Fourier Series

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Fourier Series

Periodic Functions

The Mathematic Formulation

- Any function that satisfies

$$f(t) = f(t + mT) \quad m \in \mathbb{Z}$$

where T is a constant and is called the *period* of the function.

Example:

$$f(t) = \cos \frac{t}{3} + \cos \frac{t}{4} \qquad \text{Find its period.}$$

$$f(t) = f(t+T) \implies \cos \frac{t}{3} + \cos \frac{t}{4} = \cos \frac{1}{3}(t+T) + \cos \frac{1}{4}(t+T)$$

Fact: $\cos \theta = \cos(\theta + 2m\pi)$

$$\frac{T}{3} = 2m\pi \qquad\qquad T = 6m\pi$$

$$\qquad\qquad\qquad\qquad\qquad\qquad T = 24\pi \quad \text{smallest } T$$

$$\frac{T}{4} = 2n\pi \qquad\qquad T = 8n\pi$$

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Example:

$$f(t) = \cos \omega_1 t + \cos \omega_2 t \qquad \text{Find its period.}$$

$$f(t) = f(t + T) \implies \cos \omega_1 t + \cos \omega_2 t = \cos \omega_1 (t + T) + \cos \omega_2 (t + T)$$

$$\omega_1 T = 2m\pi$$

$$\omega_2 T = 2n\pi$$

$$\implies \frac{\omega_1}{\omega_2} = \frac{m}{n} \implies \frac{\omega_1}{\omega_2} \text{ must be a rational number}$$

Example:

$$f(t) = \cos 10t + \cos(10 + \pi)t$$

Is this function a periodic one?
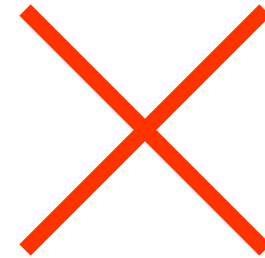
$$\frac{\omega_1}{\omega_2} = \frac{10}{10 + \pi}$$ not a rational number

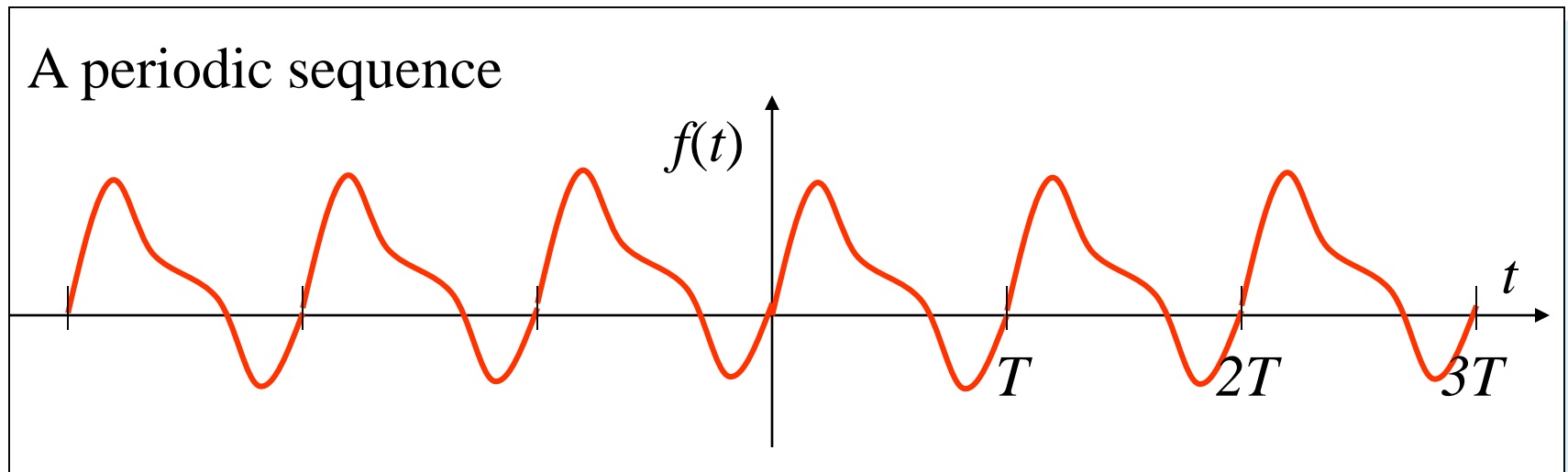# Fourier Series

Fourier Series

Introduction

- Decompose a periodic input signal into *primitive periodic components*.

A periodic sequence

$f(t)$

t

T $2T$ $3T$

Synthesis

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T}$$

DC Part Even Part Odd Part

T is a period of all the above signals

Let $\omega_0 = 2\pi/T$.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Orthogonal Functions

- Call a set of functions $\{\phi_k\}$ *orthogonal* on an interval $a < t < b$ if it satisfies

$$\int_a^b \varphi_m(t)\varphi_n^*(t)dt = \begin{cases} 0 & m \neq n \\ N & m = n \end{cases} \quad or$$

$$\left\langle \varphi_m(t), \varphi_n(t) \right\rangle = N\delta_{m,n}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi/T$.

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\,dt = 0, \qquad \forall m$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases} = \frac{T}{2}\delta_{m,n}$$

We now prove this one

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \frac{T}{2}\delta_{m,n}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0, \quad \text{for all } m \text{ and } n$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

Proof

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t) \cos(n\omega_0 t) dt \qquad m \neq n$$

$$= \frac{1}{2} \int_{-T/2}^{T/2} \cos[(m+n)\omega_0 t] dt + \frac{1}{2} \int_{-T/2}^{T/2} \cos[(m-n)\omega_0 t] dt$$

$$= \frac{1}{2} \frac{1}{(m+n)\omega_0} \sin[(m+n)\omega_0 t]\Big|_{-T/2}^{T/2} + \frac{1}{2} \frac{1}{(m-n)\omega_0} \sin[(m-n)\omega_0 t]\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{2} \frac{1}{(m+n)\omega_0} \underbrace{2\sin[(m+n)\pi]}_{0} + \frac{1}{2} \frac{1}{(m-n)\omega_0} \underbrace{2\sin[(m-n)\pi]}_{0}$$

$$= 0$$

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Proof

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\cos^2 \alpha = \frac{1}{2}[1 + \cos 2\alpha]$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t) \cos(n\omega_0 t) dt \qquad m = n$$

$$= \int_{-T/2}^{T/2} \cos^2(m\omega_0 t) dt \quad = \frac{1}{2}\int_{-T/2}^{T/2}[1 + \cos 2m\omega_0 t] dt$$

$$= \frac{1}{2} t \Big|_{-T/2}^{T/2} + \underbrace{\frac{1}{4m\omega_0} \sin 2m\omega_0 t] \Big|_{-T/2}^{T/2}}_{0}$$

$$= \frac{T}{2}$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t) \cos(n\omega_0 t) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi/T$.

$$\int_{-T/2}^{T/2} \text{c} \quad m \neq 0$$

$$\int_{-T/2}^{T/2} \text{c}$$

$$\left\{ 1, \cos \omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \cdots \\ \sin \omega_0 t, \sin 2\omega_0 t, \sin 3\omega_0 t, \cdots \right\}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt =$$

an orthogonal set.

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad \text{for all } m \text{ and } n$$

Decomposition

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\, dt \qquad n = 1, 2, \cdots$$

Convergence

Let $f(t), T$ picewise continuous periodic function

$$f_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(n\omega_0 t) + \sum_{n=1}^{N} b_n \sin(n\omega_0 t)$$

$$1 - \forall t \in \mathbb{R} \quad \lim_{N \to \infty} f_N(t) = \frac{1}{2}\left[f(t^+) - f(t^-) \right]$$

If $f(t)$ continuous at t_0 the limit is $f(t_0)$

2- if $f(t)$ continuous in $\mathbb{R}$ and $f'(t)$ picewise continuous function within (a,b) interval, $f_N(t)$ CVU to $f(t)$

(Suite)

3- if $f''(t)$ exists except on a finite number points on $\forall$ closed interval, then in a point t_0 where $f''(t_0)$ exists then

$$\lim_{N \to \infty} f'_N(t) = f'(t)$$

Convergence and existence

1- absolutely integrable over any period

$$\int\limits_{\langle T_0\rangle} \big|f(t)\big|dt\langle\infty$$

2- $f(t)$ has a finite number of extremum within any finite interval of t

3- $f(t)$ has a finite number of discontinuities within any interval of t and each of these discontinuities is finite

….sufficient but not necessary conditions

Proof

Use the following facts:

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad \text{for all } m \text{ and } n$$

Example (Square Wave)
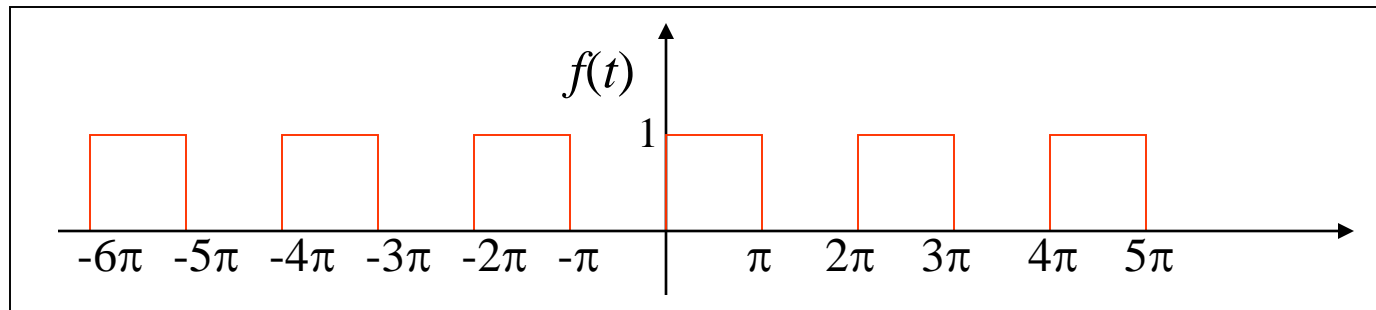
$$a_0 = \frac{2}{2\pi} \int_0^\pi 1 dt = 1 \quad : \quad T = 2\pi$$

$$a_n = \frac{2}{2\pi} \int_0^\pi \cos nt\, dt = \frac{1}{n\pi} \sin nt \Big|_0^\pi = 0 \quad n = 1, 2, \cdots$$

$$b_n = \frac{2}{2\pi} \int_0^\pi \sin nt\, dt = -\frac{1}{n\pi} \cos nt \Big|_0^\pi = -\frac{1}{n\pi} (\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1,3,5,\cdots \\ 0 & n = 2,4,6,\cdots \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$

