Implementation of GARCH on a Given Stock and Quantification Using Specific Metrics
LSM Investment Club - Team Quant

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Abstract

This report presents the results of a quantitative financial analysis using GARCH models for the calculation of Value at Risk (VaR) and Expected Shortfall (ES). We discuss the methodologies, the results obtained, and the implications of these risk measures.

Contents

1 Stock Presentation

For the choice of the stock studied, we selected TotalEnergies. Indeed, it is a stock that does not have unexpected or exceptional volatilities, making it suitable for our modeling. If we wanted to study more complex stocks or those subject to trends or news, we would need more information than just the stock price over a given period. TotalEnergies is often favored by individuals seeking long-term dividends, particularly retirees. Therefore, unless there is a major shock in the world related to energy or oil, it seems unlikely that investors would suddenly withdraw.

1.1 Price Evolution

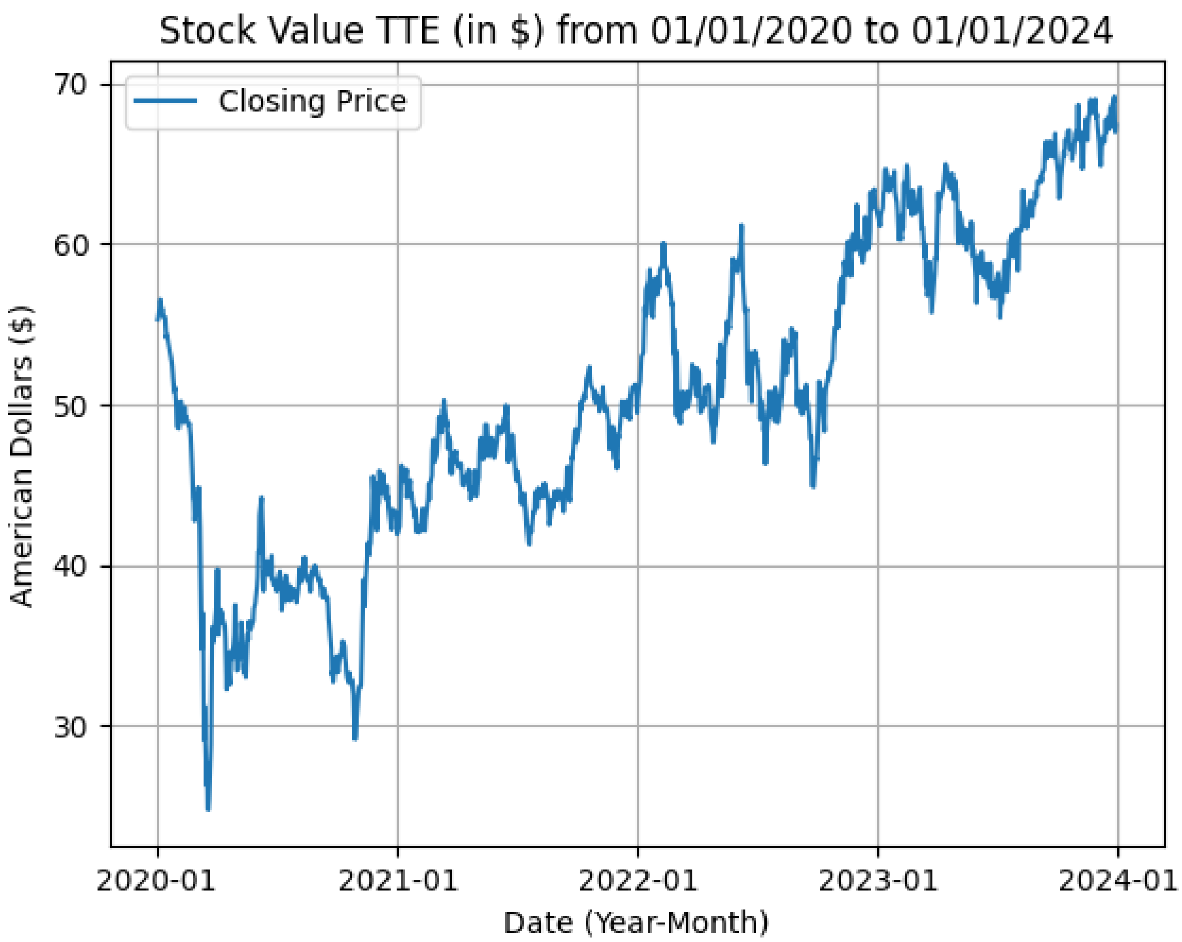


Figure 1: Evolution of the closing price of the analyzed stock

We chose to use data going back 4 years. There is no consensus on the necessary duration for analysis, but it is important to be aware that this will influence our results, especially if we capture global events in our data such as the COVID-19 pandemic and the war in Ukraine, which are unusual.

