5.

Intro

Before "solving" this question, is important to say that it has a error in its formulation. Since there is an error in the hypothesis, all the derivation is wrong. Nevertheless, I will show how to arrive at the desired result in the Solution section. In the Conclusion section I will point out the mistake.

That being said, let's have a look at the exercise. The question gives us a logit relation between ϕ and θ and one expression for the probability of ϕ (i.e. $p(\phi) \propto 1 \implies p(\phi) = k$). Starting with this information, we need to derive an expression for the probability of θ. As the author hints, we will need to use change of variables.

Solution

If $p_\phi(\phi) \propto 1$ then $p_\phi(\phi) = k$, where k is a constant. Let's use the author's hint and work with change of variables.

$$\begin{aligned} &p_\theta(\theta) = \left|\frac{d\phi}{d\theta}\right| p_\phi(\phi) = \\ &k\left|\frac{d}{d\theta}\left(log\frac{\theta}{1-\theta}\right)\right| = \\ &k\frac{1-\theta}{\theta}\left[\frac{1}{1-\theta} + \frac{\theta}{(1-\theta)^2}\right] = \\ &k\theta^{-1}(1-\theta)^{-1} \end{aligned} \tag{1}$$

Equation (1) tell us that $p_\theta(\theta) \propto \theta^{-1}(1-\theta)^{-1}$. Remember that distributions of the form $\theta^a(1-\theta)^b$ are in the Beta distribution territory. In particular, $Beta(\theta|0,0) \propto \theta^{-1}(1-\theta)^{-1}$. Therefore:

$$p_\theta(\theta) \propto Beta(\theta|0,0) \tag{2}$$

Conclusion

As we saw, p(ϕ) $\propto 1 \implies p_\theta(\theta) \propto Beta(\theta|0,0)$. The demonstration was a direct use of change of variables. There is no problem with the math itself. However, the idea that $p(\phi)$ and $p(\theta)$ are probability functions is wrong. To see this, we just need to look at the "model" of θ. For those that recall the Beta model $B(\theta|a,b)$ definition in Chapter 2, we need $a, b > 0$ in order to the distribution be integrable. This can be quick checked using a numerical package to calculate $\int_0^1 Beta(\theta|0,0)d\theta$. The result goes to infinty, instead of converging to one.

Furthermore, the argument of the logit function is in the $[0,1]$ interval. Thus, $\phi \in (-\infty, \infty)$. At the same time, $p(\phi) = k$. So this is also not a probability function. Since the proof of the change of variables uses specific properties of probability functions and we are not working with them here, we cannot use change of variables on this question. I already e-mailed the author about this question.