Example (Square Wave)
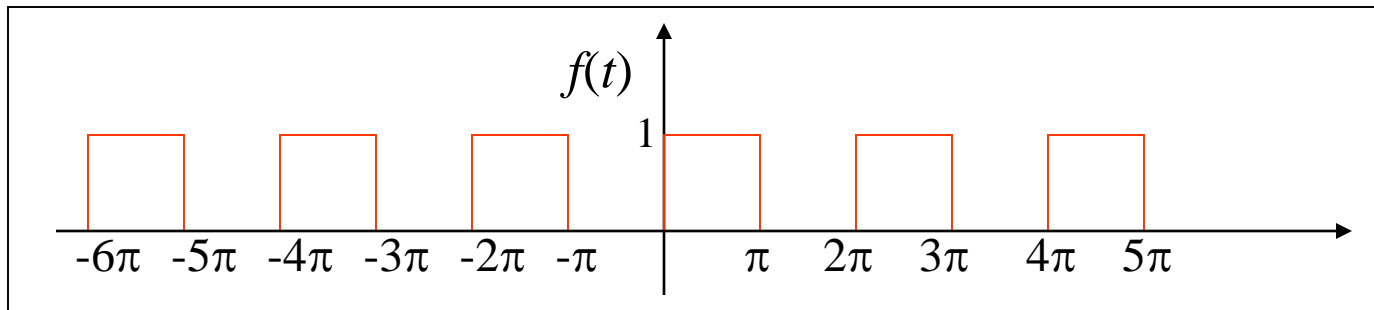
$$a_0 = \frac{2}{2\pi}\int_0^\pi 1\,dt = 1$$

$$a_n = \frac{2}{2\pi}\int_0^\pi \cos nt\,dt = \frac{1}{n\pi}\sin nt\Big|_0^\pi = 0 \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{2\pi}\int_0^\pi \sin nt\,dt = -\frac{1}{n\pi}\cos nt\Big|_0^\pi = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1,3,5,\cdots \\ 0 & n = 2,4,6,\cdots \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots \right)$$

Example (Square Wave)

$$c \quad \frac{2}{\pi}\int^{\pi} 1 dt = 1$$

a

b

$n = 1,3,5,\cdots$

$n = 2,4,6,\cdots$

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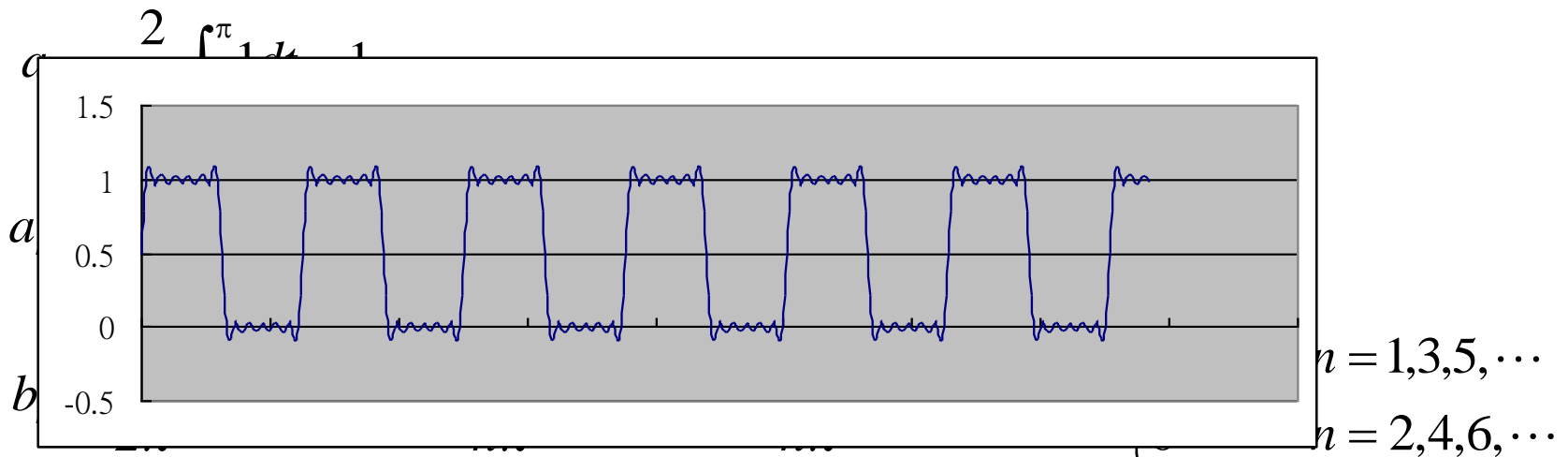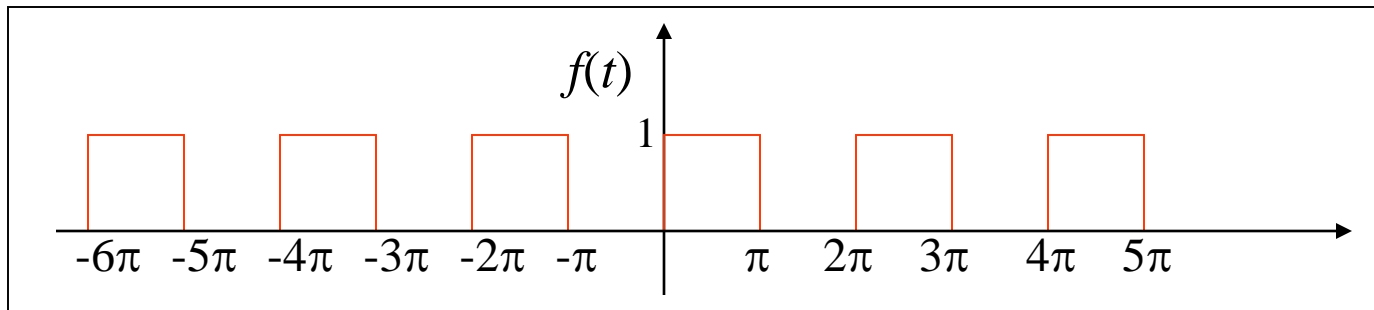
Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi nt}{T}$$

$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)}_{\text{Odd Part}}$$

T is a period of all the above signals

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Harmonics

Define $\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T}$, called the *fundamental angular frequency*.

Define $\omega_n = n\omega_0$, called the *n*-th *harmonic* of the periodic function.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \omega_n t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \omega_n t \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\cos \theta_n \cos \omega_n t + \sin \theta_n \sin \omega_n t \right)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

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Amplitudes and Phase Angles

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

harmonic amplitude   phase angle

$$C_0 = \frac{a_0}{2}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

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Fourier Series

Complex Form of the Fourier Series

Complex Exponentials

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j \sin n\omega_0 t$$

$$e^{-jn\omega_0 t} = \cos n\omega_0 t - j \sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t = \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

Complex Form of the Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n \left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - \frac{j}{2}\sum_{n=1}^{\infty} b_n \left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t}\right]$$

$$= c_0 + \sum_{n=1}^{\infty}\left[c_n e^{jn\omega_0 t} + c_{-n}e^{-jn\omega_0 t}\right]$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$
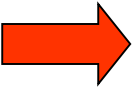
$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = c_0 + \sum_{n=1}^{\infty} \left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} \right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{T} \left[\int_{-T/2}^{T/2} f(t) \cos n\omega_0 t \, dt - j \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t \, dt \right]$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j \sin n\omega_0 t) dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{jn\omega_0 t} dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

If $f(t)$ is real,

$$c_{-n} = c_n^*$$

hermitian symmetry

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$n = \pm 1, \pm 2, \pm 3, \cdots$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

$$c_0 = \frac{1}{2}a_0$$

Complex Frequency Spectra

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) \qquad n = \pm 1, \pm 2, \pm 3, \cdots$$

