4.1

Intro

In this exercise we will show that uncorrelated r.v's are not necessarily independent. Remembering chapter 2, the correlation coefficient can only express linear relationships between r.v's. Therefore, for our case where we have $Y = X^2$, it is possible to have $\rho(X, Y) = 0$.

Solution

$$\rho(X, Y) = \frac{cov(X, Y)}{\sqrt{var(X)var(Y)}} \tag{1}$$

$$cov(X, Y) = E[XY] - \mu_x \mu_y \tag{2}$$

where $\mu_x, \mu_y = E[X], E[Y]$. Substituting $Y = X^2$ in $E[XY]$:

$$E[XY] = E[X^3] = \int_{-1}^{1} x^3 p(x) dx = \frac{1}{2} \int_{-1}^{1} x^3 dx = 0 \tag{3}$$

In the integral of Equation 3 we used the fact that x^3 is an odd function and the integration interval was symmetrical.

Since $X \sim U(-1, 1)$, we have $\mu_x = 0$. Therefore:

$$cov(X, Y) = E[XY] - \mu_x \mu_y = 0 \implies \rho(X, Y) = 0 \tag{4}$$

Conclusion

In this exercise, we have shown that it is possible for two random variables to be dependent and have correlation coefficient equal 0. The null correlation coefficient $\rho(X, Y) = 0$ only allows us to say with certanty that if X and Y are dependent, the depedence is non-linear.