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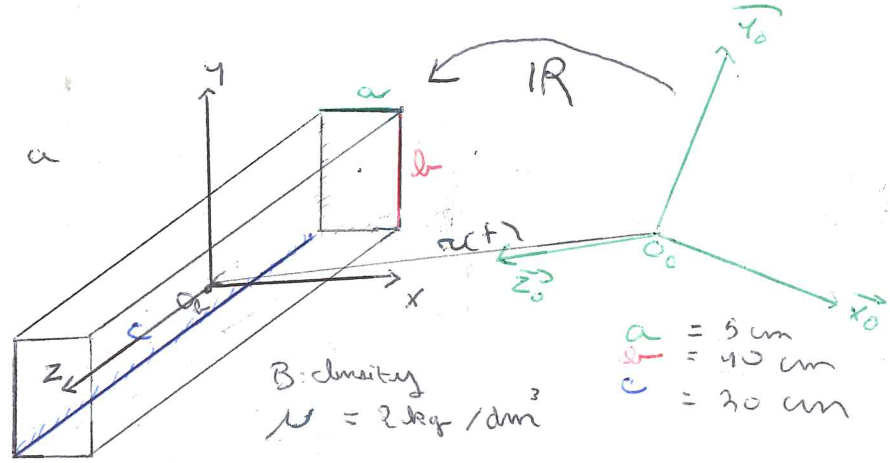
Term Project for TEK4040 Mathematical Modelling of Dynamic systems

November 2022

1.

Let us consider a brick B with affine space $\mathcal{B}$



with center of mass b such that

$$\mathcal{B} : \mathcal{F}^{b}_{\mathcal{B}} = \{ O_b , \vec{x}, \vec{y}, \vec{z} \} = \{b\}$$

Introducing inertial frame with affine space $\mathcal{R}$

$$\mathcal{R} : \mathcal{F}^{0}_{\mathcal{R}} = \{ O_0 , \vec{x}_0, \vec{y}_0, \vec{z}_0 \} = \{0\}$$

introducing position vector $r(t) = \overrightarrow{O_0 O_b}(t)$
and Rotation operator $\mathbb{R}$ between $\{0\} \to \{b\}$
such that $\underbrace{\mathbb{R} = \mathbb{R}_{0b} = \mathbb{R}^{0}_{0b} = \mathbb{R}^{b}_{0b}}_{\text{rotational operator}} = \underbrace{R^{0}_{b}}_{\text{DCM}}$

introducing position vector $\underline{p}^{b}_{c} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

such that $\underline{p}^{b}_{c} = \begin{bmatrix} \overrightarrow{O_b C} \cdot \vec{x} \\ \overrightarrow{O_b C} \cdot \vec{y} \\ O_b C \cdot \vec{z} \end{bmatrix}$

$$\underline{J}^{b}_{b} = \iiint_{C \in B} \begin{bmatrix} p_2^2 + p_3^2 & -p_1 p_2 & -p_1 p_3 \\ -p_2 p_1 & p_1^2 + p_3^2 & -p_2 p_3 \\ -p_3 p_1 & -p_3 p_2 & p_1^2 + p_2^2 \end{bmatrix} \mu(c)\, dx\, dy\, dz = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{bmatrix}$$

it can be shown that product of inertia are null because of symmetry :

② $\forall i \neq j$

$$J^b_{ij} = -\iiint p_i p_j \, \mu \, dp_i \, dp_j \, dp_k$$

$$L_i = \int_{\in B} dp_i$$

$$= \int \left[\int \left[\int p_i p_j \, \mu \, dp_j \right] dp_i \right] dp_k$$

$$= \iint \mu p_i \left[\left[\frac{p_j^2}{2} \right]_{-L_i/2}^{L_i/2} \right] dp_i \, dp_k$$

$$= 0$$

because for all ~~$\in B$~~ l p_l is centered

$$= 0$$

$\forall i$, $j \neq i$, $k \neq i$, $j \neq k$

$$J^b_{ii} = \iiint \mu \left[p_j^2 + p_k^2 \right] \mu \, dp_i \, dp_j \, dp_k$$

$$= \mu L_i \iint \left[p_j^2 + p_k^2 \right] dp_j \, dp_k$$

$$= \mu L_i \int \left[\frac{p_j^3}{3} + p_k^2 p_j \right]_{-L_j/2}^{L_j/2} dp_k$$

