Capítulo 1

Verosimilitudes Extendidas

1.1. $\pi(\alpha_d|(\theta_{ij})_{j=1}^{n_i}{}_{i=1}^{I})$

$$\begin{aligned}\pi(\alpha_d|(\theta_{ij})_{j=1}^{n_i}{}_{i=1}^{I}) &\propto \Big(\prod_{i=1}^{I} Gamma(\theta_{i1}|\alpha_d+\alpha_\theta, d_{i1}+\beta_\theta \times \prod_{i=1}^{I}\prod_{j=2}^{n_i} Gamma(\theta_{ij}|2\alpha_d+\alpha_\theta, d_{ij}+d_{ij-1}+\beta_\theta)\Big)\\ &\times \quad Gamma(\alpha_d|\alpha_0,\beta_0)\end{aligned}$$

$$\begin{aligned}&= \Big(\prod_{i=1}^{I}\frac{(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta}}{\Gamma(\alpha_d+\alpha_\theta)} \quad \theta_{i1}^{\alpha_d+\alpha_\theta} e^{-\theta_{i1}(d_{i1}+\beta_\theta)}\\ &\times \prod_{i=1}^{I}\prod_{j=2}^{n_i}\frac{(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta}}{\Gamma(2\alpha_d+\alpha_\theta)} \quad \theta_{ij}^{2\alpha_d+\alpha_\theta-1} e^{-\theta_{ij}(d_{ij}+d_{ij-1}+\beta_\theta)}\Big)\\ &\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\alpha_d^{\alpha_0-1}e^{-\alpha_d\beta_0}\end{aligned}$$

$$\begin{aligned}&\propto \Big(\frac{1}{\Gamma(\alpha_d+\alpha_\theta)}\Big)^I\Big(\frac{1}{\Gamma(2\alpha_d+\alpha_\theta)}\Big)^{\sum_{i=1}^I n_i} \quad \alpha_d^{\alpha_0-1}e^{-\alpha_d\beta_0} \times\\ &\times \Big(\prod_{i=1}^{I}(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta} \quad \theta_{i1}^{\alpha_d+\alpha_\theta-1}\prod_{i=1}^{I}\prod_{j=2}^{n_i}(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta} \quad \theta_{ij}^{2\alpha_d+\alpha_\theta-1}\Big)\end{aligned}$$

1.2. $\pi\big(\alpha_\theta\big|(\theta_{ij})_{i=1\cdot j=1}^{I\quad n_i}\big)$

$$\begin{aligned}\pi(\alpha_\theta|(\theta_{ij})_{j=1\cdot i=1}^{n_i\quad I}) &\propto \Big(\prod_{i=1}^{I} Gamma(\theta_{i1}|\alpha_d+\alpha_\theta, d_{i1}+\beta_\theta \times \prod_{i=1}^{I}\prod_{j=2}^{n_i} Gamma(\theta_{ij}|2\alpha_d+\alpha_\theta, d_{ij}+d_{ij-1}+\beta_\theta)\Big)\\ &\times \quad Gamma(\alpha_\theta|\alpha_0,\beta_0)\end{aligned}$$

$$\begin{aligned}&= \quad \Bigg(\prod_{i=1}^{I}\frac{(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta}}{\Gamma(\alpha_d+\alpha_\theta)} \quad \theta_{i1}^{\alpha_d+\alpha_\theta-1}e^{-\theta_{i1}(d_{i1}+\beta_\theta)} \times\\ &\times \quad \prod_{i=1}^{I}\prod_{j=2}^{n_i}\frac{(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta}}{\Gamma(2\alpha_d+\alpha_\theta)} \quad \theta_{ij}^{2\alpha_d+\alpha_\theta-1}e^{-\theta_{ij}(d_{ij}+d_{ij-1}+\beta_\theta)}\Bigg) \times\\ &\times \quad \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \quad \alpha_\theta^{\alpha_0-1}e^{-\alpha_\theta\beta_0}\end{aligned}$$

