17.

Intro

In this question, we have two r.v's $X, Y \sim \mathcal{U}(0,1)$. We want to know the expected value of the leftmost point (i.e. the expected value of the minimum). Since $E[X] = E[Y] = \frac{1}{2}$, it is reasonable to assume that if we repeat this experiment several times, then $max(X,Y) > \frac{1}{2}$ and $min(X,Y) < \frac{1}{2}$. Let's check if this is true.

Solution

Random variables:
$X \sim \mathcal{U}(0,1)$
$Y \sim \mathcal{U}(0,1)$
The mininum between the two r.v's can be computed by the *min* function:

$$min(x,y) = \frac{x+y-|x-y|}{2} \tag{1}$$

Now, we just have to compute the expected value of this function.

$$E[min(x,y)] = \int_0^1 \int_0^1 min(x,y)p(x,y)dxdy \tag{2}$$

Since X and Y are independent, we can easily decompose the joint distribution of (2):

$$\begin{aligned} E[min(x,y)] &= \int_0^1 \int_0^1 min(x,y)p(x)p(y)dxdy = \int_0^1 \int_0^1 min(x,y)dxdy = \\ &\int_0^1 \int_0^1 \frac{x+y-|x-y|}{2}dxdy = \frac{1}{2}\int_0^1 \int_0^1 x+y-|x-y|dxdy \end{aligned} \tag{3}$$

For the final step, we have to decompose the double integral in (3), in order to strip of the absolute value of the integrand.

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left(\int_0^y x+y-(-x+y)dx + \int_y^1 x+y-(x-y)dx \right) dy = \\ &\frac{1}{2}\int_0^1 \left(x^2\Big|_0^y + 2xy\Big|_y^1 \right) dy = \frac{1}{2}\int_0^1 \left(y^2+2y-2y^2\right) dy = \\ &\frac{1}{2}\left[-\frac{y^3}{3}\Big|_0^1 + y^2\Big|_0^1 \right] = \frac{1}{3} \end{aligned} \tag{4}$$

Therefore, the expectation of the leftmost variable is $\frac{1}{3}$.

Conclusion As we expected, the expecetd value of the leftmost point is smaller then the expected value of X and Y. But why is it $\frac{1}{3}$? There is a alternative line of reason to the previous calculation, which helps to explain it. Be aware that I am 100% certain of its rigor.

By the symmetry of the problem, we can conclude that $|E[min(X,Y)]-0| = |1-E[max(X,Y)]|$. So, we know there are two line segments of the inverval [0, 1] with the same lenght. All that there is left is the distance between $E[min(X,Y)]$ and $E[max(X,Y)]$. For determine this, we should note that $min(X,Y)$ is a uniform random variable in the inverval $[0, max(X,Y)]$. Now, by the Law of Large Numbers if we perform this experiment several times, then the average of the results should be close to the expected value. Let Z_i be the leftmost value of the i-th experiment and n a large number of trials. Then:

$$\begin{aligned} &E[Z_1 + Z_2 + ... + Z_n] = E[Z_1] + E[Z_2] + ... + E[Z_n] = \\ &\frac{n}{n}\left[\frac{max(X_1,Y_1)}{2} + \frac{max(X_2,Y_2)}{2} + ...\frac{max(X_n,Y_n)}{2}\right] = n\frac{E[max(X,Y)]}{2} \end{aligned} \tag{5}$$

Since, $E[Z_1+Z_2+...+Z_n] = nE[min(X,Y)]$ by definition, Equation (5) tell us that $E[min(X,Y)] = \frac{E[max(X,Y)]}{2}$. So, $|E[min(X,Y)]-0| = |E[max(X,Y)]-E[min(X,Y)]|$. Therefore, the three segments have the same size and $E[min(X,Y)] = \frac{1}{3}$.