2 Value at Risk and Expected Shortfall

2.1 Mathematical Definitions of Risk Measures

The Value at Risk (**VaR**) is the minimum loss associated with a risk level $\alpha \in]0,1[$ of a loss/gain distribution X. More formally, we define VaR by

$$VaR_\alpha(X) = -\inf\{x \in \mathbb{R} | F_X(x) > \alpha\}.$$

Note that we have a negative sign, so VaR is generally a positive value, as the sign is absorbed in the definition by the associated loss. Typically, α is 5% or 1%. The limitation of VaR lies in the fact that it does not describe the shape of the loss distribution tails in the worst cases, thus it does not describe the distribution of losses in the α% worst cases. This is why we introduce the next concept.

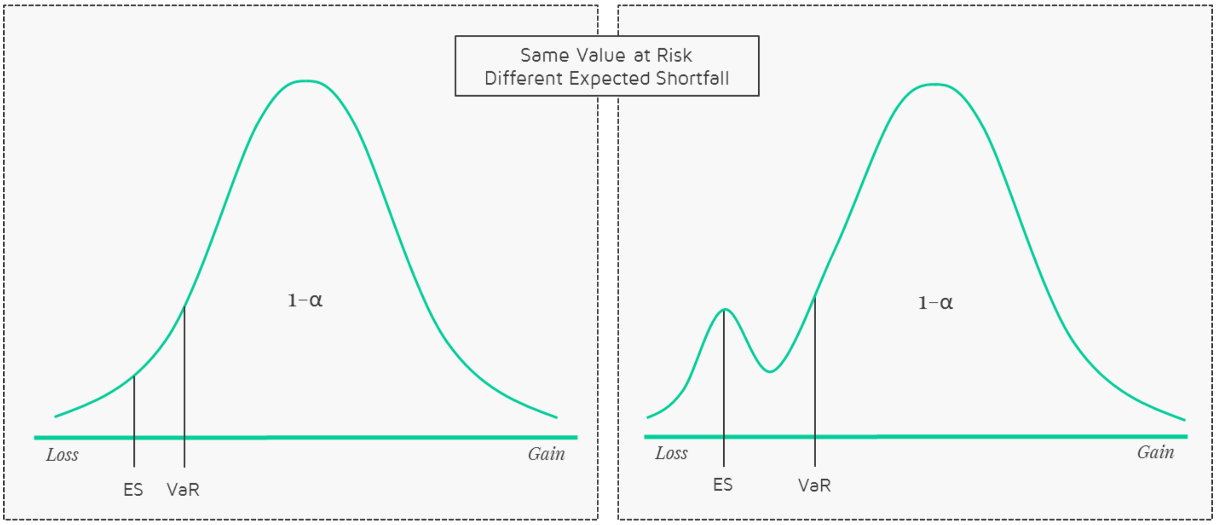


Figure 2: Example of two distributions with equal VaR but different ES.

The Expected Shortfall (**ES**) is the average expected VaR over the α% worst cases, or in other words,

$$ES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha VaR_\gamma(X)\mathrm{d}\gamma.$$

This allows us to better understand the differences between several distributions, and combining the two metrics is often useful.

2.2 Implementation and Analysis of Results

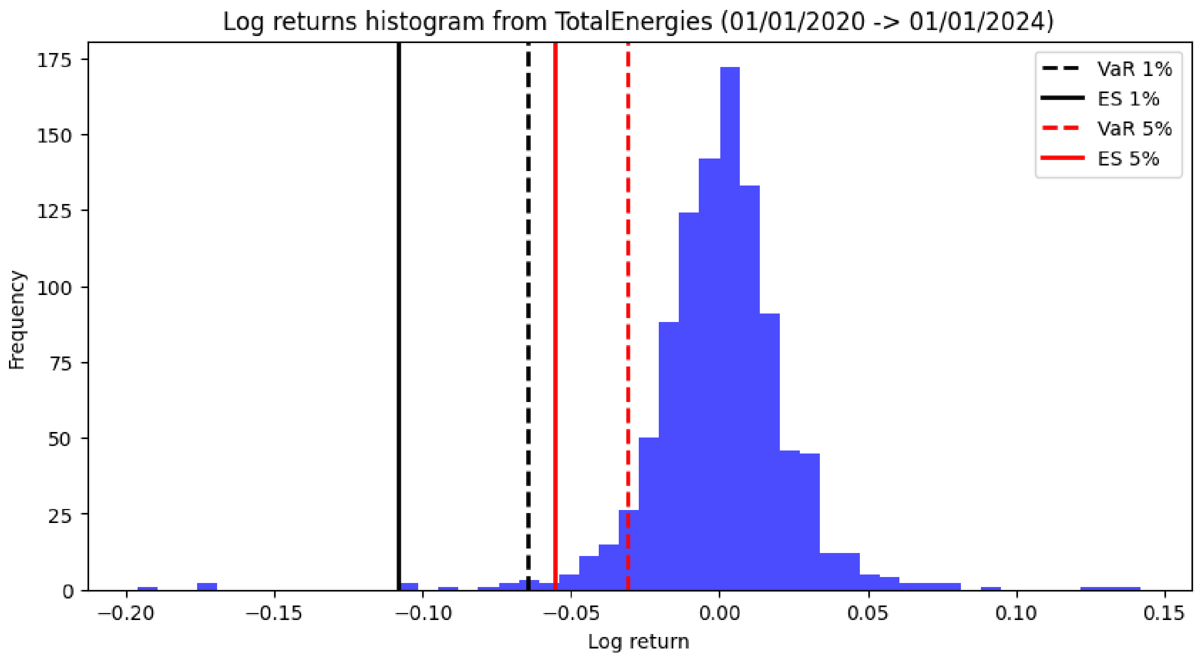


Figure 3: Histogram of VaR and ES

We calculated the two metrics based on the log returns of TotalEnergies' price at two different risk levels ($\alpha = 0.01/0.05$). We can observe that the ES is always lower than the VaR (on the graph), which is quite consistent with our intuition of the average over the worst cases. The 1% VaR is located to the left (even though both are positive values according to the definition) because the 1% worst cases will result in at least more losses than the 5% worst cases.

