4.5

Intro

In this exercise, we will derive the expression for the normalization consant of the multivariate Gaussian distribution. The normalization constant $(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}$ that we want to find is is a generalization of the expression we derived for the 1-D Gaussian in chapter 2. To derive the expression, we will use the hints given by the author.

Solution

We will start by performing the eigendecomposition of $\Sigma = U\Lambda U^t$.

$$\int exp(-\frac{1}{2}(x-\mu)^t\Sigma^{-1}(x-\mu))dx = \int exp(-\frac{1}{2}(x-\mu)^tU\Lambda^{-1}U^t(x-\mu))dx \tag{1}$$

where, we used the fact that U is an orthogonal matrix ($U^{-1} = U^t$) to derive Equation 1. Now, we will perform the following change of coordinates: $y = U^t(x-\mu)$.

$$\begin{aligned} &\int exp(-\frac{1}{2}(x-\mu)^tU\Lambda^{-1}U^t(x-\mu))dx = \int exp(-\frac{1}{2}y^t\Lambda^{-1}y)dx = \\ &\int exp(-\frac{1}{2}\sum_i \frac{y_i^2}{\lambda_i})dx = \int exp(-\frac{1}{2}\sum_i \frac{y_i^2}{\lambda_i})\frac{\partial(x_1,x_2,...,x_n)}{\partial(y_1,y_2,...,y_n)}dy \end{aligned} \tag{2}$$

To change the vector of integration from dx to dy, we need to compute the Jacobian.

$$\begin{aligned} &y = U^t(x-\mu) \implies x = Uy+\mu \\ &J_{ij} = \frac{\partial x_i}{\partial yj} = u_{ij} \end{aligned} \tag{3}$$

Thus, $J = U$ and $\frac{\partial(x_1,x_2,...,x_n)}{\partial(y_1,y_2,...,y_n)} = det(U)$. Since U is orthogonal, we have $det(U) = \pm 1$. Furthermore, as we know that the original integral had positive value and our new function $exp(f(y))$ is always postive, we can be sure that $det(U) = 1$. Therefore, our simplified expression becomes:

$$\int exp(-\frac{1}{2}\sum_i \frac{y_i^2}{\lambda_i})dy = \prod_i \int exp(-\frac{1}{2}\frac{y_i^2}{\lambda_i})dy_i \tag{4}$$

Equation 4 says we have a product of several one dimensional Gaussians. Remember from Exercise 2.11 that $\int exp(-\frac{x^2}{2\sigma^2})dx = \sqrt{(2\pi\sigma^2)}$. Using this result in Equation 4, we get:

$$\begin{aligned} \prod_i \int exp(-\frac{1}{2}\frac{y_i^2}{\lambda_i})dy_i = \prod_{i=1}^{D}\sqrt{(2\pi\lambda_i)} = \sqrt{(2\pi)}^D \prod_{i=1}^{D}\sqrt{\lambda_i} = \\ (2\pi)^{\frac{D}{2}}|\Sigma|^{\frac{1}{2}} \end{aligned} \tag{5}$$

where we used the fact that $|\Sigma| = \prod \lambda_i$.

Conclusion

In this question, we derived the expression for the normalization constant of the multivariate Gaussian distribution. The proof required performing the eigendecomposition of the covariance matrix and changing the coordinates to be centered in the mean μ and aligned with the eigenvectors of Σ. We also needed the result of the normalization constant for the 1-D case to perform one of the last steps in the calculation. The final result $(2\pi)^{\frac{D}{2}}|\Sigma|^{\frac{1}{2}}$ is a generalization of the expression for the 1-D case, involving the constant 2π and the covariance matrix.