4.6

Intro

In this exercise, we will show how to express the pdf of the bivariate Gaussian distribution in a way that highlights the importance of the correlation coefficient. Remember that for jointly Gaussian distribution, the correlation coefficient is enough to state if the two r.v's are independent or not. We must see this property in the final equation.

Solution

The general expression for a multivariate Gaussian is given by:

$$\frac{1}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}} exp(-\frac{1}{2}(x-\mu)^t\Sigma^{-1}(x-\mu)) \tag{1}$$

To derive the desired expression, we have to expand the quadractic form inside the *exp* function and calculate the determinant of the covariance matrix Σ. Let's start with the determinant:

$$|\Sigma| = (\sigma_1\sigma_2)^2 - (\rho\sigma_1\sigma_2)^2 = (\sigma_1\sigma_2)^2(1-\rho^2) \tag{2}$$

Now, to calculate the quadractic expression, we need to calculate the inverse of the covariance matrix. Fortunatly, there is a very well know expression for the inverse of a 2x2 matrixes.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{3}$$

Using Equation 3, we arrive at:

$$\Sigma^{-1} = \frac{1}{(\sigma_1\sigma_2)^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \tag{4}$$

Substituting (2) and (4) in (1), and expanding the terms we arrive at:

$$\begin{aligned} &\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} exp\left[-\frac{1}{2det(\Sigma)}\left(\sigma_2^2(x_1-\mu_1)^2 - 2\rho\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2(x_2-\mu_2)^2\right)\right] = \\ &\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(x_1-\mu_1)}{\sigma_1}\frac{(x_2-\mu_2)}{\sigma_2}\right)\right] \end{aligned} \tag{5}$$

Conclusion

In this exercise, we have shown that the pdf can expressed in a form that highlights the importance of the correlation coefficient for the relationship between the r.v's. Note that if $\rho = 0$ it is straighfoward to see that the X_1 and X_2 are independent.