Nuclear Engineering 101: Midterm 2 Study Guide

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Note: Everything in this guide is from the text (Krane) or lecture, or office hours. I have tried my best to cite everything as completely as possible, but some of the info may be from office hours. I didn't think of any of this on my own.

1 Nuclear Electromagnetic Moments

Ref: [1, pp. 71-75],[2, Lec 10]
A distribution of electric charge and/or current will create an electric potential. This can be described by the "multipole moment".

Note the values of L are the *order of the moment*, not angular momentum.

1.1 Electric multipoles

- Monopole moment: $L = 0$, this is just the spherical nucleus, the electric potential looks like $V = \frac{Q}{R}$.
- Dipole moment: $L = 1$. The electric dipole is just linear translation of the nucleus. $V = 0$, see below.
- Quadrupole moment: $L = 2$, the electric potential is from the nucleus squishing in one direction and then the other. The amount of quadrupole moment can be calculated:

$$eQ = e\int \psi^*(3z^2 - r^2)\psi \; d^3\vec{r}$$

  If the nucleus is spherical, $\langle z\rangle^2 = \langle x\rangle^2 = \langle y\rangle^2$, so $eQ = 0$. The higher the value of eQ the more deformed the nucleus is.

All electric moments have parity determined by $(-1)^L$. If you want to do some kind of electromagnetic operation on a state (ψ) with some operator O, then you have to evaluate $\int \psi^* O\psi \; dv$. It doesn't matter what the parity of ψ is because they will always multiply (even times even or odd times odd are both always even). If O has negative parity, the whole thing is an odd function and the integral goes to 0. Therefore, to keep things based in reality, all moments with odd angular momentum must not be a thing because they have negative parity ($L = 1, 3, 5$ etc). This is why the dipole doesn't happen.

1.2 Magnetic multipoles

- Monopole moment: $L = 0$ doesn't happen, as far as we know.
- Dipole moment: $L = 1$, nucleons and nuclei have dipole magnetic moments due to their angular momentum (they are charges moving in a loop, therefore magnetic field).

$$\mu = \frac{e\hbar}{2m}l = g_l l \mu_n$$

  Here l **is** the angular momentum, g_l is a constant (1 for protons, 0 for neutrons) and μ_n is the "nuclear magneton" (just a constant).
- Quadrupole moment: $L = 3$ doesn't happen.

Just like before, all the moments with odd parity have to not exist. For magnetic moments, the parity is determined by $(-1)^{L+1}$, so all the *even* order moments can't exist.

2 Shell Model

2.1 Evidence

- Ionization energies "jump" at "magic numbers" like 2, 10, 18, 36, 54, 86. Something about these make the nucleus more tightly bound, this wouldn't be seen in a liquid drop [2, Lec. 12].
- α-decay: see a big jump in the α-decay of Radon after $N = 128$. This is because the daughter has $N = 126$ (magic number) [2, Lec. 12].

2.2 Shells

- You can use the 3D square well to get a pretty good approximation of what we see, or a parabolic ($1/r^2$) one to get better. But the simple harmonic oscillator gives the best approximation but it's still wrong. [2, Lec. 12].
- Spin-orbit interaction: due to the interaction of the spin and the angular momentum ($\vec{l} \bullet \vec{s}$), you can get two different values of j, $l \pm \frac{1}{2}$. Now two different nucleons will see a *different* potential, so it splits all the states in two. [2, Lec. 13-16], [1, pp. 123-125].
- The size of the split gets bigger with increasing l. The potential is negative so the $J = l + \frac{1}{2}$ states will occur at *lower* energies. High spin states that enter a different lower shell are called *the intruder*. [2, Lec. 13-16]
- Evidence: Magnetic dipole $\mu = \mu_N(g_l l_z + g_s s_z)/\hbar$ should be different for the different spins in the same l level. Observations show this is true. [2, Lec. 13-16].
- Works well for **spherical** nuclei and ones that are **close to magic numbers**.
- The ratio of the 4^+ and 2^+ states is usually close to 1:

