2^2/2}\right)$$

$$Z_1^2 Z_3 = Z_1\left(Z_1^2 \gamma e + Z_2^2/2\right) \qquad Z_1 Z_3 = Z_1^2 \gamma e + Z_2^2/2$$

$$Z_1 Z_3 - \frac{Z_2^2}{2} = Z_1^2 \gamma e \quad \rightarrow \quad \boxed{e = \frac{Z_3}{Z_1 \gamma} - \frac{Z_2^2}{2 Z_1^2 \gamma}} \quad (1)$$

from EOS, $P = e(\gamma-1)\rho =$

$$P = (\gamma-1) Z_1^2 \left[\frac{Z_3}{Z_1 \gamma} - \frac{Z_2^2}{2 Z_1^2 \gamma}\right] = (\gamma-1)\left[\frac{Z_1 Z_3}{\gamma} - \frac{Z_2^2}{2\gamma}\right]$$

$$\boxed{P = Z_1 Z_3 - \frac{Z_2^2}{2} - \frac{Z_1 Z_3}{\gamma} + \frac{Z_2^2}{2\gamma}} \quad (2)$$

$\rightarrow$

Now, consider each element of Q and F:

$$Q_1 = \rho = \sqrt{\rho}^2 \rightarrow \boxed{Q_1 = Z_1^2}$$

$$Q_2 = \rho u = \sqrt{\rho}\sqrt{\rho}\,u \rightarrow \boxed{Q_2 = Z_1 Z_2}$$

$$Q_3 = \rho e_t = \rho(e + u^2/2) = \underbrace{\rho}_{Z_1^2} e + \underbrace{\rho u^2/2}_{(\sqrt{\rho}u)^2/2}$$

substitute ①

$$Q_3 = Z_1^2\left[\frac{Z_3}{Z_1\gamma} - \frac{Z_2^2}{2Z_1^2\gamma}\right] + \frac{Z_2^2}{2}\cdot\frac{\gamma}{\gamma}$$

$$Q_3 = \frac{Z_1 Z_3}{\gamma} + \frac{Z_2^2\gamma}{2\gamma} - \frac{Z_2^2}{2\gamma} \rightarrow \boxed{Q_3 = \frac{Z_1 Z_3}{\gamma} + \frac{(\gamma-1)}{2\gamma} Z_2^2}$$

$$\boxed{F_1 = \rho u = Z_1 Z_2}$$

$$F_2 = \underbrace{\rho u^2}_{(\sqrt{\rho}u)^2} + \underbrace{P}_{\text{substitute in ②}} = Z_2^2 + Z_1 Z_3 - \frac{Z_2^2}{2} - \frac{Z_1 Z_3}{\gamma} + \frac{Z_2^2}{2\gamma}$$

Group Z_2^2, $Z_1 Z_3$ terms:

$$F_2 = Z_2^2\left(1 - \frac{1}{2} + \frac{1}{2\gamma}\right) + Z_1 Z_3\left(1 - \frac{1}{\gamma}\right)$$

$$= Z_2^2\left(\frac{1}{2}\cdot\frac{\gamma}{\gamma} + \frac{1}{2\gamma}\right) + Z_1 Z_3\left(1\cdot\frac{\gamma}{\gamma} - \frac{1}{\gamma}\right)$$

$$\boxed{F_2 = \left(\frac{\gamma - 1}{\gamma}\right) Z_1 Z_3 + \left(\frac{\gamma+1}{2\gamma}\right) Z_2^2}$$

$\rightarrow$

$$F_3 = \rho u H = \underbrace{\sqrt{\rho}\, u}_{Z_2} \underbrace{\sqrt{\rho}\, H}_{Z_3} \rightarrow \boxed{F_3 = Z_2 Z_3}$$

Now, assemble Q and F in terms of the Z Roe parameter vector:

$$Q = \begin{bmatrix} Z_1^2 \\ Z_1 Z_2 \\ \frac{Z_1 Z_3}{\gamma} + \frac{(\gamma-1)}{2\gamma} Z_2^2 \end{bmatrix}$$

$$F = \begin{bmatrix} Z_1 Z_2 \\ \left(\frac{\gamma-1}{\gamma}\right) Z_1 Z_3 + \left(\frac{\gamma+1}{2\gamma}\right) Z_2^2 \\ Z_2 Z_3 \end{bmatrix}$$

