Lec 2.

We observe "Data" $X = \langle 0, 0, 1, 0, 1, 0 \rangle$ realization from a random process

Assume a parametric model $f(x) = f(x; n) = \sin(ax)$

$\mathcal{F}$: iid Binomial $= \{ p(x;\theta) : \theta \in \Theta \} = \{ \theta^{x}(1-\theta)^{1-x} : \theta \in (0,1) \}$

($\underset{\text{pmf}}{p(x)} = p(x;\theta)$ constant θ needed to compute the prob)

We'd like to learn about θ (inference)

$$P(X:\theta) = P(\langle 0,0,1,0,1,0 \rangle ; \theta) \overset{\mathcal{F}}{=} \left(\theta^{0}(1-\theta)^{1-0}\right) \cdot \left(\theta^{0}(1-\theta)^{1-0}\right) \cdots \left(\theta^{1}(1-\theta)^{1-1}\right) \cdot \left(\theta^{0}(1-\theta)^{1-0}\right)$$

6 data "~" 6 times

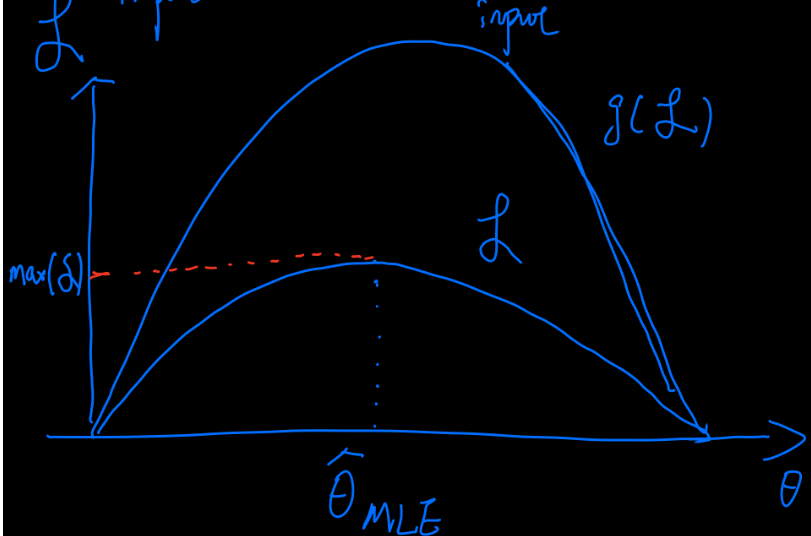
$= \theta^{2}(1-\theta)^{4}$

if $\theta = 0.5$ $p(x;\theta) = 0.0156$ $\theta = 0.25$ .. $= 0.0198$

"likelihood function" "probability"

$$\mathcal{L}(\theta; x) = P(x; \theta)$$

θ: input, x: constant; x: input, θ: constant



$\int_{\Theta} \mathcal{L}(\theta; x)\, d\theta \overset{?}{=} 1$ No

$P(x;\theta) \in (0,1)$

$\Rightarrow \mathcal{L}(\theta; x) \in (0,1)$

$\sum_{x} P(x;\theta) = 1$

$\qquad //$

$\int_{x} p(x;\theta)\, d\theta$

$$\hat{\theta}_{MLE} := \underset{\theta \in \Theta}{\arg\max} \left\{ \mathcal{L}(\theta, x) \right\} = \underset{\theta \in \Theta}{\arg\max} \left\{ g(\mathcal{L}(\theta; x)) \right\}$$

MLE: maximum likelihood estimate

argument that in the maximum

Consider g in a strictly increasing function

$$\ln\left(\prod_{i=1}^{6} \theta^{x_i} (1-\theta)^{1-x_i} \right)$$

six

$$= \sum_{i=1}^{6} \ln\left(\theta^{x_i}(1-\theta)^{1-x_i}\right) = \sum_{i=1}^{6}\left[x_i \ln(\theta) + (1-x_i)\ln(1-\theta)\right] = \left(\ln(\theta)\sum_{i=1}^{6} x_i\right) + \ln(1-\theta)\sum_{i=1}^{6}(1-x_i)$$

let $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ in our case $\bar{x} = \frac{1}{6}\sum_{i=1}^{6} x_i$ $6\bar{x} = \sum_{i=1}^{6} x_i$

$$= 6\left(\ln(\theta)\bar{x} + (1-\bar{x})\ln(1-\theta) \right)$$

① find $l(\theta; x)$ ② Find $\frac{d}{d\theta}[l(\theta, x)] = l'(\theta, x)$ ③ $l'(\theta, x) = 0$ ④ solve $\hat{\theta}_{MLE}$

$$l'(\theta; x) = 6\left(\frac{\bar{x}}{\theta} - \frac{1-\bar{x}}{1-\theta}\right) = 0 \Rightarrow \frac{\bar{x}}{\theta} = \frac{1-\bar{x}}{1-\theta} \quad \bar{x} = \theta \quad \hat{\theta}_{MLE} = \bar{x}$$

$$x = (0,0,1,0,1,0) \Rightarrow \bar{x} = \frac{1}{6}\sum_{i=1}^{6} x_i = \frac{0+0+1+0+1+0}{6} = \frac{1}{3} = \hat{\theta}_{MLE}$$

$$X_1, \ldots, X_n \mid \theta \overset{iid}{\sim} \mathrm{Bern}(\theta) \Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

MLE isn't the only point estimate strategy but it in common as it has nice proportion