$$\propto \quad \Bigg(\prod_{i=1}^{I}\frac{(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta}}{\Gamma(\alpha_d+\alpha_\theta)} \quad \theta_{i1}^{\alpha_d+\alpha_\theta-1} \quad \prod_{i=1}^{I}\prod_{j=2}^{n_i}\frac{(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta}}{\Gamma(2\alpha_d+\alpha_\theta)} \quad \theta_{ij}^{2\alpha_d+\alpha_\theta-1}\Bigg) \times \alpha_\theta^{\alpha_0-1} \quad e^{-\alpha_\theta\beta_0}$$

$$\begin{aligned}&= \quad \Big(\frac{1}{\Gamma(\alpha_d+\alpha_\theta)}\Big)^{I}\Big(\frac{1}{\Gamma(2\alpha_d+\alpha_\theta)}\Big)^{\sum_{i=1}^{I} n_i} \quad \alpha_\theta^{\alpha_0-1}e^{-\alpha_\theta\beta_0} \times\\ &\times \quad \Bigg(\prod_{i=1}^{I}(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta} \quad \theta_{i1}^{\alpha_d+\alpha_\theta-1} \quad \prod_{i=1}^{I}\prod_{j=2}^{n_i}(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta} \quad \theta_{ij}^{2\alpha_d+\alpha_\theta-1}\Bigg)\end{aligned}$$

1.3. $\pi\big(\beta_\theta\big|(\theta_{ij})_{i=1\cdot j=1}^{I\quad n_i}\big)$

$$\begin{aligned}\pi(\beta_\theta|(\theta_{ij})_{j=1\cdot i=1}^{n_i\quad I}) &\propto \Big(\prod_{i=1}^{I} Gamma(\theta_{i1}|\alpha_d+\alpha_\theta, d_{i1}+\beta_\theta \times \prod_{i=1}^{I}\prod_{j=2}^{n_i} Gamma(\theta_{ij}|2\alpha_d+\alpha_\theta, d_{ij}+d_{ij-1}+\beta_\theta)\Big)\\ &\times \quad Gamma(\beta_\theta|\alpha_0,\beta_0)\end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{I} \frac{(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta}}{\Gamma(\alpha_d+\alpha_\theta)} \quad \theta_{i1}^{\alpha_d+\alpha_\theta-1} e^{-\theta_{i1}(d_{i1}+\beta_\theta)} \times \right. \\
&\times \left. \prod_{i=1}^{I}\prod_{j=2}^{n_i} \frac{(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta}}{\Gamma(2\alpha_d+\alpha_\theta)} \quad \theta_{ij}^{2\alpha_d+\alpha_\theta-1} e^{-\theta_{ij}(d_{ij}+d_{ij-1}+\beta_\theta)} \right) \times \\
&\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \quad \beta_\theta^{\alpha_0-1} e^{-\beta_\theta \beta_0}
\end{aligned}
$$

$$
\propto \left(\prod_{i=1}^{I} (d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta} \times \prod_{i=1}^{I}\prod_{j=2}^{n_i} (d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta} \right) \times \beta_\theta^{\alpha_0-1} e^{-\beta_\theta\left(\beta_0+\sum_{i=1}^{I}\sum_{j=1}^{n_i}\theta_{ij}\right)}
$$

1.4. $\pi\left(\alpha_\gamma \middle| (\gamma_{ij})_{i=1\cdot j=1}^{I\ n_i}\right)$

