MAP 531: Homework

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Problem 1: Estimating parameters of a Poisson distribution

We recall that the Poisson distribution with parameter $\theta > 0$ has a pdf given by $(p(\theta, k), k \in \mathbb{N})$ w.r.t the counting measure on $\mathbb{N}$:

$$p(\theta, k) = e^{-\theta}\frac{\theta^k}{k!}$$

Question 1

The poisson distribution is a discrete distribution since it has a countable number of possible values ($\mathbb{N}$).

In statistics, we use this distribution to compute the probability of a given number of (rare) events in a time period.

For example a poisson distribution can model:

- The number of patients arriving in an emergency room between 9 and 10am.
- The number of network failures per day.
- In quality control, the number of manufacturing defect.

Question 2

We assume that $\mathbb{X}$ follows a Poisson distribution with parameter $\theta > 0$.

We will use the fact that $e^{\theta} = \sum_{i=0}^{\infty}(\frac{\theta^i}{i!}), \forall \theta \in \mathbb{R}$

$$\mathbb{E}[\mathbb{X}] = \sum_{i=0}^{\infty}(i * p(\theta, i)) = \sum_{i=0}^{\infty}(i * e^{-\theta}\frac{\theta^i}{i!}) = \theta * e^{-\theta}\sum_{i=1}^{\infty}(\frac{\theta^{i-1}}{(i-1)!}) = \theta * e^{-\theta}\sum_{i=0}^{\infty}(\frac{\theta^i}{i!}) = \theta * e^{-\theta} * e^{\theta} = \theta$$

$$\mathbb{E}[\mathbb{X}^2] = \sum_{i=0}^{\infty}(i^2 * p(\theta, i)) = \sum_{i=0}^{\infty}(i^2 * e^{-\theta}\frac{\theta^i}{i!}) = \theta * e^{-\theta}\sum_{i=1}^{\infty}(i\frac{\theta^{i-1}}{(i-1)!}) = \theta * e^{-\theta}\sum_{i=0}^{\infty}((i+1)\frac{\theta^i}{i!})$$
$$= \theta * e^{-\theta}[\sum_{i=0}^{\infty}(i\frac{\theta^i}{i!}) + \sum_{i=0}^{\infty}(\frac{\theta^i}{i!})] = \theta * e^{-\theta}[\theta\sum_{i=0}^{\infty}(\frac{\theta^i}{i!}) + e^{\theta}] = \theta * e^{-\theta}[\theta * e^{\theta} + e^{\theta}] = \theta(\theta + 1)$$

$$\mathbb{V}(\mathbb{X}) = \mathbb{E}[\mathbb{X}^2] - \mathbb{E}[\mathbb{X}]^2 = \theta(\theta + 1) - \theta^2 = \theta$$

Question 3

We are provided with n independent observations of a Poisson random variable of parameter $\theta \in \Theta = \mathbb{R}_+^*$.
Our observations are $X_k \sim Pois(\theta), \forall k \in 1, ..., n$.
The corresponding statistical model is:

$$\mathcal{M}^n = (\mathbb{N}^n, \mathcal{P}(\mathbb{N}^n),\ \{\mathbb{P}_\theta^n,\ \theta \in \Theta\})$$

with $\mathbb{P}_\theta^n = \mathbb{P}_\theta \otimes ... \otimes \mathbb{P}_\theta$ (n times)
We are trying to estimate the parameter θ.

Question 4

The likelihood function is the function on θ that makes our n observations most likely.

Using the independance of the X_k:

$$l(\theta) = \prod_{k=1}^{n} e^{-\theta} \frac{\theta^{X_k}}{X_k!}$$

$$L(\theta) = log(l(\theta)) = \sum_{k=1}^{n}(-\theta + X_k log(\theta) - log(X_k!)) = -n\theta + log(\theta)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} log(X_k!)$$

By derivating with respect to θ, we have:

$$L'(\theta) = -n + \frac{\sum_{k=1}^{n} X_k}{\theta}$$
$$L''(\theta) = -\frac{\sum_{k=1}^{n} X_k}{\theta^2} < 0$$

Since, the second derivative of the log-likelihood function is negative, the function is concave and admits a global maximum given by:

$$L'(\theta) = 0 \Leftrightarrow -n + \frac{\sum_{k=1}^{n} X_k}{\theta} = 0 \Leftrightarrow \hat{\theta}_{MLE} = \overline{X}$$

So, the maximum likelihood estimator is:

$$\hat{\theta}_{MLE} = \overline{X}$$

Question 5

Since the X_k are iid, we have that:

$$\mathbb{E}[\overline{X}] = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[X_k] = \mathbb{E}[X_1] = \theta$$

$$\mathbb{V}(\overline{X}) = \frac{1}{n^2}\sum_{k=1}^{n} \mathbb{V}(X_k) = \frac{1}{n}\mathbb{V}[X_1] = \frac{\theta}{n}$$

Applying the central limit theorem, we have that $\sqrt{n}(\hat{\theta}_{MLE} - \theta)$ converges towards a Gaussian $\mathcal{N}(0, \theta)$.

