15.

Intro

In this exercise, we will set the hyper parameters of the Beta distribution $\beta(a, b)$, based on the mean m and variance v. This is a very nice property to have, since direct estimation of the mean and the variance is much more simpler than the estimation of the hyper parameters. After expressing them as functions of m and v, we will apply the result in a example where $m = 0.7$ and $v = 0.2^2$.

Solution

$$\begin{aligned} mean = m &= \frac{a}{a+b} \\ b &= a\left(\frac{1-m}{m}\right) \end{aligned} \tag{1}$$

$$\begin{aligned} &var = v = \frac{ab}{(a+b)^2(a+b+1)} = \frac{a^2\left(\frac{1-m}{m}\right)}{a^2\left(\left(1+\frac{1-m}{m}\right)\right)^2(a+b+1)} \therefore \\ &a+b+1 = \left(\frac{1}{v}\right)\left(\frac{1-m}{m}\right)\left(\frac{m^2}{m^2+2m(1-m)+(1-m)^2}\right) = \frac{m(1-m)}{v} \therefore \\ &a\left(1+\frac{1-m}{m}\right) = \frac{m(1-m)}{v} - 1 \therefore \\ &a = m\left(\frac{m(1-m)}{v} - 1\right) \end{aligned} \tag{2}$$

Substituting (2) in (1):

$$b = a\left(\frac{1-m}{m}\right) = \left(\frac{m(1-m)}{v} - 1\right)(1-m) \tag{3}$$

For $m = 0.7$ and $v = 0.2^2 = 0.04$ we have:

$$\begin{aligned} a &= 0.7\left[\frac{0.7 \times 0.3}{0.04} - 1\right] = 2.975 \\ b &= 2.975\frac{0.3}{0.7} = 1.275 \end{aligned} \tag{4}$$

Conclusion

In this exercise, we derived an expression for the hyper parameters of the Beta distribution as a function of the mean m and the variance v. This is a very interesting expression to have in hands, since we can direct estimate the mean and the variance more easily than the hyper parameteres. By the way, one of the reasons the Gaussian is so popuplar is that its hyper parameters are also its mean and variance, which makes infering its parameters much more easy.