3 GARCH Model

The GARCH(p,q) model is a volatility model that attempts to describe observations by a mean and a volatility term, which itself depends on past volatilities and past errors. In other words, for observations r_t at time t,

$$r_t = \mu + \epsilon_t$$

where the error term is described by

$$\epsilon_t = \sigma_t z_t \quad z_t \sim N(0,1)$$

where z_t often follows a standard normal distribution but can be a Student/Skewed Student distribution. And the term we can construct,

$$\sigma_t^2 = \omega + \sum_{i=0}^{p} \beta_i \sigma_{t-i}^2 + \sum_{i=0}^{q} \alpha_i \epsilon_{t-i}^2$$

so our estimate of σ_t, our conditional volatility, depends on our p previous conditional variances σ_{t-i} and our q previous error terms ϵ_{t-i}.

3.1 Assumptions of the GARCH Model

The GARCH model is built on certain assumptions. Their invalidation in real life could question the validity of this model and justify more complex models that take these unmet assumptions into account. Below are the different assumptions of the GARCH model:

- Stationarity of returns: This assumes that the different moments of returns are constant.
- Conditional independence of residuals: This requires that the residuals, given past information, be independent.
- Distribution of errors: Each GARCH model makes an assumption about the distribution followed by past errors.
- Absence of structural biases: We simply assume that no extreme event impacting the economy from a structural point of view occurs. A typical example would be the COVID crisis, the 2008 crisis, or the 1987 crisis, which structurally change the behavior of volatility.
- Market rationality assumption: We assume that returns reflect all available information.

3.2 Mathematical Approach

For the modeling of the GARCH model, we each time modeled the 3 types of distributions considered for the term z_t.

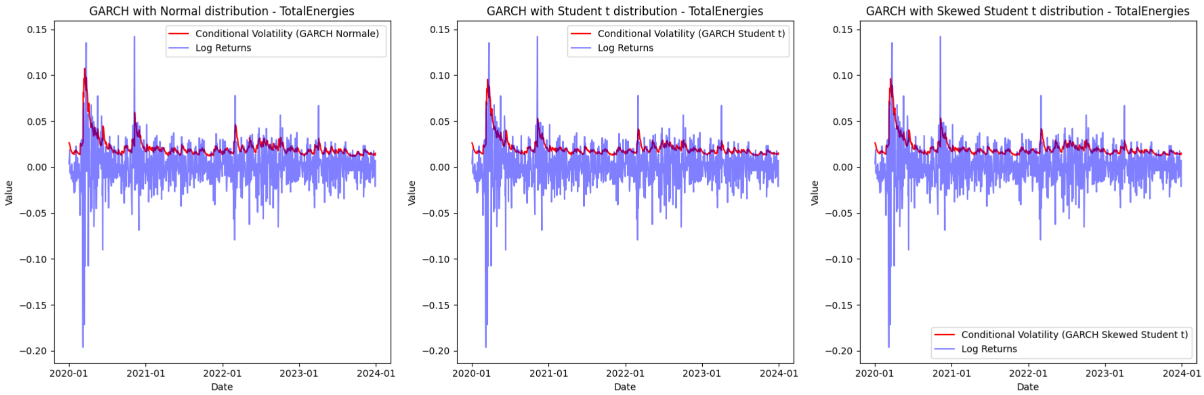


Figure 4: Distribution of volatility with the GARCH model

3.3 Autocorrelation and Partial Autocorrelation

Although the theoretical justification seems elegant, the choice of parameters (p, q) seems less obvious. To do this, consider the autocorrelation and partial autocorrelation graphs; they indicate which $p = q$ are bad choices. A point outside the blue confidence interval indicates that a model with a larger number of lagged terms (a larger lag) might be suitable. The convention is to choose parameters (p, q) just before a drop in our graph. Here, a suitable choice could be 2 up to 4; beyond that, it is unlikely that our model will be suitable. In practice, the choice is made by minimizing the AIC (balance between the quality of the fit and the number of parameters to avoid overfitting). In practice, the AIC was minimized at each stage, and our models were around parameters p, q of 1 or 2.