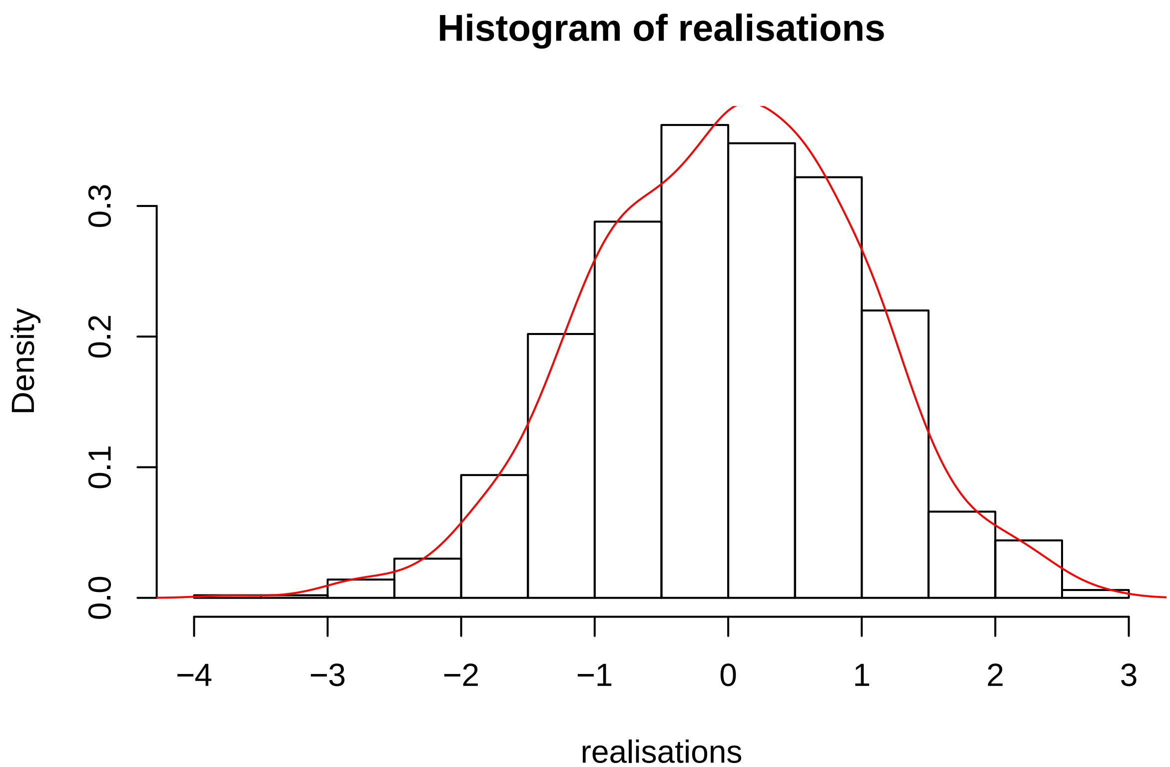
Question 6

The weak law of large numbers gives us that $\hat{\theta}_{MLE}$ converges in probability towards θ.
By continuous mapping, $\sqrt{\hat{\theta}_{MLE}}$ converges in probability towards $\sqrt{\theta}$. Then, by Slutsky's theorem, we have that $\sqrt{n}\frac{(\hat{\theta}_{MLE}-\theta)}{\sqrt{\hat{\theta}_{MLE}}}$ converges in law towards a gaussian $\mathcal{N}(0, 1)$.

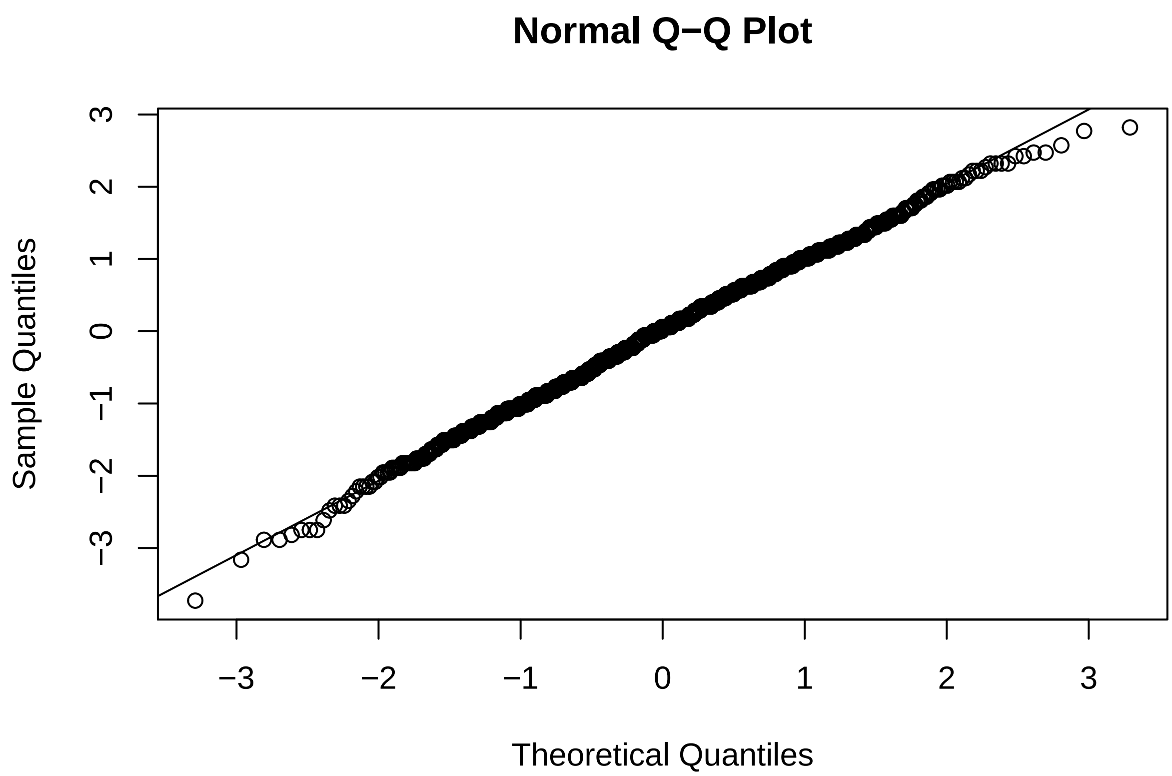
Let's check this result in R by simulating 1000 times our random variable $\sqrt{n}\frac{(\hat{\theta}_{MLE}-\theta)}{\sqrt{\hat{\theta}_{MLE}}}$ with a sample size of 100:

```r
estim <- function(x, theta){
  n <- length(x)
  est <- sqrt(n) * (mean(x) - theta) / sqrt(mean(x))
  return(est)}
```

```r
set.seed(42)
Nattempts = 1e3
nsample = 100
theta = 3

samples <- lapply(1:Nattempts, function(i) rpois(nsample, theta))
realisations <- sapply(samples, function(x) estim(x, theta))

hist(realisations, probability = TRUE)
d = density(realisations, kernel='gaussian')
lines(d, col = 'red')
```




```r
qqnorm(realisations)
qqline(realisations)
```



This confirms what we found theoretically: the random variable $\sqrt{n}\frac{(\hat{\theta}_{MLE}-\theta)}{\sqrt{\hat{\theta}_{MLE}}}$ follows a standard gaussian distribution.