5) First, it is important to note that Q and F are quadratic in z, which means that the jump in states at $i+\frac{1}{2}$ are exactly equal to Δz times the Jacobian $\left(\frac{\partial Q}{\partial z} \text{ and } \frac{\partial F}{\partial z} \text{ respectively}\right)$ evaluated at the arithmetic average of z (denoted as $\bar{z}$) using z_L (left state) and z_R (right state).

$$\underbrace{\Delta Q}_{\text{jump in } Q} = \underbrace{\left.\frac{\partial Q}{\partial z}\right|_{\bar{z}}}_{\text{Jacobian}} \Delta z = \underset{\sim}{B}\,\Delta z \qquad \Delta z = \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \\ \Delta z_3 \end{bmatrix}$$

$$\underbrace{\Delta F}_{\text{jump in } F} = \left.\frac{\partial F}{\partial z}\right|_{\bar{z}} \Delta z = \underset{\sim}{C}\,\Delta z$$

where $\bar{z} = \frac{1}{2}\left(z_L + z_R\right)$

$$\frac{\partial Q}{\partial z} = \begin{bmatrix} \frac{\partial Q_1}{\partial z_1} & \frac{\partial Q_1}{\partial z_2} & \frac{\partial Q}{\partial z_1} \\ \frac{\partial Q_2}{\partial z_1} & \frac{\partial Q_2}{\partial z_2} & \frac{\partial Q_2}{\partial z_3} \\ \frac{\partial Q_3}{\partial z_1} & \frac{\partial Q_3}{\partial z_2} & \frac{\partial Q_3}{\partial z_3} \end{bmatrix}$$

$$\frac{\partial Q_1}{\partial z_1} = \frac{\partial}{\partial z_1}\left(z_1^2\right) = 2z_1 \qquad \frac{\partial Q_1}{\partial z_2} = \frac{\partial}{\partial z_2}\left(z_1^2\right) = 0$$

→

$$\frac{\partial Q_1}{\partial z_3} = \frac{\partial}{\partial z_3}\left(z_1^2\right) = 0 \qquad \frac{\partial Q_2}{\partial z_1} = \frac{\partial}{\partial z_1}\left(z_1 z_2\right) = z_2$$

$$\frac{\partial Q_2}{\partial z_2} = \frac{\partial}{\partial z_2}\left(z_1 z_2\right) = z_1 \qquad \frac{\partial Q_2}{\partial z_3} = \frac{\partial}{\partial z_3}\left(z_1 z_2\right) = 0$$

$$\frac{\partial Q_3}{\partial z_1} = \frac{\partial}{\partial z_1}\left(\frac{z_1 z_3}{\gamma} + \frac{(\gamma-1)}{2\gamma} z_2^2\right) = \frac{z_3}{\gamma}$$

$$\frac{\partial Q_3}{\partial z_2} = \frac{\partial}{\partial z_2}\left(\frac{z_1 z_3}{\gamma} + \frac{(\gamma-1)}{2\gamma} z_2^2\right) = \frac{(\gamma-1)}{\gamma} z_2$$

$$\frac{\partial Q_3}{\partial z_3} = \frac{\partial}{\partial z_3}\left(\frac{z_1 z_3}{\gamma} + \frac{(\gamma-1)}{2\gamma} z_2^2\right) = \frac{z_1}{\gamma}$$

Now, assemble $\frac{\partial Q}{\partial z}$

$$\frac{\partial Q}{\partial z} = \begin{bmatrix} 2z_1 & 0 & 0 \\ z_2 & z_1 & 0 \\ z_3/\gamma & (\gamma-1)z_2/\gamma & z_1/\gamma \end{bmatrix}$$

evaluate $\frac{\partial Q}{\partial z}$ at $\bar{z}$

$$z_1 = \bar{z}_1 = \frac{1}{2}\left(z_{1L} + z_{1R}\right) \qquad z_3 = \bar{z}_3 = \frac{1}{2}\left(z_{3L} + z_{3R}\right)$$

$$z_2 = \bar{z}_2 = \frac{1}{2}\left(z_{2L} + z_{2R}\right)$$

$$\boxed{\underset{\sim}{B} = \left.\frac{\partial Q}{\partial z}\right|_{\bar{z}} = \begin{bmatrix} 2\bar{z}_1 & 0 & 0 \\ \bar{z}_2 & \bar{z}_1 & 0 \\ \bar{z}_3/\gamma & (\gamma-1)\bar{z}_2/\gamma & \bar{z}_1/\gamma \end{bmatrix}}$$