$$= \mu L_i \int p_k^2 L_j + \frac{L_j^3}{12} \, dp_k$$

$$= \mu L_i L_j \left[\frac{p_k^3}{3} + \frac{L_j^2}{12} \right]_{-L_k/2}^{L_k/2}$$

$$= \mu L_i L_j L_k \left[\frac{L_j^2}{12} + \frac{L_k^2}{12} \right]$$

$$= m \left[\frac{L_j^2}{12} + \frac{L_k^2}{12} \right]$$

$m = 2\text{e-}06$ Kg

$J^b_{11} \simeq 8{,}3\text{e-}09$ ~~…~~ $Kg \cdot m^2$

$J^b_{22} \simeq 7{,}083\text{e-}09$ $Kg \cdot m^2$

$J^b_{33} \simeq 2{,}083\text{e-}09$ $Kg \cdot m^2$

3 2) Given constant energy J, the inertia ellipsoid is defined wrt to all $\underline{x}^b = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ such that

not needed for the answer

$$\frac{1}{2} \underline{x}^{bT} J^b \underline{x}^b = J$$

$$\Leftrightarrow \frac{1}{2} [p_1, p_2, p_3] \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = J$$

$$\Leftrightarrow \frac{J_{11} p_1^2 + J_{22} p_2^2 + J_{33} p_3^2}{2} = J$$

$$\Leftrightarrow \frac{p_1^2}{\frac{2J}{J_{11}}} + \frac{p_2^2}{\frac{2J}{J_{22}}} + \frac{p_3^2}{\frac{2J}{J_{33}}} = 1$$

$\hookrightarrow$ this is an ellipsoid with half axis

$$a_i = \sqrt{\frac{2J}{J_{ii}}}$$

Given free torque motion the kinetic rotation energy ellipsoid is derived from euler equations for all (i, j, k), $i \neq j \neq k$, $k \equiv j+1 \, [3]$ (using $J = \begin{bmatrix} J_{00} & & \\ & J_{11} & \\ & & J_{22} \end{bmatrix} = \begin{bmatrix} J_{11} & & \\ & J_{22} & \\ & & J_{33} \end{bmatrix}$)

$$m_i = 0 = J^b_{ii} \dot{\omega}_i + \omega_j \omega_k (J^b_{kk} - J^b_{jj})$$

$$\Leftrightarrow 0 = J^b_{ii} \dot{\omega}_i \omega_i + \omega_i \omega_j \omega_k (J^b_{kk} - J^b_{jj})$$

$$\Rightarrow \sum_{\substack{i=0 \\ (i \neq j \neq k \\ k \equiv j+1 [3])}}^{2} J^b_{ii} \dot{\omega}_i \omega_i + \sum_{\substack{i=0 \\ i \neq j \neq k \\ k \equiv j+1 [3]}}^{2} \omega_i \omega_j \omega_k \underbrace{(J^b_{kk} - J_{jj})}_{=0}$$

$$\Rightarrow \sum_{i=0}^{2} J^b_{ii} \dot{\omega}_i \omega_i = 0$$

$$\Rightarrow \int_{t_0}^{t} \sum_{i=1}^{2} J^b_{ii} \dot{\omega}_i \omega_i \, dt = K$$

$$\sum_{k=0}^{2} \left[\frac{J^b_{ii} \omega_i^2}{2} \right] = K$$

$$\sum_{k=0}^{2} \frac{\omega_i^2}{\frac{2K}{J^b_{ii}}} = 1 \Rightarrow$$ ellipsoid with half axis $a_i = \sqrt{\frac{2K}{J^b_{ii}}}$

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Given free torque motion, the angular momentum ellipsoid is derived from

$$\dot{\vec{h}}_b^{ii} = \vec{0}$$

$$\Rightarrow \|\vec{h}_b^{i}\| = h_0 \quad \text{(constant lenght (}$$

$$\underline{h}_b^{ib} = \underline{J}_b^{b} \underline{\omega}_b^{ib} = \begin{bmatrix} J_{xx}\,\omega_x \\ J_{yy}\,\omega_y \\ J_{zz}\,\omega_z \end{bmatrix}$$

$$(\underline{h}_b^{ib})^T (\underline{h}_b^{ib}) = h_0^2 = J_{xx}^2\omega_x^2 + J_{yy}^2\omega_y^2 + J_{zz}^2\omega_z^2$$

$$\Leftrightarrow \quad 1 = \frac{\omega_x^2}{\frac{h_0^2}{J_{xx}^2}} + \frac{\omega_y^2}{\frac{h_0^2}{J_{yy}^2}} + \frac{\omega_z^2}{\frac{h_0^2}{J_{zz}^2}}$$