$$R_{42} = \frac{E(4^+)}{E(2^+)} \approx 1$$

2.3 Independent Particle Model

- All shells are completely full or empty except for a single particle in the lowest energy of an otherwise empty shell. The nucleus J^π depends only on that last nucleon. [2, Lec. 13-16].
- The total angular momentum of the nuclei J is equal to the J value of the shell with the single particle in it. The parity is equal to the parity of the shell with the single particle in it, $(-1)^l$. For example, if there is a single particle in the $1p_{3/2}$ state, then $J^\pi = (\frac{3}{2})^-$ because $j = \frac{3}{2}$ and $l = 1$. [2, Lec. 13-16].
- The value J^π is referred to as the spin-parity **or** the total angular momentum of the nucleus. Both of those things mean the same thing *when you're talking about the nucleus.* For a nucleon, spin is the intrinsic angular momentum; the nucleus doesn't have any **intrinsic** angular momentum, only angular momentum due to it's components. We call it spin anyway when referring to the nucleus just to be confusing.
- The independent nucleon always gives the total angular momentum of the shell to the nucleus when in the **ground state.**
- When you are in an **excited state**, you can line up the nucleons in more interesting way. If you have three nucleons in the $f_{7/2}$ shell and we're in an excited state, we use the notation $(f_{7/2)})^3$. You can add up their angular momentum any way you want, remember each can have angular momentum from $-j$ to $+j$. So these can have $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}$. They can't have the **same** value of m, but they can all add to get, for example $\frac{7}{2} + \frac{5}{2} + \frac{3}{2} = \frac{15}{2}$. [1, pp. 149-151]
- In excited states, the single nucleons can also jump around to other states, changing the J^π of the nucleus. [2, Lec. 13-16]

3 Collective Motion

- Collective nuclear motion comes from the interplay between nucleons making the nuclear potential, and nucleons moving in the nuclear potential. [2, Lec. 13-16]
- Collective motion lowers the energy for the first excited state (2^+ for vibration and rotation). Instead of having to yank a nucleon from a low, tightly bound state, to a high state, you just put enough energy into the system to get all the nucleons vibrating or rotating together. So nuclei where collective motion is seen have much, much lower first excited states.
- Rotations and vibrations and not mutually exclusive. A rotational band can "ride" on top of a vibration, or vice versa.
- The collective model is useful and good when there are a large number of valence nucleons, promiscuity:

$$P = \frac{N_p N_n}{N_p + N_n}$$

  Where N_p is the number of valence protons, and N_n is the number of valence neutrons. Valence nucleons are the extra nucleons after the last large shell. Around 4, you see collective motion. [2, Lec. 13-16]

3.1 Vibration

- A vibration can move over the surface of the nucleus. The order of this vibration is given by lambda (λ), this determines the order of the spherical harmonic that describes the collective vibration.
  - The $\lambda = 1$ vibration is just dipole movement, which is just linear movement of the nucleus. This is not a collective motion state.
  - The $\lambda = 2$ vibration is the quadrupole vibration. The energy is quantized in the *phonon* which is just a discrete unit of vibrational energy. For $\lambda = 2$, each carries exactly $2\hbar\omega$ units of vibrational energy.
  - The $\lambda = 3$ vibration is octopole and I'm pretty sure Lee said this hasn't been observed.

  [2, Lec 13-16]
- A vibration collective structure on top of the 0^+ ground state will be a 2^+ level, followed by a 4^+ level, followed by 6^+, etc. They will be *evenly spaced* in energy, because each level represents the addition of one phonon and each phonon has the same energy. The parity doesn't change because $\lambda = 2$ and parity goes like $(-1)^\lambda$. [2, Lec 13-16]
- Therefore, he ratio of the 4^+ and 2^+ states should almost always be:

