Plasma axisymmetric control

Adriano Mele

Università degli Studi della Tuscia

adriano.mele@unitus.it

November 20, 2023

Overview

1. Introduction to magnetic control
   - Tokamak coils system
   - Electromagnetic modelling for control
2. Coil currents control
3. Vertical stabilization
   - Rigid filament model of the vertical dynamics
   - Vertical stabilization
4. Plasma current control
5. Position control
   - Interlude: radial force balance in toroidal devices
   - Radial control
6. Shape control

What to expect from this class

In this class we will see

- An overview of the **main problems** tackled by magnetic control
- The physics behind these problems with a survey of different **modelling** approaches (basically a recap of yesterday's lecture - calculations are there but we just want to grasp the basic idea)
- Some **examples** from past (and present) experiments
- **A model to rule them all**: how to design a basic (but complete) magnetic control system based on a single linearized plasma model (*hands on!* Have your matlab ready)

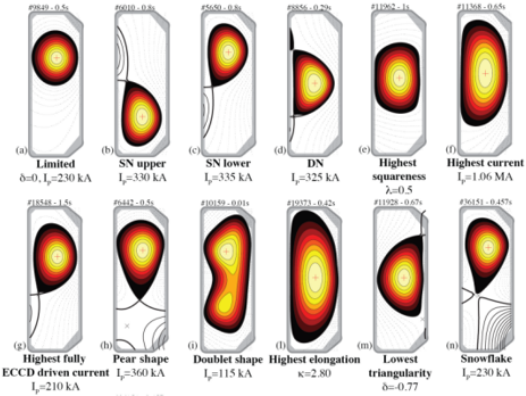
What **not** to expect from this class

In this class we will **not** see

- **Control theory**: you already had enough yesterday (except something at the end of the lecture)
- **Technology/implementation**: these include
  - control hardware & software
  - computational aspects
  - power supplies and actuators in general
  - magnetic diagnostics
    (you should already have a dedicated course on that!)
- Equilibrium reconstruction (would need a course on its own)
- Numerical modelling techniques (ditto)
- Breakdown optimization
- **Advanced** magnetic control (e.g. ADCs)

Introduction to magnetic control

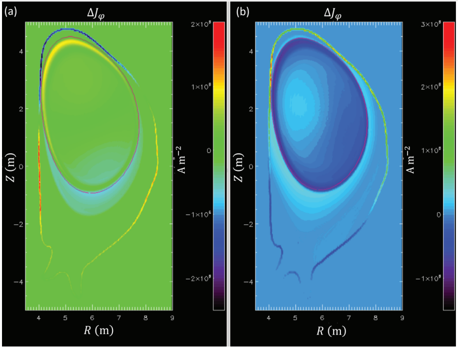
- **Magnetic control** is the exploitation of dedicated **coils** to manipulate the **magnetic field/flux** inside a tokamak's vacuum chamber in order to achieve different objectives
- It can be divided into different sub-problems



Plasma magnetic control in tokamaks

Vertical Stabilization

Elongated plasmas are **vertically unstable** and call for an active stabilization system

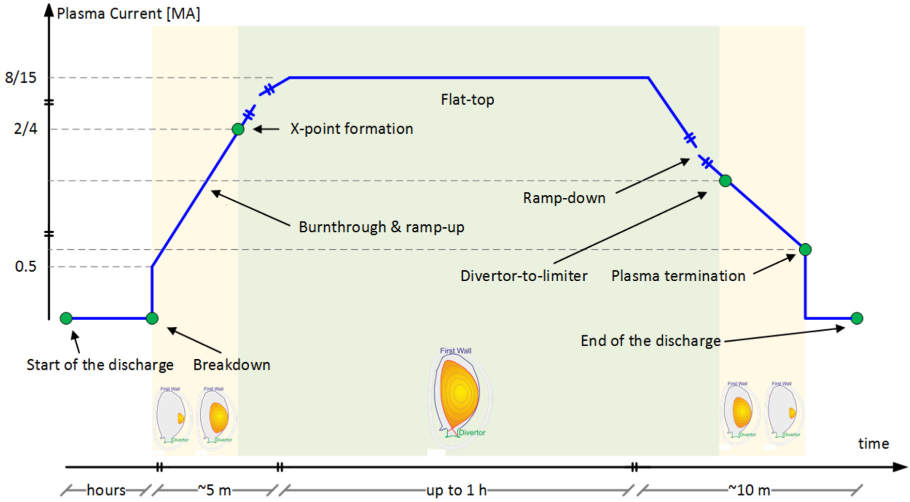


Variation of toroidal current density during a Thermal Quench (image from Clauser et al. Nucl. Fus., 2019)

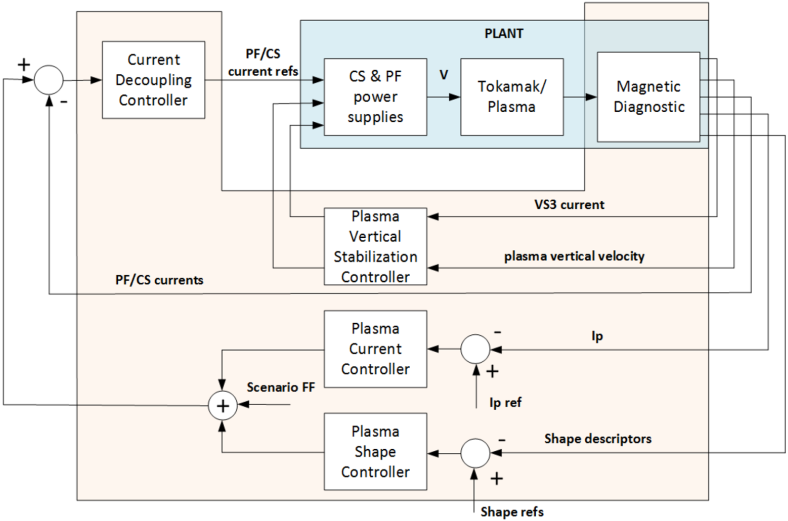
Plasma Current Control

Compensates for **ohmic drop/current drive** sources, which can be difficult to

A typical discharge

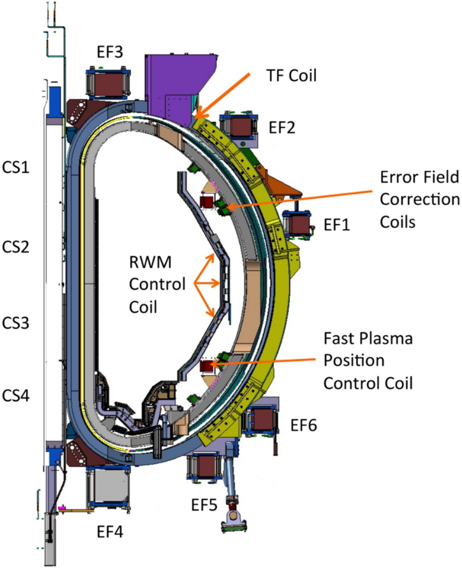


The big picture



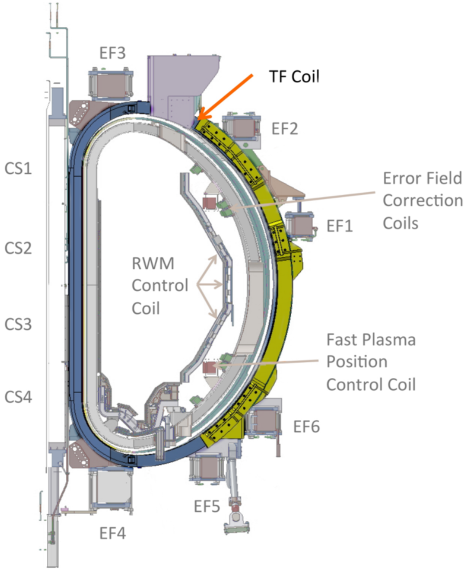
Tokamak coils

The **coils system** of a tokamak can be conceptually divided into sets that are designed for different functions