Question 7

Let $Z_n = \sqrt{n}\frac{(\hat{\theta}_{MLE}-\theta)}{\sqrt{\hat{\theta}_{MLE}}}$ be our random variable.

Denote z_{alpha} the α-quantile for the standard Normal distribution for $\alpha \in (0,\ 1)$.

$$\lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2} \leq Z_n \leq z_{1-\alpha/2}) \geq 1-\alpha \Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2}\sqrt{\frac{\hat{\theta}_{MLE}}{n}} \leq \hat{\theta}_{MLE}-\theta \leq z_{1-\alpha/2}\sqrt{\frac{\hat{\theta}_{MLE}}{n}}) \geq 1-\alpha$$

For $\alpha \in (0,1)$, an asymptotic confidence interval for θ of level α is therefore :

$$[\hat{\theta}_{MLE} - z_{1-\alpha/2}\frac{\sqrt{\hat{\theta}_{MLE}}}{\sqrt{n}};\ \hat{\theta}_{MLE} + z_{1-\alpha/2}\frac{\sqrt{\hat{\theta}_{MLE}}}{\sqrt{n}}]$$

Question 8

We apply the δ-method with $g(x) = 2\sqrt{x}$
We have: $g'(x) = \frac{1}{\sqrt{x}}$
So,

$$\sqrt{n}(g(\hat{\theta}_{MLE}) - g(\theta)) \xrightarrow{d} \mathcal{N}(0,\ g'(\theta)^2 \times \theta) \Leftrightarrow \sqrt{n}(g(\hat{\theta}_{MLE}) - g(\theta)) \xrightarrow{d} \mathcal{N}(0,1)$$

Question 9

Let $W_n = \sqrt{n}(2\sqrt{\hat{\theta}_{MLE}} - 2\sqrt{\theta})$ be our random variable.

We know by the last question that $W_n \xrightarrow{d} \mathcal{N}(0,1)$.

$$\lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2} \leq W_n \leq z_{1-\alpha/2}) \geq 1-\alpha \Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(-\frac{z_{1-\alpha/2}}{2\sqrt{n}} \leq \sqrt{\hat{\theta}_{MLE}} - \sqrt{\theta} \leq \frac{z_{1-\alpha/2}}{2\sqrt{n}}) \geq 1-\alpha$$

$$\Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(\sqrt{\hat{\theta}_{MLE}} - \frac{z_{1-\alpha/2}}{2\sqrt{n}} \leq \sqrt{\theta} \leq \sqrt{\hat{\theta}_{MLE}} + \frac{z_{1-\alpha/2}}{2\sqrt{n}}) \geq 1-\alpha$$

Since all the quantities in the inequalities are positive:

$$\Leftrightarrow \lim_{n\to+\infty} \mathbb{P}((\sqrt{\hat{\theta}_{MLE}} - \frac{z_{1-\alpha/2}}{2\sqrt{n}})^2 \leq \theta \leq (\sqrt{\hat{\theta}_{MLE}} + \frac{z_{1-\alpha/2}}{2\sqrt{n}})^2) \geq 1-\alpha$$

For $\alpha \in (0,1)$, an asymptotic confidence interval for θ of level α is therefore:

$$[(\sqrt{\hat{\theta}_{MLE}} - \frac{z_{1-\alpha/2}}{2\sqrt{n}})^2;\ (\sqrt{\hat{\theta}_{MLE}} + \frac{z_{1-\alpha/2}}{2\sqrt{n}})^2]$$

Question 10

Based on the first moment of a poisson distribution, we easily have that:

$$\hat{\theta}_{MME} = \overline{X}$$

We can remark that $\hat{\theta}_{MME} = \hat{\theta}_{MLE}$

Based on the second moment of a poisson distribution, we have:

$$n^{-1}\sum_{k=1}^{n} X_k^2 = \hat{\theta}_2(\hat{\theta}_2 + 1)$$

Let's define the function $h(x) = x(x+1)$
Its inverse on $\mathbb{R}_+^*$ is $h^{-1}(x) = \frac{1}{2}[-1+\sqrt{4x+1}]$ and this gives us:

$$\hat{\theta}_2 = \frac{1}{2}[-1+\sqrt{(4n^{-1}\sum_{k=1}^{n} X_k^2) + 1}]$$

Question 11

$\mathbb{E}[\hat{\theta}_{MLE}] = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[X_k]$ by linearity of the expectation. So,

$$\mathbb{E}[\hat{\theta}_{MLE}] = \frac{1}{n} * n * \theta = \theta$$

Therefore, $\hat{\theta}_{MLE}$ is an unbiased estimator of θ, ie. $b_\theta{}^*(\hat{\theta}_{MLE}) = 0$

$\mathbb{V}(\hat{\theta}_{MLE}) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}(X_i)$ by independance of the X_k.

$$\mathbb{V}(\hat{\theta}_{MLE}) = \frac{1}{n^2} * n * \theta = \frac{\theta}{n}$$

The quadratic risk Q is:

$$Q = b_\theta{}^*(\hat{\theta}_{MLE})^2 + \mathbb{V}^*(\hat{\theta}_{MLE}) = 0 + \frac{\theta}{n} = \frac{\theta}{n}$$

Question 12

$\hat{\theta}_{MLE}$ is an unbiased estimator. So the Cramer-Rao bound is given by:

$$\frac{1}{I_n(\theta^*)} = \frac{1}{\mathbb{E}[-L''(\theta^*)]}$$

By derivating the log-likelihood function with respect to θ, we have:

$$L'(\theta^*) = -n + \frac{\sum_{i=1}^n X_k}{\theta}$$

$$-L''(\theta^*) = \frac{\sum_{i=1}^n X_k}{\theta^2}$$

Therefore,

$$\mathbb{E}[-L''(\theta^*)] = \frac{\sum_{i=1}^n \mathbb{E}[X_k]}{\theta^2} = \frac{n}{\theta}$$

Finally,

$$\frac{1}{I_n(\theta^*)} = \frac{\theta}{n} = \mathbb{V}(\hat{\theta}_{MLE})$$

We can conclude that our estimator $\hat{\theta}_{MLE}$ is efficient.

