Introduction to Fourier Series and Fourier Transforms

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1 Fourier Series

The core idea behind Fourier transforms is that

> "Any" periodic function can be expressed as a sum of sines and cosines.

If this set of notes were written for mathematicians, we would probably want to qualify the word "any", since there *do* exist periodic functions that can be expressed as a sum of sines and cosines. But from a physicist/astronomer's perspective, pretty much any periodic function that we deal with can be expressed as a sum of sines and cosines.

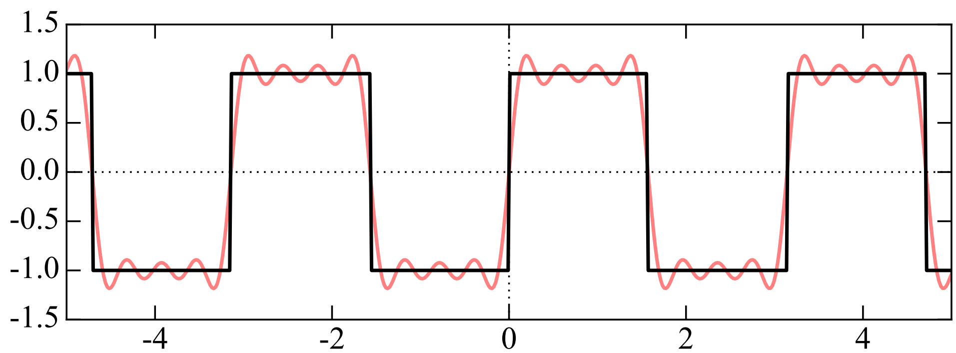

Figure 1: A square wave (black) approximated by a sum of cosines (red).

Let's consider a quick example. In Figure 1 above, we show in black a periodic function $g(t)$ with period π. In red we show an approximation of the function $g_{\rm approx}(t)$ given by

$$g_{\rm approx}(t) = \frac{4}{\pi}\sin(2t) + \frac{4}{3\pi}\sin(6t) + \frac{4}{5\pi}\sin(10t) + \frac{4}{7\pi}\sin(14t). \tag{1}$$

This approximate expression, built out of a sum of sines, does pretty well in mimicking the original function! To get some intuition for how this works, we show in Figure 2 the first two terms in this sum. The way the approximation works is that the first sine term (blue) captures the rough periodicity and ups and downs of the function. But this first term alone is not enough—the peaks of the sine are too high, and on either side of each peak the function drops off too quickly compared to the function we want to mimic. The next sine term (orange) fixes this by lowering the peak a little and shoring up the shoulders a little. Note that since the first term roughly got thing right, this is a small correction. As one can see from examining the coefficients in Equation 1, the higher order terms die off pretty quickly, which means that by utilizing only a small handful of sines, it's possible to get a great approximation like we had in Figure 1. But the more terms we add, the more corrections we can make, and the more closely we mimic the original function. What Fourier proved was that if we add an *infinite* number of sines and cosines with just the right coefficients, we can approximate any periodic function to arbitrary accuracy.

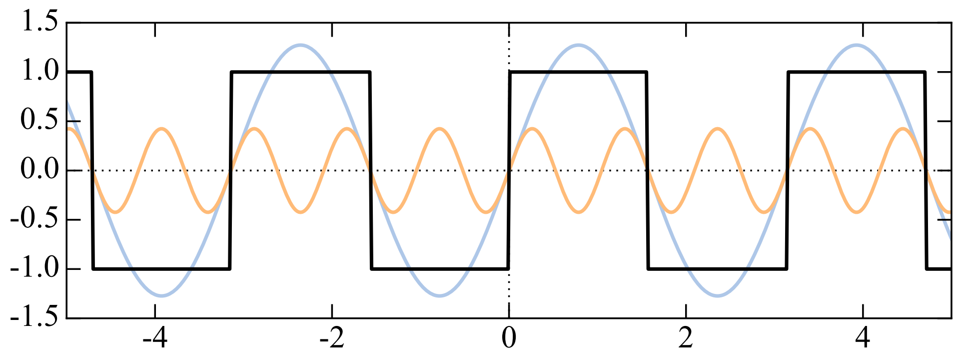

Figure 2: The first two terms of Equation 1.