$$
\begin{aligned}
\pi(\alpha_\gamma|(\gamma_{ij})_{i=1\cdot j=1}^{I\ n_i}) &\propto \left(\prod_{i=1}^{I} \left(\frac{1}{\gamma_{i1}}\right)^{d_{i1}+\alpha_\gamma+1} \quad e^{-\left((\frac{\beta_\gamma}{\gamma_{i1}})+(\frac{c_{i1}}{\gamma_{i1}})^{d_{i1}}\right)} \times \right. \\
&\times \left. \prod_{i=1}^{I}\prod_{j=2}^{n_i} \left(\frac{1}{\gamma_{ij}}\right)^{2d_{ij}+\alpha_\gamma+1} \quad e^{-\left((\frac{\beta_\gamma}{\gamma_{ij}})+(\frac{c_{ij}+c_{ij-1}}{\gamma_{ij}})^{d_{ij}}\right)} \right) \times \\
&\times Gamma(\alpha_\gamma|\alpha_0,\beta_0)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{I} \left(\frac{1}{\gamma_{i1}}\right)^{d_{i1}+\alpha_\gamma+1} \quad e^{-\left((\frac{\beta_\gamma}{\gamma_{i1}})+(\frac{c_{i1}}{\gamma_{i1}})^{d_{i1}}\right)} \times \right. \\
&\times \left. \prod_{i=1}^{I}\prod_{j=2}^{n_i} \left(\frac{1}{\gamma_{ij}}\right)^{2d_{ij}+\alpha_\gamma+1} \quad e^{-\left((\frac{\beta_\gamma}{\gamma_{ij}})+(\frac{c_{ij}+c_{ij-1}}{\gamma_{ij}})^{d_{ij}}\right)} \right) \times \\
&\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \quad \alpha_\gamma^{\alpha_0-1} e^{-\alpha_\gamma \beta_0}
\end{aligned}
$$

$$\propto \left(\prod_{i=1}^{I}\left(\frac{1}{\gamma_{i1}}\right)^{\alpha_\gamma} \times \prod_{i=1}^{I}\prod_{j=2}^{n_i}\left(\frac{1}{\gamma_{ij}}\right)^{\alpha_\gamma}\right) \quad \alpha_\gamma^{\alpha_0-1}e^{-\alpha_\gamma\beta_0}$$

$$= \left(\prod_{i=1}^{I}\prod_{j=1}^{n_i}\left(\frac{1}{\gamma_{ij}}\right)^{\alpha_\gamma}\right) \quad \alpha_\gamma^{\alpha_0-1}e^{-\alpha_\gamma\beta_0}$$

1.5. $\pi(\beta_\gamma|(\gamma_{ij})_{i=1\cdot j=1}^{I\ n_i})$

$$\begin{aligned}\pi(\alpha_\gamma|(\gamma_{ij})_{i=1\cdot j=1}^{I\ n_i}) \propto & \left(\prod_{i=1}^{I}\left(\frac{1}{\gamma_{i1}}\right)^{d_{i1}+\alpha_\gamma+1} e^{-\left((\frac{\beta_\gamma}{\gamma_{i1}})+(\frac{c_{i1}}{\gamma_{i1}})^{d_{i1}}\right)} \times\right.\\ & \left.\times \prod_{i=1}^{I}\prod_{j=2}^{n_i}\left(\frac{1}{\gamma_{ij}}\right)^{2d_{ij}+\alpha_\gamma+1} e^{-\left((\frac{\beta_\gamma}{\gamma_{ij}})+(\frac{c_{ij}+c_{ij-1}}{\gamma_{ij}})^{d_{ij}}\right)}\right) \times\\ & \times Gamma(\beta_\gamma|\alpha_0,\beta_0)\end{aligned}$$

$$\begin{aligned}= & \left(\prod_{i=1}^{I}\left(\frac{1}{\gamma_{i1}}\right)^{d_{i1}+\alpha_\gamma+1} e^{-\left((\frac{\beta_\gamma}{\gamma_{i1}})+(\frac{c_{i1}}{\gamma_{i1}})^{d_{i1}}\right)} \times\right.\\ \times & \left.\prod_{i=1}^{I}\prod_{j=2}^{n_i}\left(\frac{1}{\gamma_{ij}}\right)^{2d_{ij}+\alpha_\gamma+1} e^{-\left((\frac{\beta_\gamma}{\gamma_{ij}})+(\frac{c_{ij}+c_{ij-1}}{\gamma_{ij}})^{d_{ij}}\right)}\right) \times\\ \times & \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \quad \beta_\gamma^{\alpha_0-1}e^{-\beta_\gamma\beta_0}\end{aligned}$$