Question 13

$$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X_n})^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \theta + \theta - \overline{X_n})^2 = \frac{1}{n}\sum_{i=1}^n [(X_i - \theta)^2 + (\theta - \overline{X_n})^2 + 2(X_i - \theta)(\theta - \overline{X_n})]$$

$$= \frac{1}{n}\sum_{i=1}^n (X_i - \theta)^2 + (\theta - \overline{X_n})^2 + \frac{2}{n}(\theta - \overline{X_n})\sum_{i=1}^n (X_i - \theta) = \frac{1}{n}\sum_{i=1}^n (X_i - \theta)^2 + (\theta - \overline{X_n})^2 + 2(\theta - \overline{X_n})(\overline{X_n} - \theta)$$

$$= \frac{1}{n}\sum_{i=1}^n (X_i - \theta)^2 - (\theta - \overline{X_n})^2$$

Question 14

$$\mathbb{E}[(\theta - \overline{X_n})^2] = \mathbb{E}[\theta^2 - 2\theta\overline{X_n} + \overline{X_n}^2] = \theta^2 - 2\theta\mathbb{E}[\overline{X_n}] + \mathbb{E}[\overline{X_n^2}]$$
$$= -\theta^2 + \mathbb{V}(\overline{X_n}) + \mathbb{E}[\overline{X_n}^2] = -\theta^2 + \frac{\theta}{n} + \theta^2 = \frac{\theta}{n}$$

$$\mathbb{E}[\hat{\theta}_2] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^n (X_i - \theta)^2 - (\theta - \overline{X_n})^2] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[(X_i - \theta)^2] - \mathbb{E}[(\theta - \overline{X_n})^2]$$
$$= \frac{1}{n}\sum_{i=1}^n \mathbb{V}(X_i) - \frac{\theta}{n} = \theta(1 - \frac{1}{n})$$

Therefore the bias is:

$$b_{\hat{\theta}_2} = -\frac{\theta}{n}$$

We can get an unbiased estimator $\hat{\theta}_3$ by defining $\hat{\theta}_3 = (1 - \frac{1}{n})^{-1}\hat{\theta}_2$

Question 15

Using the previous questions, we know that:

$$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\theta)^2 - (\theta - \overline{X_n})^2$$

therefore, we have:

$$\sqrt{n}(\hat{\theta}_2 - \theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i-\theta)^2 - \sqrt{n}(\theta - \overline{X_n})^2 - \sqrt{n}\theta = \sqrt{n}(\overline{Y_n} - \theta) - \sqrt{n}(\theta - \overline{X_n})^2$$

where:

$$\forall i \in [\![1, n]\!], Yi = (X_i - \theta)^2$$

$$\bar{Y_n} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Since:

$$\mathbb{E}[Y_i] = \mathbb{V}(X_i) = \theta$$

and

$$\mathbb{V}(Y_i) = 2\theta^2 + \theta$$

We can apply the central limit theorem, and we have that $\sqrt{n}(\overline{Y_n} - \theta)$ converges towards a Gaussian $\mathcal{N}(0, 2\theta^2 + \theta)$.

We also have that:

$$\sqrt{n}(\theta - \overline{X_n})^2 = \sqrt{n}(\overline{X_n} - \theta)^2 = \sqrt{n}(\overline{X_n} - \theta)(\overline{X_n} - \theta)$$

Applying the central limit theorem, we have that $\sqrt{n}(\overline{X_n} - \theta)$ converges towards a Gaussian $\mathcal{N}(0, \theta)$.
On the other hand, applying the law of large numbers: $(\theta - \overline{X_n})$ converges in probability towards 0.

Applying Slutsky's theorem, $\sqrt{n}(\theta - \overline{X_n})^2$ converges in distribution towards the constant 0. Therefore, it converges in probability towards 0.

Now, we can apply Slutsky's theorem to $\sqrt{n}(\overline{Y_n} - \theta) - \sqrt{n}(\theta - \overline{X_n})^2$ which gives us finally that $\sqrt{n}(\hat{\theta}_2 - \theta)$ converges in distribution towards a Gaussian $\mathcal{N}(0, 2\theta^2 + \theta)$.

We can now compute an other asymptotic confidence interval centered in $\hat{\theta}_2$.

Let $V_n = \sqrt{n}(\hat{\theta}_2 - \theta)$ be our random variable.

We know by the last question that $\frac{V_n}{\sqrt{2\theta^2+\theta}} \xrightarrow{d} \mathcal{N}(0, 1)$.

$$\lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2} \leq \frac{V_n}{\sqrt{2\theta^2+\theta}} \leq z_{1-\alpha/2}) \geq 1-\alpha \Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(** \leq ** \leq **) \geq 1-\alpha$$

$$\Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(** \leq ** \leq **) \geq 1-\alpha$$

For $\alpha \in (0, 1)$, an asymptotic confidence interval for θ of level α is therefore:

$$[**;\ **]$$

Question 16

Let $s \in \mathbb{R}$. The probability generating function of the Poisson distribution is given by:

$$G_{\mathbb{X}}(s) = \mathbb{E}[exp(s\mathbb{X})] = \sum_{k=0}^{\infty} e^{ks} e^{-\theta} \frac{\theta^k}{k!} = e^{-\theta} \sum_{k=0}^{\infty} \frac{(\theta e^s)^k}{k!} = e^{-\theta} e^{\theta e^s} = e^{\theta(e^s - 1)}$$

In order to compute the first and second moment of the Poisson distribution, we can now use the moment generating function. Let's compute its first and second order derivatives.

$$G'_{\mathbb{X}}(s) = \theta e^s e^{\theta(e^s-1)}$$
$$G''_{\mathbb{X}}(s) = \theta[e^s e^{\theta(e^s-1)} + \theta e^{2s} e^{\theta(e^s-1)}] = \theta e^s [e^{\theta(e^s-1)} + \theta e^s e^{\theta(e^s-1)}]$$

Then, we have:

$$\mathbb{E}[\mathbb{X}] = G'_{\mathbb{X}}(0) = \theta$$
$$\mathbb{E}[\mathbb{X}^2] = G''_{\mathbb{X}}(0) = \theta(1+\theta)$$
$$\mathbb{V}(\mathbb{X}) = \mathbb{E}[\mathbb{X}^2] - \mathbb{E}[\mathbb{X}]^2 = \theta(1+\theta) - \theta^2 = \theta$$

