3.

Intro

This problem ask us to demonstrate the identity of the variance of a sum of random variables (r.v.'s). Before showing the steps, is it worth to mention that the final expression is intuitive. It says that the variance of the sum depends on the variance of the components and in their linear relationship (given by the correlation).

Solution

var[X + Y] = E[(X + Y - $\mu_x - \mu_y)^2] =$
$E[((X - \mu_x) + (Y - \mu_y))^2] =$
$E[(X - \mu_x)^2 + (Y - \mu_y)^2 + 2(X - \mu_x)(Y - \mu_y)] =$
$E[(X - \mu_x)^2] + E[(Y - \mu_y)^2] + E[2(X - \mu_x)(Y - \mu_y)] =$
$var[X] + var[Y] + 2cov[X, Y]$

Conclusion

I think the result speaks for itself.Now that the identity is proven, we can analyze two extreme cases, just to get some insight.

First, if the r.v.'s are uncorrelated, then

$$var[X + Y] = var[X] + var[Y].$$

The variance of the sum is the sum of the variances.

Second, if there is a positive linear relationship between the r.v.'s (Y = aX + b), then

var[X + Y] = var[X] + var[Y] + 2cov[X,Y] =
var[X] + var[Y] + 2$\sqrt{var[X]}\sqrt{var[Y]} =$
$(\sqrt{var[X]} + \sqrt{var[Y]})^2 =$
$(\sigma_x + \sigma_y)^2$

var[X + Y] = $\sigma^2_{x+y} = (\sigma_x + \sigma_y)^2$

$\sigma_{x+y} = \sigma_x + \sigma_y$

Thus, we have a linear relationship between the standard deviations. The standard deviation of the sum is the sum of the standard deviations.

Following the same steps as before, we can show that for a negatively linear relationship between the r.v's, the relation becomes:

$$\sigma_{x+y} = |\sigma_x - \sigma_y|.$$