$\hookrightarrow$ ellipsoid with half axis $\frac{h_0}{J_{ii}}$, $i = 1,2,3$

We will perform rotation along axis "new" $\vec{x}, \vec{y}, \vec{z}$, with the same kinetic energy as a pure rotation along $\vec{z}$ of periode $T = 1\,s$.

$$\rightarrow \underline{\omega}_{b\,ref_z}^{ib} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\omega}_z \end{bmatrix} \qquad \tilde{\omega}_z = \frac{2\pi}{T} = 2\pi$$

$$\hookrightarrow K = \frac{1}{2}\tilde{\omega}_z^2 J_{zz} \simeq 4{,}44e-08 \quad \underbrace{kg \cdot m^2 \cdot rad^2 \cdot s^{-2}}_{N \cdot m \cdot s^{-1} \cdot rad^2}$$

We can find similarly

$$\begin{cases} \tilde{\omega}_x = \sqrt{\frac{2K}{J_{11}}} \simeq 3{,}1446 \text{ rad/s} \\ \tilde{\omega}_y = \sqrt{\frac{2K}{J_{22}}} \simeq 3{,}4075 \text{ rad/s} \end{cases}$$

such that $\underline{\omega}_{b\,ref_x}^{ib} = \begin{bmatrix} \tilde{\omega}_x \\ 0 \\ 0 \end{bmatrix}$ and $\underline{\omega}_{b\,ref_y}^{ib} = \begin{bmatrix} 0 \\ \tilde{\omega}_y \\ 0 \end{bmatrix}$ gives same kinetic energy as $\underline{\omega}_{b\,ref_z}^{ib}$

5) We will choose

$$\underline{W}_{b\ near\ z}^{ib} = \begin{bmatrix} \omega_{x\ near\ z} \\ 0 \\ \omega_z \end{bmatrix} \qquad \omega_{x\ near\ z} = \frac{\tilde{\omega}_x}{10}$$

$$\underline{\omega}_{b\ near\ y}^{ib} = \begin{bmatrix} 0 \\ \omega_y \\ \omega_{z\ near\ y} \end{bmatrix} \qquad \omega_{z\ near\ y} = \omega_{z\ near\ x} = \frac{\tilde{\omega}_z}{10}$$

$$\underline{W}_{b\ near\ x}^{ib} = \begin{bmatrix} \omega_x \\ 0 \\ \omega_{z\ near\ x} \end{bmatrix}$$

such that the Kinetic Energy induced is equal to K_0

it can be shown that
$$\begin{cases} \omega_z = \tilde{\omega}_z \sqrt{1 - \dfrac{\omega_{x\ near\ z}}{\omega_x^{2}}} \quad 1/10^2 \\ \omega_y = \tilde{\omega}_y \sqrt{1 - \dfrac{1}{100}} \\ \omega_x = \tilde{\omega}_x \sqrt{1 - \dfrac{1}{100}} \end{cases}$$

$$\Rightarrow \begin{cases} \omega_z \simeq 6{,}2517 \text{ rad/s} \\ \omega_y \simeq 3{,}3905 \text{ rad/s} \\ \omega_x \simeq 3{,}1258 \text{ rad/s} \end{cases}$$

