5.4

Intro

In this exercise we will solve a problem very similar to exercise 5.3. Both problems focus on optimal bayesian decision in a classification problem with the possibility of rejection. The core difference lies on the practical nature of the present problem. Beign more specific, in this problem we will solve a binary classification problem with the possibility of rejection. We have a loss matrix at our disposal and we shall see how to use it to make optimal choices. Item a and b focus on particular scenarios, where the probability $P(y = 1|x)$ is given. Item c encodes our intuition acquired in the previous itens and come up with a rule of optimal classification.

Solution

a)

Knowing that $P(y = 1|x) = 0.2$, we can write the expected loss as:

$$\rho(a|x) = 0.8L(0, a) + 0.2L(1, a) \tag{1}$$

If we choose action $a = 0$

$$\rho(a|x) = 0.2 * 10 = 2 \tag{2}$$

If we choose action $a = 1$

$$\rho(a|x) = 0.8 * 10 = 8 \tag{3}$$

If we choose action $a = rejection$

$$\rho(a|x) = 0.8 * 3 + 0.2 * 3 = 3 \tag{4}$$

Therefore, the best decision is to choose class 0. This makes sense, because $P(y = 0|x) = 1 - P(y = 1|x) = 0.8$ is very high.

b)

Know, suppose $P(y = 1|x) = 0.4$. Then we can write the expected loss as:

$$\rho(a|x) = 0.6L(0, a) + 0.4L(1, a) \tag{5}$$

If we choose action $a = 0$

$$\rho(a|x) = 0.4 * 10 = 4 \tag{6}$$

If we choose action $a = 1$

$$\rho(a|x) = 0.6 * 10 = 6 \tag{7}$$

If we choose action $a = rejection$

$$\rho(a|x) = 0.8 * 3 + 0.2 * 3 = 3 \tag{8}$$

Therefore, the best decision is to choose rejection. This makes sense, because $P(y = 0|x) = 1 - P(y = 1|x) = 0.6$ is not so high.

c)

From the previous itens, we can infer the following pattern: if we strongly believe that class 1 or 0 are right, we should choose the respective class. Else, we choose the rejection. The question is, what probabilities threshold θ_0 and θ_1 corresponds to a "strong belief" for our particular loss matrix? Let's find out:

$$\begin{aligned}&\rho(a|x) = P(y = 0|x)L(0, a) + P(y = 1|x)L(1, a) =\\&(1 - p_1)L(0, a) + p_1 L(1, a)\end{aligned} \tag{9}$$

Know let's go through the actions one last time:
If we choose action $a = 0$

$$\rho(a|x) = 10p_1 \tag{10}$$

If we choose action $a = 1$

$$\rho(a|x) = 10(1 - p_1) \tag{11}$$

If we choose action $a = rejection$

$$\rho(a|x) = 3p_1 + 3(1 - p_1) = 3 \tag{12}$$

For $a = 0$ to be the best choice, we must have:

$$\begin{aligned}&10p_1 \leq 10(1 - p_1) \iff p_1 \leq 0.5\\&10p_1 \leq 3 \iff p_1 \leq 0.3\end{aligned} \tag{13}$$

For $a = 1$ to be the best choice, we must have:

$$\begin{aligned}&10(1 - p_1) \leq 10p_1 \iff p_1 \geq 0.5\\&10(1 - p_1) \leq 3 \iff p_1 \geq 0.7\end{aligned} \tag{14}$$

Therefore, the thresholds are $\theta_0 = 0.3$ and $\theta_1 = 0.7$.

Conclusion

In this exercise, we performed a more practical analysis of optimal bayesian decision. We explored in which circunstances, we should choose each of our three possibilities $\{rejection, 0, 1\}$. The final result in item c shows us that we only choose a class, if it has more than 70% of being true. Else, the safer choice is the rejection.