$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$c_0 = \frac{1}{2} a_0$$

$|c_n|$ *amplitude spectrum* ω

ϕ_n *phase spectrum* ω
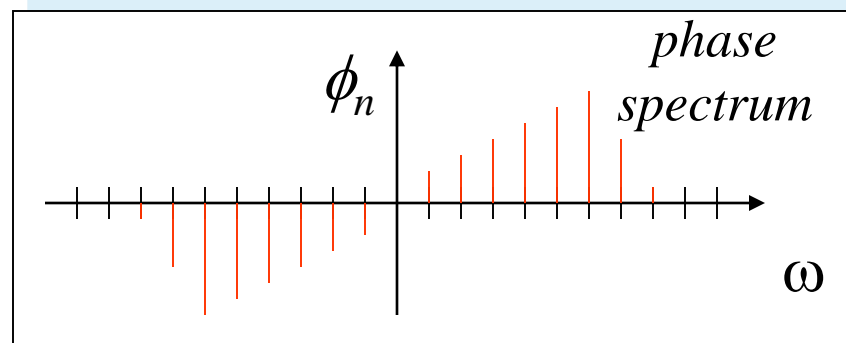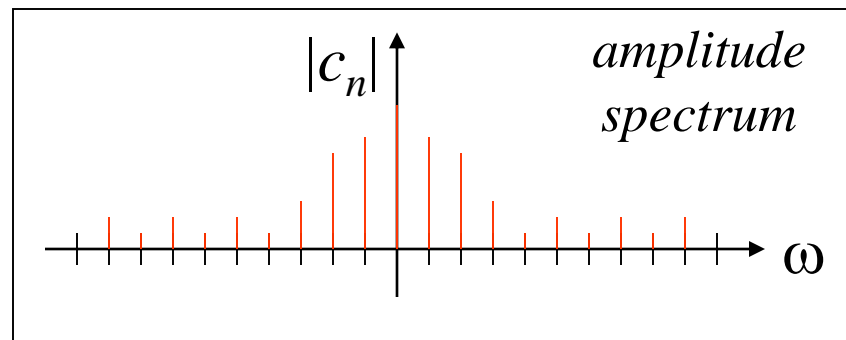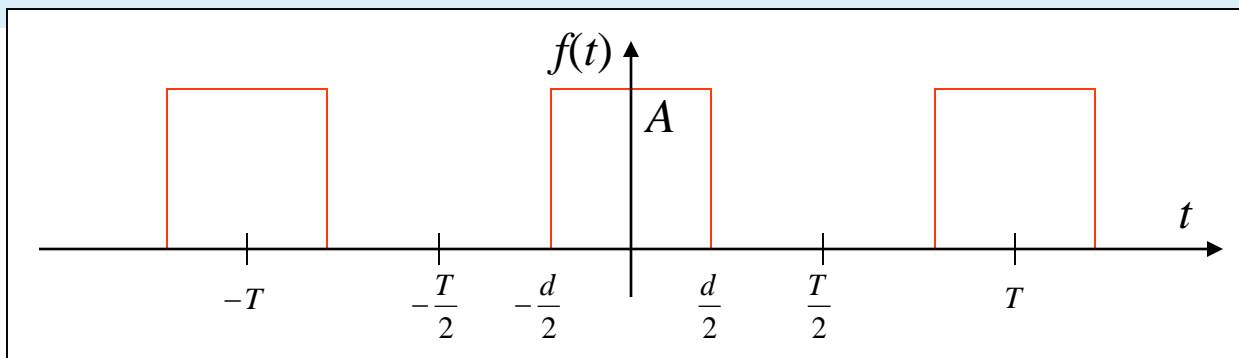
Example

$$c_n = \frac{A}{T} \int_{-d/2}^{d/2} e^{-jn\omega_0 t} dt$$

$$= \frac{A}{T} \frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \Big|_{-d/2}^{d/2}$$

$$= \frac{A}{T} \left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 d/2} - \frac{1}{-jn\omega_0} e^{jn\omega_0 d/2} \right)$$

$$= \frac{A}{T} \frac{1}{-jn\omega_0} (-2j \sin n\omega_0 d/2)$$

$$= \frac{A}{T} \frac{1}{\frac{1}{2} n\omega_0} \sin n\omega_0 d/2$$

$$= \frac{Ad}{T} \frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)} = \frac{Ad}{T} \sin c \frac{n\pi d}{T}$$
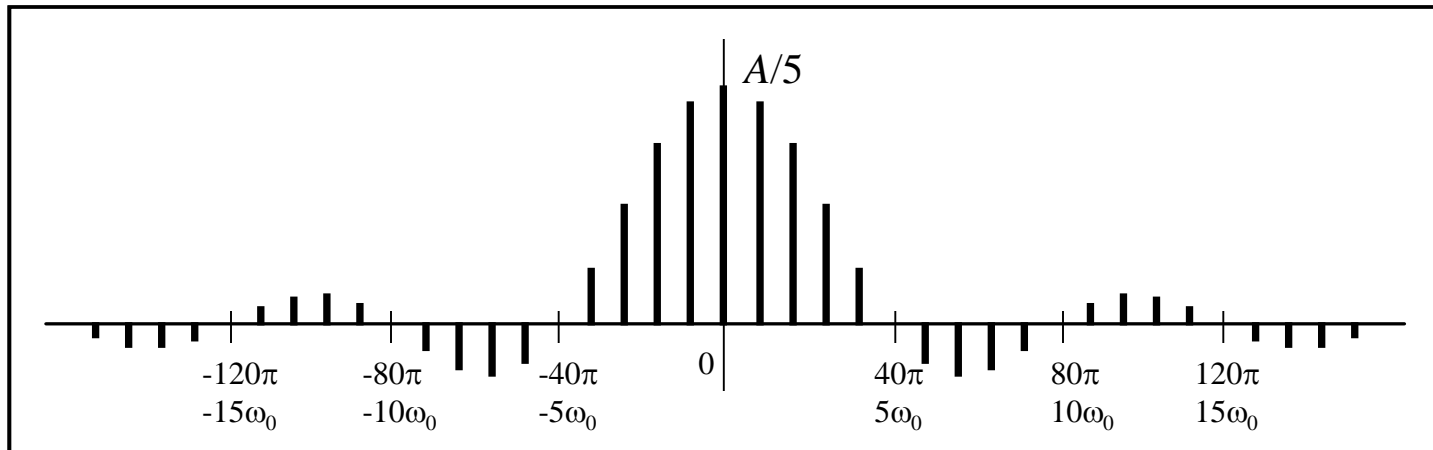
Example

$$c_n = \frac{Ad}{T} \frac{\sin\left(\dfrac{n\pi d}{T}\right)}{\left(\dfrac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20}, \quad \boxed{T = \frac{1}{4}}, \quad \frac{d}{T} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 8\pi$$

Example
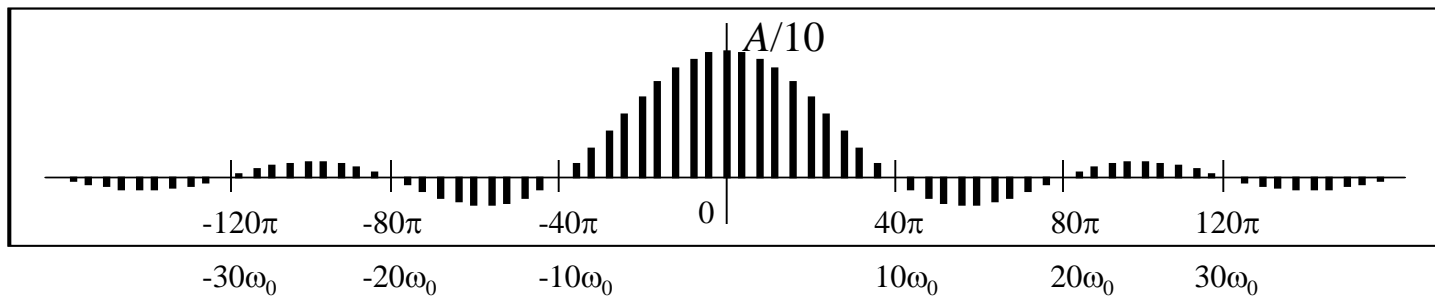
$$c_n = \frac{Ad}{T} \frac{\sin\left(\dfrac{n\pi d}{T}\right)}{\left(\dfrac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20}, \quad \boxed{T = \frac{1}{2},} \quad \frac{d}{T} = \frac{1}{10}$$

$$\omega_0 = \frac{2\pi}{T} = 4\pi$$

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Example
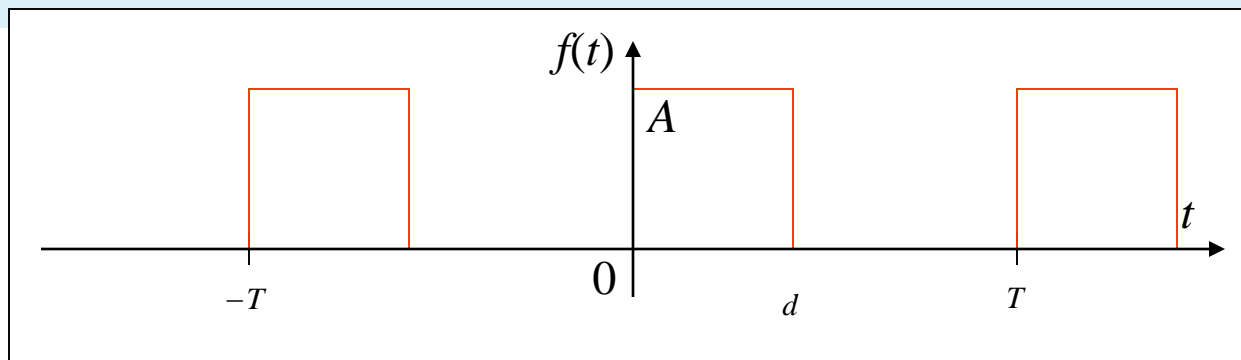
$$c_n = \frac{A}{T} \int_0^d e^{-jn\omega_0 t} dt$$

$$= \frac{A}{T} \frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \Big|_0^d$$

$$= \frac{A}{T} \left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 d} - \frac{1}{-jn\omega_0} \right)$$

$$= \frac{A}{T} \frac{1}{jn\omega_0} (1 - e^{-jn\omega_0 d})$$

$$= \frac{A}{T} \frac{1}{jn\omega_0} e^{-jn\omega_0 d/2} (e^{jn\omega_0 d/2} - e^{-jn\omega_0 d/2})$$

$$= \frac{Ad}{T} \frac{\sin\left(\dfrac{n\pi d}{T}\right)}{\left(\dfrac{n\pi d}{T}\right)} e^{-jn\omega_0 d/2}$$

Fourier Series

Impulse Train

Dirac Delta "Function"

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1 \quad \forall \varepsilon \rangle 0$$
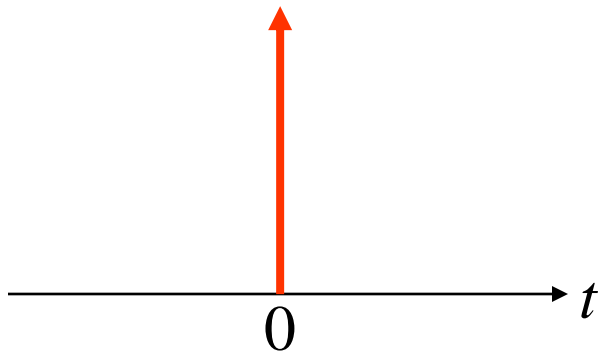
Also called *unit impulse function*.