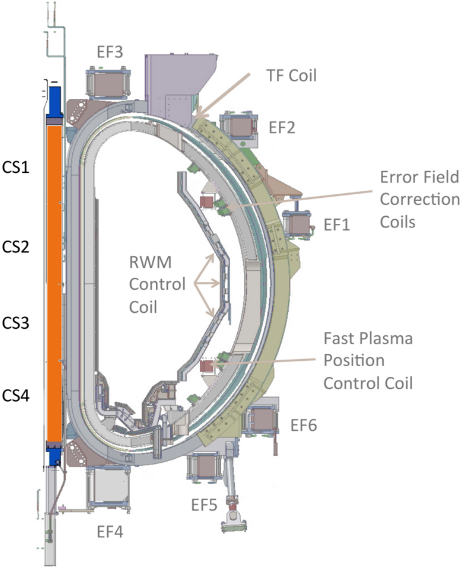
Toroidal field coils

Provide the strong toroidal field needed for magnetic confinement



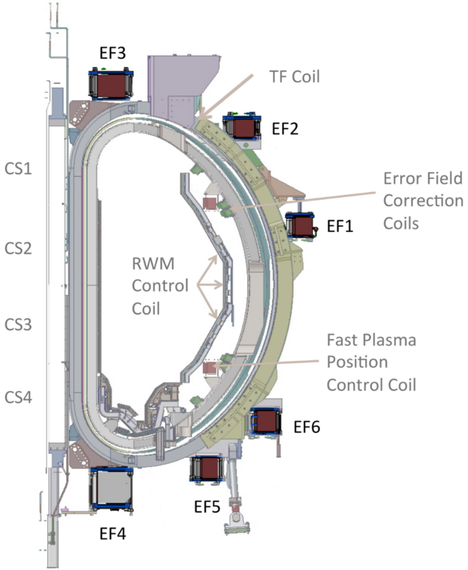
Central Solenoid

Used (mainly) for plasma current control



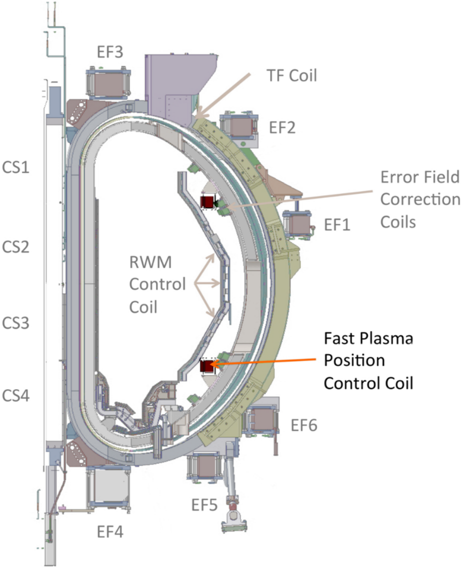
Poloidal Field Coils

Used (mainly) for position & shape control



In-Vessel Coils

Often dedicated to Vertical Stabilization, do not suffer shielding from passive structures

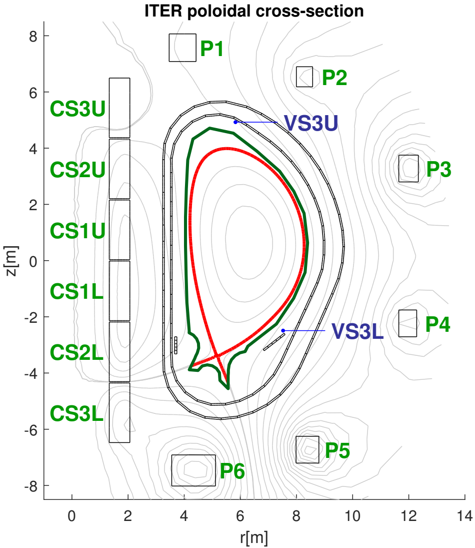


Grad-Shafranov equation

$$\underbrace{r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial z^2}}_{\Delta^*\psi(r,z)} = -\mu_0 r \underbrace{j_\phi(r,z)}_{\substack{\text{toroidal}\\ \text{current}\\ \text{density}}} \qquad (1)$$

- Describes the ideal MHD equilibrium of a tokamak axisymmetric plasma in cylindrical coordinates (r, z)
  (see also yesterday's lecture by prof. F. Villone)
- $\psi = \psi(r, z)$ is the **poloidal magnetic flux** function (normalized to 2π - it is strictly related to the vector potential)
- often with B.C. $\psi(0, z) = \lim_{(r^2+z^2)\to\infty} \psi(r, z) = 0$

Grad-Shafranov equation



- j_ϕ is **known** in the conductors
- in the plasma we assume

$$j_\phi = \frac{f}{\mu_0 r} \frac{df}{d\psi} + r \frac{dp}{d\psi}$$

  - p: plasma pressure
  - $f = rB_\phi$: poloidal current function

Grad-Shafranov equation

- often parameterized expressions are used for j_ϕ in the plasma
- a possible choice [Luxon and Brown, 1982]:

$$j_\phi(\psi) = \lambda \left(\frac{r}{R_0}\beta_0 + \frac{R_0}{r}(1-\beta_0) \right) (1-\bar{\psi}^{\alpha_m})^{\alpha_n}$$

- $\bar{\psi}$ is a normalized flux coordinate (0=axis, 1=boundary)
- the $[\lambda, \beta_0, \alpha_m, \alpha_n]$ parameters are related to $[I_p, \beta_p, l_i, q]$

J.L. Luxon, B.B. Brown (1982)
Magnetic analysis of non-circular cross-section tokamaks
Nuclear Fusion 22(813).

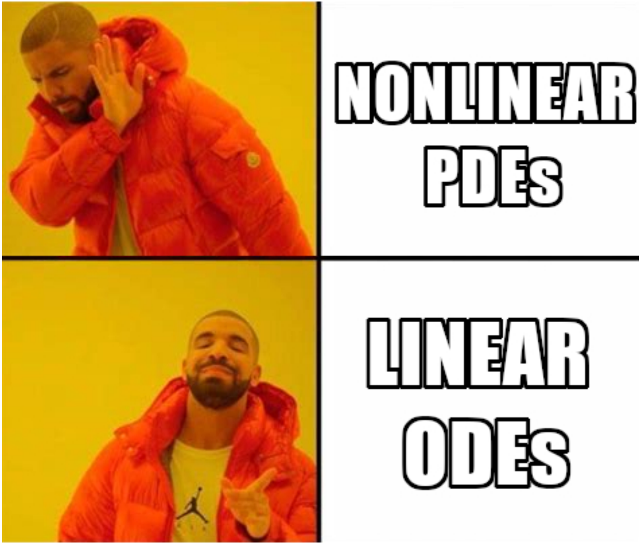
Evolutionary equilibrium

- GS equation can be coupled with **circuit dynamics** (coils, passive structures, plasma)

$$L\dot{I} + 2\pi\dot{\psi}_p + RI = V$$

- $2\pi\dot{\psi}_p$ is the flux contribution from the plasma to the considered circuits
- Notice that ψ_p depends on the solution of the GS problem, which depends on I, which depend on ψ, etc.
  - also: *no plasma = linear problem!*
- In principle, we should **solve the GS PDE at every time-step**

However...



Linearization

- GS equation is a **nonlinear, elliptic PDE**
- Control-theory is largely based on **linear ODEs**

"Classic" approach:

- **discretize** our problem in space
  (finite no. of conductors + finite-elements/finite-differences/other)
- **linearize** it around an equilibrium
  (i.e. a reference GS solution)

In our case, linearization is also done with respect to the β_p and l_i parameters, which are treated as **additional, inaccessible inputs** that represent variations in the j_ϕ profile

Linearization

$$\delta\dot{x} = A\delta x + B\delta u + E\delta\dot{w}$$
$$\delta y = C\delta x + D\delta u + F\delta w$$

- δx: **current** variations (coils, passive structures, plasma)
- δw: **profile** parameters variations (β_p, l_i)
- δu: applied **voltages**
- δy: basically **anything we want to control!**

A final note about control-oriented modelling

What we really want to control

What we simulate

What we use to design the controller

Coil currents control

Introduction

- Usually, plasma scenarios are designed by assigning suitable **currents** in the **PF coils**
- However, power supplies usually are capable of providing **voltages**
- For this reason, usually it is desirable to control such currents
- We will see an easy (but effective!) way

Introduction

- We have seen that the plasma dynamics can be put in the form of **circuit equations**
- However, we have two **main problems**
  - L depends in general on the plasma configuration
  - the eddy currents usually are not measurable
- Hence we make the following **approximations**
  - design the controller based on the **plasmaless model**, hoping that the effect of the plasma on the L matrix is negligible
  - neglect the effect of the eddy currents, and use **reduced L and R matrices**

(we will come back to these approximations later)

