4.3 In this exercise, we will show that the correlation coeffcient satisfies the following bound: $-1 \leq \rho(X,Y) \leq 1$. Since $\rho(X,Y) = \frac{cov(X,Y)}{\sigma_x \sigma_y}$, we will also be showing the the product of the standard deviations bounds the possible values of the covariance. The key insight to perform this demonstration is to expresse the expected value as a inner product and apply the Cauchy-Schwarz inequality.
Intro

Solution

Let's define the inner product $< X, Y >= E[XY]$. Also, let's remember Cauchy-Schwarz inequality:

$$< X, Y >^2 \leq < X, X >< Y, Y > \tag{1}$$

Now, let's write the covariance between X and Y as a inner product:

$$\begin{aligned} &Cov(X,Y)^2 = E[(X-\mu_x)(Y-\mu_y)]^2 = \\ &< X-\mu_x, Y-\mu_y >^2 \leq < X-\mu_x, X-\mu_x >< Y-\mu_y, Y-\mu_y >= \\ &E[(X-\mu_x)^2]E[(Y-\mu_y)^2] = Var(X)Var(Y) \end{aligned} \tag{2}$$

With the upper bound for $Cov(X,Y)^2$, we can find the bound for the correlation coefficient:

$$Cov(X,Y)^2 \leq Var(X)Var(Y) \implies -1 \leq \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \rho(X,Y) \leq 1 \tag{3}$$

Thus, the min/max value of $\rho(X,Y)$ is $-1/+1$.