The key, then, is to figure out "just the right coefficients". Mathematically, if we presuppose (as Fourier tells us we're allowed to), that a periodic function $g(t)$ with period T can be expressed as

$$g(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right) \right], \tag{2}$$

then our goal is to find out what all the A_ns and B_ns are. To do so, it is helpful to remember that

$$\int_0^T dt \cos\left(\frac{2\pi nt}{T}\right) \sin\left(\frac{2\pi mt}{T}\right) = 0, \tag{3}$$

and

$$\int_0^T dt \cos\left(\frac{2\pi nt}{T}\right) \cos\left(\frac{2\pi mt}{T}\right) = \int_0^T dt \sin\left(\frac{2\pi nt}{T}\right) \sin\left(\frac{2\pi mt}{T}\right) = \frac{T}{2}\delta_{mn}, \tag{4}$$

where δ_{mn} is the *Kronecker delta function*, which takes the value 1 if $m = n$ and 0 if $m \neq n$. In words, these relations tell us that if we integrate a sine and a cosine over a full period, the answer is zero. The same is true if we integrate the product of two cosines or a product of two sines, as long as they have different periods. The only time when the integral is nonzero is when $n = m$, i.e., when we integrate a cosine or a sine against itself.

Notice the following trick. Suppose, as an example, I multiplied both sides of Equation 2 by $\cos(4\pi t/T)$ and then integrated from 0 to T. This gives

$$\begin{aligned}\int_0^T dt\,\cos\left(\frac{4\pi t}{T}\right) g(t) = \frac{A_0}{2}\int_0^T dt\,\cos\left(\frac{4\pi t}{T}\right) + \sum_{n=1}^{\infty}\Bigg[A_n \int_0^T dt\,\cos\left(\frac{4\pi t}{T}\right)\cos\left(\frac{2\pi n t}{T}\right) \\ + B_n \int_0^T dt\,\cos\left(\frac{4\pi t}{T}\right)\sin\left(\frac{2\pi n t}{T}\right)\Bigg].\end{aligned} \tag{5}$$

Using the integrals we established above, we see that the first and third terms on the right hand side integrate to zero. The second integral on the right is also zero *unless* $n = 2$. In other words, of the infinite number of terms on the right hand side, all but one of them drops out! The result is

$$\int_0^T dt\,\cos\left(\frac{4\pi t}{T}\right) g(t) = \frac{A_2 T}{2}, \tag{6}$$

which allows us to solve for A_2. From this, one can imagine what it takes to find any of the A_n or B_ns—simply integrate $f(t)$ against the corresponding sine or cosine and multiply by $2/T$. Explicitly, we can say

$$\boxed{A_n = \frac{2}{T}\int_0^T dt\,\cos\left(\frac{2\pi n t}{T}\right) f(t)} \tag{7}$$

and

$$\boxed{B_n = \frac{2}{T}\int_0^T dt\,\sin\left(\frac{2\pi n t}{T}\right) f(t)}. \tag{8}$$

We've thus accomplished our goal of figuring out how to express any periodic function as a sum of sines and cosines.

2 Complex Fourier Series

It turns out that sines and cosines aren't the only building blocks that can be used to reconstruct a periodic function. Other building blocks are possible, and one important alternative is to use complex exponentials. That this is possible is unsurprising because of Euler's relation, which says that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{9}$$

In other words, complex exponentials are just linear combinations of sines and cosines. Thus, a linear combination of complex exponentials is also a linear combination of sines and cosines, and if it's possible to reconstruct a periodic function using the latter, it must also be possible using the former. Here, we say

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T}, \tag{10}$$

and to solve for the coefficients, we compute

$$c_n = \frac{1}{T} \int_0^T dt\, e^{-i2\pi nt/T} g(t). \tag{11}$$

There are several important things to note here:

- Suppose $g(t)$ is a real-valued (as opposed to a complex-valued) function. Decomposing this function into sines and cosines (i.e., using Equation 2) will give A_n and B_n that are real numbers. In general, however, the c_n in Equation 10 will be complex even if $g(t)$ is real.

- The prescription with the sine/cosine Fourier transform was to solve for each coefficient by integrating the original function against the sine/cosine that is attached to the coefficient in question. Here, however, the prescription is to integrate the original function against the *complex conjugate* of the complex exponential (notice the difference in the signs of the exponents between Equations 10 and 11). Can you prove that this is correct?

