

# Capítulo 1

## Verosimilitudes Extendidas

1.1.  $\pi(\alpha_d | (\theta_{ij})_{j=1 \cdot i=1}^{n_i \cdot I})$

$$\begin{aligned} \pi(\alpha_d | (\theta_{ij})_{j=1 \cdot i=1}^{n_i \cdot I}) &\propto \left( \prod_{i=1}^I \text{Gamma}(\theta_{i1} | \alpha_d + \alpha_\theta, d_{i1} + \beta_\theta) \times \prod_{i=1}^I \prod_{j=2}^{n_i} \text{Gamma}(\theta_{ij} | 2\alpha_d + \alpha_\theta, d_{ij} + d_{ij-1} + \beta_\theta) \right) \\ &\times \text{Gamma}(\alpha_d | \alpha_0, \beta_0) \end{aligned}$$

$$\begin{aligned} &= \left( \prod_{i=1}^I \frac{(d_{i1} + \beta_\theta)^{\alpha_d + \alpha_\theta}}{\Gamma(\alpha_d + \alpha_\theta)} \theta_{i1}^{\alpha_d + \alpha_\theta - 1} e^{-\theta_{i1}(d_{i1} + \beta_\theta)} \right. \\ &\times \left. \prod_{i=1}^I \prod_{j=2}^{n_i} \frac{(d_{ij} + d_{ij-1} + \beta_\theta)^{2\alpha_d + \alpha_\theta}}{\Gamma(2\alpha_d + \alpha_\theta)} \theta_{ij}^{2\alpha_d + \alpha_\theta - 1} e^{-\theta_{ij}(d_{ij} + d_{ij-1} + \beta_\theta)} \right) \\ &\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \alpha_d^{\alpha_0 - 1} e^{-\alpha_d \beta_0} \end{aligned}$$

$$\begin{aligned} &\propto \left( \frac{1}{\Gamma(\alpha_d + \alpha_\theta)} \right)^I \left( \frac{1}{\Gamma(2\alpha_d + \alpha_\theta)} \right)^{\sum_{i=1}^I n_i} \alpha_d^{\alpha_0 - 1} e^{-\alpha_d \beta_0} \times \\ &\times \left( \prod_{i=1}^I (d_{i1} + \beta_\theta)^{\alpha_d} \theta_{i1}^{\alpha_d} \prod_{i=1}^I \prod_{j=2}^{n_i} (d_{ij} + d_{ij-1} + \beta_\theta)^{2\alpha_d} \theta_{ij}^{2\alpha_d} \right) \end{aligned}$$

### 1.2. $\pi(\alpha_\theta | (\theta_{ij})_{i=1}^I \cdot j=1^{n_i})$

$$\begin{aligned}
\pi(\alpha_\theta | (\theta_{ij})_{j=1}^{n_i} \cdot i=1^I) &\propto \left( \prod_{i=1}^I \text{Gamma}(\theta_{i1} | \alpha_d + \alpha_\theta, d_{i1} + \beta_\theta) \times \prod_{i=1}^I \prod_{j=2}^{n_i} \text{Gamma}(\theta_{ij} | 2\alpha_d + \alpha_\theta, d_{ij} + d_{ij-1} + \beta_\theta) \right) \\
&\times \text{Gamma}(\alpha_\theta | \alpha_0, \beta_0) \\
\\
&= \left( \prod_{i=1}^I \frac{(d_{i1} + \beta_\theta)^{\alpha_d + \alpha_\theta}}{\Gamma(\alpha_d + \alpha_\theta)} \theta_{i1}^{\alpha_d + \alpha_\theta - 1} e^{-\theta_{i1}(d_{i1} + \beta_\theta)} \times \right. \\
&\times \left. \prod_{i=1}^I \prod_{j=2}^{n_i} \frac{(d_{ij} + d_{ij-1} + \beta_\theta)^{2\alpha_d + \alpha_\theta}}{\Gamma(2\alpha_d + \alpha_\theta)} \theta_{ij}^{2\alpha_d + \alpha_\theta - 1} e^{-\theta_{ij}(d_{ij} + d_{ij-1} + \beta_\theta)} \right) \times \\
&\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \alpha_\theta^{\alpha_0 - 1} e^{-\alpha_\theta \beta_0} \\
\\
&\propto \left( \prod_{i=1}^I \frac{(d_{i1} + \beta_\theta)^{\alpha_\theta}}{\Gamma(\alpha_d + \alpha_\theta)} \theta_{i1}^{\alpha_\theta} \prod_{i=1}^I \prod_{j=2}^{n_i} \frac{(d_{ij} + d_{ij-1} + \beta_\theta)^{\alpha_\theta}}{\Gamma(2\alpha_d + \alpha_\theta)} \theta_{ij}^{\alpha_\theta} \right) \times \alpha_\theta^{\alpha_0 - 1} e^{-\alpha_\theta \beta_0} \\
\\
&= \left( \frac{1}{\Gamma(\alpha_d + \alpha_\theta)} \right)^I \left( \frac{1}{\Gamma(2\alpha_d + \alpha_\theta)} \right)^{\sum_{i=1}^I n_i} \alpha_\theta^{\alpha_0 - 1} e^{-\alpha_\theta \beta_0} \times \\
&\times \left( \prod_{i=1}^I (d_{i1} + \beta_\theta)^{\alpha_\theta} \theta_{i1}^{\alpha_\theta} \prod_{i=1}^I \prod_{j=2}^{n_i} (d_{ij} + d_{ij-1} + \beta_\theta)^{\alpha_\theta} \theta_{ij}^{\alpha_\theta} \right)
\end{aligned}$$