PFC decoupling controller

- Start from the active coils circuit equations (plasmaless)

$$L_{PF}\dot{I}_{PF}(t) + R_{PF}I_{PF}(t) = V_{PF}(t)$$

- Choose the applied voltage as
  (assume R_{PF} known and I_{PF} measurable)

$$V_{PF}(t) = \underbrace{V_{ff}}_{\substack{\text{feedforward}\\ \text{voltages}}} + \underbrace{R_{PF}I_{PF}(t)}_{\substack{\text{resistive}\\ \text{compensation}}} + \underbrace{U_{PF}(t)}_{\substack{\text{"virtual"}\\ \text{control}}}$$

PFC decoupling controller

- Neglect the feedforward term for now.
  In this way, we have pure (coupled) integrators

$$L_{PF}\dot{I}_{PF}(t) = U_{PF}(t)$$

- If the L matrix is known/measurable, we can in principle assign the current dynamics by choosing a suitable **feedback controller**
  In the Laplace domain

$$U(s) = \underbrace{L_{PF}}_{\substack{\text{decoupling}\\ \text{matrix}}} \quad \underbrace{K_{PF}(s)}_{\substack{\text{dynamic}\\ \text{controller}}} \underbrace{\left(I_{ref}(s) - I_{PF}(s)\right)}_{\substack{\text{feedback}\\ \text{error}}}$$

- We are left with the open-loop tf

$$I_{PF}(s) = \frac{K_{PF}(s)}{s}(I_{ref} - I_{PF}(s))$$

PFC decoupling controller

- Simplest choice: **proportional controller** $K_{PF}(s) = \Lambda = \lambda \mathbb{I}$

$$U(t) = L_{PF}\Lambda(I_{ref} - I_{PF}(t))$$

$-\lambda$ being the desired eigenvalue for the currents dynamics

- With this choice, for each current $I_{PF,k}$ we get the **error dynamics**

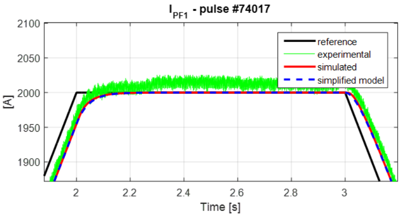
$$\frac{d}{dt}\underbrace{(I_{ref,k} - I_{PF,k}(t))}_{\text{control error}} = -\lambda(I_{ref,k} - I_{PF,k}(t))$$

(where we considered a constant reference for simplicity)

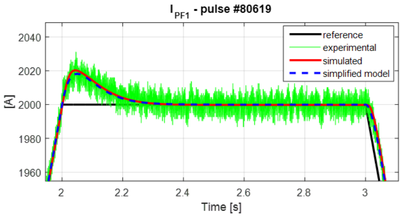
- Let's see an example

PFC decoupling at EAST

- The figures show the results obtained during a dry-run on EAST
- The simplified model (dashed) is a pure integrator

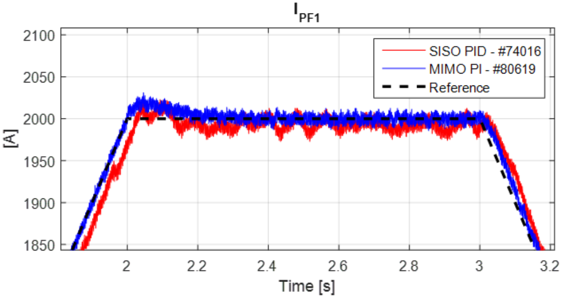


Purely proportional controller



PI controller

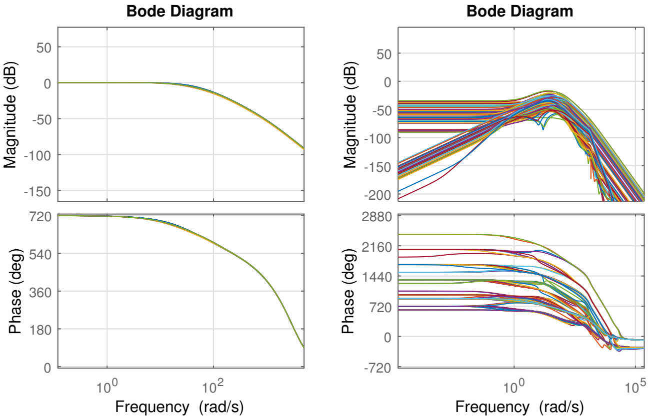
PFC decoupling at EAST



MIMO control **improves decoupling** with respect to the previous SISO solution and provides a **faster response**

PFC decoupling at EAST

Decoupling remains satisfactory even when plasma and eddy currents are taken into account



Linearized model of EAST pulse #78289 at $t = 3$ s, P controller

An alternative approach

- An alternative, proposed at TCV, is to choose the **PI** action

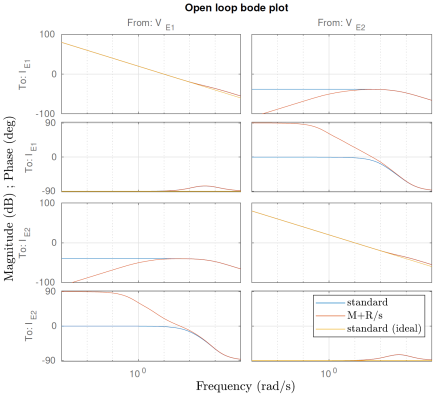
$$V_{PF}(t) = V_{ff} + \left(L_{PF} + \frac{R_{PF}}{s}\right) K_{PF}(s)(I_{ref} - I_{PF}(s))$$

- In this way, the controller is acting on the current **error** only
- We obtain (neglecting again the feedforward term)

$$I_{PF}(s) = \cancel{(sL_{PF} + R_{PF})^{-1}} \cancel{(sL_{PF} + R_{PF})} \frac{K_{PF}(s)}{s} (I_{ref} - I_{PF}(s))$$

An alternative approach

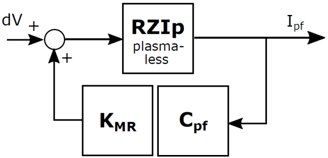
- With the standard approach, off-diagonal coupling appears when eddy currents are included
- Note that off-diagonal channels have **nonzero dc-gain**
- This effect can be reduced on long time scales using the PI approach
- This approach should also be less sensitive to **inexact resistivity** estimates

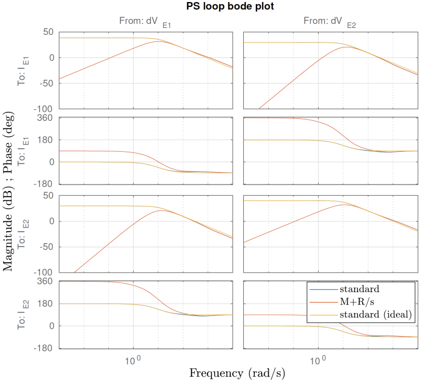


Bode plots for TCV E1 and E2 coils using a plasmaless RZIP model (the yellow trace shows the respons without vessel)

(image from F. Pesamosca PhD Thesis, EPFL 2021)

An alternative approach





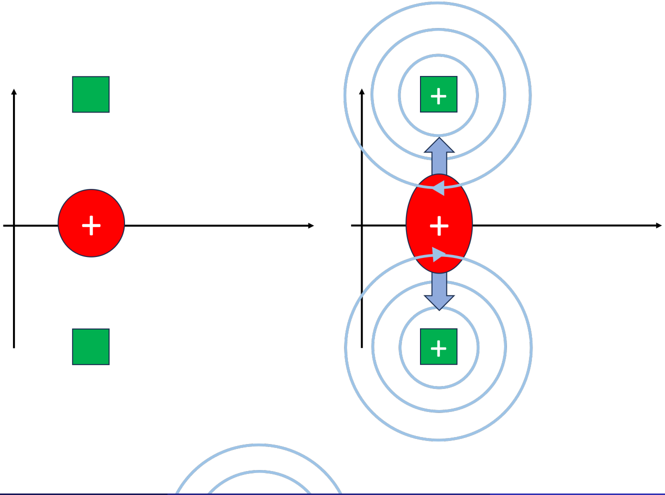
Closed-loop response for TCV E1 and E2 coils using a plasmaless RZIP model (the yellow trace shows the respons without vessel)

(image from F. Pesamosca PhD Thesis, EPFL 2021)