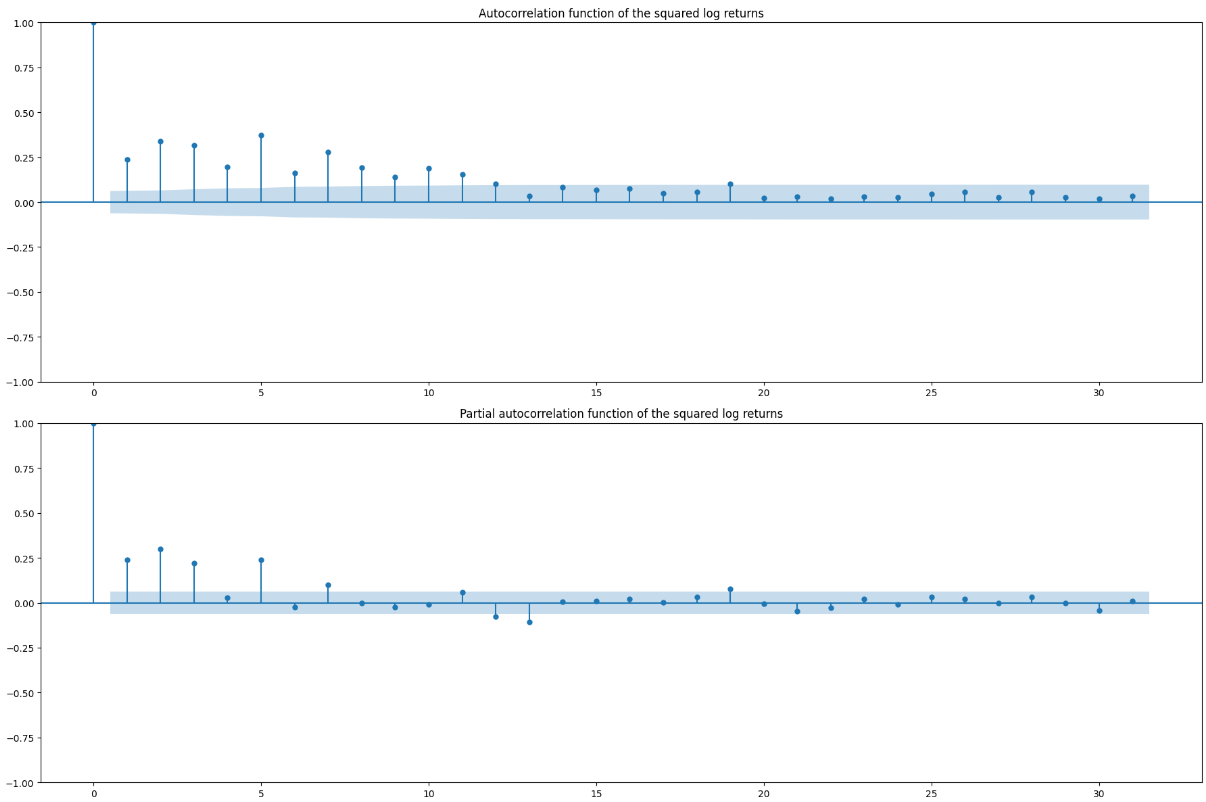


Figure 5: Autocorrelation of squared returns

3.4 Different Fit Metrics

In this section, we considered different metrics to quantify the quality of our fit. We are accustomed to the MSE for regression models, for example, but since we are dealing with a volatility model related to time series, it would be good to broaden our horizon.

3.4.1 Directional Accuracy as a Function of Data Percentage

Directional accuracy is a simple measure; it tells us whether our model has modeled the volatility in the same direction as our data, meaning that when the variation is of the same sign in the model and in the data, we consider that our model has good directional accuracy. However, we have a problem because applying it directly is not a good choice since the GARCH model is a model with lag, which is why we can observe an accuracy tending towards 0 for a small percentage of test data.