### 1.3. $\pi(\beta_\theta | (\theta_{ij})_{i=1}^I \cdot j=1^{n_i})$

$$\begin{aligned}
\pi(\beta_\theta | (\theta_{ij})_{j=1}^{n_i} \cdot i=1^I) &\propto \left( \prod_{i=1}^I \text{Gamma}(\theta_{i1} | \alpha_d + \alpha_\theta, d_{i1} + \beta_\theta) \times \prod_{i=1}^I \prod_{j=2}^{n_i} \text{Gamma}(\theta_{ij} | 2\alpha_d + \alpha_\theta, d_{ij} + d_{ij-1} + \beta_\theta) \right) \\
&\times \text{Gamma}(\beta_\theta | \alpha_0, \beta_0)
\end{aligned}$$

$$\begin{aligned}
&= \left( \prod_{i=1}^I \frac{(d_{i1} + \beta_\theta)^{\alpha_d + \alpha_\theta}}{\Gamma(\alpha_d + \alpha_\theta)} \theta_{i1}^{\alpha_d + \alpha_\theta - 1} e^{-\theta_{i1}(d_{i1} + \beta_\theta)} \times \right. \\
&\times \left. \prod_{i=1}^I \prod_{j=2}^{n_i} \frac{(d_{ij} + d_{ij-1} + \beta_\theta)^{2\alpha_d + \alpha_\theta}}{\Gamma(2\alpha_d + \alpha_\theta)} \theta_{ij}^{2\alpha_d + \alpha_\theta - 1} e^{-\theta_{ij}(d_{ij} + d_{ij-1} + \beta_\theta)} \right) \times \\
&\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \beta_\theta^{\alpha_0 - 1} e^{-\beta_\theta \beta_0} \\
&\propto \left( \prod_{i=1}^I (d_{i1} + \beta_\theta)^{\alpha_d + \alpha_\theta} e^{-\theta_{i1} \beta_\theta} \times \prod_{i=1}^I \prod_{j=2}^{n_i} (d_{ij} + d_{ij-1} + \beta_\theta)^{2\alpha_d + \alpha_\theta} e^{-\theta_{ij} \beta_\theta} \right) \\
&\times \beta_\theta^{\alpha_0 - 1} e^{\beta_\theta \beta_0} \\
&= \left( \prod_{i=1}^I (d_{i1} + \beta_\theta)^{\alpha_d + \alpha_\theta} \times \prod_{i=1}^I \prod_{j=2}^{n_i} (d_{ij} + d_{ij-1} + \beta_\theta)^{2\alpha_d + \alpha_\theta} \right) \times \beta_\theta^{\alpha_0 - 1} e^{-\beta_\theta (\beta_0 + \sum_{i=1}^I \sum_{j=1}^{n_i} \theta_{ij})}
\end{aligned}$$