Math 621

Ⅰ $\hat{\theta}_{MLE} \xrightarrow{P} \theta$ "consistence" i.e. it converge to the value in n

(II) $\hat{\theta}_{MLE} \overset{\sim}{\sim} N(\theta, SE[\hat{\theta}_{MLE}]^2)$ Asymptotic Normality

standard error

(III) Among all consistent continuous, $\hat{\theta}_{MLE}$ has ____ variable

(Efficiency

Geometric case

$\tilde{F}$ = iid Geometric $\quad X \sim Geom(\theta) = (1-\theta)^x\theta \quad \mathcal{L}(\theta; x) = \prod_{i=1}^{n}(1-\theta)^{x_i}\theta$

take log $\Rightarrow l(\theta; x) = \sum_{i=1}^{n}\left[\ln\left[(1-\theta)^{x_i}\theta\right]\right] = n\ln(\theta) + \ln(1-\theta)\sum_{i=1}^{n}X_i = n\left(\ln(\theta) + \bar{X}\ln(1-\theta)\right)$

take derivative $(d\theta)$

$$l'(\theta; X) = n\left(-\frac{\bar{X}}{1-\theta} + \frac{1}{\theta}\right) \overset{set}{=} 0 \quad \theta = \frac{1}{1+\bar{X}}$$

$\bar{X} = 99 \quad \theta_{MLE} = 0.01$ $\quad 1\%$

$\bar{X} = 1 \quad \theta_{MLE} = 50\%$

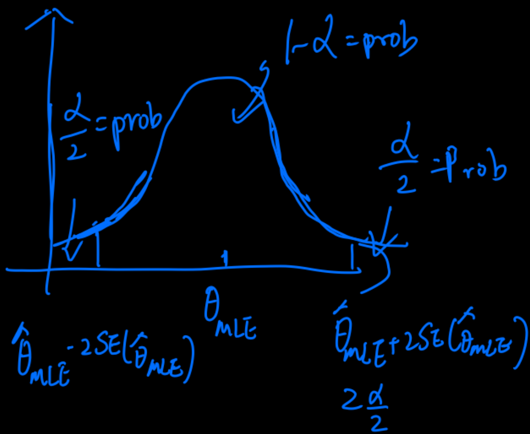
if $\tilde{F}$ is iid Bern $\quad \hat{\theta}_{MLE} = \bar{X} \quad SE(\hat{\theta}_{MLE}) = SE[\bar{X}] = \sqrt{\frac{\theta(1-\theta)}{n}}$

$$\hat{\theta}_{MLE} \overset{d}{\underset{\sim}{\approx}} N(\theta, SE(\hat{\theta}_{MLE})^2) \overset{(I)}{\approx} N\left(\hat{\theta}_{MLE}, SE[\hat{\theta}_{MLE}]^2\Big|_{\theta=\hat{\theta}_{MLE}}\right) \approx N\left(\bar{x}, \sqrt{\frac{\bar{x}(1-\bar{x})}{n}}^2\right)$$

Asymptotic

$\tilde{F}$ = iid Geometric $\quad \hat{\theta}_{MLE} = \frac{1}{1+\bar{x}}, \; SE[\hat{\theta}_{MLE}] = SE\left[\frac{1}{1+\bar{x}}\right] = ?$

goal inference "____" ie "provide a range of possible value of θ.



math 242, Econ382

Confidence interval $(CI)_{\theta, 1-\alpha} = \left[\hat{\theta}_{MLE} \pm Z_{\frac{\alpha}{2}} SE(\hat{\theta}_{MLE})\right]$

$1-\alpha$ is probability value

Third goal of Inference: Testing

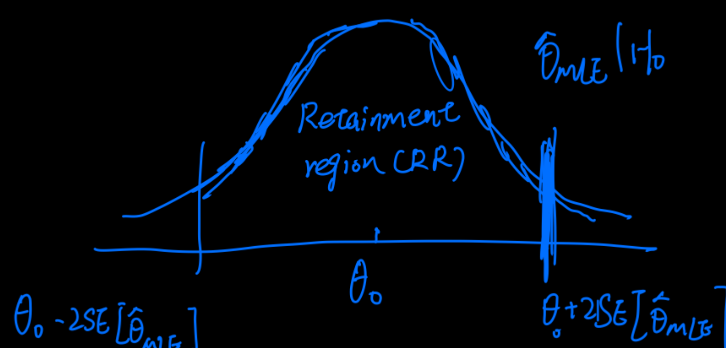
also called Hypothesis Theory (half of math 242 or ~~and~~ Econ382 will teach)

"null hypothesis" H_0: $\theta = \theta_0$ ← some specific value

"a hypothesis" H_a: $\theta \neq \theta_0$

Assume my theory ____ and ____ tell me if I'm right or wrong

$$\Rightarrow \hat{\theta}_{MLE} \sim N\left(\theta_0, SE(\hat{\theta}_{MLE})^2\Big|_{\theta=\theta_0}\right)$$



$$RR_{\theta_0, 1-\alpha} := \left[\theta_0 \pm Z_{\frac{\alpha}{2}} SE(\hat{\theta}_{MLE})\Big|_{\theta=\theta_0}\right]$$

if $\hat{\theta}_{MLE} \in RR \Rightarrow$ Retain H_0

if $\hat{\theta}_{MLE} \notin RR \Rightarrow$ Reject H_0 / Accept H_a

We have a strategy for all 3 inference goals We've done "frequence inference" which in all historical class