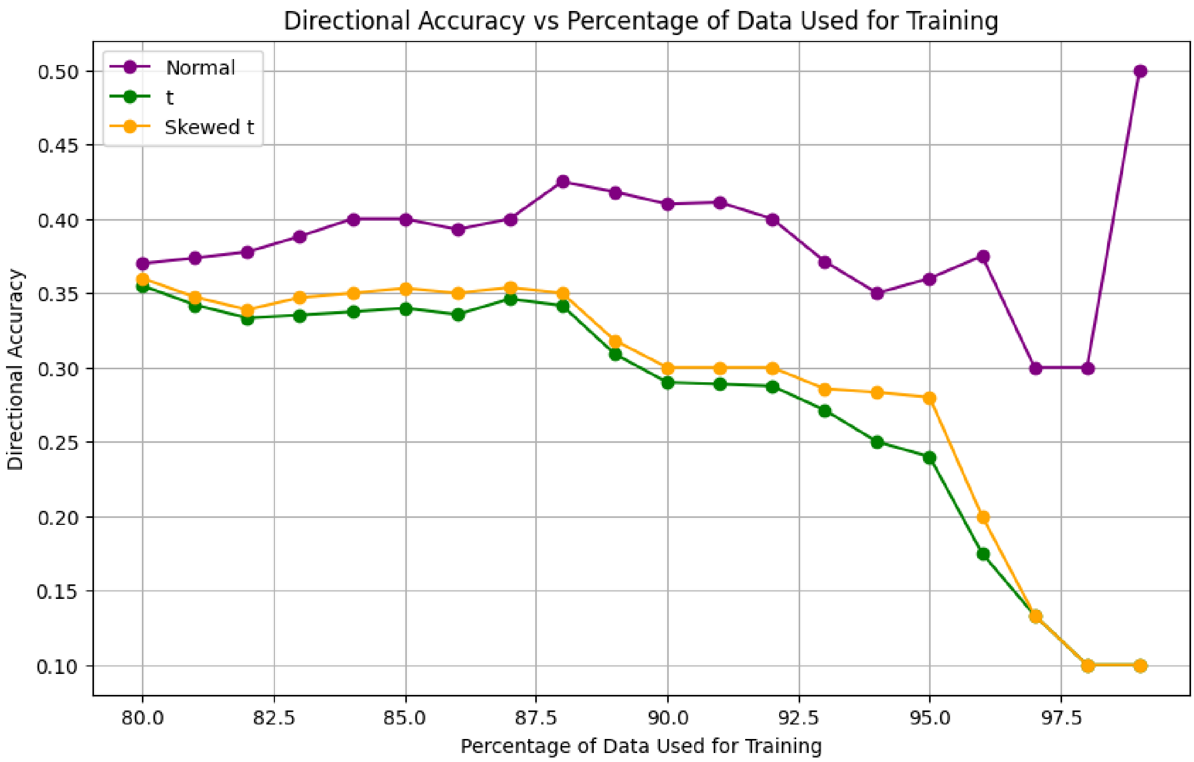


Figure 6: Directional accuracy of the GARCH model as a function of data

3.4.2 Likelihood as a Function of Data Percentage

The likelihood describes the plausibility of the parameter choice given our data. Our goal is then to maximize it, and the normal law seems to achieve this best.

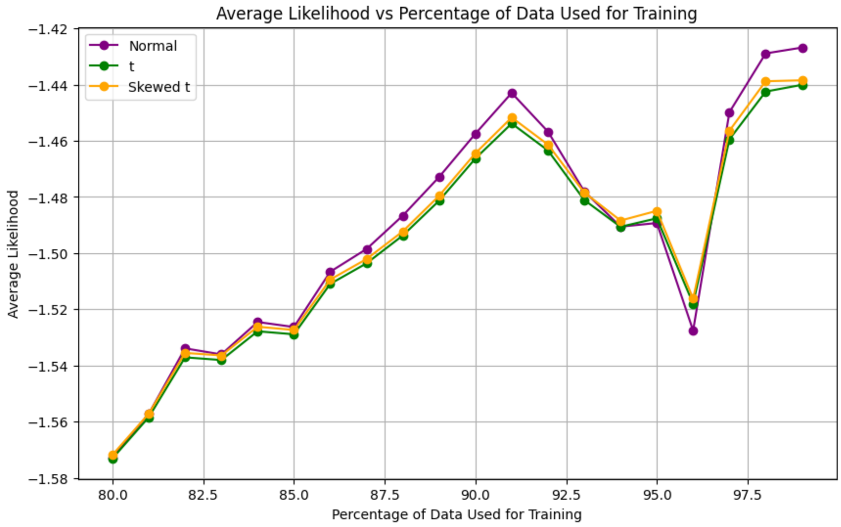


Figure 7: Likelihood of the GARCH model as a function of data

3.5 Prediction

In general, this model can be adapted for forecasts over a few days, but beyond that, its relevance seems to fade. Note that in the following, we will use test data up to 20% while our total data covers 4 years day by day.

3.5.1 95% Prediction for Different Distributions

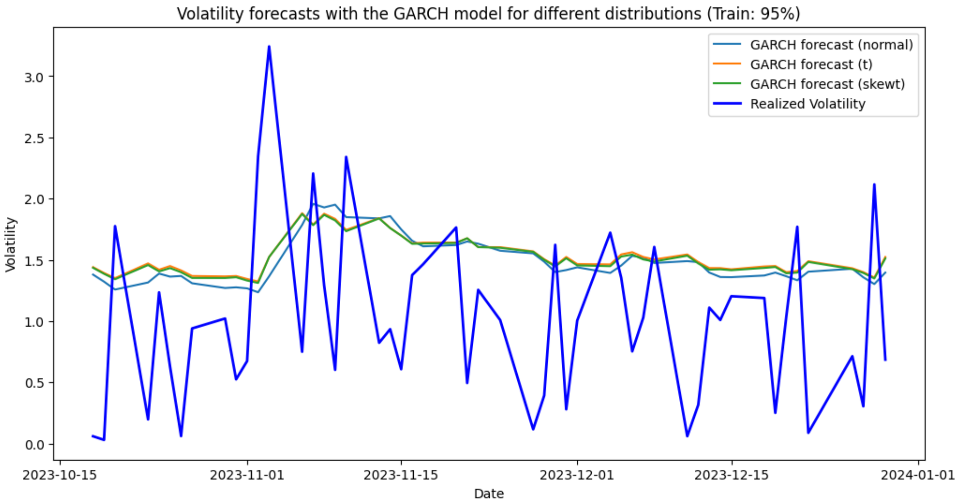


Figure 8: 95% prediction for different distributions in the GARCH model

This graph might be shocking; we expect better prediction when we use this term, but it is quite typical for a GARCH model; we only trace volatility trends, which is exhibited.

