FAETERJ-Rio
Cálculo I
Professor DSc. Wagner Zanco

Solução dos Exercícios 2.11 – 2.12

Exercício 2.11: Encontre os valores máximo e mínimo absolutos de f no intervalo dado.

a) $f(x) = 3x^2 - 12x + 5$, [0,3]

$$f'(x) = 6x - 12$$

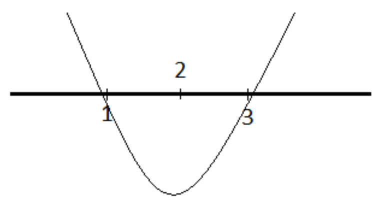
$$6x - 12 = 0$$

$$6x = 12$$

$$x = \frac{12}{6} = 2$$

$$f'(1) = 6.1 - 12 = -6 \text{ (decrescente)}$$

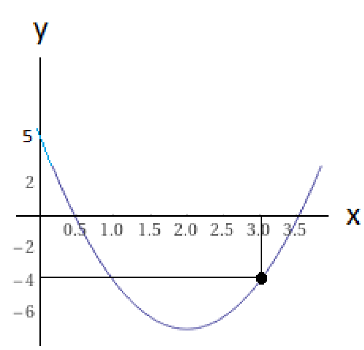
$$f'(3) = 6.3 - 12 = 6 \text{(crescente)}$$



$$f(0) = 3.0^2 - 12.0 + 5 = 5$$

$$f(2) = 3.2^2 - 12.2 + 5 = -7$$

$$f(3) = 3.3^2 - 12.3 + 5 = 27 - 36 + 5 = -4$$



Máximo (0, 5); Mínimo (2, -7)

b) $f(x) = x^3 - 3x + 1$, [0,3]

$$f'(x) = 3x^2 - 3$$

$$3x^2 - 3 = 0$$

$$x^2 = \frac{3}{3} = 1$$

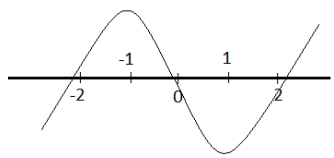
$$x = \pm\sqrt{1}$$

$$x' = 1 \;\; e \;\; x'' = -1$$

$$f'(-2) = 3(-2)^2 - 3 = 9 \text{ (crescente)}$$

$$f'(0) = 3(0)^2 - 3 = -3 \text{ (descrescente)}$$

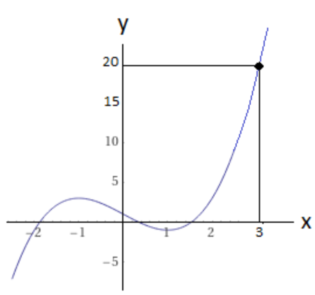
$$f'(2) = 3(2)^2 - 3 = 9 \text{ (crescente)}$$



$$f(0) = 0^3 - 3.0 + 1 = 1$$

$$f(1) = 1^3 - 3.1 + 1 = -1$$

$$f(3) = 3^3 - 3.3 + 1 = 27 - 9 + 1 = 19$$



Máximo (3, 19); Mínimo (1, -1)

c) $f(x) = 2x^3 - 3x^2 - 12x + 5$, [-2,3]

$$f'(x) = 6x^2 - 6x - 12$$

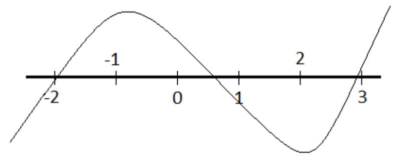
$$6x^2 - 6x - 12 = 0$$

$$x' = -1 \;\; e \;\; x'' = 2$$

$$f'(-2) = 6(-2)^2 - 6(-2) - 12 = 24 + 12 - 12 = 24 \text{ (crescente)}$$

$$f'(0) = 6(0)^2 - 6(0) - 12 = 0 + 0 - 12 = -12 \text{ (decrescente)}$$

$$f'(3) = 6(3)^2 - 6(3) - 12 = 54 - 18 - 12 = 24 \text{ (crescente)}$$



$$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 5$$

$$= -16 - 12 + 24 + 5 = 1$$

$$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 5$$

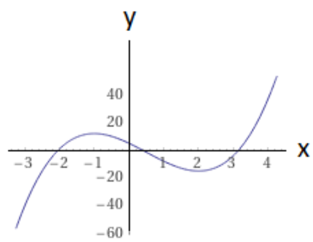
$$= 54 - 27 - 36 + 5 = 59 - 63 = -4$$

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5$$

$$= -2 - 3 + 12 + 5 = 12$$

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 5$$

$$= 16 - 12 - 24 + 5 = -15$$



Máximo (-1, 12); Mínimo (2, -15)