0   t

Generalized function

- But an ordinary function which is everywhere 0 except at a single point must have integral 0 (in the Riemann integral sense).

  – Thus $\delta(t)$ cannot be an ordinary function and mathematically it is defined by

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \varphi(0)$$

Where $\varphi(t)$ is an regular function continuous at $t = 0$

Generalized function (suite)

- An alternative definition of $\delta(t)$ is given by

$$\int_a^b \delta(t)\varphi(t)dt = \begin{cases} \varphi(0) & a\langle 0 \langle b \\ 0 & a\langle b \langle 0 \quad or \quad 0\langle a \langle b \\ undefined & a = 0 \quad or \quad b = 0 \end{cases}$$

- It is why, $\delta(t)$ is often called a generalized function and $\varphi(t)$ a testing function $\in \mathcal{S}$

- Also

$$\delta(t - t_0) \quad is \ defined \ by$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\varphi(t)dt = \varphi(t_0)$$

Some properties of $\delta(t)$

Some properties

$$\delta(at) = \frac{1}{|a|}\delta(t) \qquad \delta(-t) = \delta(t)$$

$$x(t)\delta(t) = x(0)\delta(t) \qquad x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) = x(t)*\delta(t)$$

Generalized derivatives

$$\int_{-\infty}^{\infty} g^{(n)}(t)\varphi(t)dt = (-1)^n \int_{-\infty}^{\infty} g(t)\varphi^{(n)}(t)dt$$

$and\ (because\ \varphi(t)\ vanished\ out\ side\ some\ fixed\ \mathrm{int}\,erval)$

$$\int_{-\infty}^{\infty} \delta'(t)\varphi(t)dt = -\varphi'(0)$$

$example \quad \delta(t) = \dfrac{du(t)}{dt}$

Property

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0)$$

$\phi(t)$: Test Function

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\phi(0)\,dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)\,dt = \phi(0)$$

Impulse Train
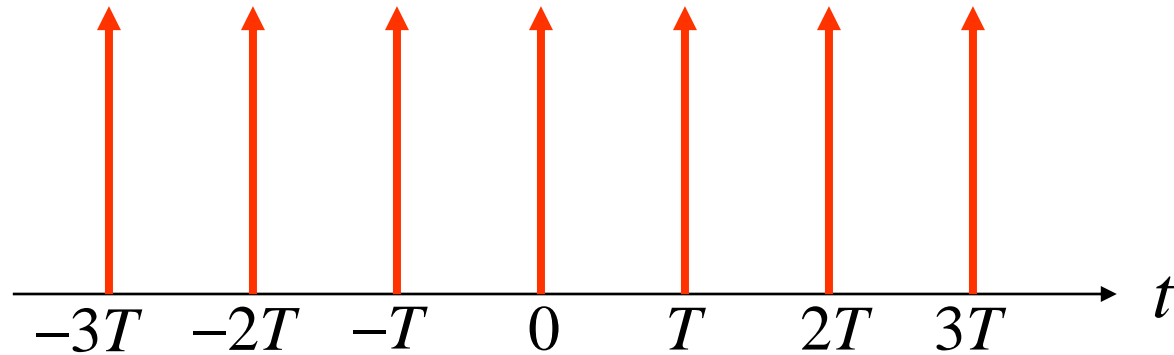
$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Fourier Series of the Impulse Train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} \delta_T(t)\,dt = \frac{2}{T}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} \delta_T(t) \cos(n\omega_0 t)\,dt = \frac{2}{T}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} \delta_T(t) \sin(n\omega_0 t)\,dt = 0$$

$$\delta_T(t) = \frac{1}{T} + \frac{2}{T} \sum_{n=-\infty}^{\infty} \cos n\omega_0 t$$

Complex Form
Fourier Series of the Impulse Train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t)\,dt = \frac{1}{T}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_0 t}\,dt = \frac{1}{T}$$

$$\delta_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

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Fourier Series

Analysis of Periodic Waveforms

Waveform Symmetry

- Even Functions

$$f(t) = f(-t)$$
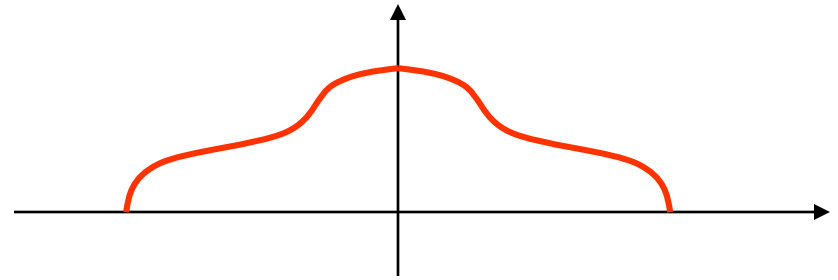
- Odd Functions

$$f(t) = -f(-t)$$
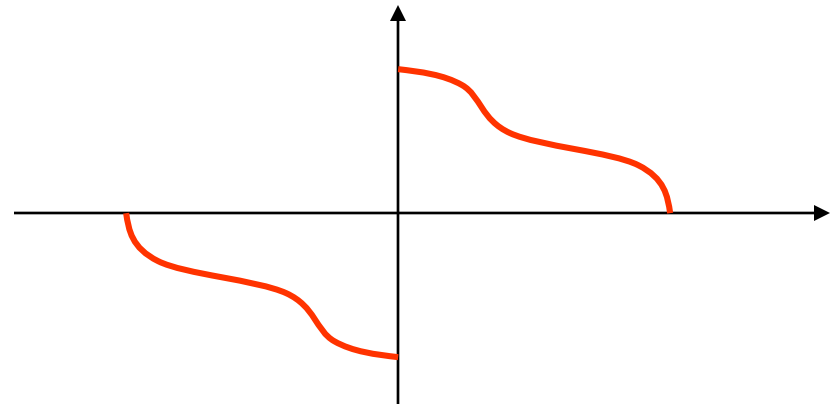
Decomposition

- Any function $f(t)$ can be expressed as the sum of an even function $f_e(t)$ and an odd function $f_o(t)$.

$$f(t) = f_e(t) + f_o(t)$$

$$f_e(t) = \tfrac{1}{2}[f(t) + f(-t)] \qquad \text{Even Part}$$

$$f_o(t) = \tfrac{1}{2}[f(t) - f(-t)] \qquad \text{Odd Part}$$

Example

$$f(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & t < 0 \end{cases}$$
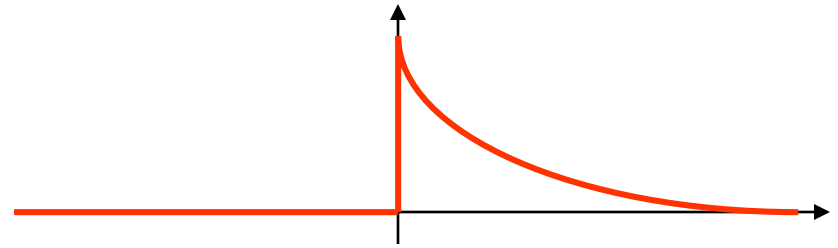
$$f_e(t) = \begin{cases} \frac{1}{2}e^{-t} & t > 0 \\ \frac{1}{2}e^{t} & t < 0 \end{cases}$$

Even Part
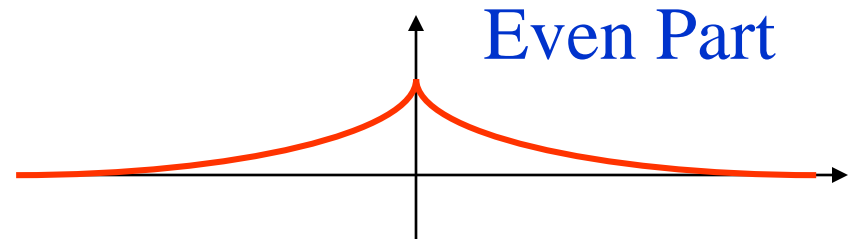
$$f_o(t) = \begin{cases} \frac{1}{2}e^{-t} & t > 0 \\ -\frac{1}{2}e^{t} & t < 0 \end{cases}$$