Vertical stabilization

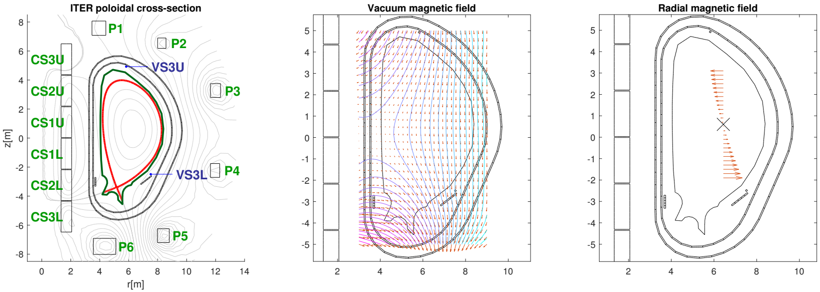
Vertical instability in tokamaks

- Elongated plasmas are unstable, due to the configuration of the magnetic field used to produce the elongated shape
- Cartoon picture:



Vertical instability in tokamaks

- Elongated plasmas are obtained through a **quadrupole** field
- The (vacuum) **radial** magnetic field component has a **downward gradient**



A small displacement results into a net force in the **same** direction: the equilibrium position ($B_r = 0$) is **unstable**

Rigid filament model

- This instability can be detected already by using a *very* crude model of the plasma: a **rigid filamentary current**
  (see also yesterday's lecture by prof. F. Villone)
- Assume plasma current and radial position fixed. This filament is subject to a force:

$$m_p \ddot{z}_p = -2\pi r_p I_p B_r(z_p, I_a, I_e) \tag{2}$$

Rigid filament model

- Eq. (2) can be **linearized**

$$m_p \ddot{z}_p \approx -2\pi r_p I_p \left[\left(\frac{\partial B_r}{\partial z_p} \right) \delta z_p + \left(\frac{\partial B_r}{\partial I_a} \right)^T \delta I_a + \left(\frac{\partial B_r}{\partial I_e} \right)^T \delta I_e \right] \quad (3)$$

- and coupled with the **circuit equations** (remember, we assumed constant I_p)

$$\begin{aligned} L_a \dot{I}_a + R_a I_a + M_{ae} \dot{I}_e + \dot{\psi}_{ap}(z_p) &= V_a \\ L_e \dot{I}_e + R_e I_e + M_{ea} \dot{I}_a + \dot{\psi}_{ep}(z_p) &= 0 \end{aligned} \quad (4)$$

Rigid filament model: linearization wrt $I_{(a,e)}$

- The radial field $B_r^{(a,e)}$ generated by the active and passive currents and affecting the plasma filament can be expressed as

$$B_r^{(a,e)} = -\frac{1}{2\pi r_p}\frac{\partial \psi_{p(a,e)}}{\partial z_p} \tag{5}$$

- The flux contributions from the coils to the plasma are

$$\psi_{p(a,e)} = M_{p(a,e)} I_{(a,e)}$$

where $M_{p(a,e)}$ are geometric mutual inductances that depend on z_p

- Note that $B_r^{(a,e)}$ is **linear** wrt the external currents

$$\frac{\partial B_r^{(a,e)}}{\partial I_{(a,e)}} = -\frac{1}{2\pi r_p}\frac{\partial M_{p(a,e)}}{\partial z_p} \tag{6}$$

Rigid filament model: field decay index

- Since the instability is linked to the **gradient of** B_r along the vertical direction, it is standard practice to define the **radial field decay index**

$$n := -\frac{r_p}{B_V}\frac{\partial B_r}{\partial z_p} \tag{7}$$

- B_V is the vertical field needed to fulfill the radial force balance condition (see later)
- The (linearized) force on the plasma filament due to a vertical displacement δz_p reads

$$\delta F_z \approx -2\pi r_p I_p \frac{\partial B_r}{\partial z_p}\delta z_p = (2\pi I_p B_V n)\delta z_p = \tilde{F}\delta z_p \tag{8}$$

- The vertical instability is present whenever $n > 0$

Rigid filament model: linearization wrt z_p

Using the symmetry of the mutual inductance matrix (for fixed z_p)

$$\psi_{(a,e)p} = M_{(a,e)p} I_p = M_{p(a,e)} I_p \tag{9}$$

and combining (3), (6) and (8) we obtain the **linearized rigid displacement model**

$$m_p \delta \ddot{z}_p = \underbrace{I_p \left(\frac{\partial M_{pa}}{\partial z_p} \right)^T}_{g_a^T} \delta I_a + \underbrace{I_p \left(\frac{\partial M_{pe}}{\partial z_p} \right)^T}_{g_e^T} \delta I_e + \underbrace{(2\pi I_p B_V n)}_{\tilde{F}} \delta z_p \tag{10a}$$

$$L_a \delta \dot{I}_a + R_a \delta I_a + M_{ae} \delta \dot{I}_e + I_p \frac{\partial M_{pa}}{\partial z_p} \delta \dot{z}_p = \delta V_a \tag{10b}$$

$$L_e \delta \dot{I}_e + R_e \delta I_e + M_{ea} \delta \dot{I}_a + I_p \frac{\partial M_{pe}}{\partial z_p} \delta \dot{z}_p = 0 \tag{10c}$$

Rigid filament model: wrap-up

- Apply the **singular perturbation** $m_p \approx 0$

$$\delta z_p \approx -\frac{1}{\tilde{F}} \underbrace{\begin{bmatrix} g_a^T & g_e^T \end{bmatrix}}_{G^T} \begin{bmatrix} \delta I_a \\ \delta I_e \end{bmatrix}$$

- Substitute in (10b)-(10c)

$$\underbrace{\left(\begin{bmatrix} L_a & M_{ae} \\ M_{ea} & L_e \end{bmatrix} - \frac{G G^T}{\tilde{F}} \right)}_{L^*} \begin{bmatrix} \delta \dot{I}_a \\ \delta \dot{I}_e \end{bmatrix} + \begin{bmatrix} R_a & 0 \\ 0 & R_e \end{bmatrix} \begin{bmatrix} \delta I_a \\ \delta I_e \end{bmatrix} = \begin{bmatrix} \delta V_a \\ 0 \end{bmatrix}$$

- $\tilde{F}$ is a **destabilizing** force
- g_a, g_e are the **stabilizing** efficiencies of the active and passive structures

Rigid filament model: wrap-up

- In state-space form

$$\begin{bmatrix} \delta \dot{I}_a \\ \delta \dot{I}_e \end{bmatrix} = -L^{*-1} R \begin{bmatrix} \delta I_a \\ \delta I_e \end{bmatrix} + L^{*-1} \begin{bmatrix} \delta V_a \\ 0 \end{bmatrix} \tag{11}$$

- Observe that the **passive structures** affect the characteristic time of the vertical instability through $-L^{*-1} R$

Vertical stabilization controller

- Usually, a fixed combination of coils connected in anti-series is used to produce the **radial field** needed to control the plasma in the vertical direction
- Often, dedicated **in-vessel** coils are used to this aim
  - These must be **copper** coils (cannot use superconductors inside the vessel)
  - Their action is not filtered by the vessel, and hence it is **faster**
- It can be shown that the resulting system has an **unstable pole** and an **"unstable" zero** (bad for performance)
- A **PD** action can be used to stabilize the system
  - However, the **proportional** action is undesired: it may interact with The position/shape loop

Vertical stabilization controller

- A possible solution to the VS problem that uses **in-vessel coils**

$$U_{IC}(s) = F_{VS}(s) \cdot \left(K_v \cdot \bar{I}_{p_{ref}} \cdot V_p(s) + K_{ic} \cdot I_{IC}(s)\right)$$
$$U_{EC}(s) = K_{ec} \cdot I_{IC}(s)$$

- The **VS controller**
  - takes the centroid **vertical speed** and the **in-vessel** circuit current
  - generates the voltage references for both the **in-vessel** and **ex-vessel** circuits

G. Ambrosino et al.
Plasma vertical stabilization in the ITER tokamak via constrained static output feedback
IEEE Trans. Contr. System Tech., 2011