3 Fourier Transforms

One shortcoming of Fourier series is that they require $g(t)$ to be a periodic function. The concept of a *Fourier transform* removes this restriction by thinking of a non-periodic function as a periodic function with $T \to \infty$. Said differently, a function that never repeats itself is just a function that takes infinitely long before it repeats itself.

Consider again the Fourier series of the square wave (similar to that shown in Figure 1), but with a period of $T = 3$:

$$g_{T=3}(t) = \frac{4}{\pi} \sin\left(\frac{2\pi t}{3}\right) + \frac{4}{3\pi} \sin\left(\frac{2\pi 3t}{3}\right) + \frac{4}{5\pi} \sin\left(\frac{2\pi 5t}{3}\right) + \frac{4}{7\pi} \sin\left(\frac{2\pi 7t}{3}\right) + \dots \tag{12}$$

(Our reason for writing this expression in blue—and in such a bizarre way without simplifying some of the fractions— will soon be evident). Note that each term is of the form $\sin(2\pi f t)$. We may thus represent the Fourier series graphically by plotting the coefficient of each term against its value of f (i.e., the frequencies of the constituent sines and cosines). This is shown in the top panel of Figure 3.

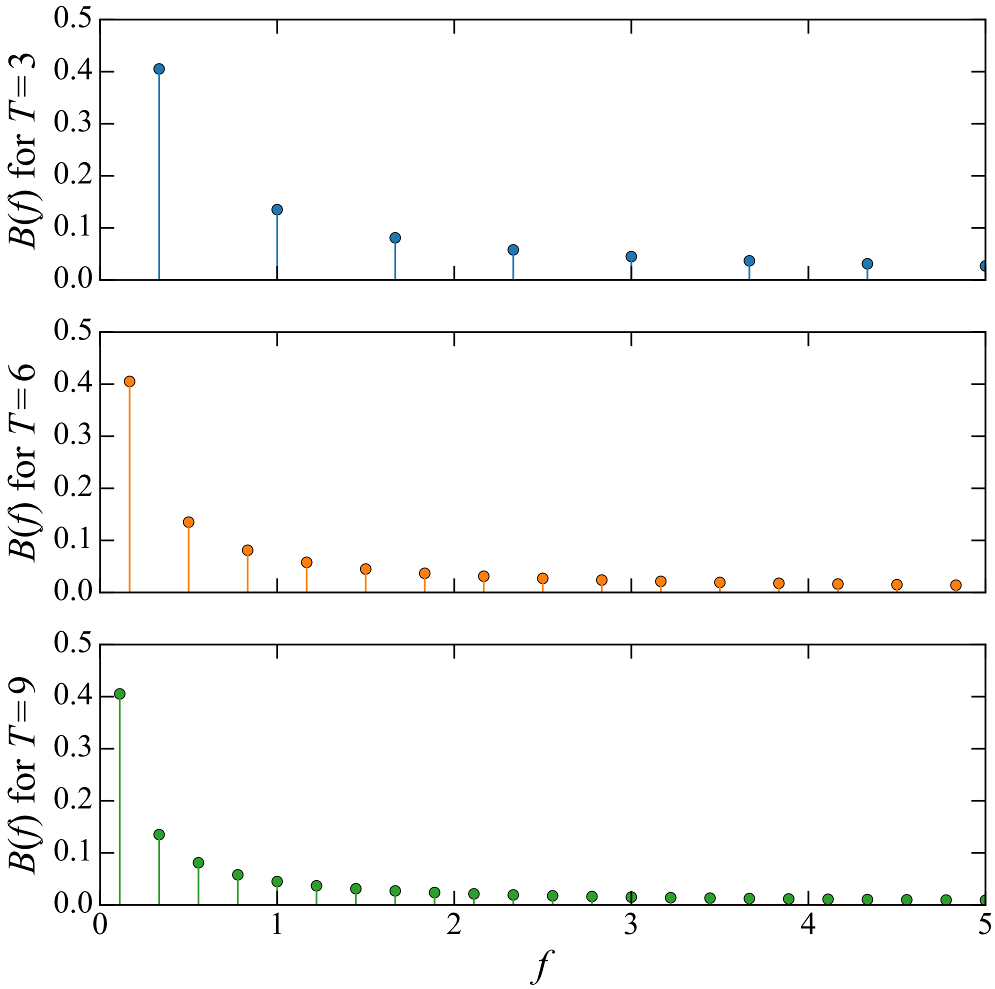

Figure 3: The coefficients (shown as a function of frequency f) for the Fourier series of square waves with periods $T = 3$ (blue), $T = 6$ (orange), and $T = 9$ (green).