Now, calculate $\frac{\partial F}{\partial z}$ and $\underset{\sim}{C} = \left.\frac{\partial F}{\partial z}\right|_{\bar{z}}$ $\rightarrow$

$$\frac{\partial F}{\partial Z} = \begin{bmatrix} \frac{\partial F_1}{\partial Z_1} & \frac{\partial F_1}{\partial Z_2} & \frac{\partial F_3}{\partial Z_3} \\ \frac{\partial F_2}{\partial Z_1} & \frac{\partial F_2}{\partial Z_2} & \frac{\partial F_2}{\partial Z_3} \\ \frac{\partial F_3}{\partial Z_1} & \frac{\partial F_3}{\partial Z_2} & \frac{\partial F_3}{\partial Z_3} \end{bmatrix}$$

$$\frac{\partial F_1}{\partial Z_1} = \frac{\partial}{\partial Z_1}\left(Z_1 Z_2\right) = Z_2 \qquad \frac{\partial F_1}{\partial Z_2} = Z_1$$

$$\frac{\partial F_1}{\partial Z_3} = 0 \qquad \frac{\partial F_2}{\partial Z_1} = \frac{\partial}{\partial Z_1}\left(\frac{(\gamma-1)}{\gamma} Z_1 Z_3 + \frac{(\gamma+1)}{2\gamma} Z_2^2\right) = \frac{(\gamma-1)}{\gamma} Z_3$$

$$\frac{\partial F_2}{\partial Z_2} = \frac{(\gamma+1)}{\gamma} Z_2 \qquad \frac{\partial F_2}{\partial Z_3} = \frac{(\gamma-1)}{\gamma} Z_1 \qquad \frac{\partial F_3}{\partial Z_1} = \frac{\partial}{\partial Z_1}\left(Z_2 Z_3\right) = 0$$

$$\frac{\partial F_3}{\partial Z_2} = Z_3 \qquad \frac{\partial F_3}{\partial Z_3} = Z_2$$

Assemble $\frac{\partial F}{\partial Z}$:

$$\frac{\partial F}{\partial Z} = \begin{bmatrix} Z_2 & Z_1 & 0 \\ \frac{(\gamma-1)}{\gamma} Z_3 & \frac{(\gamma+1)}{\gamma} Z_2 & \frac{(\gamma-1)}{\gamma} Z_1 \\ 0 & Z_3 & Z_2 \end{bmatrix}$$

evaluate at $\bar{Z}$

$$\boxed{\underset{\sim}{C} = \left.\frac{\partial F}{\partial Z}\right|_{\bar{Z}} = \begin{bmatrix} \bar{Z}_2 & \bar{Z}_1 & 0 \\ \frac{(\gamma-1)}{\gamma} \bar{Z}_3 & \frac{(\gamma+1)}{\gamma} \bar{Z}_2 & \frac{(\gamma-1)}{\gamma} \bar{Z}_1 \\ 0 & \bar{Z}_3 & \bar{Z}_2 \end{bmatrix}}$$

Finally, assemble full expressions for jumps in Q and F at interface:

$$\Delta Q = \underset{\sim}{B} \Delta Z = \begin{bmatrix} 2\bar{Z}_1 & 0 & 0 \\ \bar{Z}_2 & \bar{Z}_1 & 0 \\ \frac{\bar{Z}_3}{\gamma} & \frac{(\gamma-1)}{\gamma}\bar{Z}_2 & \frac{\bar{Z}_1}{\gamma} \end{bmatrix} \begin{bmatrix} \Delta Z_1 \\ \Delta Z_2 \\ \Delta Z_3 \end{bmatrix}$$

$$\Delta F = \underset{\sim}{C} \Delta Z = \begin{bmatrix} \bar{Z}_2 & \bar{Z}_1 & 0 \\ \frac{(\gamma-1)}{\gamma}\bar{Z}_3 & \frac{(\gamma+1)}{\gamma}\bar{Z}_2 & \frac{(\gamma-1)}{\gamma}\bar{Z}_1 \\ 0 & \bar{Z}_3 & \bar{Z}_2 \end{bmatrix} \begin{bmatrix} \Delta Z_1 \\ \Delta Z_2 \\ \Delta Z_3 \end{bmatrix}$$