d) $f(x) = 2x^3 - 6x^2 + 5$, [-3,5]

$$f'(x) = 6x^2 - 12x$$

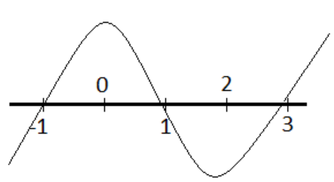
$$6x^2 - 12x = 0$$

$$x' = 0 \;\; e \; x'' = 2$$

$$f'(-1) = 6(-1)^2 - 12(-1) = 6 + 12 = 18 \text{ (crescente)}$$

$$f'(1) = 6(1)^2 - 12(1) = 6 - 12 = -6 \text{ (decrescente)}$$

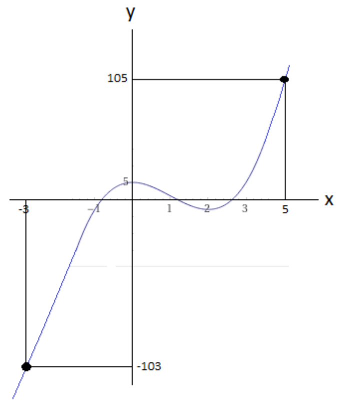
$$f'(3) = 6(3)^2 - 12(3) = 54 - 36 = 18 \text{ (crescente)}$$



$$f(-3) = 2(-3)^3 - 6(-3)^2 + 5 = -54 - 54 + 5 = -103$$

$$f(0) = 2(0)^3 - 6(0)^2 + 5 = 0 + 0 + 5 = 5$$

$$f(5) = 2(5)^3 - 6(5)^2 + 5 = 250 - 150 + 5 = 105$$



Máximo $(5, 105)$; Mínimo $(-3, -103)$

e) $f(x) = x^3 - 6x^2 + 5, [-2, 3]$

$$f'(x) = 3x^2 - 12x$$

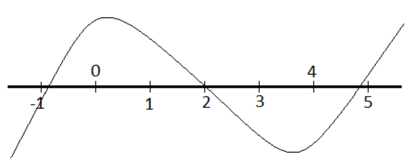
$$3x^2 - 12x = 0$$

$$x' = 0 \ e \ x'' = 4$$

$$f'(-1) = 3(-1)^2 - 12(-1) = 15 \text{ (crescente)}$$

$$f'(1) = 3(1)^2 - 12(1) = -9 \text{ (decrescente)}$$

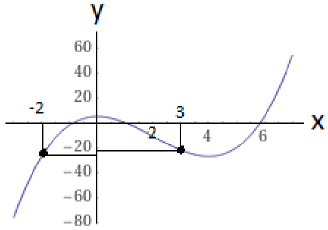
$$f'(5) = 3(5)^2 - 12(5) = 75 - 60 = 15 \text{ (crescente)}$$



$$f(-2) = (-2)^3 - 6(-2)^2 + 5 = -8 - 24 + 5 = -27$$

$$f(0) = (0)^3 - 6(0)^2 + 5 = 5$$

$$f(3) = (3)^3 - 6(3)^2 + 5 = 27 - 54 + 5 = -22$$



Máximo $(0,5)$; Mínimo $(-2,-27)$

f) $f(t) = (t^2 - 4)^3, [-2,3]$

$$f'(t) = 3(t^2 - 4)^2.(2t) = 6t(t^2 - 4)^2$$

$$= 6t((t^2)^2 - 2t^2 4 + 4^2) = 6t(t^4 - 8t^2 + 16)$$

$$= 6t^5 - 48t^3 + 96t$$

$$6t^5 - 48t^3 + 96t = 0$$

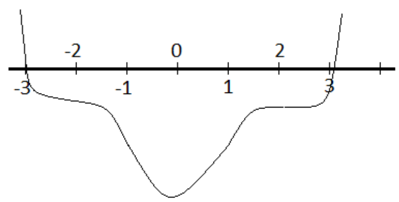
$$t' = -2; \ \ t'' = 0; \ \ t''' = 2$$

$f'(-3) = 6(-3)((-3)^2 - 4)^2 = -18(9-4)^2 = -18(25) = -450$ (decrescente)