Odd Part
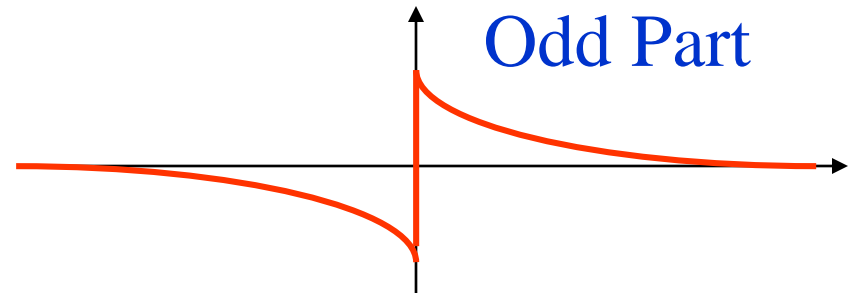
Half-Wave Symmetry

$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t + T/2\right)$$
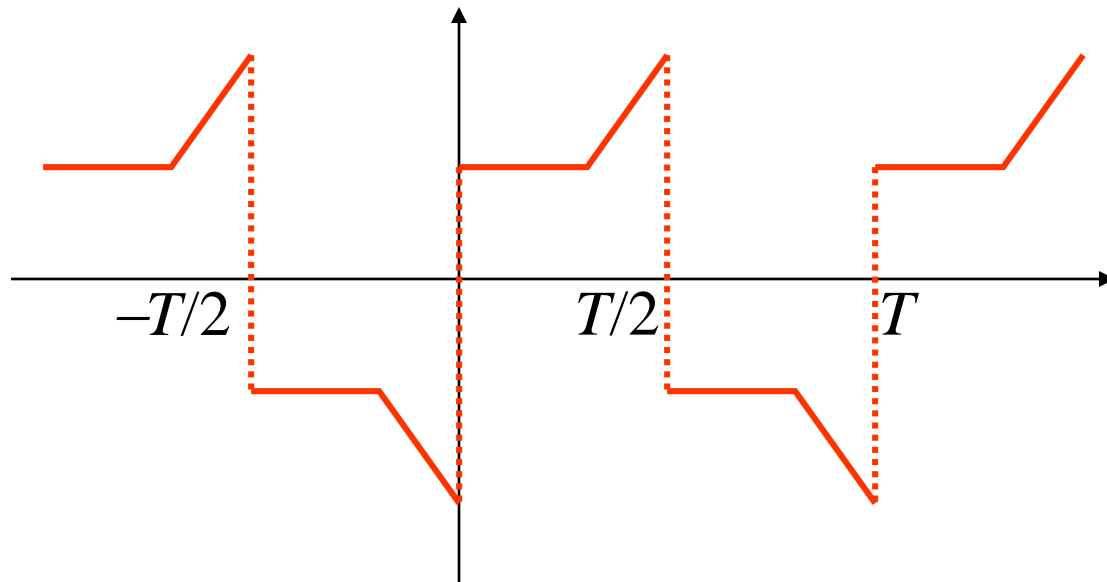
Quarter-Wave Symmetry
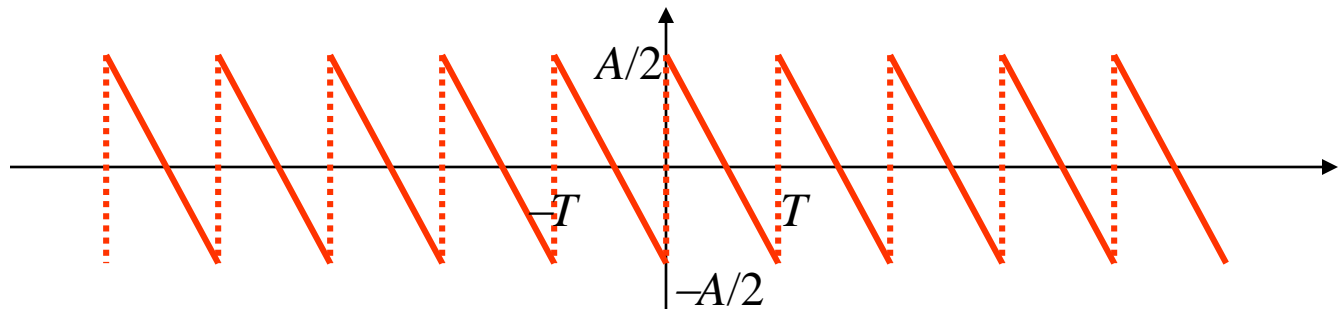
Even Quarter-Wave Symmetry
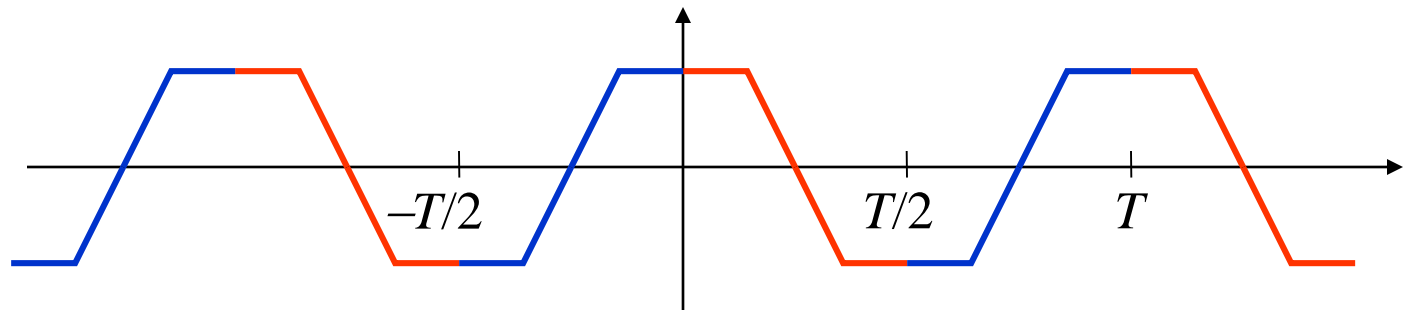
Odd Quarter-Wave Symmetry
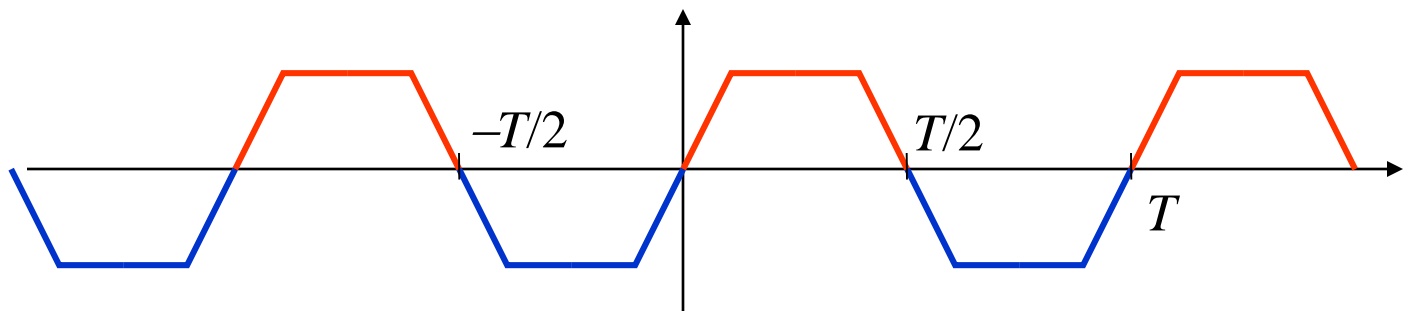
Hidden Symmetry

- The following is a asymmetry periodic function:
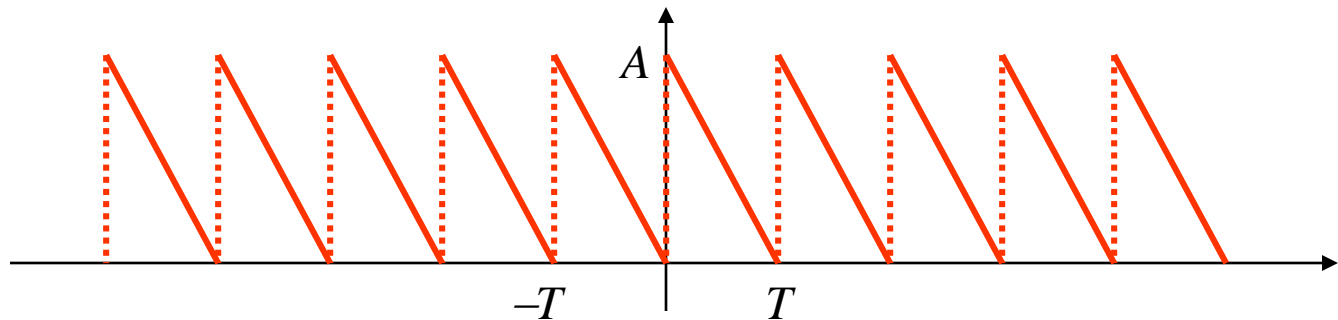
- Adding a constant to get symmetry property.

Fourier Coefficients of Symmetrical Waveforms

● The use of symmetry properties simplifies the calculation of Fourier coefficients.

  – Even Functions
  – Odd Functions
  – Half-Wave
  – Even Quarter-Wave
  – Odd Quarter-Wave
  – Hidden

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Fourier Coefficients of Even Functions
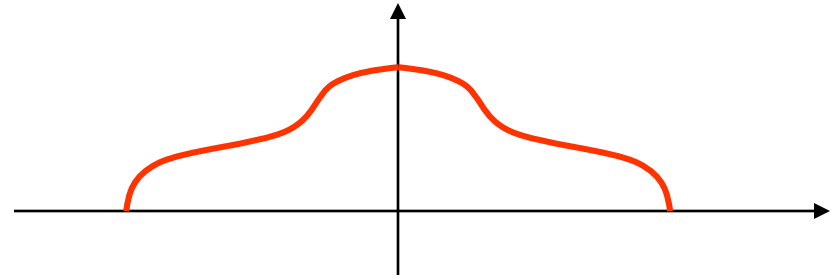
$$f(t) = f(-t)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos(n\omega_0 t) dt$$