#### 1.4. $\pi(\alpha_\gamma | (\gamma_{ij})_{i=1}^I, j=1}^{n_i})$

$$\begin{aligned}
\pi(\alpha_\gamma | (\gamma_{ij})_{i=1}^I, j=1}^{n_i}) &\propto \left( \prod_{i=1}^I \left( \frac{1}{\gamma_{i1}} \right)^{d_{i1} + \alpha_\gamma + 1} e^{-\left( \left( \frac{\beta_\gamma}{\gamma_{i1}} \right) + \left( \frac{c_{i1}}{\gamma_{i1}} \right) d_{i1} \right)} \times \right. \\
&\times \left. \prod_{i=1}^I \prod_{j=2}^{n_i} \left( \frac{1}{\gamma_{ij}} \right)^{d_{ij} + d_{ij-1} + \alpha_\gamma + 1} e^{-\left( \left( \frac{\beta_\gamma}{\gamma_{ij}} \right) + \left( \frac{c_{ij-1}}{\gamma_{ij}} \right) d_{ij-1} + \left( \frac{c_{ij}}{\gamma_{ij}} \right) d_{ij} \right)} \right) \times \\
&\times \text{Gamma}(\alpha_\gamma | \alpha_0, \beta_0)
\end{aligned}$$

$$\begin{aligned}
&= \left( \prod_{i=1}^I \left( \frac{1}{\gamma_{i1}} \right)^{d_{i1} + \alpha_\gamma + 1} e^{-\left( \left( \frac{\beta_\gamma}{\gamma_{i1}} \right) + \left( \frac{c_{i1}}{\gamma_{i1}} \right)^{d_{i1}} \right)} \times \right. \\
&\times \left. \prod_{i=1}^I \prod_{j=2}^{n_i} \left( \frac{1}{\gamma_{ij}} \right)^{d_{ij} + d_{ij-1} + \alpha_\gamma + 1} e^{-\left( \left( \frac{\beta_\gamma}{\gamma_{ij}} \right) + \left( \frac{c_{ij-1}}{\gamma_{ij}} \right)^{d_{ij-1}} + \left( \frac{c_{ij}}{\gamma_{ij}} \right)^{d_{ij}} \right)} \right) \times \\
&\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \alpha_\gamma^{\alpha_0 - 1} e^{-\alpha_\gamma \beta_0} \\
&\propto \left( \prod_{i=1}^I \left( \frac{1}{\gamma_{i1}} \right)^{\alpha_\gamma} \times \prod_{i=1}^I \prod_{j=2}^{n_i} \left( \frac{1}{\gamma_{ij}} \right)^{\alpha_\gamma} \right) \alpha_\gamma^{\alpha_0 - 1} e^{-\alpha_\gamma \beta_0} \\
&= \left( \prod_{i=1}^I \prod_{j=1}^{n_i} \left( \frac{1}{\gamma_{ij}} \right)^{\alpha_\gamma} \right) \alpha_\gamma^{\alpha_0 - 1} e^{-\alpha_\gamma \beta_0}
\end{aligned}$$

**1.5.**  $\pi(\beta_\gamma | (\gamma_{ij})_{i=1}^I \cdot j=1^{n_i})$

$$\begin{aligned}
\pi(\alpha_\gamma | (\gamma_{ij})_{i=1}^I \cdot j=1^{n_i}) &\propto \left( \prod_{i=1}^I \left( \frac{1}{\gamma_{i1}} \right)^{d_{i1} + \alpha_\gamma + 1} e^{-\left( \left( \frac{\beta_\gamma}{\gamma_{i1}} \right) + \left( \frac{c_{i1}}{\gamma_{i1}} \right)^{d_{i1}} \right)} \times \right. \\
&\times \left. \prod_{i=1}^I \prod_{j=2}^{n_i} \left( \frac{1}{\gamma_{ij}} \right)^{d_{ij} + d_{ij-1} + \alpha_\gamma + 1} e^{-\left( \left( \frac{\beta_\gamma}{\gamma_{ij}} \right) + \left( \frac{c_{ij-1}}{\gamma_{ij}} \right)^{d_{ij-1}} + \left( \frac{c_{ij}}{\gamma_{ij}} \right)^{d_{ij}} \right)} \right) \times \\
&\times \textit{Gamma}(\beta_\gamma | \alpha_0, \beta_0)
\end{aligned}$$