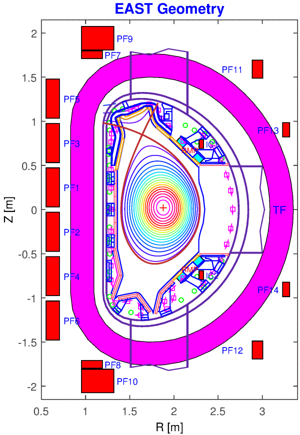
Vertical stabilization controller

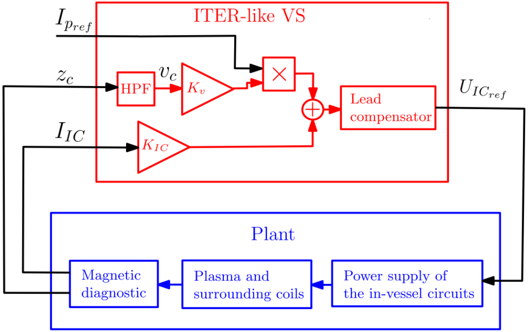
- Vertical stabilization is achieved acting on the **in-vessel** circuit
- The voltage applied to the **ex-vessel** circuit is used to **reduce the effort** on the in-vessel coils
- The *velocity* gain is scaled according to the value of $I_p \rightarrow K_v \cdot \bar{I}_{p_{ref}}$

Vertical stabilization controller

- The proposed approach includes (just) three **gains** and (if needed) a **lead compensator** $F_{VS}(s)$
  - the *speed* gain K_v
  - the gain on the in-vessel current K_{ic}
  - the gain on the imbalance current K_{ec}
- the proposed structure is rather *simple*, as there are **few parameters** to be tuned
- ...**but how to tune these (few) parameters?**
- Let's see how to design the gains for the EAST tokamak following a **model-based approach**

ITER-like VS for the EAST tokamak

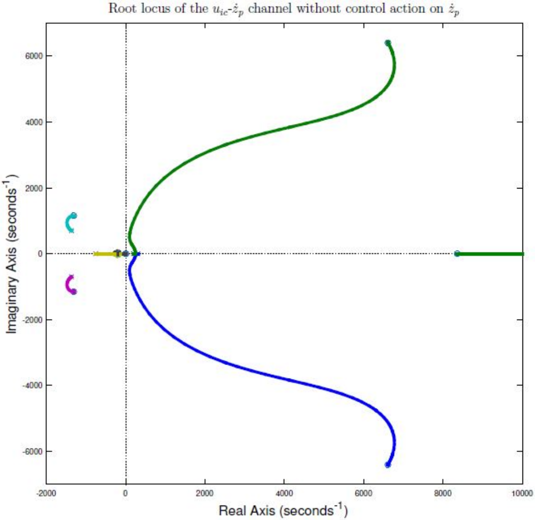




$$U_{IC_{ref}}(s) = \frac{1+s\tau_1}{1+s\tau_2} \cdot \left(K_v \cdot \bar{I}_{p_{ref}} \cdot \frac{s}{1+s\tau_z} \cdot Z_c(s) + K_{IC} \cdot I_{IC}(s) \right)$$

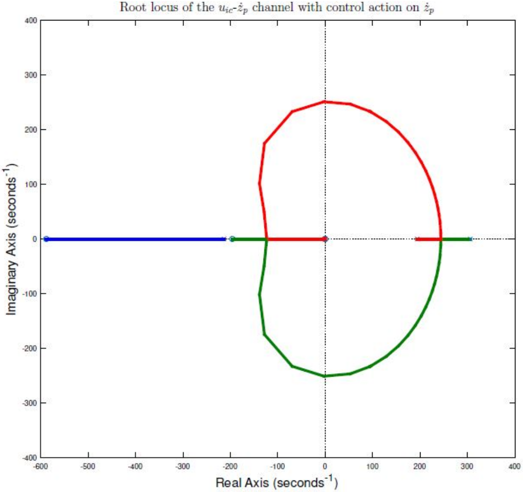
ITER-like VS for the EAST tokamak

- First close a **positive** loop on $I_{IC}(s)$
  - this introduces another unstable pole in the $u_{ic} - \dot{z}_p$ channel
  - the rhp zero comes from the power supplies!



ITER-like VS for the EAST tokamak

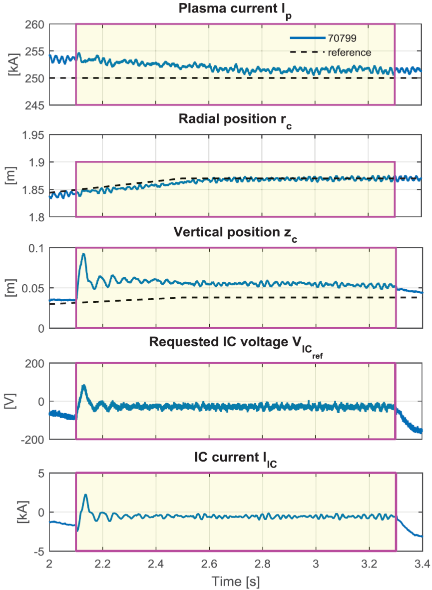
- First close a **positive** loop on $I_{IC}(s)$
  - this introduces another unstable pole in the $u_{ic} - \dot{z}_p$ channel
  - the rhp zero comes from the power supplies!
- Then close a **stable** loop on the vertical speed



ITER-like VS for the EAST tokamak

- EAST pulse #70799
- *ITER-like* VS system enabled from $t = 2.1$s to $t = 3.3$s
- R_p, I_p are also controlled
- Note that there is **no vertical position control** (indeed, the dashed black reference is not tracked)

R. Albanese et al.
ITER-like vertical stabilization system for the EAST Tokamak
Nucl. Fus., 2017



Plasma current control

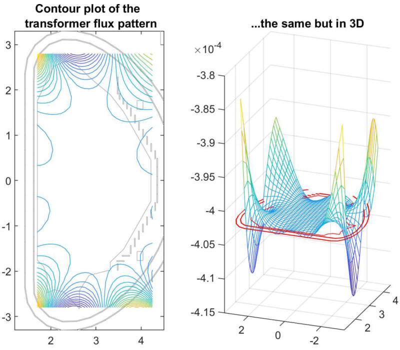
Plasma current control

- A SISO control loop is often used to control the plasma current
- This is useful to compensate for terms that may be hard to model, like
  - resistive drops
  - external current sources
- Traditionally, the CS is used to this aim
  - e.g. **OH coils** on TCV, directly controlled in **voltage**: faster response but imperfect decoupling with other control loops
- For instance, in **ITER** it is important to track the I_p reference also during the **ramp-up** and **ramp-down** phases
  - a **double integral action** is foreseen in the controller

Plasma current control

- An alternative is to find a **current pattern** that does not affect shape
  - in practice, we need a set of currents that produce a **flat** flux variation over the plasma region *(we will see an example later)*
  - this **current combination** is then used as an actuator

$$\delta I_{PF}(s) = \underbrace{\mathbf{I}_{transf}}_{\text{MISO, static}} \quad \underbrace{K_{I_p}(s)}_{\text{SISO, dynamic}} \quad I_{p_e}(s)$$

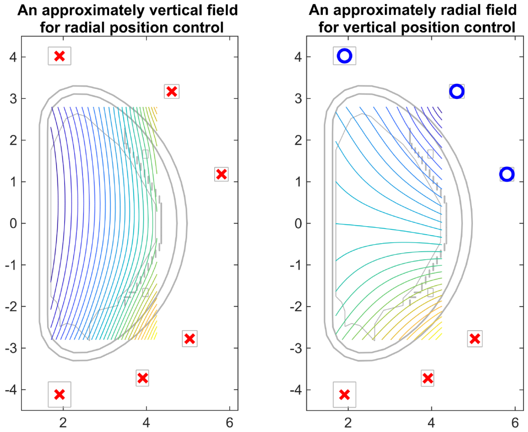


Position control

Position control

- As we have seen for I_p, an effecive approach is to find suitable **current combinations** that can reduce the problem to a **SISO** one

- Force $\sim I_p \times B$, so
  - Vertical position $\rightarrow$ **radial field**
  - Radial position $\rightarrow$ **vertical field**



Position control

- We have also seen a very simple model of the **vertical dynamics** and a possible VS system
- The same model can also be used for **vertical** position control (using the outer PF coils)
- Now let's turn our attention to the problem of **radial** position control

- Early (linear) plasma confinement devices were based on the **pinch** effect
- When a large current is driven into a conductor, it "pinches" it: **useful for confinement!**
- the easiest arrangement is the **Z-pinch**: a current flows in the $\hat{z}$ direction
- **main problems**:
  - the plasma current eroded the **electrodes** at the ends of the machine
  - **poor stability properties** (prone to kink instabilities)

Interlude: toroidal Z-pinch

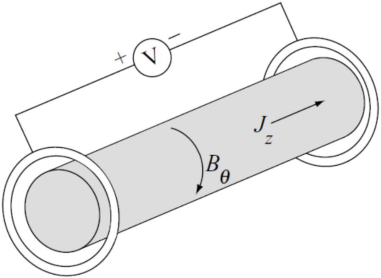
- **Toroidal configurations** emerged as a way to try to avoid the **loss of plasma at the ends** of a pinch device
- To drive the toroidal current in the pinch a **transformer core** was used (similarly to a tokamak)
- However, 'vanilla' (toroidal) Z-pinches showed significant limitations (e.g. the Perhapsatron - the name was not a good omen anyway...)