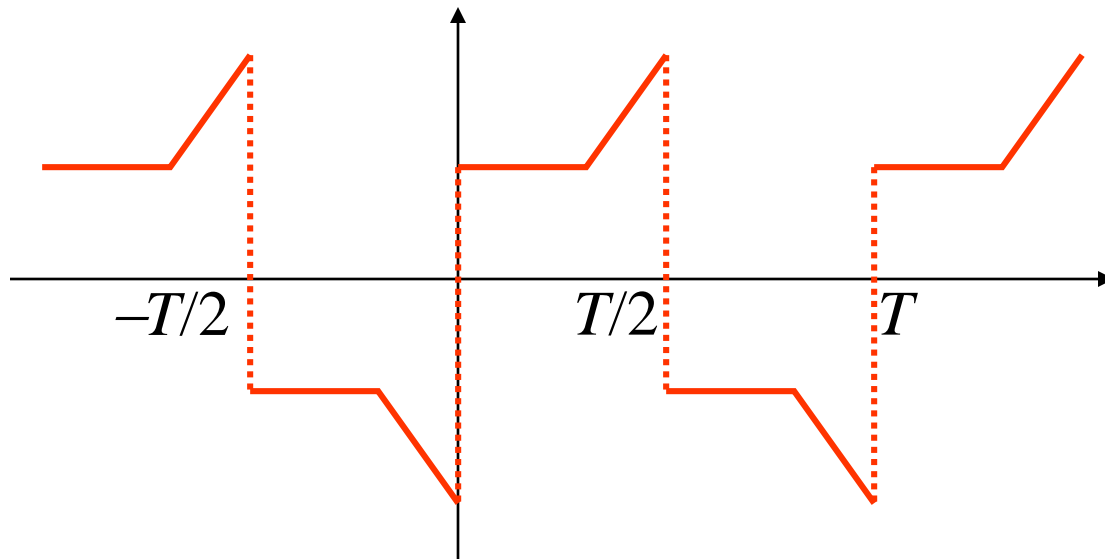# Fourier Coefficients of Even Functions
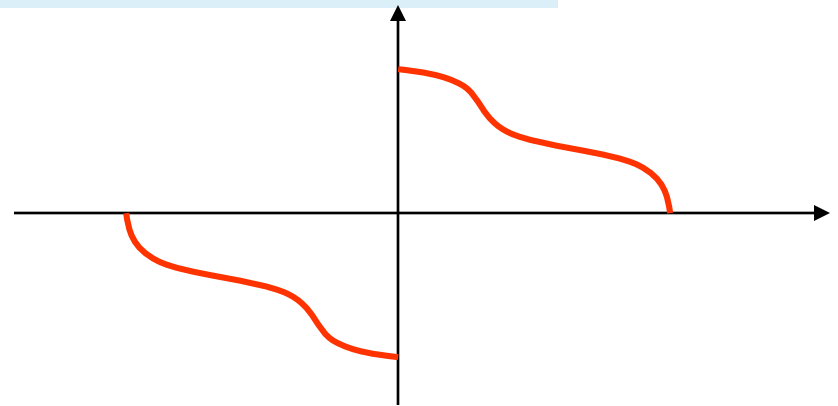
$$f(t) = -f(-t)$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin(n\omega_0 t)dt$$

Fourier Coefficients for Half-Wave Symmetry

$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f(t+T/2)$$

The Fourier series contains only odd harmonics.

Fourier Coefficients for Half-Wave Symmetry

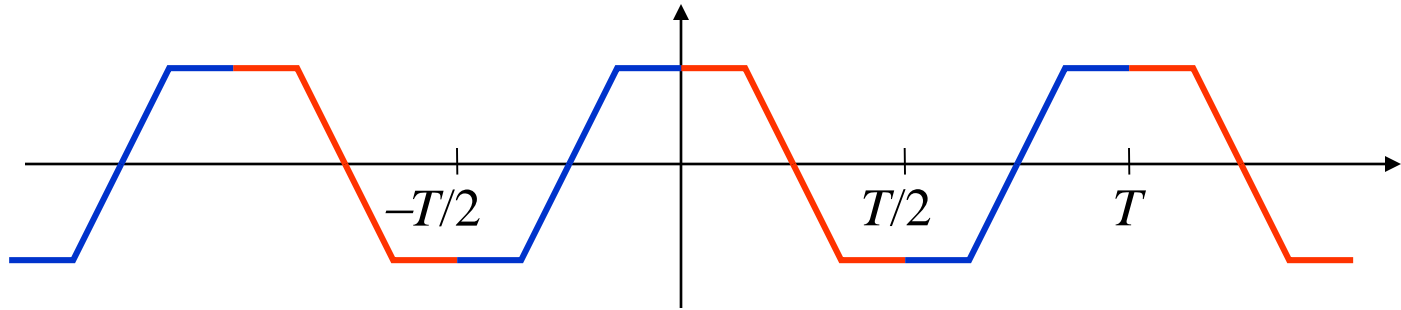$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t+T/2\right)$$

$$f(t) = \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{T} \displaystyle\int_0^{T/2} f(t)\cos(n\omega_0 t)dt & \text{for } n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{T} \displaystyle\int_0^{T/2} f(t)\sin(n\omega_0 t)dt & \text{for } n \text{ odd} \end{cases}$$

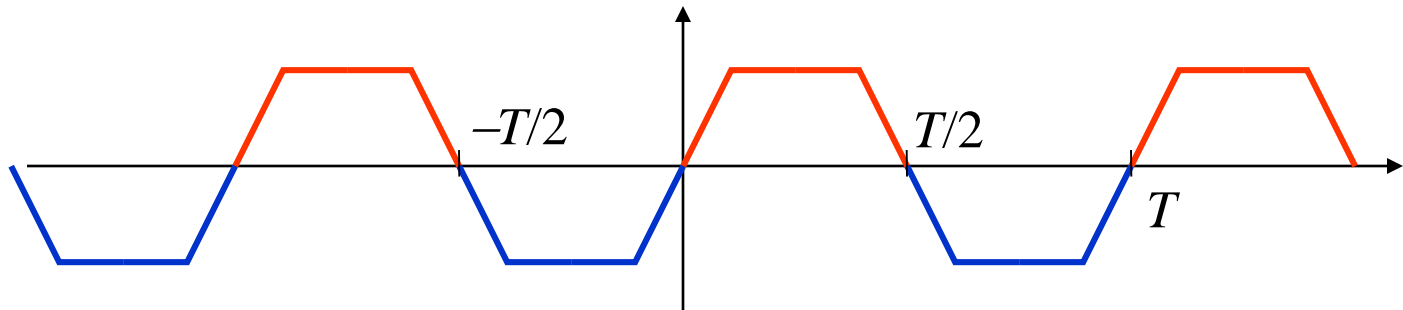# Fourier Coefficients for Even Quarter-Wave Symmetry

$$f(t) = \sum_{n=1}^{\infty} a_{2n-1} \cos[(2n-1)\omega_0 t]$$

$$a_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \cos[(2n-1)\omega_0 t] dt$$

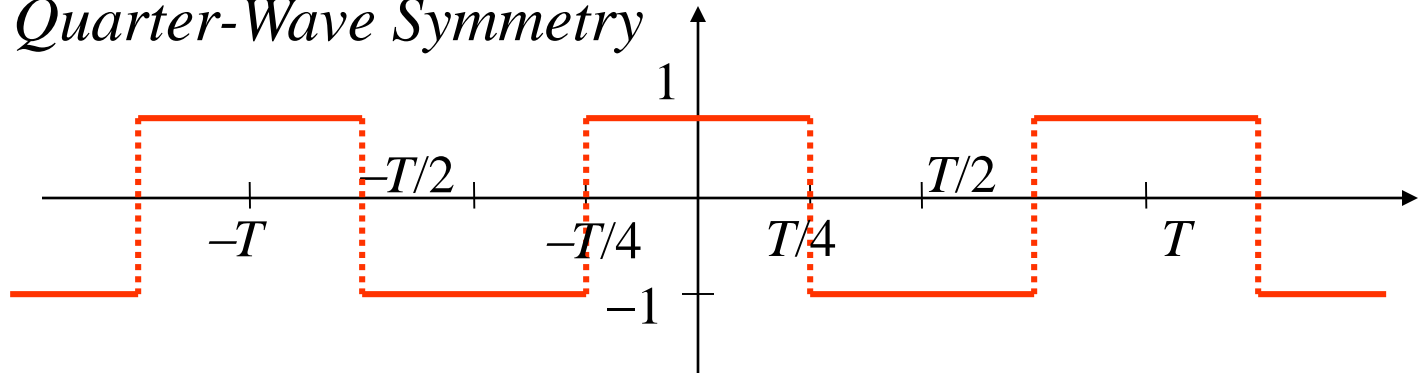# Fourier Coefficients for Odd Quarter-Wave Symmetry

$$f(t) = \sum_{n=1}^{\infty} b_{2n-1} \sin[(2n-1)\omega_0 t]$$

$$b_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \sin[(2n-1)\omega_0 t] dt$$

Example

Even Quarter-Wave Symmetry

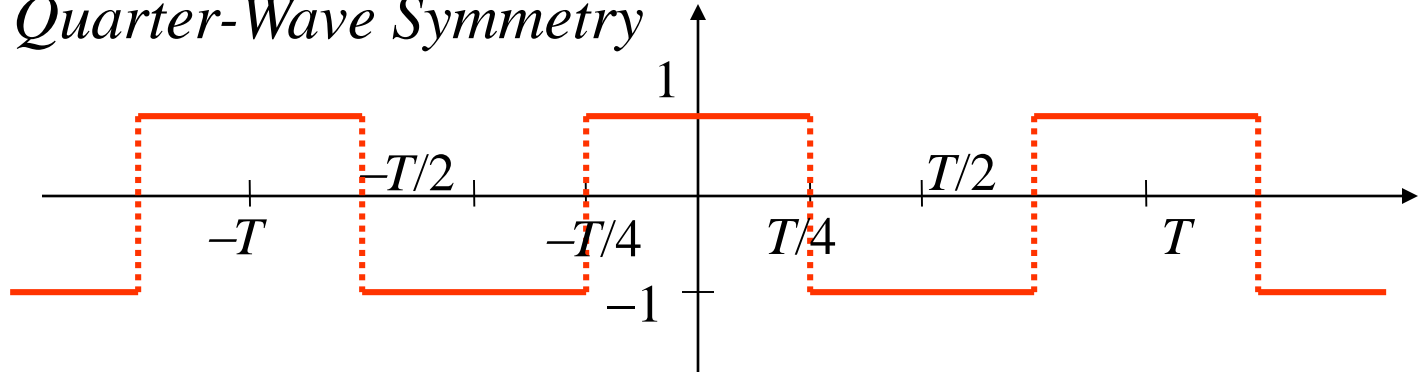$$a_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \cos[(2n-1)\omega_0 t]\, dt = \frac{8}{T} \int_0^{T/4} \cos[(2n-1)\omega_0 t]\, dt$$

$$= \frac{8}{(2n-1)\omega_0 T} \sin[(2n-1)\omega_0 t]\Big|_0^{T/4} = (-1)^{n-1}\frac{4}{(2n-1)\pi}$$