$f'(-1) = 6(-1)((-1)^2 - 4)^2 = -6(1-4)^2 = -6(9) = -54$ (decrescente)

$f'(1) = 6(1)((1)^2 - 4)^2 = 6(1-4)^2 = 6(9) = 54$ (crescente)

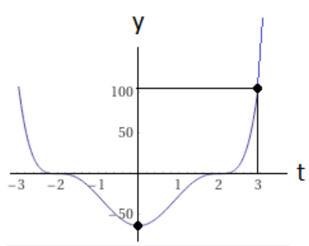
$f'(3) = 6(3)((3)^2 - 4)^2 = 18(9-4)^2 = 18(25) = 450$ (crescente)



$$f(-2) = ((-2)^2 - 4)^3 = (4-4)^3 = 0$$

$$f(0) = ((0)^2 - 4)^3 = (-4)^3 = -64$$

$$f(3) = ((3)^2 - 4)^3 = (9-4)^3 = 125$$



Máximo $(3,125)$; Mínimo $(0,-64)$

g) $f(x) = x + \frac{1}{x}$, [0,2, 4]

$$f'(x) = 1 - \frac{1}{x^2}$$

$$1 - \frac{1}{x^2} = 0$$

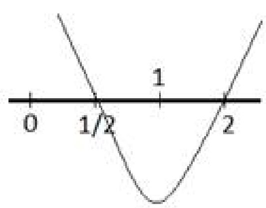
$$1 = \frac{1}{x^2}$$

$$x^2 = \frac{1}{1} = 1$$

$$x = \pm 1$$

$$f'\left(\frac{1}{2}\right) = 1 - \frac{1}{\left(\frac{1}{2}\right)^2} = 1 - \frac{1}{\frac{1}{4}} = 1 - 4 = -3 \text{(decrescente)}$$

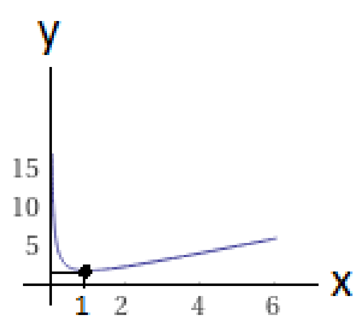
$$f'(2) = 1 - \frac{1}{(2)^2} = 1 - \frac{1}{4} = \frac{3}{4} \text{ (crescente)}$$



$$f\left(\frac{2}{10}\right) = \frac{2}{10} + \frac{1}{\frac{2}{10}} = \frac{2}{10} + \frac{10}{2} = \frac{4 + 100}{20} = \frac{104}{20} = \frac{26}{5}$$

$$f(1) = 1 + \frac{1}{1} = 2$$

$$f(4) = 1 + \frac{1}{4} = \frac{5}{4}$$



Máximo $(0{,}2, \frac{26}{5})$; Mínimo $(1, 2)$

h) $f(x) = \frac{x}{x^2 - x + 1}$, [0,3]

$$f'(x) = \frac{(x^2 - x + 1)1 - (2x - 1)x}{(x^2 - x + 1)^2}$$

$$= \frac{x^2 - x + 1 - x(2x - 1)}{(x^2 - x + 1)^2} = \frac{x^2 - x + 1 - 2x^2 + x}{(x^2 - x + 1)^2}$$