We will now show that: $\mathbb{V}[(\mathbb{X}_i - \theta)^2] = 2\theta^2 + \theta$

$$G^{(3)}_{\mathbb{X}}(s) = (1 + 3\theta e^s + \theta^2 e^{2s})\theta e^{s+\theta(e^s-1)}$$
$$G^{(4)}_{\mathbb{X}}(s) = (1 + \theta^3 e^{3s} + 6\theta^2 e^{2s} + 7\theta e^s)\theta e^{s+\theta(e^s-1)}$$
$$\mathbb{V}[(\mathbb{X}_i - \theta)^2] = \mathbb{E}[(\mathbb{X} - \theta)^4] - \mathbb{E}[(\mathbb{X} - \theta)^2]^2 = ... = 2\theta^2 + \theta$$

Problem 2: Analysis of the USJudgeRatings dataset

This exercise is open. You are asked to use the tools we have seen together to analyze the USJudgeRatings data set. This data set is provided in the package datasets. Your analysis should be reported here and include:

- an introduction
- a general description of the data
- the use of descriptive statistics
- the use of all techniques we have seen together that might be relevant
- a conclusion

Overall, your analysis, including the graphs and the codes should not exceed 15 pages in pdf.

Introduction

We have to analyse a dataset, named USJudgeratings, containing various ratings of state judges in the US Superior Court. The different informations given help us to determine if a judge is worthy staying in the US Superior Court or not. To be continued…

General description

First, let's see how the dataset is organized.

```
data(USJudgeRatings)
```

The data is stored in a dataframe.

We can have a view of the dataset:

```
library(knitr)
kable(USJudgeRatings)
```

	CONT	INTG	DMNR	DILG	CFMG	DECI	PREP	FAMI	ORAL	WRIT	PHYS
AARONSON,L.H.	5.7	7.9	7.7	7.3	7.1	7.4	7.1	7.1	7.1	7.0	8.3
ALEXANDER,J.M.	6.8	8.9	8.8	8.5	7.8	8.1	8.0	8.0	7.8	7.9	8.5
ARMENTANO,A.J.	7.2	8.1	7.8	7.8	7.5	7.6	7.5	7.5	7.3	7.4	7.9
BERDON,R.I.	6.8	8.8	8.5	8.8	8.3	8.5	8.7	8.7	8.4	8.5	8.8
BRACKEN,J.J.	7.3	6.4	4.3	6.5	6.0	6.2	5.7	5.7	5.1	5.3	5.5
BURNS,E.B.	6.2	8.8	8.7	8.5	7.9	8.0	8.1	8.0	8.0	8.0	8.6
CALLAHAN,R.J.	10.6	9.0	8.9	8.7	8.5	8.5	8.5	8.5	8.6	8.4	9.1
COHEN,S.S.	7.0	5.9	4.9	5.1	5.4	5.9	4.8	5.1	4.7	4.9	6.8
DALY,J.J.	7.3	8.9	8.9	8.7	8.6	8.5	8.4	8.4	8.4	8.5	8.8
DANNEHY,J.F.	8.2	7.9	6.7	8.1	7.9	8.0	7.9	8.1	7.7	7.8	8.5
DEAN,H.H.	7.0	8.0	7.6	7.4	7.3	7.5	7.1	7.2	7.1	7.2	8.4
DEVITA,H.J.	6.5	8.0	7.6	7.2	7.0	7.1	6.9	7.0	7.0	7.1	6.9
DRISCOLL,P.J.	6.7	8.6	8.2	6.8	6.9	6.6	7.1	7.3	7.2	7.2	8.1
GRILLO,A.E.	7.0	7.5	6.4	6.8	6.5	7.0	6.6	6.8	6.3	6.6	6.2
HADDEN,W.L.JR.	6.5	8.1	8.0	8.0	7.9	8.0	7.9	7.8	7.8	7.8	8.4
HAMILL,E.C.	7.3	8.0	7.4	7.7	7.3	7.3	7.3	7.2	7.1	7.2	8.0
HEALEY.A.H.	8.0	7.6	6.6	7.2	6.5	6.5	6.8	6.7	6.4	6.5	6.9
HULL,T.C.	7.7	7.7	6.7	7.5	7.4	7.5	7.1	7.3	7.1	7.3	8.1
LEVINE,I.	8.3	8.2	7.4	7.8	7.7	7.7	7.7	7.8	7.5	7.6	8.0
LEVISTER,R.L.	9.6	6.9	5.7	6.6	6.9	6.6	6.2	6.0	5.8	5.8	7.2
MARTIN,L.F.	7.1	8.2	7.7	7.1	6.6	6.6	6.7	6.7	6.8	6.8	7.5
MCGRATH,J.F.	7.6	7.3	6.9	6.8	6.7	6.8	6.4	6.3	6.3	6.3	7.4
MIGNONE,A.F.	6.6	7.4	6.2	6.2	5.4	5.7	5.8	5.9	5.2	5.8	4.7
MISSAL,H.M.	6.2	8.3	8.1	7.7	7.4	7.3	7.3	7.3	7.2	7.3	7.8
MULVEY,H.M.	7.5	8.7	8.5	8.6	8.5	8.4	8.5	8.5	8.4	8.4	8.7
NARUK,H.J.	7.8	8.9	8.7	8.9	8.7	8.8	8.9	9.0	8.8	8.9	9.0
O'BRIEN,F.J.	7.1	8.5	8.3	8.0	7.9	7.9	7.8	7.8	7.8	7.7	8.3
O'SULLIVAN,T.J.	7.5	9.0	8.9	8.7	8.4	8.5	8.4	8.3	8.3	8.3	8.8
PASKEY,L.	7.5	8.1	7.7	8.2	8.0	8.1	8.2	8.4	8.0	8.1	8.4
RUBINOW,J.E.	7.1	9.2	9.0	9.0	8.4	8.6	9.1	9.1	8.9	9.0	8.9
SADEN.G.A.	6.6	7.4	6.9	8.4	8.0	7.9	8.2	8.4	7.7	7.9	8.4
SATANIELLO,A.G.	8.4	8.0	7.9	7.9	7.8	7.8	7.6	7.4	7.4	7.4	8.1
SHEA,D.M.	6.9	8.5	7.8	8.5	8.1	8.2	8.4	8.5	8.1	8.3	8.7
SHEA,J.F.JR.	7.3	8.9	8.8	8.7	8.4	8.5	8.5	8.5	8.4	8.4	8.8
SIDOR,W.J.	7.7	6.2	5.1	5.6	5.6	5.9	5.6	5.6	5.3	5.5	6.3
SPEZIALE,J.A.	8.5	8.3	8.1	8.3	8.4	8.2	8.2	8.1	7.9	8.0	8.0
SPONZO,M.J.	6.9	8.3	8.0	8.1	7.9	7.9	7.9	7.7	7.6	7.7	8.1
STAPLETON,J.F.	6.5	8.2	7.7	7.8	7.6	7.7	7.7	7.7	7.5	7.6	8.5
TESTO,R.J.	8.3	7.3	7.0	6.8	7.0	7.1	6.7	6.7	6.7	6.7	8.0
TIERNEY,W.L.JR.	8.3	8.2	7.8	8.3	8.4	8.3	7.7	7.6	7.5	7.7	8.1
WALL,R.A.	9.0	7.0	5.9	7.0	7.0	7.2	6.9	6.9	6.5	6.6	7.6
WRIGHT,D.B.	7.1	8.4	8.4	7.7	7.5	7.7	7.8	8.2	8.0	8.1	8.3