Example

$$f(t) = \frac{4}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t + \cdots\right)$$
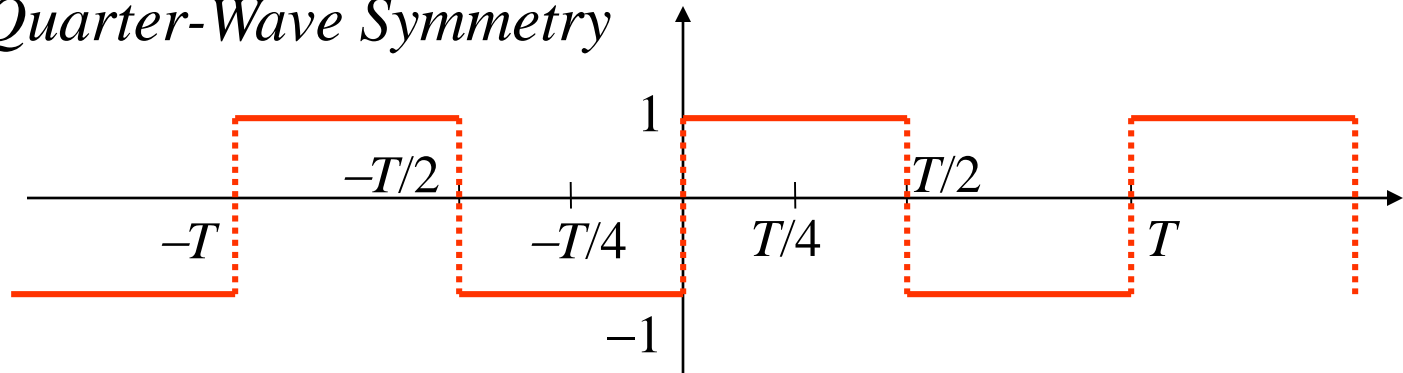
Even Quarter-Wave Symmetry

$$a_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\cos[(2n-1)\omega_0 t]\,dt = \frac{8}{T}\int_0^{T/4}\cos[(2n-1)\omega_0 t]\,dt$$

$$= \frac{8}{(2n-1)\omega_0 T}\sin[(2n-1)\omega_0 t]\Big|_0^{T/4} = (-1)^{n-1}\frac{4}{(2n-1)\pi}$$
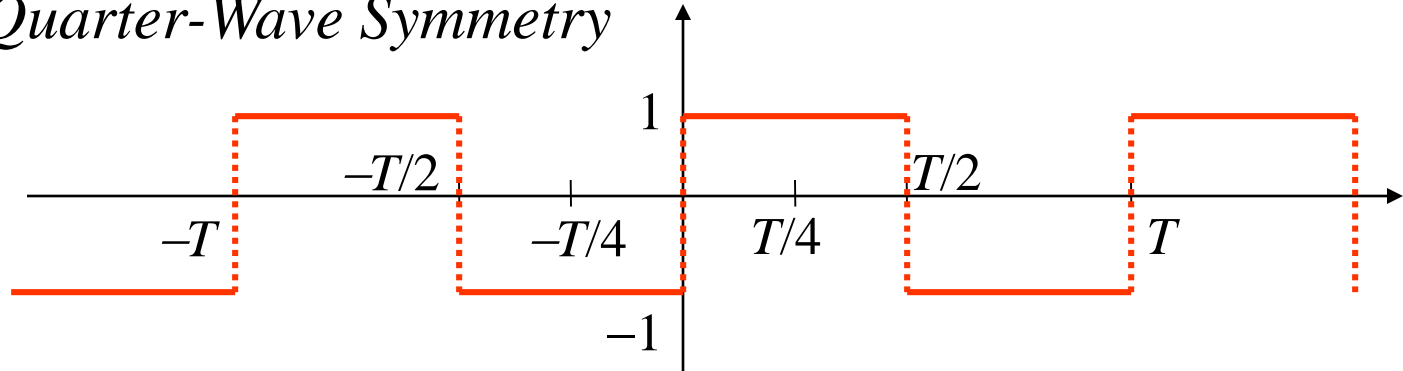
Example

Odd Quarter-Wave Symmetry

$$b_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\sin[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4}\sin[(2n-1)\omega_0 t]dt$$

$$= \frac{-8}{(2n-1)\omega_0 T}\cos[(2n-1)\omega_0 t]\Big|_0^{T/4} = \frac{4}{(2n-1)\pi}$$

$$f(t) = \frac{4}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right)$$
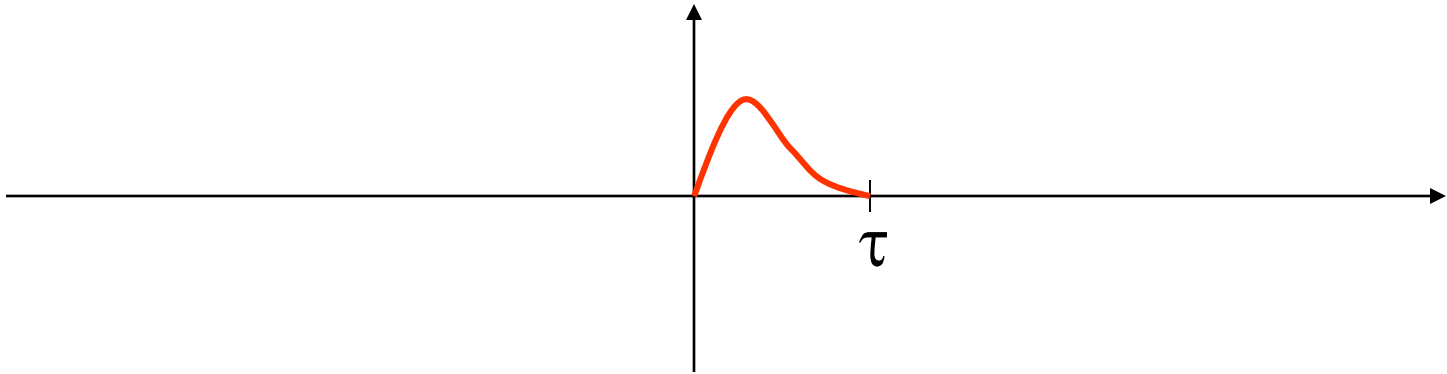
Example

Odd Quarter-Wave Symmetry

$$b_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\sin[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4}\sin[(2n-1)\omega_0 t]dt$$

$$= \frac{-8}{(2n-1)\omega_0 T}\cos[(2n-1)\omega_0 t]\Big|_0^{T/4} = \frac{4}{(2n-1)\pi}$$
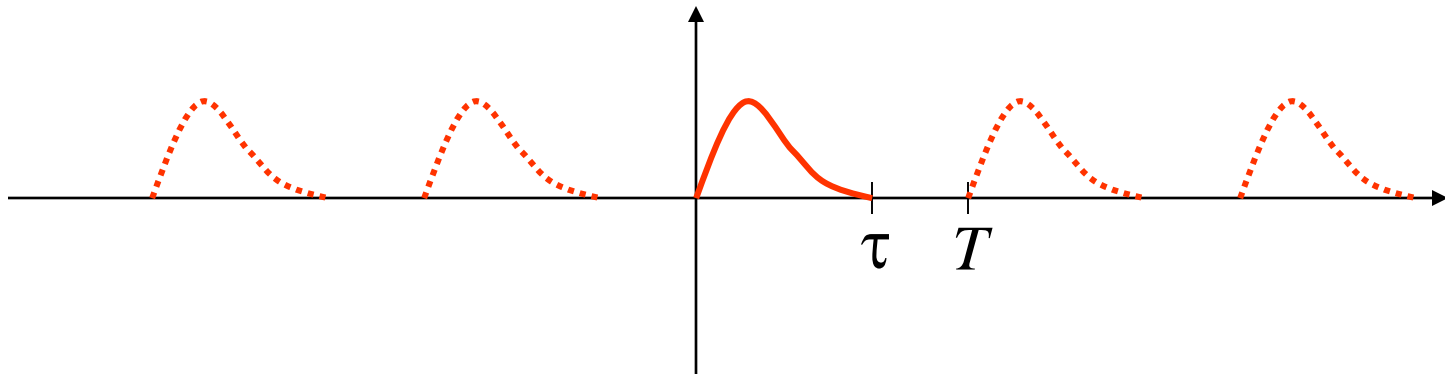
Fourier Series

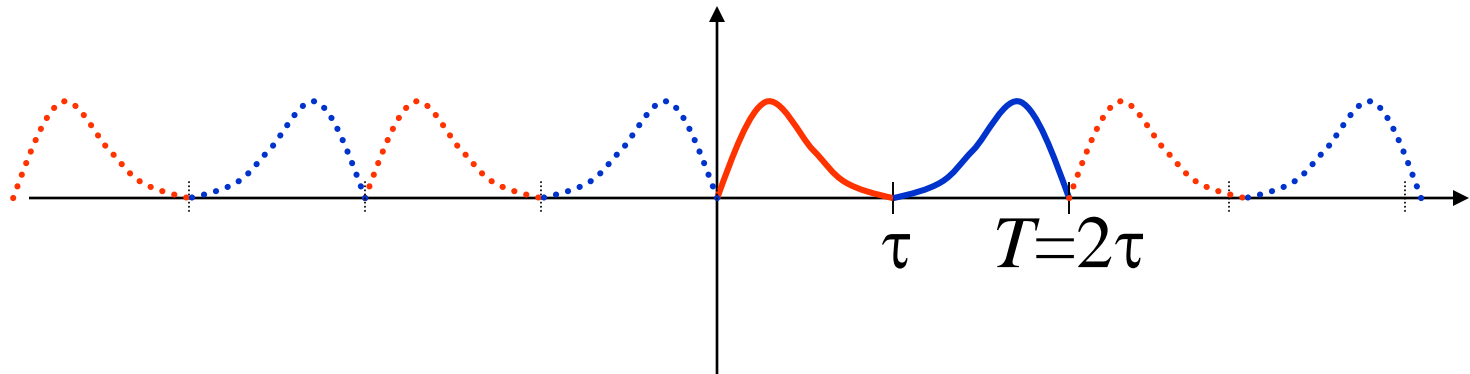## Half-Range Expansions

Non-Periodic Function Representation

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Without Considering Symmetry