3.5.2 Prediction as a Function of Data Percentage

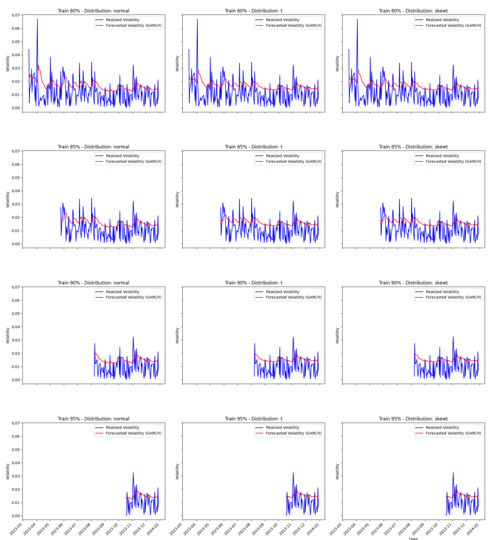


Figure 9: Prediction of the GARCH model as a function of data percentage and distribution

3.5.3 MSE as a Function of Training Data Percentage

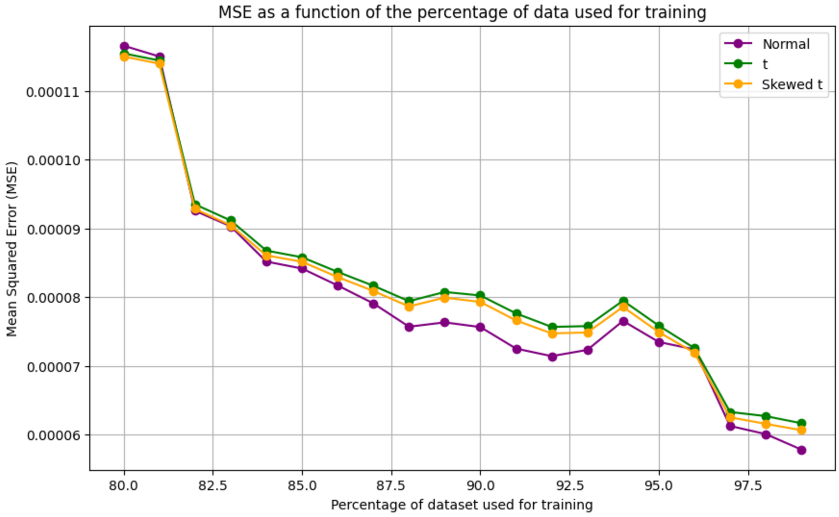


Figure 10: Mean squared error as a function of training data

Overall, the normal law seems to minimize the MSE (mean squared error), which seems to be a good point.

3.5.4 MSE as a Function of Time Horizon

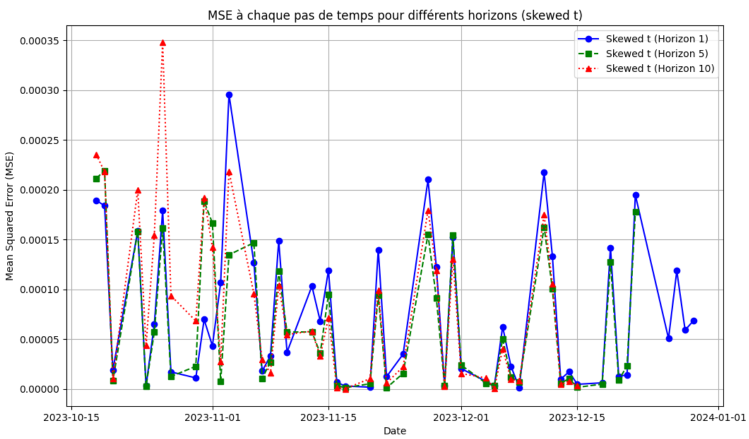


Figure 11: Mean squared error as a function of time horizon

Here, we plotted the MSE at each forecast time horizon for the skewed t; we observe that overall, we are practically at 0, which is a good sign. A priori, we do not observe a "good horizon" for prediction.