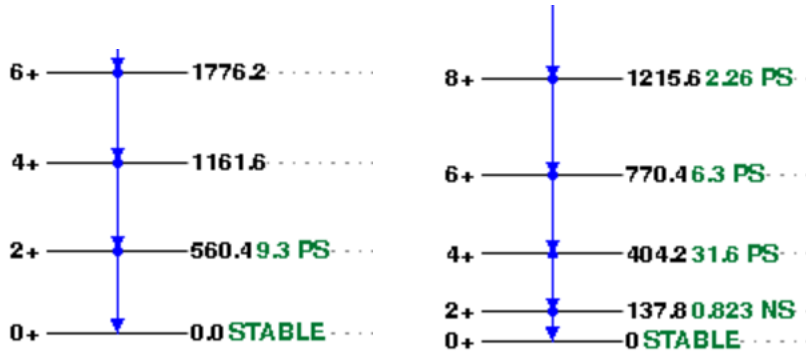
$$R_{42} = \frac{E(4^+)}{E(2^+)} \approx 2$$

- For vibration, we expect the quadropole moment (the measure of nuclear deformation) to be $Q(2^+) = 0$. This is because the nucleus is just a sphere with a surface wave moving over it, so it all averages out to 0. [2, Lec 13-16]

3.2 Rotation

- There is an enhanced probability of decay between rotational bands because the wave functions are all the same, or very similar.
- The ratio of the 4^+ and 2^+ states should almost always be:

$$R_{42} = \frac{E(4^+)}{E(2^+)} \approx 3.3$$



(a) (b)

Figure 1: (a) Shows vibrational structure in Te-120. Notice that the 2^+, 4^+, and 6^+ states are all evenly spaced. (b) Shows a rotational structure in Dy-156. The levels are not evenly spaced, but get larger with I

4 α-decay

- The α-decay half-life can be estimated using a barrier tunneling model. You can calculate the frequency of collision with the potential well and use the transmission probability to calculate a half-life. This isn't the real model and only represents an electromagnetic baseline to give us insight into what is happening. It isn't *nuclear*. [2, Lec. 18]

- Any angular momentum carried away by the α-particle is **purely orbital**. The spins all couple pairwise to 0. The angular momentum change from α-decay can range from:

$$|I_i - I_f| \leq l_\alpha \leq I_i + I_f$$

  With parity change:

$$\Delta\pi = (-1)^{l_\alpha}$$

  The simplest decay is any $0 \rightarrow X$ state, because the only value for l_α is X. If that doesn't give the right parity change, the decay is *forbidden*. In this case it really means *forbidden*, unlike β-decay, which plays fast and loose with the term. [1, pp. 257-258].

- α-decay can populate many different states. Each one has a different Q-value (given by the Q value for decay to the ground state, minus the excitation energy). [1, pp. 257-258].

- The intensity depends on how similar the initial and final wave functions are, and the angular momentum (l_α). [1, pp. 257-258].

- The hinderence factor (HF) tells us about the similarity between the parent and daughter states.:
  - $1 < H < 4$ Favored: α-particle is from two low-lying pairs of nucleons.
  - $4 < H < 10$: favorable overlap between initial and final state.
  - $10 < H < 100$: spin projections are parallel. Unfavorable overlap between initial and final states.
  - $100 < H < 1000$: spin projections are parallel, change in parity.
  - $H > 1000$: spin-flip, change in parity.

  Any spin flip $\rightarrow$ over 1000. Any parity change $\rightarrow$ over 100. Non-favorable overlap $\rightarrow$ over 10. See Table. 1. [2, Lec. 18]

- α-decay spectrum has sharp peaks, each represents a decay to different excited states. Higher E_α, lower excitation energy. The excited states then usually release γ-ray photons. [1, pp. 262-264]

H	**Nuclear State Overlap**	**Parity Change**	**Spins**
<10	Favorable	None	Parallel
10–100	Unfavorable	None	Parallel
100–1000	Unfavorable	Yes	Parallel
>1000	Unfavorable	Yes	Spin-flip

Table 1: Summary of α-decay hinderance factors.