$$
\begin{aligned}
&\propto \left(\prod_{i=1}^{I} e^{-\left(\frac{\beta_\gamma}{\gamma_{i1}}\right)} \prod_{i=1}^{I} \prod_{j=2}^{n_i} e^{-\left(\frac{\beta_\gamma}{\gamma_{ij}}\right)} \right) \quad \beta_\gamma^{\alpha_0 - 1} e^{-\beta_\gamma \beta_0} \\
&= e^{\{-\sum_{i=1}^{I} \sum_{j=1}^{n_i} \left(\frac{\beta_\gamma}{\gamma_{ij}}\right)\}} \quad \beta_\gamma^{\alpha_0 - 1} e^{-\beta_\gamma \beta_0} \\
&= \beta_\gamma^{\alpha_0 - 1} e^{-\beta_\gamma \left(\beta_0 + \sum_{i=1}^{I} \sum_{j=1}^{n_i} \left(\frac{\beta_\gamma}{\gamma_{ij}}\right)\right)}
\end{aligned}
$$

$$
\propto \quad \beta_\gamma \sim Gamma\left(\alpha_0, \beta_0 + \sum_{i=1}^{I} \sum_{j=1}^{n_i} \left(\frac{\beta_\gamma}{\gamma_{ij}}\right) \right)
$$

Capítulo 2

Log-verosimilitudes

2.1. $ln(\pi(\alpha_d|(\theta_{ij})_{j=1}^{n_i}._{i=1}^{I}))$

$$
\begin{aligned}
ln(\pi(\alpha_d|(\theta_{ij})_{j=1}^{n_i}._{i=1}^{I})) &= ln\Bigg(\Big(\frac{1}{\Gamma(\alpha_d+\alpha_\theta)}\Big)^{I}\Big(\frac{1}{\Gamma(2\alpha_d+\alpha_\theta)}\Big)^{\sum_{i=1}^{I} n_i} \quad \alpha_d^{\alpha_0-1}e^{-\alpha_d\beta_0} \times \\
&\times \quad \Bigg(\prod_{i=1}^{I}(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta} \quad \theta_{i1}^{\alpha_d+\alpha_\theta-1}\prod_{i=1}^{I}\prod_{j=2}^{n_i}(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta} \quad \theta_{ij}^{2\alpha_d+\alpha_\theta-1}\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -I\times ln(\Gamma(\alpha_d+\alpha_\theta)) - \sum_{i=1}^{I} n_i \times ln(\Gamma(2\alpha_d+\alpha_\theta)) + (\alpha_0-1)\times ln(\alpha_d) - \alpha_d\beta_0 + \\
&+ \quad \Bigg(\sum_{i=1}^{I}(\alpha_d+\alpha_\theta)\times ln(d_{i1}+\beta_\theta) + (\alpha_d+\alpha_\theta-1)\times ln(\theta_{i1}) + \\
&+ \quad \sum_{i=1}^{I}\sum_{j=2}^{n_i}(2\alpha_d+\alpha_\theta)\times ln(d_{ij}+d_{ij-1}+\beta_\theta) + (2\alpha_d+\alpha_\theta-1)\times ln(\theta_{ij})\Bigg)
\end{aligned}
$$