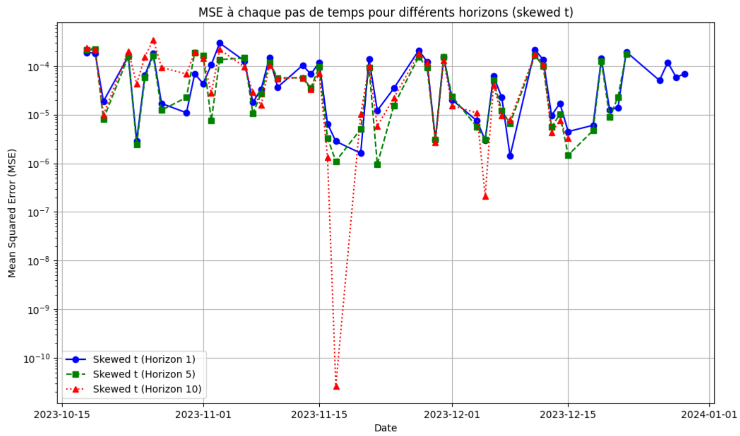


Figure 12: Mean squared error (log scale) as a function of time horizon

3.6 Analysis and Critique of Results

3.6.1 Forecast Error Percentage

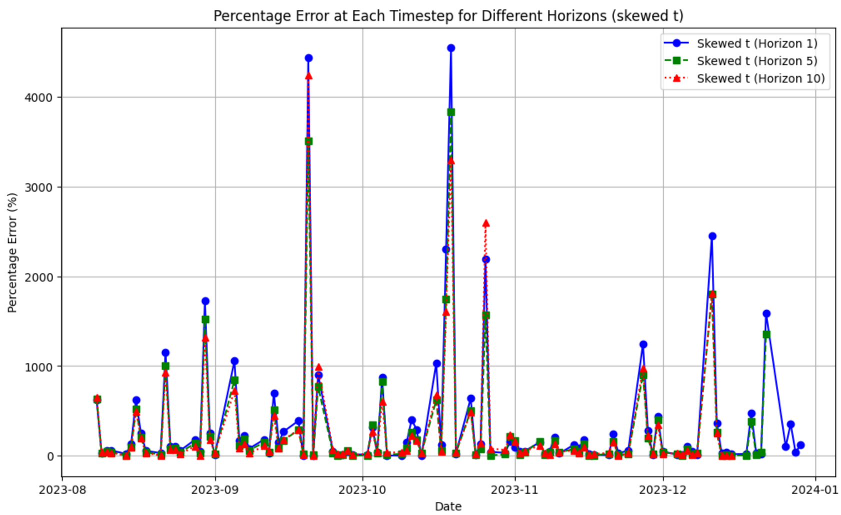


Figure 13: Forecast error percentage as a function of time horizon

The error percentage gives us a bit more information about the scale of these errors; the MSE, although we seek to minimize it, does not directly indicate the magnitude of these errors relative to the observations. Despite some peaks, the error seems to be contained below 100% (better visible with the logarithmic scale), which does not seem to be a bad prediction for volatility.

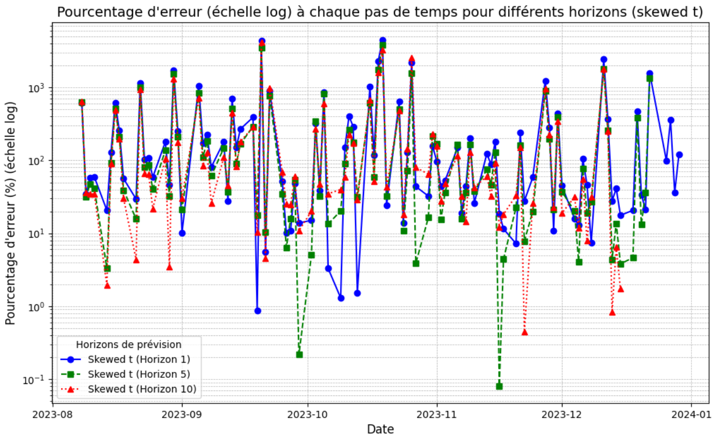


Figure 14: Forecast error percentage (log scale) as a function of time horizon

3.6.2 QQ-Plot

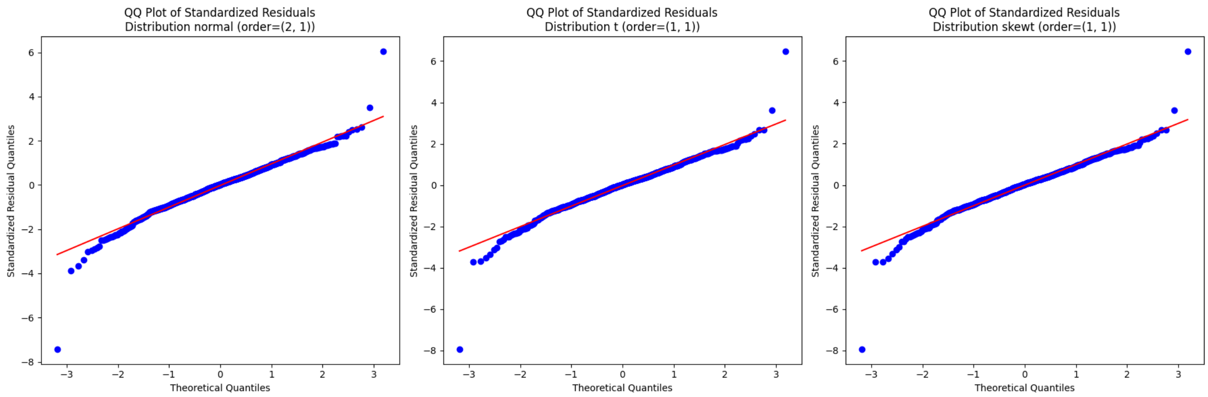


Figure 15: QQ-Plot for the GARCH model

Here, we can see with the QQ-Plot of standardized residuals, depending on the distributions, that our residuals seem to be normally distributed, and if we ignore the extreme values, the Student/skewed Student seem to better smooth our data in this direction. Note that given the amount of data we use, the normal law is sufficiently well approximated by a Student law, but with slightly heavier tails than the normal law.

