Sum of squares

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1 Motzkin polynomial

The form

$$p = z^6 - 3x^2y^2z^2 + x^2y^4 + x^4y^2 \in \mathbb{R}[x, y, z]_6$$

is globally nonnegative (by the arithmetic-geometric mean inequality [1]), but it is easily seen by looking at its Newton polytope that it is not a sum of squares of polynomials (SOS). Let us perturb p with a term of the form $\varepsilon(x^2 + y^2 + z^2)^3$, where ε is an unknown parameter. This gives

$$\begin{aligned} p_\varepsilon &= p + \varepsilon(x^2 + y^2 + z^2)^3 \\ &= \varepsilon x^6 + (1 + 3\varepsilon)x^4y^2 + (1 + 3\varepsilon)x^2y^4 + \varepsilon y^6 + 3\varepsilon x^4z^2 + 3\varepsilon x^2z^4 + 3\varepsilon y^4z^2 + 3\varepsilon y^2z^4 + (6\varepsilon - 3)x^2y^2z^2 + (1 + \varepsilon)z^6. \end{aligned}$$

Now, p_ε is SOS if and only if $p_\varepsilon = q_1^2 + \cdots + q_r^2$, with $q_i \in \mathbb{R}[x, y, z]_3$, and moreover, such that

$$2\mathrm{NP}(q_i) = \mathrm{NP}(p_\varepsilon) = \mathrm{conv}(\{(6, 0, 0), (4, 2, 0), (2, 4, 0), (0, 6, 0), (4, 0, 2), (2, 0, 4), (0, 4, 2), (0, 2, 4), (2, 2, 2), (0, 0, 6)\}).$$

In other words, we can choose the q_i satisfying

$$\begin{aligned} \mathrm{NP}(q_i) &= \mathrm{conv}(\{(3, 0, 0), (2, 1, 0), (1, 2, 0), (0, 3, 0), (2, 0, 1), (1, 0, 2), (0, 2, 1), (0, 1, 2), (1, 1, 1), (0, 0, 3)\}) = \\ &= \mathrm{conv}(\{(3, 0, 0), (1, 2, 0), (0, 3, 0), (2, 0, 1), (0, 2, 1), (0, 0, 3)\}). \end{aligned}$$

because

$$\begin{aligned} (1, 1, 1) &= \frac{1}{3}(3, 0, 0) + \frac{1}{3}(0, 3, 0) + \frac{1}{3}(0, 0, 3) \\ (2, 1, 0) &= \frac{1}{2}(3, 0, 0) + \frac{1}{2}(1, 2, 0) \\ (0, 1, 2) &= \frac{1}{2}(0, 0, 3) + \frac{1}{2}(0, 2, 1) \\ (1, 0, 2) &= \frac{1}{2}(0, 0, 3) + \frac{1}{2}(2, 0, 1). \end{aligned}$$

Consider now the monomial basis

$$\beta = (x^3, xy^2, y^3, x^2z, y^2z, z^3).$$

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One has p_ε SOS if and only if there exists a matrix $W = (w_{a,b})_{a,b\in\beta} \in \mathbb{S}_6(\mathbb{R})$, such that $W \succeq 0$, and that $p_\varepsilon = \beta^T W \beta$. This yields the following 6×6 parametric LMI to be solved:

$$W = \begin{bmatrix} w_{300,300} & w_{300,120} & w_{300,030} & w_{300,201} & w_{300,021} & w_{300,003} \\ \star & w_{120,120} & w_{120,030} & w_{120,201} & w_{120,021} & w_{120,003} \\ \star & \star & w_{030,030} & w_{030,201} & w_{030,021} & w_{030,003} \\ \star & \star & \star & w_{201,201} & w_{201,021} & w_{201,003} \\ \star & \star & \star & \star & w_{021,021} & w_{021,003} \\ \star & \star & \star & \star & \star & w_{003,003} \end{bmatrix} \succeq 0$$

$$p_\varepsilon = \beta^T W \beta \qquad \Leftarrow \text{ linear conditions on } W.$$

After reducing variables according to linear equations, we get the following parametric LMI:

$$\begin{bmatrix} \varepsilon & \frac{1}{2}+\frac{3}{2}\varepsilon & 0 & 0 & w_{300,021} & 0 \\ \star & 1+3\varepsilon & 0 & -w_{300,021} & 0 & 0 \\ \star & \star & \varepsilon & 0 & 0 & 0 \\ \star & \star & \star & 3\varepsilon & 3\varepsilon-\frac{3}{2} & \frac{3}{2}\varepsilon \\ \star & \star & \star & \star & 3\varepsilon & \frac{3}{2}\varepsilon \\ \star & \star & \star & \star & \star & 1+\varepsilon \end{bmatrix} \succeq 0.$$

For $\varepsilon = 1$, one has the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & w_{300,021} & 0 \\ 2 & 4 & 0 & -w_{300,021} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -w_{300,021} & 0 & 3 & \frac{3}{2} & \frac{3}{2} \\ w_{300,021} & 0 & 0 & \frac{3}{2} & 3 & \frac{3}{2} \\ 0 & 0 & 0 & \frac{3}{2} & \frac{3}{2} & 2 \end{bmatrix}$$