	CONT	INTG	DMNR	DILG	CFMG	DECI	PREP	FAMI	ORAL	WRIT	PHYS
ZARRILLI,K.J.	8.6	7.4	7.0	7.5	7.5	7.7	7.4	7.2	6.9	7.0	7.8

```
dim(USJudgeRatings)
```

```
## [1] 43 12
```

We are provided with n = 43 observations and p = 12 quantitative variables.

```
str(USJudgeRatings)
```

```
## 'data.frame':    43 obs. of  12 variables:
##  $ CONT: num  5.7 6.8 7.2 6.8 7.3 6.2 10.6 7 7.3 8.2 ...
##  $ INTG: num  7.9 8.9 8.1 8.8 6.4 8.8 9 5.9 8.9 7.9 ...
##  $ DMNR: num  7.7 8.8 7.8 8.5 4.3 8.7 8.9 4.9 8.9 6.7 ...
##  $ DILG: num  7.3 8.5 7.8 8.8 6.5 8.5 8.7 5.1 8.7 8.1 ...
##  $ CFMG: num  7.1 7.8 7.5 8.3 6 7.9 8.5 5.4 8.6 7.9 ...
##  $ DECI: num  7.4 8.1 7.6 8.5 6.2 8 8.5 5.9 8.5 8 ...
##  $ PREP: num  7.1 8 7.5 8.7 5.7 8.1 8.5 4.8 8.4 7.9 ...
##  $ FAMI: num  7.1 8 7.5 8.7 5.7 8 8.5 5.1 8.4 8.1 ...
##  $ ORAL: num  7.1 7.8 7.3 8.4 5.1 8 8.6 4.7 8.4 7.7 ...
##  $ WRIT: num  7 7.9 7.4 8.5 5.3 8 8.4 4.9 8.5 7.8 ...
##  $ PHYS: num  8.3 8.5 7.9 8.8 5.5 8.6 9.1 6.8 8.8 8.5 ...
##  $ RTEN: num  7.8 8.7 7.8 8.7 4.8 8.6 9 5 8.8 7.9 ...
```

The variables are all of numeric nature. An observation in this dataset represents the different ratings received by a judge (given by his name) in the US Superior Court.

```
colnames(USJudgeRatings)
```

```
##  [1] "CONT" "INTG" "DMNR" "DILG" "CFMG" "DECI" "PREP" "FAMI" "ORAL" "WRIT"
## [11] "PHYS" "RTEN"
```

The variables are:

- *CONT* : The number of contacts of lawyer with judge.
- *INTG* : The judicial integrity of the judge.
- *DMNR* : Demeanor of the judge.
- *DILG* : Diligence of the judge.
- *CFMG* : Case flow managed by the judge.
- *DECI* : Prompt decisions taken by the judge.
- *PREP* : How the judge is prepared trial.
- *FAMI* : His familiarity with law.
- *ORAL* : His sound oral rulings.
- *WRIT* : His sound written rulings.
- *PHYS* : His physical ability.
- *RTEN* : Scaling if the judge is worthy to retain in the US Superior court.

```
sum(is.na(USJudgeRatings))
```

```
## [1] 0
```

There are no missing values in the dataframe.