$$f'(x) = \frac{-x^2 + 1}{(x^2 - x + 1)^2}$$

$$f'(x) = 0$$

$$\frac{-x^2 + 1}{(x^2 - x + 1)^2} = 0$$

$$-x^2 + 1 = 0$$

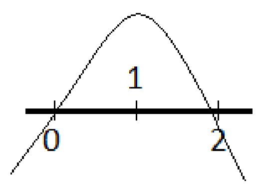
$$-x^2 = -1 \ (-1)$$

$$x^2 = 1$$

$$x = \pm 1$$

$$f'(0) = \frac{-0^2+1}{(0^2 - 0 + 1)^2} = \frac{1}{1} = 1 \text{ (crescente)}$$

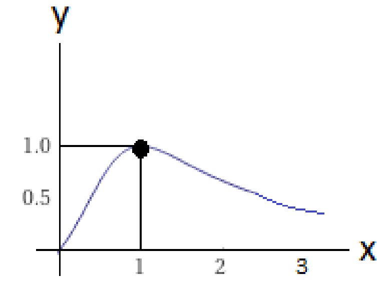
$$f'(2) = \frac{-(2^2)+1}{(2^2 - 2 + 1)^2} = -\frac{3}{3^2} = -\frac{1}{3} \text{ (decrescente)}$$



$$f(0) = \frac{0}{0^2 - 0 + 1} = 0$$

$$f(1) = \frac{1}{1^2 - 1 + 1} = \frac{1}{1} = 1$$

$$f(3) = \frac{3}{3^2 - 3 + 1} = \frac{3}{7}$$



Máximo $(1, 1)$; Mínimo $(0, 0)$.

i) $f(t) = t - \sqrt[3]{t}, [-1, 4]$

$$f(t) = t - \left(t^{\frac{1}{3}}\right)$$

$$f'(t) = 1 - \frac{1}{3}t^{-\frac{2}{3}} = 1 - \frac{1}{3t^{\frac{2}{3}}} = 1 - \frac{1}{3\sqrt[3]{t^2}}$$

$$1 - \frac{1}{3\sqrt[3]{t^2}} = 0$$

$$1 = \frac{1}{3\sqrt[3]{t^2}}$$

$$\sqrt[3]{t^2} = \frac{1}{3}$$

$$t^2 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$t = \pm\sqrt{\frac{1}{27}} = \pm\frac{1}{\sqrt{27}}$$

$$f'\left(\frac{1}{10}\right) = 1 - \frac{1}{3\sqrt[3]{\left(\frac{1}{10}\right)^2}} = 1 - \frac{1}{3\sqrt[3]{\frac{1}{100}}} =$$

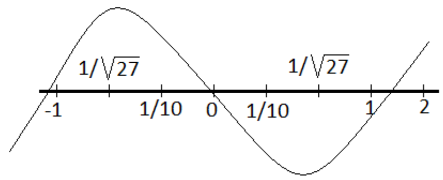
$$1 - \frac{1}{3\frac{1}{\sqrt[3]{100}}} = 1 - \frac{1}{3.0{,}21} = 1 - \frac{1}{0{,}64} = \frac{-0{,}35}{0{,}64} = -0{,}55 \text{ (decrescente)}$$

$$f'(1) = 1 - \frac{1}{3\sqrt[3]{1^2}} = 1 - \frac{1}{3} = \frac{2}{3} \text{ (crescente)}$$

$$f'\left(-\frac{1}{10}\right) = 1 - \frac{1}{3\sqrt[3]{\left(-\frac{1}{10}\right)^2}} = 1 - \frac{1}{3\sqrt[3]{\frac{1}{100}}} =$$

$$1-\frac{1}{3\frac{1}{\sqrt[3]{100}}}=1-\frac{1}{3.0{,}21}=1-\frac{1}{0{,}64}=\frac{-0{,}35}{0{,}64}=-0{,}55 \text{ (decrescente)}$$

$$f'(-1)=1-\frac{1}{3\sqrt[3]{-1^2}}=1-\frac{1}{3}=\frac{2}{3} \text{ (crescente)}$$



$$f(-1)=-1-\sqrt[3]{-1}=-1+1=0$$

$$f\left(-\frac{1}{\sqrt{27}}\right)=-\frac{1}{\sqrt{27}}-\left(\sqrt[3]{-\frac{1}{\sqrt{27}}}\right)=-\frac{1}{\sqrt{27}}-\left(\frac{\sqrt[3]{-1}}{\sqrt[3]{\sqrt{27}}}\right)=-\frac{1}{\sqrt{27}}-\left(\frac{-1}{3}\right)$$