- In an even-even nucleus, the alpha decay to the ground state is usually *very strong*. The $0^+ \rightarrow 0^+$ decay is not inhibited by differences between the wave functions (they are very similar. By the same logic, the decay of odd-A nuclei may be *very weak* because the two states may be significantly different. [1, pp.264]

5 β-decay

- Energy release distribution is continuous because there are two particles emitted so they can share any proportion of the energy. [1, 2, pp. 273, Lec 19]
- Three kinds:

$$\beta^- : n \rightarrow p + e^- + \bar{\nu}_e$$
$$\beta^+ : p \rightarrow n + e^+ + \nu_e$$
$$EC : p + e^- \rightarrow n + \nu_e$$

  Each has a different energy release:

$$Q_{\beta^-} = [m(^AX) - m(^AX')]c^2$$
$$Q_{\beta^+} = [m(^AX) - m(^AX') - 2m_e]c^2$$
$$Q_\epsilon = [m(^AX) - m(^AX')]c^2 - B_n$$

  For β^-, the electron masses all cancel when using atomic masses because $-Zm_e + (Z+1)m_e - m_e = 0$, that is the electrons accounted for in the atomic masses for the original atom, final atom and leftover electron all cancel. They don't cancel for β^+ because $-Zm_e + (Z-1)m_e - m_e = -2m_e$. [1, pp. 274-276]

5.1 Fermi Theory

- There are three things that go into determining the spectrum of beta decay energy values:

$$N(p) \propto \text{Statistical factor} * \text{Coulomb Interaction (electrodynamic)} * \text{Nuclear Matrix Element}$$
$$N(p) \propto p^2(Q_\beta - T_e)^2 * F(Z_d, T_e) * |M_{fi}|^2 * S(p,q)$$

  These terms are:

  - Statistical factor: related to the number of final states accessible. Decays like to happen when there are a lot of final states that are open.
  - Fermi function (F): the influence of the atomic coulomb field. This is the **electrodynamic** part.
  - Matrix element ($M_{fi}^2 * S(p,q)$): This is the **Nuclear** part. It is kind of like the hinderance factors for α-decay and represents the effect of the initial and final states. The S function only comes into play in **forbidden** decays, when the electron and neutrino have angular momentum ($l \neq 0$, s and q are terms for the electron and neutrino angular momenta.

  [1, pp.281-282].

- You can plot:

$$(Q - T_e) \propto \sqrt{\frac{N(p)}{p^2 F(Z', p)}}$$

Which *should* give you a straight line (Fermi-Kurie plot) that intersects the x-axis at the Q value. For forbidden decays, the line is **not** straight, you have to put that $S(p, q)$ factor in on the bottom to make it straight. This is the *shape factor*, which then gives you a straight line. [1, pp. 282]

- Total decay rate: you can integrate some messy function to get a messy function for total decay rate:

$$\lambda = \text{Matrix Element Term} * \int F\text{-function and statistical factor}$$

The full equation is in Krane, page 282. What is important is that the term outside the $\int$ is the **nuclear** part, and the integral is the **electrodynamic** part. Those are the two things (and some constants) that determine the decay rate. You can just crunch the integral and look up values, which is f, or the *Fermi Integral*. Then, if you do $\frac{f}{\lambda}$ or just $ft_{1/2}$, then the integrals cancel and all you have is the Matrix Element part, the **nuclear** part. This lets us compare $ft_{1/2}$ values, which are *only dependent on nuclear properties*. It might be useful to think about the ft value as just a *corrected* half-life. Corrected to remove electrodynamic stuff. [1, 2, pp. 282-283, Lecs. 19-21].

- ft values have a huge range (10^3 to 10^{20}) so we use logft instead.

5.2 Allowed Decays

- Electron and neutrino are created at the origin ($r = 0$), so their angular momentum is $l = 0$. There are two modes based on spin:
  - **Fermi Decay**: the spins of the neutrino and electron are anti-parallel, so total $S = 0$.
  - **Gamow-Teller Decay:** the spins are parallel, so the total $S = 1$.