$$\begin{aligned}
&= \left( \prod_{i=1}^I \left( \frac{1}{\gamma_{i1}} \right)^{d_{i1} + \alpha_\gamma + 1} e^{-\left( \left( \frac{\beta_\gamma}{\gamma_{i1}} \right) + \left( \frac{c_{i1}}{\gamma_{i1}} \right)^{d_{i1}} \right)} \times \right. \\
&\times \left. \prod_{i=1}^I \prod_{j=2}^{n_i} \left( \frac{1}{\gamma_{ij}} \right)^{d_{ij} + d_{ij-1} + \alpha_\gamma + 1} e^{-\left( \left( \frac{\beta_\gamma}{\gamma_{ij}} \right) + \left( \frac{c_{ij-1}}{\gamma_{ij}} \right)^{d_{ij-1}} + \left( \frac{c_{ij}}{\gamma_{ij}} \right)^{d_{ij}} \right)} \right) \times \\
&\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \beta_\gamma^{\alpha_0 - 1} e^{-\beta_\gamma \beta_0} \\
&\propto \left( \prod_{i=1}^I e^{-\left( \frac{\beta_\gamma}{\gamma_{i1}} \right)} \prod_{i=1}^I \prod_{j=2}^{n_i} e^{-\left( \frac{\beta_\gamma}{\gamma_{ij}} \right)} \right) \beta_\gamma^{\alpha_0 - 1} e^{-\beta_\gamma \beta_0} \\
&= e^{\{-\sum_{i=1}^I \sum_{j=1}^{n_i} \left( \frac{\beta_\gamma}{\gamma_{ij}} \right)\}} \beta_\gamma^{\alpha_0 - 1} e^{-\beta_\gamma \beta_0} \\
&= \beta_\gamma^{\alpha_0 - 1} e^{-\beta_\gamma \left( \beta_0 + \sum_{i=1}^I \sum_{j=1}^{n_i} \left( \frac{1}{\gamma_{ij}} \right) \right)} \\
&\propto \beta_\gamma \sim \text{Gamma} \left( \alpha_0, \beta_0 + \sum_{i=1}^I \sum_{j=1}^{n_i} \left( \frac{\beta_\gamma}{\gamma_{ij}} \right) \right)
\end{aligned}$$

## Capítulo 2

# Log-verosimilitudes

**2.1.**  $ln(\pi(\alpha_d | (\theta_{ij})_{j=1}^{n_i} \cdot_{i=1}^I))$

$$\begin{aligned}
 ln(\pi(\alpha_d | (\theta_{ij})_{j=1}^{n_i} \cdot_{i=1}^I)) &= ln\left(\left(\frac{1}{\Gamma(\alpha_d + \alpha_\theta)}\right)^I \left(\frac{1}{\Gamma(2\alpha_d + \alpha_\theta)}\right)^{\sum_{i=1}^I n_i} \alpha_d^{\alpha_0-1} e^{-\alpha_d \beta_0} \right. \\
 &\quad \times \left. \left( \prod_{i=1}^I (d_{i1} + \beta_\theta)^{\alpha_d} \theta_{i1}^{\alpha_d} \prod_{i=1}^I \prod_{j=2}^{n_i} (d_{ij} + d_{ij-1} + \beta_\theta)^{2\alpha_d} \theta_{ij}^{2\alpha_d} \right) \right) \\
 &= -I \times ln(\Gamma(\alpha_d + \alpha_\theta)) - \sum_{i=1}^I n_i \times ln(\Gamma(2\alpha_d + \alpha_\theta)) + (\alpha_0 - 1) \times ln(\alpha_d) - \alpha_d \beta_0 + \\
 &\quad + \left( \alpha_d \times \left( \sum_{i=1}^I ln(d_{i1} + \beta_\theta) + ln(\theta_{i1}) \right) + 2\alpha_d \times \left( \sum_{i=1}^I \sum_{j=2}^{n_i} ln(d_{ij} + d_{ij-1} + \beta_\theta) + ln(\theta_{ij}) \right) \right)
 \end{aligned}$$