For $w_{300,021} = 0$, one gets the psd matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & \frac{3}{2} & 3 & \frac{3}{2} \\ 0 & 0 & 0 & \frac{3}{2} & \frac{3}{2} & 2 \end{bmatrix}$$

and from the Cholesky decompositions

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 3 & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{3}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

one gets the SOS decomposition of p_1:

$$
\begin{aligned}
p_1 &= p + (x^2+y^2+z^2)^3 = \\
&= \begin{bmatrix} x^3 & xy^2 & y^3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^3 \\ xy^2 \\ y^3 \end{bmatrix} + \begin{bmatrix} x^2z & y^2z & z^3 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{3}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^2z \\ y^2z \\ z^3 \end{bmatrix} = \\
&= (x^3+2xy^2)^2 + y^6 + \left(\sqrt{3}x^2z + \frac{\sqrt{3}}{2}y^2z + \frac{\sqrt{3}}{2}z^3\right)^2 + \left(\frac{3}{2}y^2z + \frac{1}{2}z^3\right)^2 + z^6.
\end{aligned}
$$

In the above example, we destroyed the sparsity of the Motzkin polynomial. Of course, one could cheat, by using the information that the bad monomial in the Motzkin polynomial is $-3x^2y^2z^2$ and simply perturbing along this direction:

$$p_\varepsilon = p + \varepsilon x^2y^2z^2 = z^6 + (\varepsilon - 3)x^2y^2z^2 + x^2y^4 + x^4y^2$$

and one directly sees that p_ε is SOS if and only if $\varepsilon \geq 3$. Indeed, whenever $\varepsilon < 3$, using Newton polytope arguments one gets that p_ε cannot be SOS (indeed, this is true if and only if $\varepsilon - 3$ is SOS in $\mathbb{R}$, thus nonnegative). In this case, the parametric LMI is diagonal (with monomial basis (z^3, xyz, xy^2, x^2y) for the SOS summands):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon - 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \succeq 0.$$

2 Robinson polynomial

We recall a polynomial of Robinson [2]

$$f_R = x^6 + y^6 + z^6 - x^4y^2 - x^4z^2 - y^4x^2 - y^4z^2 - z^4x^2 - z^4y^2 + 3x^2y^2z^2$$

which is non-negative, but not a sum of squares. Consider the perturbation

$$\begin{aligned} f &= (1-\varepsilon)f_R + \varepsilon(x^2+y^2+z^2)^3 \\ &= (1+\varepsilon)(x^6+y^6+z^6) + (-1+3\varepsilon)(x^4y^2+x^4z^2+y^4x^2+y^4z^2+z^4x^2+z^4y^2) + (3+6\varepsilon)x^2y^2z^2. \end{aligned}$$

We would like to check if f admits a sum of squares decomposition with respect to the monomial basis

$$\beta^T = (x^3, xy^2, y^3, x^2z, y^2z, z^3),$$

which is equivalent to checking if there exists $W \in \mathbb{S}_6(\mathbb{R})$ such that

$$W \succeq 0, \quad f = \beta^T W \beta.$$

Note that we can drop the constraint by writing $W = (x_{ij})$, and substituting the linear conditions $f = \beta^T W \beta$ into W. By doing so, we obtain a linear matrix inequality $A \succeq 0$, where

$$A = \begin{pmatrix} 1 & -\frac{1}{2}+2\varepsilon & 0 & 0 & -x_{24} & 0 \\ -\frac{1}{2}+2\varepsilon & -1+4\varepsilon & 0 & x_{24} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_{24} & 0 & -1+4\varepsilon & \frac{3}{2}+\frac{3}{2}\varepsilon & -\frac{1}{2}+2\varepsilon \\ -x_{24} & 0 & 0 & \frac{3}{2}+\frac{3}{2}\varepsilon & -1+4\varepsilon & -\frac{1}{2}+2\varepsilon \\ 0 & 0 & 0 & -\frac{1}{2}+2\varepsilon & -\frac{1}{2}+2\varepsilon & 1 \end{pmatrix}.$$

For $\varepsilon = \frac{19}{16}$, we can show that $A(\varepsilon, x_{24}) \succeq 0 \iff x_{24} = 0$, where

$$A(\varepsilon, x_{24}) = \begin{pmatrix} 1 & \frac{15}{8} & 0 & 0 & 0 & 0 \\ \frac{15}{8} & \frac{15}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{15}{4} & \frac{105}{32} & \frac{15}{8} \\ 0 & 0 & 0 & \frac{105}{32} & \frac{15}{4} & \frac{15}{8} \\ 0 & 0 & 0 & \frac{15}{8} & \frac{15}{8} & 1 \end{pmatrix},$$

from which we easily obtain a sum of squares decomposition

$$-\frac{3}{16}f_R + \frac{19}{16}(x^2+y^2+z^2)^3 = \left(x^3 + \frac{15xy^2}{8}\right)^2 + \frac{15}{4}\left(x^2z + \frac{7y^2z}{8} + \frac{z^3}{2}\right)^2 + \frac{15}{64}\left(xy^2\right)^2 + \left(y^3\right)^2 + \frac{225}{256}\left(y^2z + \frac{4z^3}{15}\right)^2.$$

References

[1] T. S. Motzkin. The arithmetic-geometric inequality. *Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965)*, 205:54, 1967.

[2] R. M. Robinson. Some definite polynomials which are not sums of squares of real polynomials. *Selected questions of algebra and logic*, pages 264–282, 1973.