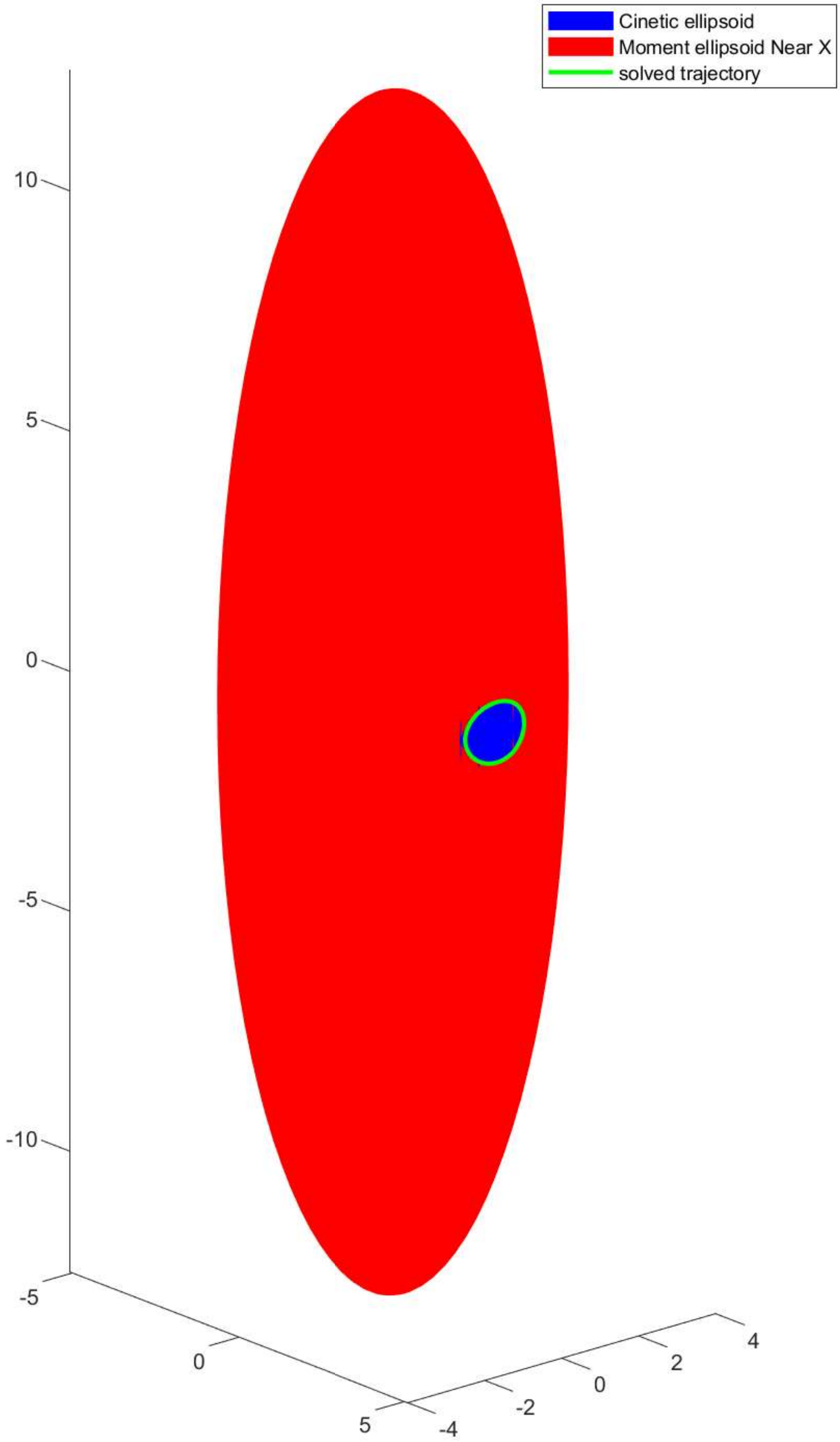
(ellipsoid available after 3))

3) $$W = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad A(W) = \begin{bmatrix} 0 & \frac{1}{2\sqrt{J_{xx}}}(J_{yy}-J_{zz})\omega_z & \frac{1}{2\sqrt{J_{xx}}}(J_{yy}-J_{zz})\omega_y \\ \frac{1}{2\sqrt{J_{yy}}}(J_{zz}-J_{xx})\omega_z & 0 & \frac{1}{2\sqrt{J_{yy}}}(J_{zz}-J_{xx})\omega_x \\ \frac{1}{2\sqrt{J_{zz}}}(J_{xx}-J_{yy})\omega_y & \frac{1}{2\sqrt{J_{zz}}}(J_{xx}-J_{yy})\omega_x & 0 \end{bmatrix}$$

