9.

Intro

In this question we have to exercise our capacity of getting some information about conditional independence and see if we can infer something more based on it. I would like to give a special shout out to github user xulongwu4 and the repository MLaPP from astahlman for pointing out a mistake in item b. The item is now corrected.

Solution

a)

We have two information about conditional independence. The first one is:

$$X \perp W|Z,Y$$

Based on this, we can state that:

$$p(x,w|z,y) = p(x|z,y)p(w|z,y) \tag{1}$$

The second one is:

$$X \perp Y|Z$$

Based on this, we can state that:

$$p(x,y|z) = p(x|z)p(y|z) \tag{2}$$

Now, we have to see if $X \perp Y, W|Z$ is true.

$$\begin{aligned} p(x,y,w|z) = p(x|z)p(y|x,z)p(w|x,y,z) = \\ p(x|z)p(y|z)p(w|y,z) = p(x|z)p(y,w|z) \end{aligned} \tag{3}$$

In (3), the first passage is the chain rule of probability. The second make use of the conditional independences given to us (Y do not depend on X given Z and W do not depend on X given Z and Y). The third passage is putting the joint distribution back together. Therefore, the proposition is **true**.

b)

Let's prove that this proposition is false with a counter example. Start by defining X, Y and Z to be i.i.d. random variables with the following distribution:

$$\begin{cases} P(X=1) = 0.5 \\ P(X=-1) = 0.5 \end{cases} \tag{4}$$

Also, define $W = XYZ$. First, we shall prove that the two conditional independences $X \perp Y|Z$ and $X \perp Y|W$ are true. Let's start with $X \perp Y|Z$:

$$p(x,y|z) = p(x|z)p(y|z,x) = p(x|z)p(y|z) \tag{5}$$

where in the last passage we used the fact that X and Y are independent by construction. As for the second conditioanl independence, start by realizing that the original sample space of (X, Y, Z) is $(-1,-1,-1),(-1,-1,1),(-1,1,-1)$ $,(-1,1,1),(1,-1,-1),(1,-1,1),(1,1,-1),(1,1,1)$ and that $P(X,Y,Z)=\frac{1}{8}$ for any of those events. Now, when we observe $W=1$, our sample space is reduced by half, because only combinations with an even number of -1 will produce this W. Therefore:

$$\begin{cases} P(X,Y,Z|W=1)=0.25 \text{ for an even number of} -1 \\ P(X,Y,Z|W=1)=0 \text{ for an odd number of} -1 \end{cases} \tag{6}$$

Notice that the same reasioning can be applied to $P(X,Y,Z|W=-1)$, but this time, the new sample space will be the complement of the previous sample space (combinations with an odd number of -1). Based on these results we can perform the following marginalization:

$$\begin{aligned} &P(X=1|W=1)=P(X=1,Y=1,Z=1|W=1)+P(X=1,Y=-1,Z=-1|W=1)+ \\ &P(X=1,Y=-1,Z=1|W=1)+P(X=1,Y=1,Z=-1|W=1)= \\ &\frac{1}{4}+\frac{1}{4}+0+0=\frac{1}{2} \end{aligned} \tag{7}$$

We can do the same for $(X,W)=(1,-1),(-1,1),(-1,-1)$ and for (Y,W) also. Thus, $P(X|W)=P(Y|W)=\frac{1}{2}$. Now, let's perform the same marginalization on the joint distribution $P(X=1,Y=1|W=1)$.

$$\begin{aligned} &P(X=1,Y=1|W=1)=P(X=1,Y=1,Z=1|W=1)+P(X=1,Y=1,Z=-1|W=1)= \\ &\frac{1}{4}+0=\frac{1}{4} \end{aligned} \tag{8}$$

We can do the same thing for the other combinations of (X,Y,W). Therefore, $P(X,Y|W)=\frac{1}{4}$. Thus, we second proposition $P(X,Y|W)=\frac{1}{4}=\frac{1}{2}\frac{1}{2}=P(X|W)P(Y|W)$ is also true.

As for the final step

$$\begin{aligned} &P(X,Y|Z,W)=P(X|Z,W)P(Y|X,Z,W)=\frac{1}{2}I(Y=\frac{W}{XZ}) \\ &P(X|Z,W)P(Y|Z,W)=\frac{1}{2}\frac{1}{2}=\frac{1}{4} \end{aligned} \tag{9}$$

So, the proposition $X\perp Y|Z,W$ does not hold true.

Conclusion

There is not a major conclusion to be mentioned. As it was said it the begin, I think the main purpose of this exercise is just to exercise the brain in inference involving conditional independence. A special notice to item b shoud be made. The counter example was based on i.i.d. random variables and a "linking" random variable W, which made the random variables dependent in the following circunstance $P(Y|X,Z,W)=I(Y=\frac{W}{XZ})$. Again, I would like to thank xulongwu4 and MLaPP repository for this correction.