Suppose we go through the same exercise for square waves with period $T = 6$ and $T = 9$. For $T = 6$, the Fourier series is given by

$$g_{T=6}(t) = \frac{4}{\pi} \sin\left(\frac{2\pi t}{6}\right) + \frac{4}{3\pi} \sin\left(\frac{2\pi 3t}{6}\right) + \frac{4}{5\pi} \sin\left(\frac{2\pi 5t}{6}\right) + \frac{4}{7\pi} \sin\left(\frac{2\pi 7t}{6}\right) + \dots \quad (13)$$

and for $T = 9$ is given by

$$g_{T=9}(t) = \frac{4}{\pi}\sin\left(\frac{2\pi t}{9}\right) + \frac{4}{3\pi}\sin\left(\frac{2\pi 3t}{9}\right) + \frac{4}{5\pi}\sin\left(\frac{2\pi 5t}{9}\right) + \frac{4}{7\pi}\sin\left(\frac{2\pi 7t}{9}\right) + \dots \quad (14)$$

The coefficients are depicted graphically in the bottom two panels of Figure 3. One sees that the overall shape of how the coefficients decay is the same, but the mix of frequencies is compressed to lower f. This makes sense—if we want to recreate square waves with longer periods, we need to build them out sines and cosines with longer periods (and thus lower frequencies). As we continue to increase T, the "trees" in Figure 3 get closer and closer together.

As we move to a non-periodic function by letting $T \to \infty$, the trees get bunched up into an infinitely dense forest. In other words, whereas the reconstruction of a periodic function can be accomplished by summing together sines and cosines at a *discrete* set of frequencies, the reconstruction of a non-periodic function requires the summation over a *continuous* set of all possible frequencies. This means that rather than using Equation 11 to work out just a discrete set of coefficients, we must work out an entire function of "coefficients":

$$\boxed{\widetilde{g}(f) = \int_{-\infty}^{\infty} dt\, e^{-i2\pi f t} g(t)}. \quad (15)$$

The function $\widetilde{g}(f)$ is known as the *Fourier transform* of $g(t)$. Note that while we used sines and cosines (as opposed to complex exponentials) to motivate the concepts, it is much more common to use complex exponentials when defining Fourier transforms.

Just as it was possible with the Fourier series to sum up the sines and cosines to reconstruct a periodic function, it is possible to do the same thing with a non-periodic function. But now that we have a continuous function instead of a discrete set of coefficients, Equation 10 goes from being a discrete sum to an integral:

$$\boxed{g(t) = \int_{-\infty}^{\infty} df\, e^{i2\pi f t} \widetilde{g}(f)}. \quad (16)$$

This operation (going from $\widetilde{g}$ back to g) is known as an *inverse Fourier transform*.

While we often talk about $g(t)$ as "the function" and $\widetilde{g}(f)$ as "its Fourier transform", the high degree of symmetry between Equations 15 and 16 suggest that it's more correct to think of g and $\widetilde{g}$ as simply being two different representations of the same function. After all, knowing one of them allows the other to be reconstructed, so they must carry the same information content. In fact, for some applications it is more natural to think of $\widetilde{g}$ as "the function". For examples, since our ears and brains allow us to distinguish different frequencies from one another, it could be argued that sound waves are more naturally expressed "in the Fourier domain", i.e., as a function of f rather than t.

4 Fourier conventions

It is important to note that there are different conventions for defining Fourier transforms. For example, in cosmology one often defines the Fourier transform to be

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \tag{17}$$

and the inverse Fourier transform to be

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \widetilde{f}(k). \tag{18}$$

In quantum mechanics textbooks, one often sees

$$\widetilde{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} h(x) \tag{19}$$

and correspondingly,

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \widetilde{h}(k). \tag{20}$$

All conventions are of course equally valid (since they're all able to reconstruct the original function as a sum of complex exponentials), and one simply has to be clear which one is being used.