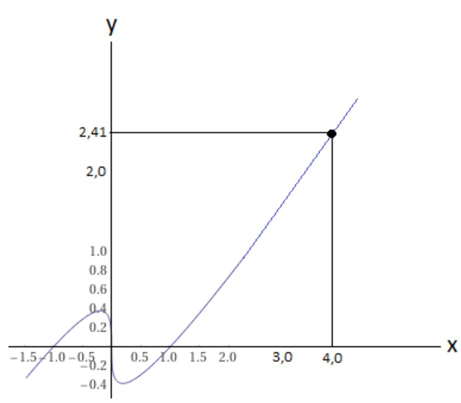
$$=-\frac{1}{\sqrt{27}}+\frac{1}{3}=\frac{-3+\sqrt{27}}{3\sqrt{27}}=\frac{2{,}2}{15{,}58}=0{,}39$$

$$f\left(\frac{1}{\sqrt{27}}\right)=\frac{1}{\sqrt{27}}-\left(\sqrt[3]{\frac{1}{\sqrt{27}}}\right)=\frac{1}{\sqrt{27}}-\left(\frac{\sqrt[3]{1}}{\sqrt[3]{\sqrt{27}}}\right)=-\frac{1}{\sqrt{27}}-\left(\frac{1}{3}\right)$$

$$=-\frac{1}{\sqrt{27}}-\frac{1}{3}=\frac{3-\sqrt{27}}{3\sqrt{27}}=\frac{-2{,}2}{15{,}58}=-0{,}39$$

$$f(1)=1-\sqrt[3]{1}=1-1=0$$

$$f(4)=4-\sqrt[3]{4}=4-1{,}58=2{,}41$$



Máximo $(4, 2{,}41)$; Mínimo $(\frac{1}{\sqrt{27}}, -0{,}39)$.

j) $f(t)=\frac{\sqrt{t}}{1+t^2}$, $[0{,}2]$

$$f'(t) = \frac{(1+t^2)\frac{1}{2\sqrt{t}} - (2t)\sqrt{t}}{(1+t^2)^2} = \frac{\frac{1+t^2}{2\sqrt{t}} - 2t\sqrt{t}}{(1+t^2)^2}$$

$$= \frac{\frac{1+t^2-4t^2}{2\sqrt{t}}}{(1+t^2)^2} = \frac{1-3t^2}{2\sqrt{t}(1+t^2)^2}$$

$$\frac{1-3t^2}{2\sqrt{t}(1+t^2)^2} = 0$$

$$1-3t^2 = 0$$

$$1 = 3t^2$$

$$t^2 = \frac{1}{3}$$

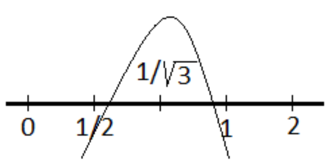
$$t = \pm\frac{1}{\sqrt{3}}$$

$$f'\left(\frac{1}{2}\right) = \frac{1-3\left(\frac{1}{2}\right)^2}{2\sqrt{\frac{1}{2}}\left(1+\left(\frac{1}{2}\right)^2\right)^2} = \frac{1-3\frac{1}{4}}{2\frac{1}{\sqrt{2}}\left(1+\frac{1}{4}\right)^2} = \frac{1-\frac{3}{4}}{\frac{2}{\sqrt{2}}\left(\frac{5}{4}\right)^2} = \frac{\frac{1}{4}}{\frac{2}{\sqrt{2}}\frac{25}{16}} =$$

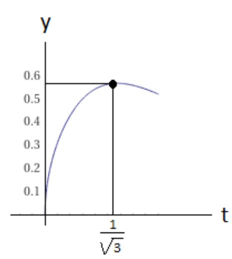
$$= \frac{\frac{1}{4}}{\frac{25}{8\sqrt{2}}} = \frac{1}{4}\frac{8\sqrt{2}}{25} = \frac{2\sqrt{2}}{25} = 0{,}11 \text{ (crescente)}$$

$$f'(2) = \frac{1-3.(2)^2}{2\sqrt{2}(1+(2)^2)^2} = \frac{1-3.4}{2\sqrt{2}(1+4)^2} = \frac{1-12}{2.\sqrt{2}.25} = -\frac{11}{50\sqrt{2}}$$