The difference in the nuclear spin parity (I^π) before and after the decay will come from the l and S that the electron and neutrino carry away. So, if you have Fermi decay ($l = 0$ and $S = 0$), there can't be any change in the nuclear spin $\Delta I = |I_i - I_t| = 0$. If you have Gamow-Teller decay ($l = 0$ and $S = 1$), you have $\Delta I = 0$ or 1.

TRICKY PART: You can have $\Delta I = 0$ in Gamow-Teller decay because you can couple a vector of length 1 onto a vector and end up with the same vector magnitude. This **does not work** if you started with $I_i = 0$ because you can't add 1 to 0 and get 0. This means that a beta decay from $I_i = 0$ to $I_f = 0$ *must* be Fermi decay.

Therefore, all allowed decays have:

$$\Delta I = 0, 1 \quad \Delta\pi = \text{no}$$

[1, 2, pp. 289, Lec. 19-21]

- **Superallowed Decays:** all super-allowed decays are allowed decays, but not all allowed decays are super-allowed. All that super-allowed means is that the ft value is 3-4. So they are just super-likely.

5.3 Forbidden Decays

- Any beta-decay that has $l \neq 0$ is **forbidden**. This doesn't mean that it doesn't happen, it just means it's not very likely. This is because angular momentum $\vec{l} = \vec{r} \bullet \vec{p}$. The radius where the β-decay is on the order of the nuclear radius (≈ 6 fm) which is *super* small. The small radius means a small angular momentum, so it's very unlikely that β-decay will occur with $l \neq 0$. But it does, rarely. [2, Lec. 19-21].

- The n$^{\text{th}}$ forbidden decay will have $l = n$, and both of the spin-arrangements (Fermi for $S = 0$ and Gamow-Teller for $S = 1$) described above. Therefore, you can have:

$$\Delta I = 0, 1, 2 \ldots (n+1) \quad \Delta\pi = (-1)^n$$

The ΔI goes up to $n + 1$ because Gamow-Teller can have the spin $S = 1$ lined up with the angular momentum. [1, 2, pp.291,Lec.19-21]

- β-decay will occur via the "lowest" decay that it can. That is, it will use an allowed decay or the lowest nth forbidden decay that can accomplish the transition. You have to look at *both* parts of the initial and final states, I **AND** $\Delta\pi$ to figure out which it is. [2, Lec 19-21]
- Based on the description above, as you get higher in the forbidden decays, 1st forbidden to 2nd forbidden etc, the decays become less likely. See Table 2. [2, Lec. 19-21]

Type of β-decay	**Log(ft)**
Superallowed	3.5
Allowed	4-7.5
1st forbidden	6-9
2nd forbidden	10-13
3rd forbidden	14-20
4th forbidden	$\approx$ 23

Table 2: Log(ft) values for types of β-decay

5.4 Electron Capture

- Almost always, the electron is captured from the inner-most S orbital, and the neutrino emitted is mono-energetic. Auger electrons may result from the cascade of electrons as they move down to fill the vacancy left by the lower shells. [2, Lec. 19-21]

5.5 Helicity

- Helicity: helicity is just a property that we define based on a particles spin and momentum:

$$h = \frac{\vec{s} \bullet \vec{p}}{|\vec{s} \bullet \vec{p}|}$$

Neutrinos have specific helicity based on if they are a neutrino or anti-neutrino:

  - **ALL** neutrinos (ν) have helicity $h = -1$
  - **ALL** anti-neutrinos ($\bar{\nu}$) have helicity $h = +1$
  - All electrons **from β-decay** have helicity $h = -\frac{v}{c}$.
  - All positrons **from β-decay** have helicity $h = +\frac{v}{c}$.

Note that the helicity rules for neutrinos are always always true, but the ones for electrons/positrons are only true *if the electron is from β-decay.*

6 γ-decay

- γ-rays are photons generated from the decay between two nuclear states.

References

[1] Kenneth S. Krane. *Introductory Nuclear Physics.* John Wiley & Sons, Inc., 3rd edition, 1988.

[2] Lee Bernstein. Nuclear engineering class lectures. Fall 2015.