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Symmetry

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.
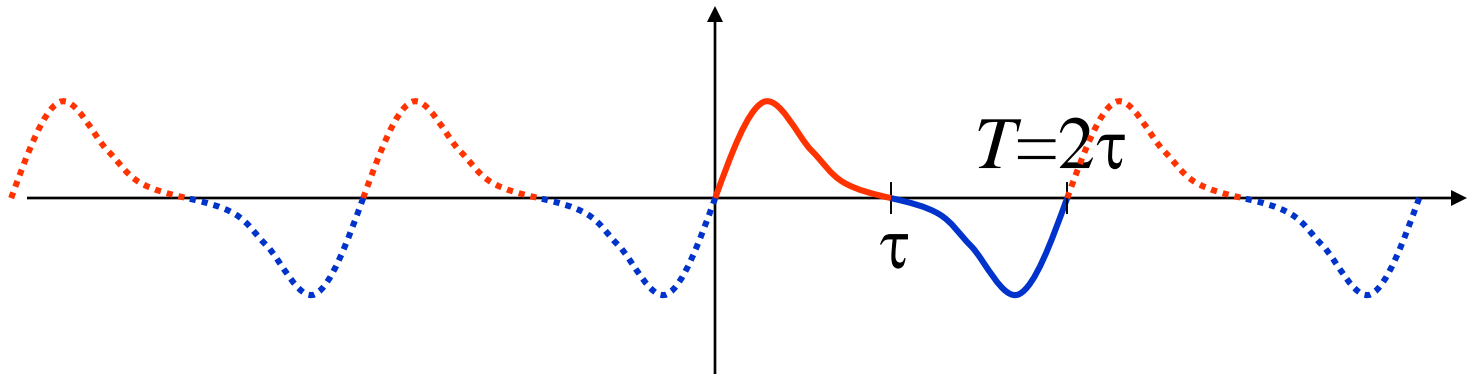
Expansion Into Odd Symmetry

$T=2\tau$

τ

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Half-Wave Symmetry

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Quarter-Wave Symmetry

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.
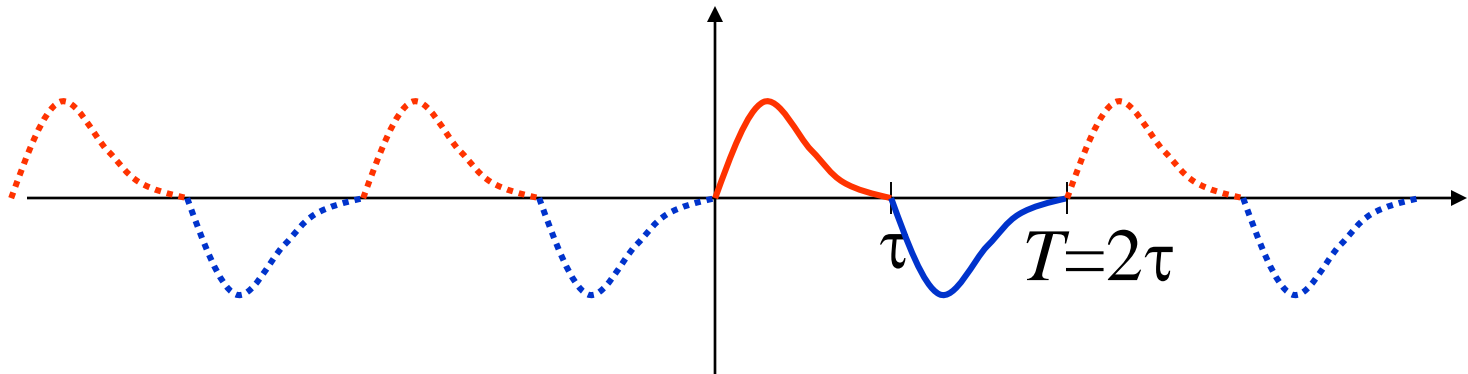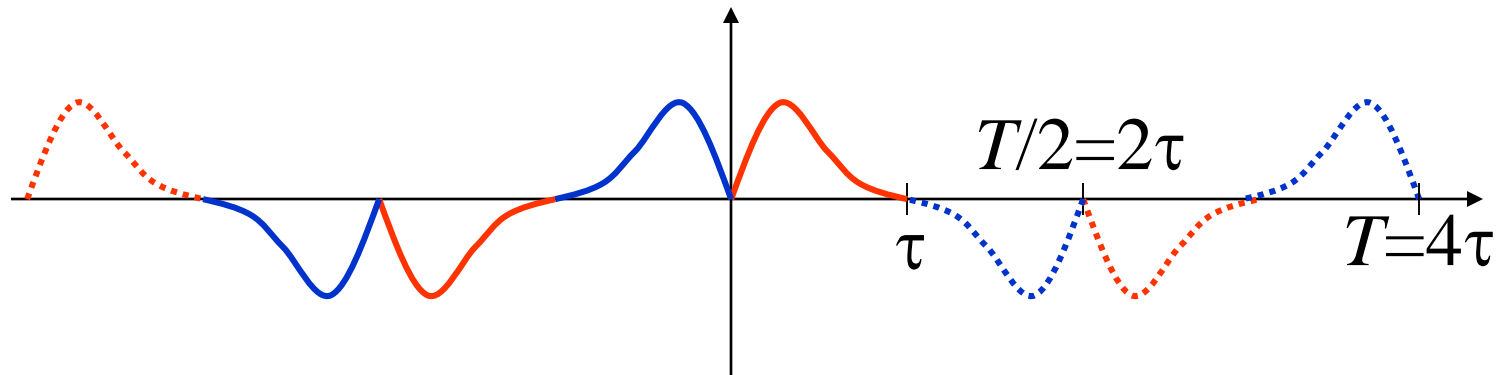
Expansion Into Odd Quarter-Wave Symmetry
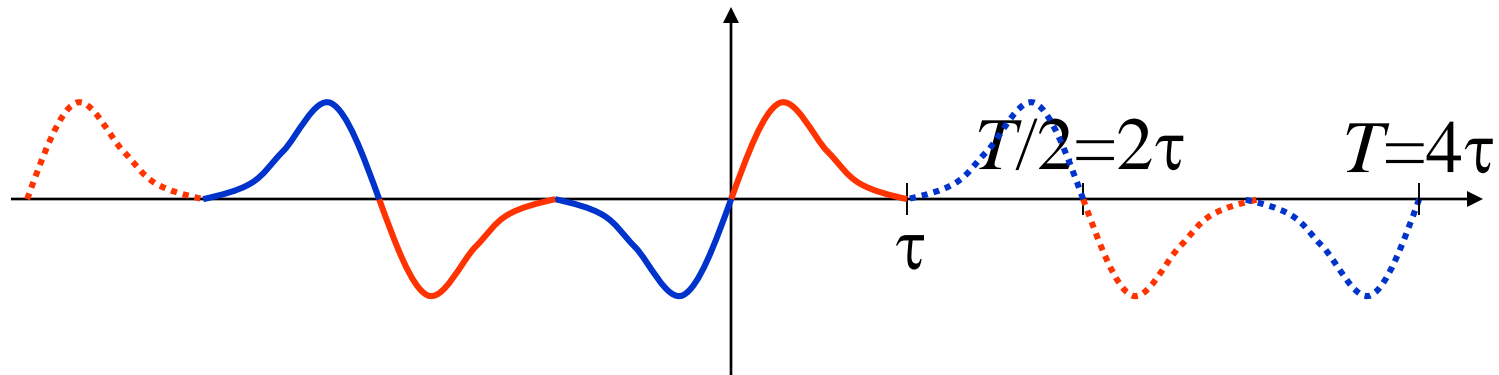

- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Fourier Series

Least Mean-Square Error Approximation

Approximation a function

Use $\quad S_N(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{N} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right)$

to represent *f(t)* on interval *−T/2 < t < T/2*.

Define $e_N(t) = f(t) - S_N(t)$

$$\overline{\varepsilon}_N^2 = \frac{1}{T} \int_{-T/2}^{T/2} \left[e_N(t) \right]^2 dt$$

Mean-Square Error

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Approximation a function

Show that using $S_N(t)$ to represent $f(t)$ has least mean-square property

$$\bar{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}\left(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)\right]^2 dt$$

Proven by setting $\partial\bar{\varepsilon}_N^2/\partial a_i = 0$ and $\partial\bar{\varepsilon}_N^2/\partial b_i = 0$.

Approximation a function

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}\left(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)\right]^2 dt$$

$$\frac{\partial\overline{\varepsilon}_N^2}{\partial a_0} = \frac{a_0}{2} - \frac{1}{T}\int_{-T/2}^{T/2}f(t)dt = 0$$

$$\frac{\partial\overline{\varepsilon}_N^2}{\partial a_n} = a_n - \frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos n\omega_0 t\,dt = 0$$

$$\frac{\partial\overline{\varepsilon}_N^2}{\partial b_n} = b_n - \frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin n\omega_0 t\,dt = 0$$

Mean-Square Error

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}\left(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)\right]^2 dt$$

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\,dt - \frac{a_0^2}{4} - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2)$$

Mean-Square Error

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}\left(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)\right]^2 dt$$

$$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\,dt \geq \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)$$

Mean-Square Error

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}\left(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)\right]^2 dt$$

$$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\,dt = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\lim_{N\to\infty}\overline{\varepsilon}_N^2 = \lim_{N\to\infty}\left\|e_N(t)\right\|^2 = 0$$