$$= -\frac{11}{50\sqrt{2}} = -0{,}15 \text{ (decrescente)}$$

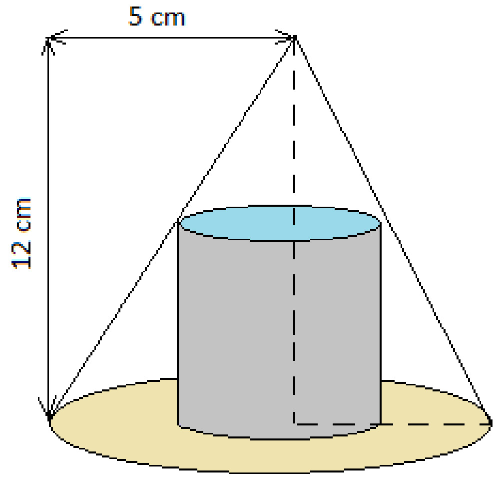


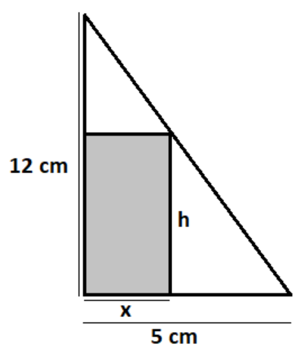
$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{\sqrt{\frac{1}{\sqrt{3}}}}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\frac{1}{\sqrt[4]{3}}}{1+\frac{1}{3}} = \frac{\frac{1}{\sqrt[4]{3}}}{\frac{4}{3}} = \frac{1}{\sqrt[4]{3}}\frac{3}{4} = \frac{3}{4\sqrt[4]{3}} = 0{,}56$$



$$\text{Máximo } \left(\frac{1}{\sqrt{3}}, 0{,}56\right); \text{ Mínimo } (0,0).$$

Exercício 2.12: Encontrar as dimensões de um cilindro circular reto de maior volume que pode ser inscrito em um cone circular reto com raio de 5 cm e altura de 12 cm.





Observando as proporcionalidades dos triângulos retângulos, podemos afirmar que

$$\frac{12}{h} = \frac{5}{5-x} = 12(5-x) = 5h$$

$$h = \frac{60-12x}{5} = 12\left(\frac{5-x}{5}\right)$$

$A = \pi.r^2$- Área do círculo

O volume do cilindro será dado por

$$V = A.h = \pi.x^2 12\left(\frac{5-x}{5}\right) = \frac{12\pi}{5}(5x^2 - x^3)$$

$$V = \frac{12\pi}{5}(-x^3 + 5x^2)$$
$$V' = \frac{12\pi}{5}(-3x^2 + 10x)$$
$$\frac{12\pi}{5}(-3x^2 + 10x) = 0$$
$$(-3x^2 + 10x) = \frac{0}{\frac{12\pi}{5}}$$

$$(-3x^2 + 10x) = 0$$

$$x' = 0 \;\; e \;\; x'' = \frac{10}{3}$$

Neste caso, o raio que fornecerá o maior volume é $r = \frac{10}{3}$.

$$h = 12\left(\frac{5 - \frac{10}{3}}{5}\right) = 12\left(\frac{\frac{5}{3}}{5}\right) = 12.\frac{5}{3}.\frac{1}{5} = 4\ cm$$

$$A = \pi.\left(\frac{10}{3}\right)^2 = \pi.\frac{100}{9} = 34{,}09\ cm^2$$

Gabarito:

2.11a) Máximo $(0, 5)$ e mínimo $(2, -7)$.
2.11b) Máximo $(3, 19)$ e mínimo $(1, -1)$. .
2.11c) Máximo $(-1, 12)$ e mínimo $(2, -15)$.
2.11d) Máximo $(5, 105)$ e mínimo $(-3, -103)$.
2.11e) Máximo $(0, 5)$ e mínimo $(-2, -27)$.
2.11f) Máximo $(3{,}125)$ e mínimo $(0, -64)$.
2.11g) Máximo $(0{,}2, \frac{26}{5})$ e mínimo $(1, 2)$.
2.11h) Máximo $(1, 1)$ e mínimo $(0, 0)$.
2.11i) Máximo $(4, 2{,}41)$ e mínimo $\left(\frac{1}{\sqrt{27}}, -0{,}39\right)$
2.11j) Máximo $\left(\frac{1}{\sqrt{3}}, 5{,}6\right)$ e mínimo$(0, 0)$.

2.12)$raio = \frac{10}{3}$ cm e altura = 4 cm