4 E-GARCH

4.1 Mathematical Approach

The E-GARCH model is a variant of the GARCH model that captures asymmetric effects on the volatility of returns. Typically, in the stock market, due to the high leverage sometimes used, a shock of -20% will have more impact on volatility than a sudden rise of +20%.
The mathematical model of the E-GARCH model is presented below:

$$\log(\sigma_t^2) = \omega + \sum_{i=1}^{q} \beta_i \log(\sigma_{t-i}^2) + \sum_{j=1}^{p} \alpha_j \left(\frac{|z_{t-j}| - E|z_{t-j}|}{\sigma_{t-j}} \right) + \sum_{j=1}^{p} \gamma_j z_{t-j}$$

where:

- σ_t^2 is the conditional variance of the series at time t,

- $z_t = \frac{\epsilon_t}{\sigma_t}$ is the standardized residual, with ϵ_t representing the error term,

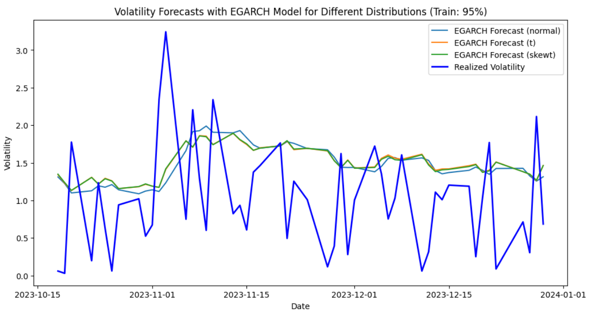
- ω is a constant modeling the expected mean of volatility,

- β_i represents the autoregressive effects of past volatility,

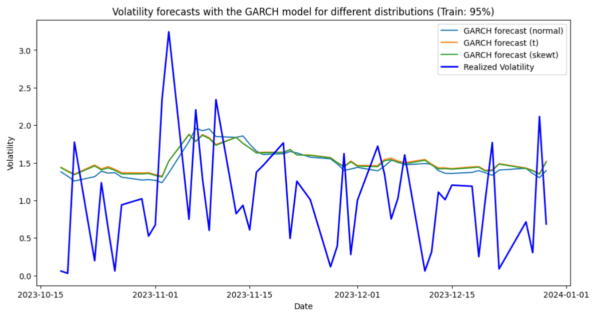
- α_j captures the impact of the magnitude of past shocks on volatility, allowing the modeling of the "leverage" effect (or asymmetry),

- γ_j is the asymmetry parameter, which differentiates the impact of positive and negative shocks on future volatility.

4.2 Prediction



(a) 95% prediction for different distributions in the E-GARCH model

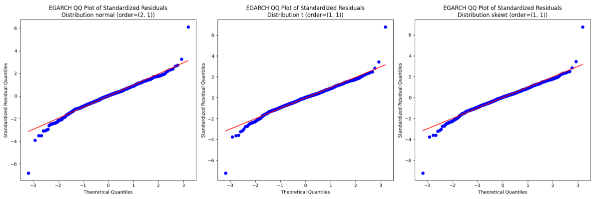


(b) 95% prediction for different distributions in the GARCH model

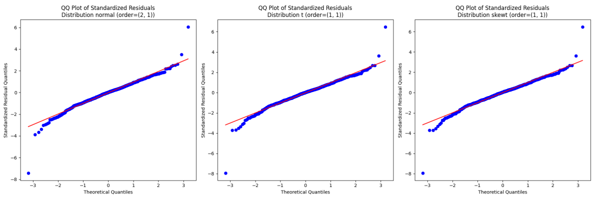
Figure 16: Comparison of 95% predictions for different distributions in the E-GARCH and GARCH models

We observe in the graph above that the E-GARCH variant of the GARCH model presents extremely similar predictions for one day. However, we can observe slightly higher or lower results for certain times. Times that we can foresee as impacts on the rise or fall.

4.3 Analysis and Critique of Results



(a) QQ-Plot for the E-GARCH model



(b) QQ-Plot for the GARCH model

Figure 17: Comparison of QQ-Plots for the E-GARCH and GARCH models

We notice extremely similar QQ-plots. The QQ-plot does not improve with this model. This is consistent considering that E-GARCH is a model that captures the asymmetry of shocks on return volatility and does not explain or better understand extreme events.

5 Conclusion

From a global perspective regarding VaR, we note that it is quickly limited since it provides no information about the distribution during extreme events, leaving the investor with a moderate VaR while being exposed to much larger losses than expected. The solution is to use Expected Shortfall coupled with VaR to have a complete view of the loss distribution. Regarding volatility models, GARCH and its asymmetric variant E-GARCH, we observe interesting predictions, albeit limited in their precision. However, these models allow us to capture volatility fluctuations and thus better adapt risk management to expected volatility.