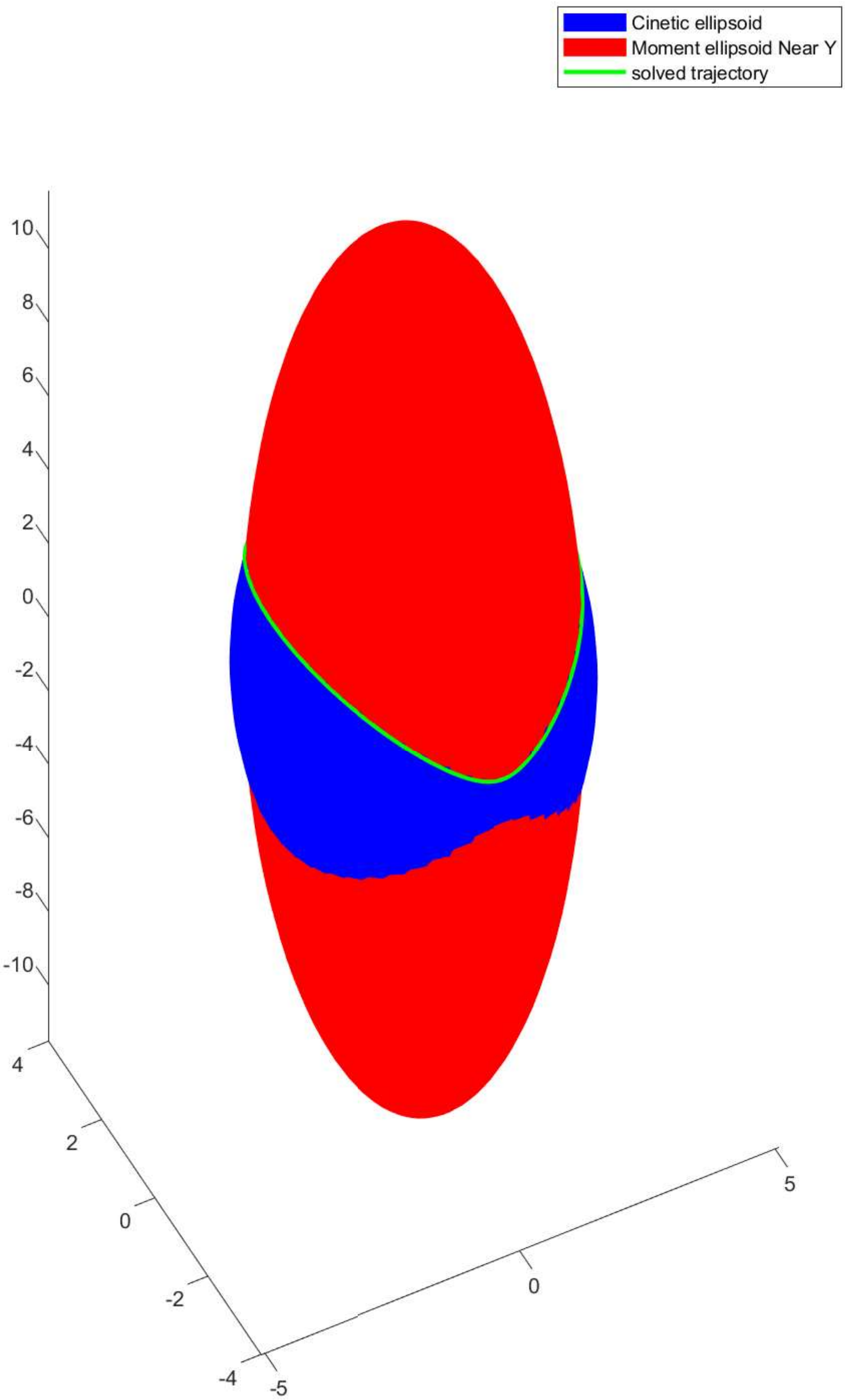
it can be shown that

$$(0) \quad \dot{\omega} = A(\omega)\,\omega \qquad \text{with euler equations}$$