**2.2.**  $ln(\pi(\alpha_\theta | (\theta_{ij})_{i=1}^I \cdot_{j=1}^{n_i}))$

$$\begin{aligned}
 ln(\pi(\alpha_\theta | (\theta_{ij})_{i=1}^I \cdot_{j=1}^{n_i})) &= ln\left(\left(\frac{1}{\Gamma(\alpha_d + \alpha_\theta)}\right)^I \left(\frac{1}{\Gamma(2\alpha_d + \alpha_\theta)}\right)^{\sum_{i=1}^I n_i} \alpha_\theta^{\alpha_0-1} e^{-\alpha_\theta \beta_0} \times \right. \\
 &\quad \times \left. \left( \prod_{i=1}^I (d_{i1} + \beta_\theta)^{\alpha_\theta} \theta_{i1}^{\alpha_\theta} \prod_{i=1}^I \prod_{j=2}^{n_i} (d_{ij} + d_{ij-1} + \beta_\theta)^{\alpha_\theta} \theta_{ij}^{\alpha_\theta} \right) \right)
 \end{aligned}$$

$$\begin{aligned}
&= -I \times \ln(\Gamma(\alpha_d + \alpha_\theta)) - \sum_{i=1}^I n_i \times \ln(\Gamma(2\alpha_d + \alpha_\theta)) + (\alpha_0 - 1) \times \ln(\alpha_\theta) - \alpha_\theta \beta_0 \\
&+ (\alpha_\theta) \times \left( \sum_{i=1}^I \ln(d_{i1} + \beta_\theta) + \ln(\theta_{i1}) + \sum_{i=1}^I \sum_{j=2}^{n_i} \ln(d_{ij} + d_{ij-1} + \beta_\theta) + \ln(\theta_{ij}) \right)
\end{aligned}$$

**2.3.**  $\ln(\pi(\beta_\theta | (\theta_{ij})_{i=1 \cdot j=1}^I))$

$$\begin{aligned}
\ln(\pi(\beta_\theta | (\theta_{ij})_{i=1 \cdot j=1}^I)) &= \ln \left( \left( \prod_{i=1}^I (d_{i1} + \beta_\theta)^{\alpha_d + \alpha_\theta} \times \prod_{i=1}^I \prod_{j=2}^{n_i} (d_{ij} + d_{ij-1} + \beta_\theta)^{2\alpha_d + \alpha_\theta} \right) \times \right. \\
&\quad \left. \times \beta_\theta^{\alpha_0 - 1} e^{-\beta_\theta (\beta_0 + \sum_{i=1}^I \sum_{j=1}^{n_i} \theta_{ij})} \right) \\
&= \sum_{i=1}^I (\alpha_d + \alpha_\theta) \times \ln(d_{i1} + \beta_\theta) + \sum_{i=1}^I \sum_{j=2}^{n_i} ((2\alpha_d + \alpha_\theta) \times \ln(d_{ij} + d_{ij-1} + \beta_\theta)) + \\
&+ ((\alpha_0 - 1) \times \ln(\beta_\theta)) - \beta_\theta \left( \beta_0 + \sum_{i=1}^I \sum_{j=1}^{n_i} \theta_{ij} \right)
\end{aligned}$$

**2.4.**  $\ln(\pi(\alpha_\gamma | (\gamma_{ij})_{i=1 \cdot j=1}^I))$

$$\begin{aligned}
\ln(\pi(\alpha_\gamma | (\gamma_{ij})_{i=1 \cdot j=1}^I)) &= \ln \left( \left( \prod_{i=1}^I \prod_{j=1}^{n_i} \left( \frac{1}{\gamma_{ij}} \right)^{\alpha_\gamma} \right) \alpha_\gamma^{\alpha_0 - 1} e^{-\alpha_\gamma \beta_0} \right) \\
&= - \sum_{i=1}^I \sum_{j=1}^{n_i} ((\alpha_\gamma) \times \ln(\gamma_{ij})) + ((\alpha_0 - 1) \times \ln(\alpha_\gamma)) - \alpha_\gamma \beta_0
\end{aligned}$$

$$\mathbf{2.5.} \quad \ln(\pi(\beta_\gamma | (\gamma_{ij})_{i=1}^I \cdot_{j=1}^{n_i}))$$

$$\begin{aligned} \ln(\pi(\beta_\gamma | (\gamma_{ij})_{i=1}^I \cdot_{j=1}^{n_i})) &= \ln\left(\beta_\gamma^{\alpha_0-1} e^{-\beta_\gamma(\beta_0 + \sum_{i=1}^I \sum_{j=1}^{n_i} (\frac{\beta_\gamma}{\gamma_{ij}}))}\right) \\ &= \left((\alpha_0 - 1) \times \ln(\beta_\gamma) - \beta_\gamma\left(\beta_0 + \sum_{i=1}^I \sum_{j=1}^{n_i} (\frac{\beta_\gamma}{\gamma_{ij}})\right)\right) \end{aligned}$$