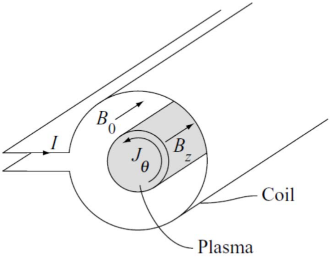
Interlude: θ-pinch

A concept that tried to solve the stability issues of the Z-pinch is the θ-**pinch**

- a (pulsed) poloidal current is driven in a **conductor wrapped around the plasma** tube
- this current induces **poloidal currents in the plasma** that tend to cancel the resulting toroidal field
- this arrangement creates an **inward force** that pinches the plasma

θ-pinches have much better **stability** properties than Z-pinches





Interlude: stabilized pinch

However, θ-pinches make for **bad toroidal devices**

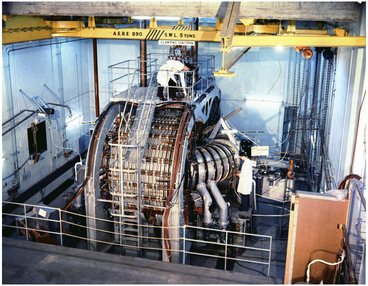
- as we will see in a moment, in a toroidal device it is necessary to apply a **radial force** to the plasma ring
- this is easy in a Z-pinch, but not in a θ-pinch
- this is why later configurations used a combination of the two concept: the so called **'screw' (or 'stabilized') pinch**

The screw pinch is also the basic recipe for a tokamak (the main difference with toroidal pinches is the **strength of the toroidal field**)

An example: ZETA

A notable example of **stabilized pinch** was the ZETA experiment

1. wrapping a **conductive** sheet of metal around the device: the induced currents would counteract **slow, large-scale** instabilities (e.g. drifts)
2. wrapping further **electromagnets** around the vacuum tube: the paths of the particles within the plasma tube were twisted like the stripes on a barber's pole. This so-called "backbone" suppressed **small-scale, localised** instabilities



Radial force balance

- The screw pinch is also the basic recipe for a tokamak (the main difference with toroidal pinches is the **strength of the toroidal field**)
- However, bending the plasma into a torus results into a number of **radial forces** that need to be compensated for
- We will give a **qualitative picture** and some expressions (without derivation)

Radial force balance

- Our considerations start from the **force-balanced MHD equilibrium**

$$J \times B = \nabla p$$

- We can combine this condition with Ampére's law

$$\nabla \times B = \mu_0 J$$

- We get

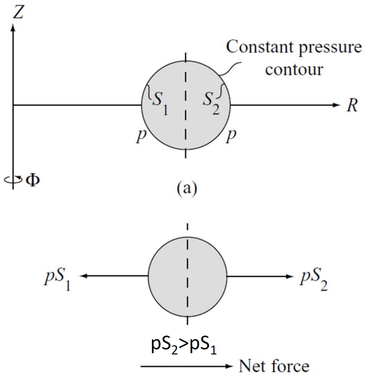
$$\left(\frac{1}{\mu_0}\nabla \times B\right) \times B = \nabla p$$

which can be rewritten as

$$\nabla \underbrace{p}_{\text{Kinetic pressure}} + \nabla \underbrace{\frac{B^2}{2\mu_0}}_{\text{Magnetic pressure}} = \underbrace{\frac{1}{\mu_0}(B \cdot \nabla)B}_{\text{Magnetic tension}}$$

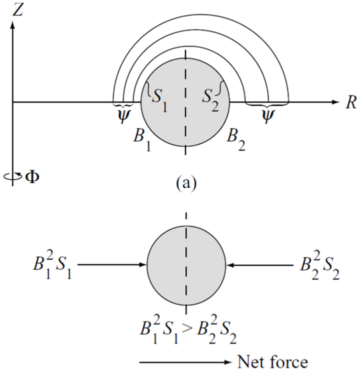
Tire-tube force

- The LCFS is an **isobaric** line
- In a toroidal plasma the external surface is larger, and since force is pressure times surface, as a result we have a **net outward force**
- Note that this effect arises bending both Z-pinches and θ-pinches, as it depends on **pressure**



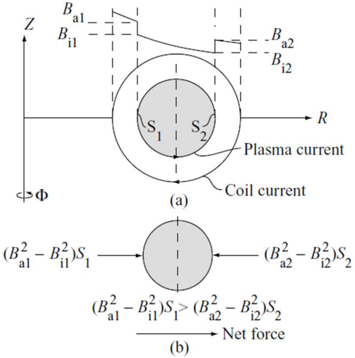
Hoop force

- The field is *stronger* in the inside of the doughnut (the so-called *high field side* of the tokamak)
- The **magnetic pressure** goes as B^2
- The quadratic dependence on B dominates, producing -again- a **net outward force**
- This effect arises in toroidal Z-pinches, as it depends on the **toroidal current**



1/R force

- In a **toroidal solenoid**, the toroidal field decays as $1/R$
- The poloidal component of the current interacts with the toroidal field (the plasma is *diamagnetic*)
- As before, the quadratic dependence on B dominates
- The result is -again!- a **net outward force**
- This effect arises in toroidal θ-pinches, as it depends on the **poloidal current**



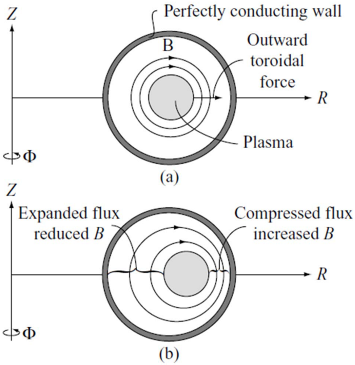
Restoring forces

We have two ways of counteracting these effects:

- surrounding the plasma with a (perfectly) **conducting wall**
- adding a **vertical magnetic field**

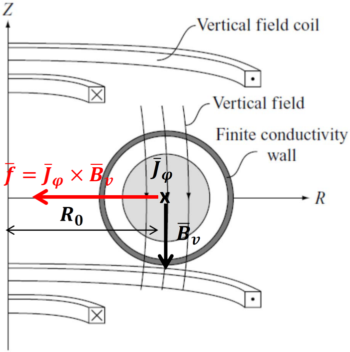
Conductive wall

- In the first case, the wall "traps" the field lines, which get compressed as the plasma moves outward
  (note: only works for the *poloidal* field, i.e. in a Z-pinch)
- The increased value of the magnetic field results in a **restoring magnetic pressure**
- However, as we have already seen perfect walls **do not exist**! The walls are always resistive, and eventually the flux diffuses across them



Equilibrium field

- In the second case, we can carefully choose the **magnitude of the applied field** in order to balance the net outward force
- This field is the B_V that we saw when talking about the VS system!
- Again, note that this can only be done for a Z-pinch (that has a *toroidal* current) force



Equilibrium field

- We can act on B_V in order to guarantee the radial equilibrium
- To do so, we need an *analytical expression* for the outward radial forces
- We will provide the expressions here, but we won't justify them - for the calculations, see e.g. [Wesson, §3.7-3.8]

Equilibrium field

- Tire-tube

$$F_1 = \frac{\mu_0 I_p^2}{4} \beta_p$$

notice how this force depends on the *pressure* through

$$\beta_p := \frac{< p >_{V_p}}{\frac{B_p^2(a)}{2\mu_0}} = \frac{4 \int_{V_p} p dV}{\mu_0 R_0 I_p^2}$$