2.2. $ln(\pi(\alpha_\theta|(\theta_{ij})_{i=1\cdot j=1}^{I\ n_i}))$

$$
\begin{aligned}
ln(\pi(\alpha_\theta|(\theta_{ij})_{i=1\cdot j=1}^{I\ n_i})) &= ln\Bigg(\Big(\frac{1}{\Gamma(\alpha_d+\alpha_\theta)}\Big)^I\Big(\frac{1}{\Gamma(2\alpha_d+\alpha_\theta)}\Big)^{\sum_{i=1}^I n_i} \quad \alpha_\theta^{\alpha_0-1}e^{-\alpha_\theta\beta_0}\times \\
&\times \quad \Bigg(\prod_{i=1}^I(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta} \quad \theta_{i1}^{\alpha_d+\alpha_\theta-1} \quad \prod_{i=1}^I\prod_{j=2}^{n_i}(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta} \quad \theta_{ij}^{2\alpha_d+\alpha_\theta-1}\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= -I\times ln(\Gamma(\alpha_d+\alpha_\theta)) - \sum_{i=1}^I n_i \times ln(\Gamma(2\alpha_d+\alpha_\theta)) + (\alpha_0-1)\times ln(\alpha_\theta) - \alpha_\theta\beta_0 + \\
&+ \quad \Bigg(\sum_{i=1}^I(\alpha_d+\alpha_\theta)\times ln(d_{i1}+\beta_\theta) + (\alpha_d+\alpha_\theta-1)\times ln(\theta_{i1}) + \\
&+ \quad \sum_{i=1}^I\sum_{j=2}^{n_i}(2\alpha_d+\alpha_\theta)\times ln(d_{ij}+d_{ij-1}+\beta_\theta) + (2\alpha_d+\alpha_\theta-1)\times ln(\theta_{ij})\Bigg)
\end{aligned}
$$

2.3. $ln(\pi(\beta_\theta|(\theta_{ij})_{i=1\cdot j=1}^{I\ n_i}))$

$$
\begin{aligned}
ln(\pi(\beta_\theta|(\theta_{ij})_{i=1\cdot j=1}^{I\ n_i})) &= ln\Bigg(\Bigg(\prod_{i=1}^I(d_{i1}+\beta_\theta)^{\alpha_d+\alpha_\theta}\times\prod_{i=1}^I\prod_{j=2}^{n_i}(d_{ij}+d_{ij-1}+\beta_\theta)^{2\alpha_d+\alpha_\theta} \quad \Bigg)\times \\
&\times \quad \beta_\theta^{\alpha_0-1}e^{-\beta_\theta\left(\beta_0+\sum_{i=1}^I\sum_{j=1}^{n_i}\theta_{ij}\right)}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^I(\alpha_d+\alpha_\theta)\times ln(d_{i1}+\beta_\theta) + \sum_{i=1}^I\sum_{j=2}^{n_i}((2\alpha_d+\alpha_\theta)\times ln(d_{ij}+d_{ij-1}+\beta_\theta)) + \\
&+ \quad ((\alpha_0-1)\times ln(\beta_\theta)) - \beta_\theta\Bigg(\beta_0+\sum_{i=1}^I\sum_{j=1}^{n_i}\theta_{ij}\Bigg)
\end{aligned}
$$

2.4. $ln(\pi(\alpha_\gamma|(\gamma_{ij})_{i=1.j=1}^{I\ n_i}))$

$$ln(\pi(\alpha_\gamma|(\gamma_{ij})_{i=1.j=1}^{I\ n_i})) = ln\left(\left(\prod_{i=1}^{I}\prod_{j=1}^{n_i}\left(\frac{1}{\gamma_{ij}}\right)^{\alpha_\gamma}\right)\ \alpha_\gamma^{\alpha_0-1}e^{-\alpha_\gamma\beta_0}\right)$$

$$= -\sum_{i=1}^{I}\sum_{j=1}^{n_i}\left((\alpha_\gamma)\times ln(\gamma_{ij})\right)+\left((\alpha_0-1)\times ln(\alpha_\gamma)\right)-\alpha_\gamma\beta_0$$

2.5. $ln(\pi(\beta_\gamma|(\gamma_{ij})_{i=1.j=1}^{I\ n_i}))$

$$ln(\pi(\beta_\gamma|(\gamma_{ij})_{i=1.j=1}^{I\ n_i})) = ln\left(\beta_\gamma^{\alpha_0-1}e^{-\beta_\gamma\left(\beta_0+\sum_{i=1}^{I}\sum_{j=1}^{n_i}\left(\frac{\beta_\gamma}{\gamma_{ij}}\right)\right)}\right)$$

$$= \left((\alpha_0-1)\times ln(\beta_\gamma)\right)-\beta_\gamma\left(\beta_0+\sum_{i=1}^{I}\sum_{j=1}^{n_i}\left(\frac{\beta_\gamma}{\gamma_{ij}}\right)\right)$$