Descriptive dataset analysis

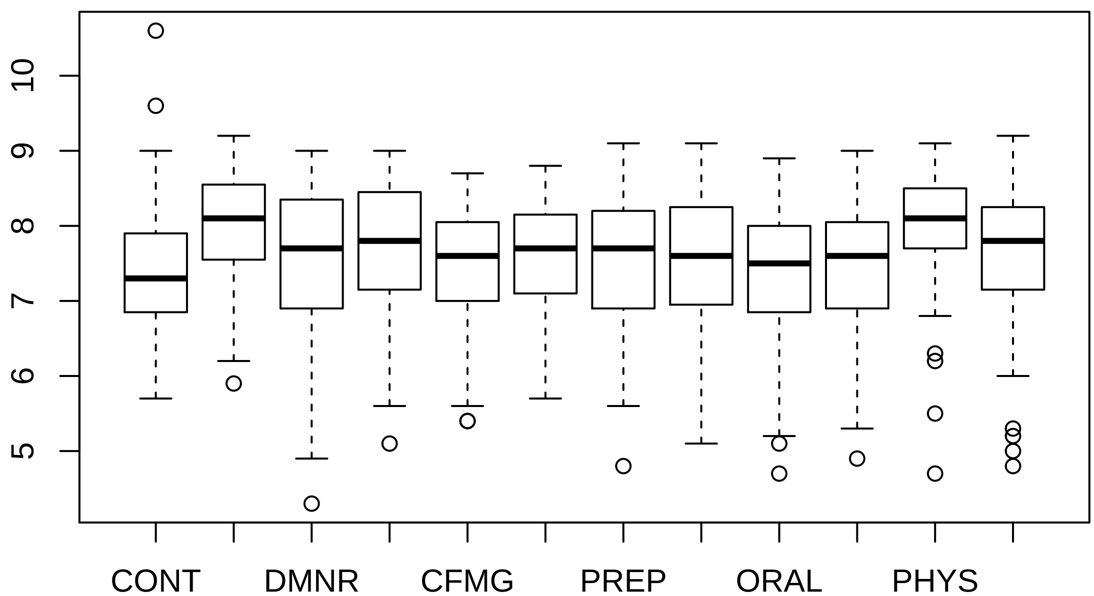
Now, let's take a deeper look on the dataset to analyze the different variables.

```r
summary(USJudgeRatings)
```

```
##       CONT             INTG            DMNR            DILG      
##  Min.   : 5.700   Min.   :5.900   Min.   :4.300   Min.   :5.100  
##  1st Qu.: 6.850   1st Qu.:7.550   1st Qu.:6.900   1st Qu.:7.150  
##  Median : 7.300   Median :8.100   Median :7.700   Median :7.800  
##  Mean   : 7.437   Mean   :8.021   Mean   :7.516   Mean   :7.693  
##  3rd Qu.: 7.900   3rd Qu.:8.550   3rd Qu.:8.350   3rd Qu.:8.450  
##  Max.   :10.600   Max.   :9.200   Max.   :9.000   Max.   :9.000  
##       CFMG            DECI            PREP            FAMI      
##  Min.   :5.400   Min.   :5.700   Min.   :4.800   Min.   :5.100  
##  1st Qu.:7.000   1st Qu.:7.100   1st Qu.:6.900   1st Qu.:6.950  
##  Median :7.600   Median :7.700   Median :7.700   Median :7.600  
##  Mean   :7.479   Mean   :7.565   Mean   :7.467   Mean   :7.488  
##  3rd Qu.:8.050   3rd Qu.:8.150   3rd Qu.:8.200   3rd Qu.:8.250  
##  Max.   :8.700   Max.   :8.800   Max.   :9.100   Max.   :9.100  
##       ORAL            WRIT            PHYS            RTEN      
##  Min.   :4.700   Min.   :4.900   Min.   :4.700   Min.   :4.800  
##  1st Qu.:6.850   1st Qu.:6.900   1st Qu.:7.700   1st Qu.:7.150  
##  Median :7.500   Median :7.600   Median :8.100   Median :7.800  
##  Mean   :7.293   Mean   :7.384   Mean   :7.935   Mean   :7.602  
##  3rd Qu.:8.000   3rd Qu.:8.050   3rd Qu.:8.500   3rd Qu.:8.250  
##  Max.   :8.900   Max.   :9.000   Max.   :9.100   Max.   :9.200  
```

All the variables (except the variable CONT) seem to be ranged between 0 and 10.
The last variable, RTEN, seems to conclude the analysis. In fact, it says if the lawyers think that the judge is worthy staying in the US Superior Court or not.
First, we can observe that each variable seems to follow a symetric distribution, since median and mean are always close. Are u sure? because sometimes the difference is big for values between 5 and 10.

```r
Outvals = boxplot(USJudgeRatings)
```



We observe the presence of outliers for 10 of the 12 variables (with larger values for CONT and with lower values for the other variables).

We can take a look at some outliers.

```r
max(USJudgeRatings$CONT)
```

```
## [1] 10.6
```

```r
rownames(USJudgeRatings)[which.max(USJudgeRatings$CONT)]
```

```
## [1] "CALLAHAN,R.J."
```

The judge with the biggest number of contacts of lawyer is judge Callahan with a a number of 10.6 contacts.

```r
min(USJudgeRatings$RTEN)
```

```
## [1] 4.8
```

```r
rownames(USJudgeRatings)[which.min(USJudgeRatings$RTEN)]
```

```
## [1] "BRACKEN,J.J."
```

The judge with the lowest rating for worthiness of retention is judge Bracken with a rating of 4.8.

```r
max(USJudgeRatings$RTEN)
```

```
## [1] 9.2
```

```r
rownames(USJudgeRatings)[which.max(USJudgeRatings$RTEN)]
```

```
## [1] "RUBINOW,J.E."
```

The judge with the highest rating for worthiness of retention is judge Rubinow with a rating of 9.2.
We are not provided with extra information and we cannot check wether the outliers correspond to mistakes. Thus, we will assume that they aren't mistakes.