- Hoop

$$F_2 = \frac{1}{2} \mu_0 I_p^2 \left[\ln \frac{8 R_0}{a} - 1 + \frac{l_i}{2} \right]$$

notice how this force depends on the *(toroidal) current distribution* through

$$l_i := \frac{< B_p^2 >_{V_p}}{B_p^2(a)} = \frac{4 \int_{V_p} \frac{B^2}{2\mu_0} dV}{\mu_0 R_0 I_p^2}$$

Equilibrium field

- 1/R

$$F_3 = 2\pi^2 a^2 \left[\frac{B_\phi^2(a)}{2\mu_0} - \frac{< B_\phi^2 >}{2\mu_0} \right]$$

notice how this force depends on the *toroidal field* B_ϕ

- It can be shown that this term can be rewritten as

$$F_3 = \frac{\mu_0}{4} I_p^2 (\beta_p - 1)$$

- The resulting force is

$$F_R = F_1 + F_2 + F_3 = \frac{\mu_0}{2} I_p^2 \left[\ln \frac{8R_0}{a} + \beta_p + \frac{l_i}{2} - \frac{3}{2} \right]$$

Equilibrium field

- We need to apply a force equal and opposite through an appropriate choice of the vertical field:

$$F_V = -I_p B_V 2\pi R_0 \hat{r}$$

(along the negative radial direction) and hence

$$B_V = \frac{\mu_0 I_p}{4\pi R_0} \underbrace{\left[\ln \frac{8R_0}{a} + \beta_p + \frac{l_i}{2} - \frac{3}{2}\right]}_{\Gamma(R_0, a, \beta_p, l_i)}$$

Radial control

- However, this expression is approximate:
  - it holds in the **large aspect ratio** limit
  - it is valid for **circular** plasmas, even though correction terms can be introduced to account for **shaped** plasmas
- Moreover, we have seen that β_p and l_i are considered as **disturbances** from the control perspective
  - The required radial force will vary depending on the plasma **pressure** and internal **current distribution**
- A feedback control loop is usually introduced to keep the plasma at the desired radial location

Radial control

- A simple model can be derived similarly to what we have seen for the vertical position

$$m_p \delta \ddot{r}_p = \frac{\mu_0 I_p^2}{2} \Gamma(r_p, a, \beta_p, l_i) + \underbrace{2\pi r_p I_p B_V(r_p, z_p, I_a, I_e)}_{\text{Radial force from PF coils}}$$

- We can assume again $m_p \approx 0$ and couple this equation with the current dynamics

A final note

- In this lecture we have seen simplified filamentary models for the r_p, z_p and I_p dynamics
- This models were used to design magnetic controllers in the early days of fusion research (often for circular plasmas)
- In the afternoon, we will take a different approach to the design of such controllers, based on a mode sophisticate model in the form seen here

Shape control

Introduction

- In modern tokamaks, accurate control of the plasma shape is desirable for several reasons
  - Keep a desired plasma-wall clearance
  - Optimize the vacuum chamber occupation
  - Achieve specific configurations to meet **scientific objectives** (e.g. negative triangularity) or for **technical reasons** (e.g. double-null plasmas or strike point sweeping for power exhaust handling)
  - etc.

M. Ariola, A. Pironti
Magnetic control of tokamak plasmas
Springer, 2008

R. Albanese et al.
Design, implementation and test of the XSC extreme shape controller in JET
Fus. Eng. Des., 2005

Introduction

- Two main approaches:
  - **Isoflux control**: the differences between poloidal flux values at various control points on the boundary are controlled to zero
  - **Gap control**: the distance between the plasma LCFS and the first wall is controlled to a desired value

MIMO shape controller

- The **output equation** of our linearized model provides us with a *linear relation* between the coil currents ($\delta x = \delta I_{PF}$) and the outputs of interest (δy)
$$\delta y = C\delta x$$
- Notice that this is a **static** relation!
- We are again **neglecting the eddy currents and the internal profile variations** - we can consider this as a steady-state condition
- When the dimension of controlled outputs (m) is larger than the dimension of available currents (n), we are basically left with a **linear regression problem**

Shape control as a linear regression problem

- Consider the **minimization problem**

$$\min_{\delta x} J(\delta x)$$

where

$$J(\delta x) = (\delta y^* - C\delta x)^T(\delta y^* - C\delta x)$$

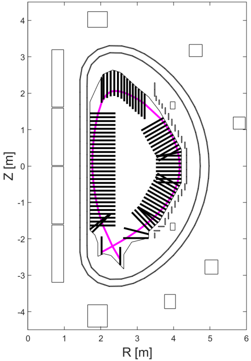
- This is an unconstrained optimization problem, so we can look for the **gradient** of J wrt δx

Shape control as a linear regression problem

- The gradient of J can be written as

$$\nabla_{\delta x} J = -2C^T \delta y^* + 2C^T C \delta x$$

- C is a $[m \times n]$ matrix
- Often we want to control **more** stuff than the number actuators we have, i.e. $m > n$
- $C^T C$ has dimensions $[n \times n]$. That means there's hope that $C^T C$ is **invertible**! (C can only have up to n independent columns)



Pseudoinversion

- In particular, if C has **full rank**, we can solve the problem by choosing

$$\delta x = (C^T C)^{-1} C^T \delta y^* \implies \nabla_{\delta x} J = 0$$

- The expression

$$C^\dagger := (C^T C)^{-1} C^T$$

  is the **Moore-Penrose (left) pseudoinverse** of C
- The name is due to the property $C^\dagger C = \mathbb{I}$

Transient response

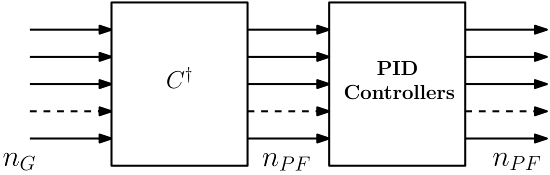
- The shape controller provides variations in the **PF coils current references**, that are fed to the PFC controller
- With this technique, we can **decouple** the shape control problem
- Each **row** of $C^\dagger$ represents a **current pattern** that is associated with one **independent combination** of the outputs of interest
- If the PF current dynamics have been **equalized** (e.g. with the controller discussed before), we can then tune a *reference governor* (on the SISO nominal PF dynamics!) to achieve the desired transient response

Additional remarks

- An **opportunely weighted** $\tilde{C}$ matrix can also be used to promote a more accurate control of some of the shape descriptors or the usage of some of the actuators (see later)

$$\tilde{C} = WCQ$$

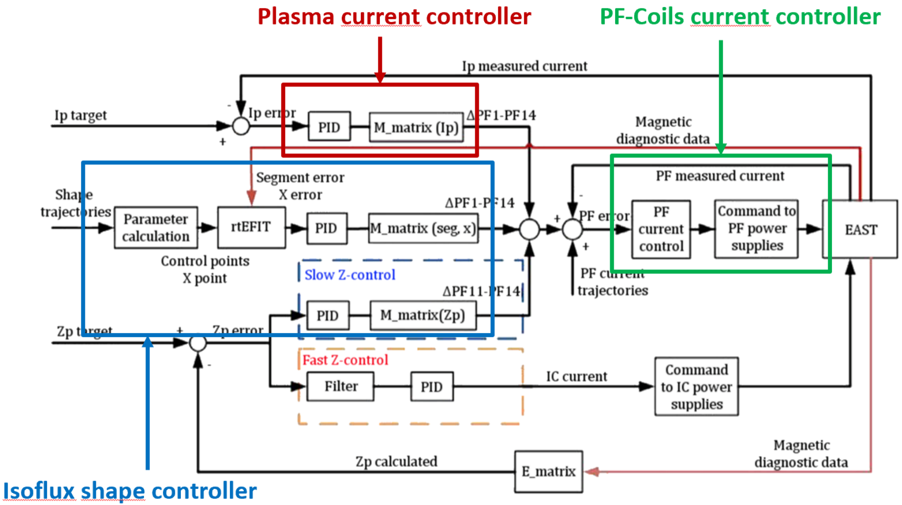
- The resulting controller block looks like this



