4.2 In this exercise we will show that uncorrelated Gaussian r.v's are not necessarily independent, unless jointly Gaussian. First, note that if the r.v's were jointly Gaussian, then $\Sigma_{i,j} = Cov[X_i, X_j] = \rho(X_i, X_j)\sigma_i\sigma_j = 0$, since $\rho(X_i, X_j) = 0$. Unfortunatly, the relationship between two Gaussian r.v's it is not obligated to be a jontly Gaussian distribution. In this exercise, for example we have the following expression linking the two variables $Y = WX$, where $W \in \{-1, 1\}$ is a discrete r.v and X is Gaussian. Now, we shall prove that Y is also Gaussian and that it is uncorrelated from X.

Intro

Solution

a)

$$p(y) = p(wx) = p(sign(w)x) \tag{1}$$

where $sign(w)$ is the sign function $\frac{w}{|w|}$. Now, let's observe an intersting propertie in the $X \sim \mathcal{N}(0, 1)$ distribution. The normal is always symmetrical around its mean. Thus, in our case we have $p(x) = p(-x)$. Therefore:

$$p(y) = p(sign(w)x) = p(x) \tag{2}$$

Hence, $Y \sim \mathcal{N}(0, 1)$.

b)

First, note that the mean of both Gaussians is zero. Second, Let's use the rule of iterated expectation shown in the hint:

$$\begin{aligned} &E[XY] = E[E[XY|W]] = 0.5E[XY|W = -1] + 0.5E[XY|W = 1] = \\ &0.5(E[-X^2] + E[X^2]) = 0.5E[X^2 - X^2] = 0 \end{aligned} \tag{3}$$

Hence, $Cov[X, Y] = E[XY] - E[X]E[Y] = 0 \implies \rho(X, Y) = 0$

Conclusion

In this exercise we have shown that it is possible to Gaussian variables to have relatinships beyond the jointly Gaussian distribution. We also shown, that for non jointly Gaussian distribution, it is possible that the correlation between the two Gaussian r.v.'s is zero. This makes sense, since the