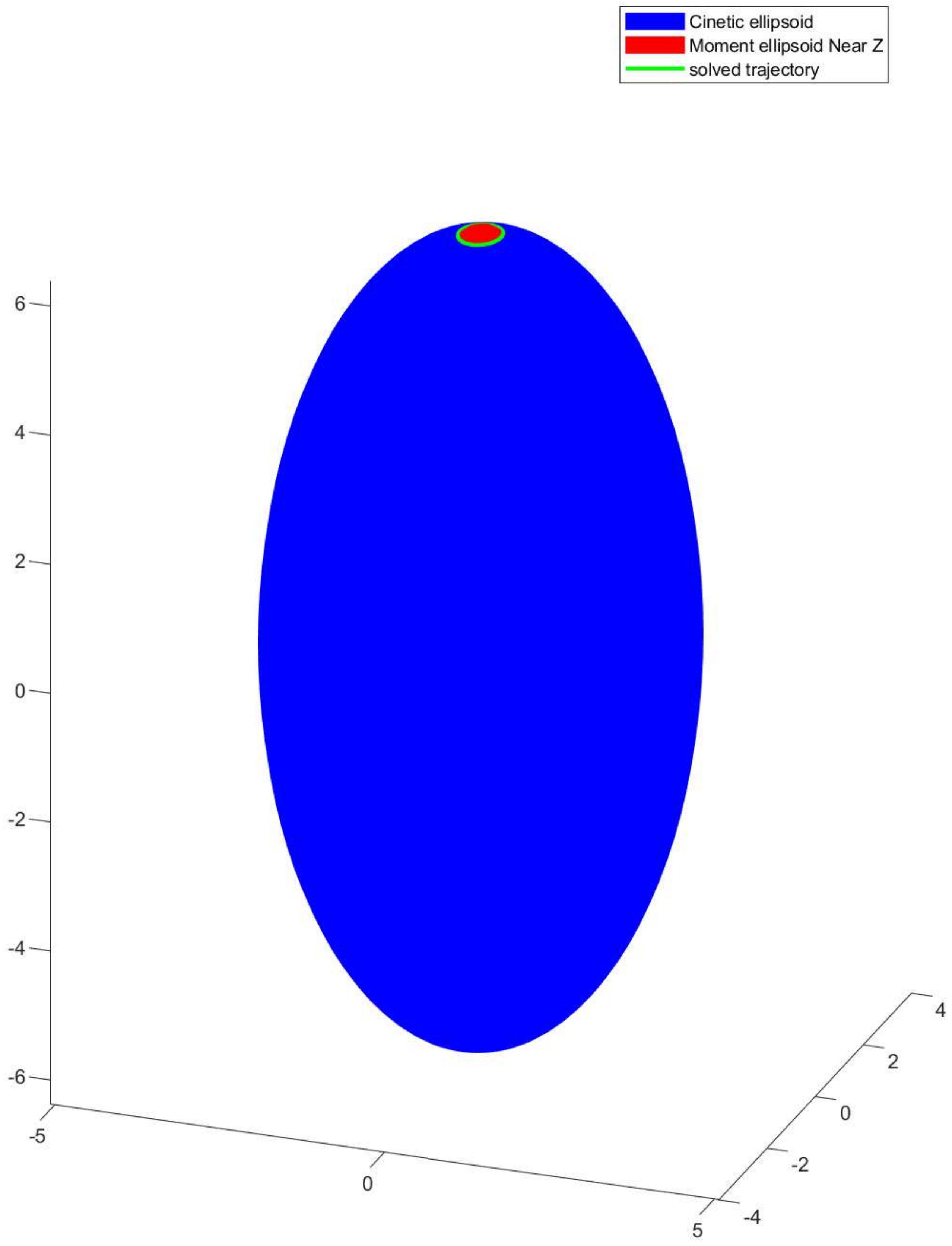
the ellipsoids with solutions from (0) are printed below



Cinetic ellipsoid
Moment ellipsoid Near X
solved trajectory
10
5
0
-5
-10
-5
0
5
-4
-2
0
2
4


Cinetic ellipsoid
Moment ellipsoid Near Y
solved trajectory
10
8
6
4
2
0
-2
-4
-6
-8
-10
4
2
0
-2
-4
-5
0
5

8



Cinetic ellipsoid
Moment ellipsoid Near Z
solved trajectory
6
4
2
0
-2
-4
-6
-5
0
5
4
2
0
-2
-4

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4) We had $\dot{\omega} = A(\omega)\,\omega$ (1)

since $\dot{\theta} = D_b^\theta(\theta)\,\omega$ (2), we can find

3-2-1 euler angles

$\theta(t)$ by resolving (1), then (2)

the corresponding theta values are to be found on simulink plot

Video :
https://www.youtube.com/watch?v=8EGT9M3CTxQab_channel=Adrien

Source code :
https://github.com/Adrien9o2/TEK4040-A1