3.

Intro

Remember that in the beginning section 3.3.4 (Posterior predictive distribution), the author derive the expression for the posterior of the next result of the coin toss. Meanwhile, as shown in section 3.3.4.2 (Predicting the outcome of multiple future trials), the posterior Beta-binomial distribution gives us the probability of getting x heads in the next n trials. This exercises want us to prove that when the number of trials is equal to one ($n = 1$), the Beta-binomail gives us the same result as the one in section 3.3.4.

Solution

The posterior predictive for the Beta-Binomial model is given by:

$$p(x|n, D) = Bb(x|\alpha_0^{'}, \alpha_1^{'}, n) = \frac{B(x + \alpha_1^{'}, n - x + \alpha_0^{'})}{B(\alpha_1^{'}, \alpha_0^{'})}\binom{n}{x} \tag{1}$$

We want to know the particular result when $n = 1$ and $x = 1$. Substituting this values on (1), we get:

$$p(1|1, D) = \frac{B(1 + \alpha_1^{'}, \alpha_0^{'})}{B(\alpha_1^{'}, \alpha_0^{'})}\binom{1}{1} = \frac{B(1 + \alpha_1^{'}, \alpha_0^{'})}{B(\alpha_1^{'}, \alpha_0^{'})} \tag{2}$$

Next, we should substitute the beta function by its definition ($B(a, b) \triangleq \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$) in (2)

$$\begin{aligned} &\frac{B(1 + \alpha_1^{'}, \alpha_0^{'})}{B(\alpha_1^{'}, \alpha_0^{'})} = \frac{\Gamma(1 + \alpha_1^{'})\Gamma(\alpha_0^{'})}{\Gamma(\alpha_1^{'} + \alpha_0^{'} + 1)}\frac{\Gamma(\alpha_1^{'} + \alpha_0^{'})}{\Gamma(\alpha_1^{'})\Gamma(\alpha_0^{'})} = \\ &\frac{\Gamma(1 + \alpha_1^{'})}{\Gamma(\alpha_1^{'})}\frac{\Gamma(\alpha_1^{'} + \alpha_0^{'})}{\Gamma(1 + \alpha_1^{'} + \alpha_0^{'})} \end{aligned} \tag{3}$$

As the last step, let's use the hint of the author $\frac{\Gamma(\alpha_0^{'}+\alpha_1^{'}+1)}{\Gamma(\alpha_0^{'}+\alpha_1^{'})} = \alpha_0^{'} + \alpha_1^{'}$ on (3):

$$\frac{\Gamma(1 + \alpha_1^{'})}{\Gamma(\alpha_1^{'})}\frac{\Gamma(\alpha_1^{'} + \alpha_0^{'})}{\Gamma(1 + \alpha_1^{'} + \alpha_0^{'})} = \frac{\alpha_1^{'}}{\alpha_1^{'} + \alpha_0^{'}} \therefore p(1|1, D) = \frac{\alpha_1^{'}}{\alpha_1^{'} + \alpha_0^{'}} \tag{4}$$

Conclusion

The result of this question was rather obvious. If I have a distribution that gives the probability of getting x heads in n trials and I make $n = 1$, then this will be the same as the probability of getting head in the next result. The calculation above was not necessary once we realize that the two events were the same! Nevertheless, since the author give a hint of how to solve the math behind this problem, it is safe to say that he wanted the readers to actually do the math. In one way or another, the result is the same.