7.

Intro

This is a classical problem of probability theory. I believe it appears as a question in every book that deals with the basics of joint distributions. As hinted by the author, in order to show that pairwise independence does not imply mutual independence, we only need to show one counter example. I will use the classical Bernstein's example.

Solution

Let X, Y and Z be three binary random variables ($X, Y, Z \in \{0, 1\}$). Let's define its joint probability distribution, using the following equation:

$$P(X, Y, Z) = \frac{1}{4}(X \oplus Y \oplus Z) \tag{1}$$

where $\oplus$ is the 'exclusive or' (XOR) operation. The XOR operation between several values outputs one if the number of ones is odd and 0 otherwise. With the joint probability defined, we can calculate all the marginals probabilities starting from the table of the joint distribution. This table is showed below.

X	Y	Z	P(X, Y, Z)
0	0	0	0
0	0	1	1/4
0	1	0	1/4
0	1	1	0
1	0	0	1/4
1	0	1	0
1	1	0	0
1	1	1	1/4

In order to calculate the marginals, we just sum the table above on the variables that we are not interested. For instance, $P(X = 1, Y = 1) = P(X = 1, Y = 1, Z = 0) + P(X = 1, Y = 1, Z = 1)$. The tables below show all the values of all the marginals.

X	Y	Z	P(X, Z)	P(Y,Z)
0	0	0	1/4	1/4
0	0	1	1/4	1/4
1	1	0	1/4	1/4
1	1	1	1/4	1/4

X	Y	P(X,Y)
0	0	1/4
0	1	1/4
1	0	1/4
1	1	1/4

X	Y	Z	P(X)	P(Y)	P(Z)
0	0	0	1/2	1/2	1/2
1	1	1	1/2	1/2	1/2

As we can see from the tables, the r.v's are pairwise independent ($P(A, B) = P(A)P(B) = \frac{1}{4}$), but not mutually independent ($P(A, B, C) = \frac{1}{4}(X \oplus Y \oplus Z) \neq P(A)P(B)P(C) = \frac{1}{8}$).

Conclusion

Some readers may think that the solution to this problem was rather artificial. I don't blame them for their opinion. After all, I did not gave any context for this distribution to happen. Instead I just wrote it down. The reason for this choice was that the joint probability of the Bernstein's example can model more than one kind of situation. So, I think the curious should look it up on Google, and discover those models by themselves.

For those curious to know how to generalize the construction of r.v's with this kind of property, see the best answer in the math forum discussion below:

http://mathoverflow.net/questions/7998/most-general-way-to-generate-pairwise-independent-random-variables.