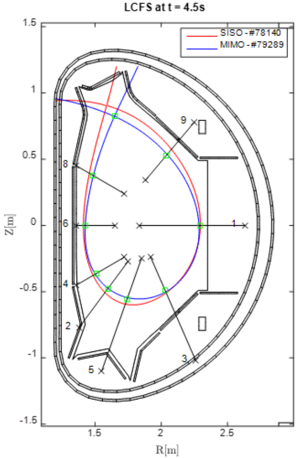
MIMO shape control at EAST

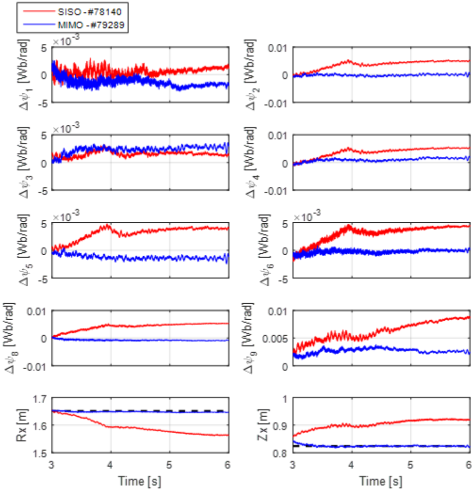
- If the PIDs are all the same, they can be moved upstream of the $C^\dagger$ block: this was convenient during our experience at EAST
- The new shape controller could be installed and tested leaving the existing software architecture basically **unaffected**

MIMO shape control at EAST

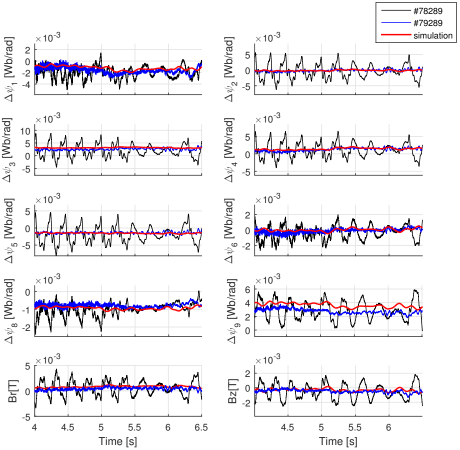


MIMO shape control at EAST





MIMO shape control at EAST



The controller in the previous slide was designed using data from a similar pulse

- black: EAST pulse #78289
  Controller taken from another shot with different I_p and plasma configuration
- red: simulation
  The controller is tuned to reduce oscillations
- blue: EAST pulse #79289
  (experimental results)

A. Mele et al.
MIMO shape control at the EAST tokamak: simulations and experiments
SOFT, 2018

A second look to pseudoinversion

- The design we have seen relies on the assumption that $C^T C$ is invertible
- With $n = m$ we can control *exactly* n independent linear combinations of shape descriptors
- With $n < m$ the controller minimizes the **steady-state mean square error**
- . . .**is this the best choice? What if C does not have full rank?**

Singular Value Decomposition

- The **svd** of $C \in \mathbb{R}^{m\times n}$ is a factorization in the form

$$C = U\Sigma V^*$$

- $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are **unitary**[1][2] matrices, whose columns are the *generalized (left and right) eigenvectors* of C
- $\Sigma \in \mathbb{R}^{m\times n}$ is a rectangular diagonal matrix, whose non-negative diagonal entries σ_i are the *singular values* of C

[1] i.e. $A^{-1} = A^*$. This also implies $\det A = 1$.

[2] The equivalent for real matrices is a **orthogonal** matrix, i.e. $A^{-1} = A^T$. For simplicity, we will always refer to the real case.

Singular Value Decomposition

- The svd generalizes the **eigendecomposition** of a square normal[3] matrix with an orthonormal eigenbasis to any $m \times n$ matrix
- In our case, for instance

$$C^T C = (V \Sigma^T U^T)(U \Sigma V^T)$$

and since U, V are orthogonal, $U^{-1} = U^T$, $V^{-1} = V^T$ and

$$\underbrace{(C^T C)}_{\substack{\text{Hermitian} \\ \text{Positive} \\ \text{semi-definite}}} \quad V = V \quad \underbrace{(\Sigma^T \Sigma)}_{\substack{\text{Diagonal} \\ \text{Non-negative} \\ \text{entries}}}$$

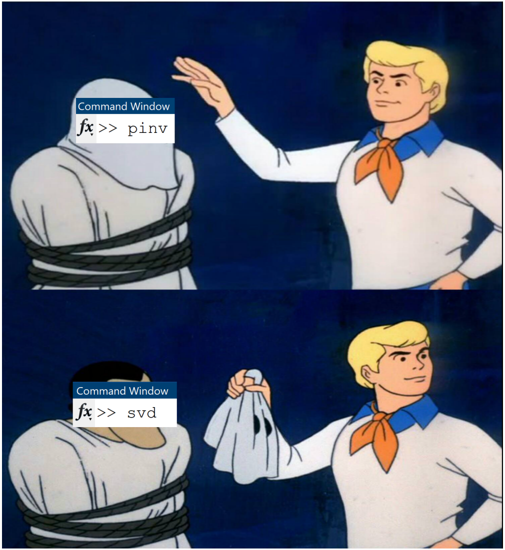
- similarly, $(CC^T)U = U(\Sigma\Sigma^T)$

[3] i.e. such that it commutes with its conjugate-transpose: $A^* A = AA^*$

Singular Value Decomposition

- To summarize:
  - the columns of V are the eigenvectors of $C^T C$
  - the columns of U are the eigenvectors of CC^T
  - the non-zero entries of Σ are the square roots of the eigenvalues of CC^T or $C^T C$
- **Why is this useful?**

Pseudoinversion through svd



Pseudoinversion through svd

- The pseudoinverse of a matrix can be computed via its **svd**

$$C = U\Sigma V^T \implies C^\dagger = V\Sigma^\dagger U^T$$

- $\Sigma^\dagger$ is obtained by replacing every **non-zero** diagonal entry by its reciprocal and transposing the result
- This procedure works also when the rank of C is not full!

Low-rank approximation

- Another advantage of using the svd is that it can be used to compute a **truncated** version of $C^\dagger$ (so-called low-rank approximation)
- Let's look at this from a geometric perspective

Low-rank approximation

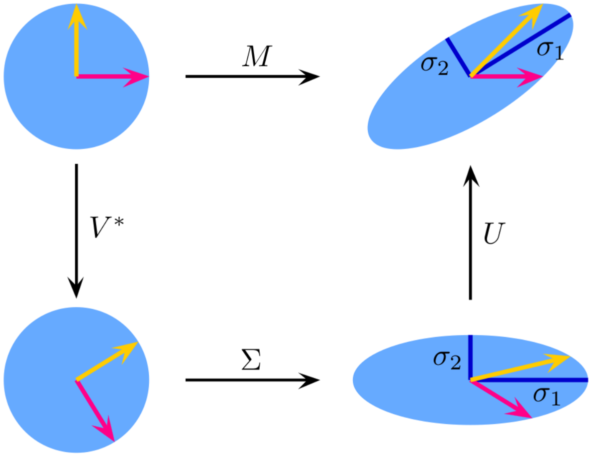
- Our cost function was

$$J = (\delta y^* - C\delta x)^T(\delta y^* - C\delta x) \sim (\delta x^* - \delta x)C^T C(\delta x^* - \delta x)$$

- The action of the positive semi-definite, symmetric matrix $C^T C$ on the error vector $\delta\tilde{x} = (\delta x^* - \delta x)$ can be visualized looking at how it transforms a **sphere** [4]

[4] i.e. every possible unit-norm $\delta\tilde{x}$.

Low-rank approximation



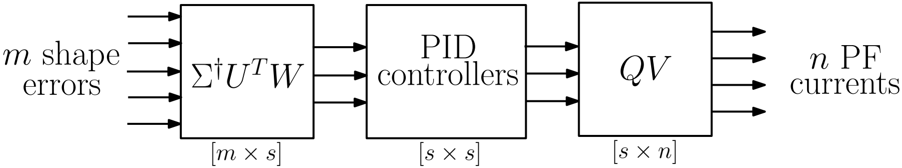
$$M = U \cdot \Sigma \cdot V^*$$

Low-rank approximation

- The columns of V associated to the **largest singular values** represent the **most controllable current directions**
- A small variations along such directions results in a **significant effect** on the shape control errors
- On the other hand, to affect the directions in U associated to **small** σ_i, we need **large current variations**
- These could stress the actuators, so it is common practice to **discard the smallest singular values**
  (usually below some tolerance, e.g. 5% of σ_1)
- In practice, we just set them to zero when computing $\Sigma^\dagger$

Control in the reduced space

- Finally, we only need to control the amplitude of the **generalized eigenmodes**
- The final scheme looks like this (including weight matrices)



Shape control at TCV