4.7

Intro

In this exercise, we will take a closer look on the conditional distribution of a bivariate Gaussian distribution.

Solution

a)

Since x_1, x_2 have a multivariate Gaussian distribution, the conditional $p(x_2|x_1)$ it is also Gaussian. Using the expressions derived in this chapter, we can state that:

$$\begin{aligned} p(x_2|x_1) &= \mathcal{N}(x_2|\mu_{2|1}, \Sigma_{2|1}) \\ \mu_{2|1} &= \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\ \Sigma_{2|1} &= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{aligned} \tag{1}$$

For the bivariate Gaussian, the vectors μ_2, μ_1 and the matrix partitions Σ_{ij} become scalars. Therefore:

$$\begin{aligned} \mu_{2|1} &= \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1) \\ \Sigma_{2|1} &= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} = \sigma_2^2 - \frac{\sigma_{21}\sigma_{12}}{\sigma_1^2} = \sigma_2^2(1 - \rho^2) \end{aligned} \tag{2}$$

Using (2), we arrive at the distribution for the conditional $p(x_2|x_1)$

$$p(x_2|x_1) = \mathcal{N}(x_2|\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1), \sigma_2^2(1 - \rho^2)) \tag{3}$$

b)

Assuming $\sigma_1 = \sigma_2 = 1$, we can write Equation 3 as follows:

$$p(x_2|x_1) = \mathcal{N}(x_2|\mu_2 + \rho(x_1 - \mu_1), 1 - \rho^2) \tag{4}$$

Conclusion

In this question, we analyzed the conditional of one random variable of a bivariate Gaussian distribution. From the mean and variance of the conditional distribution we discovered two things. First, if $x_1 \neq \mu_1$ then the mean of the conditinal suffers a linear shift, which is based on the values ρ, σ_1 and σ_2. This can be though as follow: $\frac{\sigma_2}{\sigma_1}$ substitute the scale of X_1 by the scale of X_2 and ρ express the degree of influence of X_1 over X_2. Second, the variance decreases according with the term $1 - \rho^2$. Therefore, bigger correlations representing strong linear relationships between X_1 and X_2 make the uncertainty on X_2 decrease significantly.

For $\sigma_1 = \sigma_2 = 1$, there is no more need for a change in scale in the mean of the conditional.