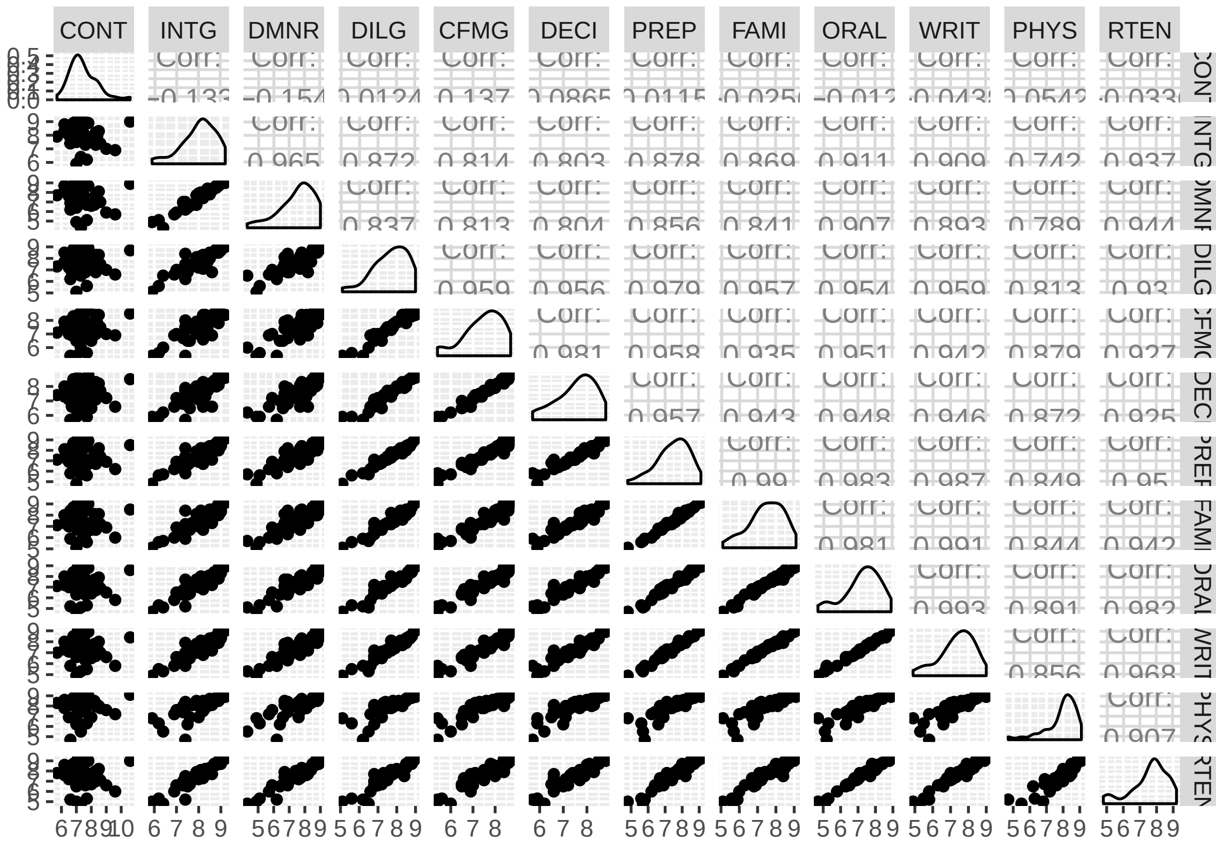
```r
library(ggplot2)
library(dplyr)
```

```
##
## Attaching package: 'dplyr'

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

```r
library(GGally)
```

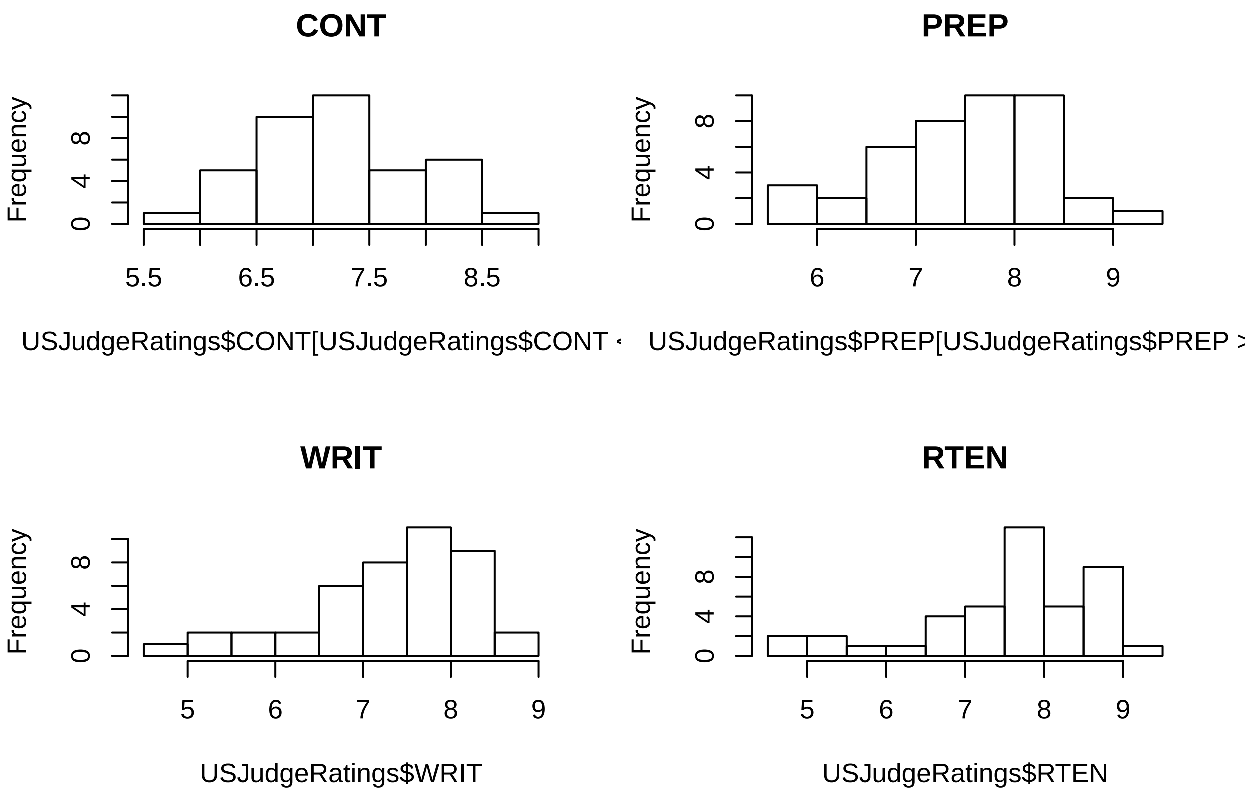
```
## Registered S3 method overwritten by 'GGally':
##   method from
##   +.gg   ggplot2

##
## Attaching package: 'GGally'

## The following object is masked from 'package:dplyr':
##
##     nasa
```

```r
ggpairs(USJudgeRatings)
```



Descriptive statistics analysis of the dataset

```
par(mfrow=c(2,2))
hist(USJudgeRatings$CONT[USJudgeRatings$CONT<9], main="CONT")
hist(USJudgeRatings$PREP[USJudgeRatings$PREP>5], main="PREP" )
hist(USJudgeRatings$WRIT, main="WRIT")
hist(USJudgeRatings$RTEN, main="RTEN")
```




```r
round(var(USJudgeRatings), 2)
```

```
##       CONT  INTG  DMNR DILG CFMG DECI PREP  FAMI  ORAL  WRIT PHYS  RTEN
## CONT  0.89 -0.10 -0.17 0.01 0.11 0.07 0.01 -0.02 -0.01 -0.04 0.05 -0.03
## INTG -0.10  0.59  0.85 0.60 0.54 0.50 0.64  0.64  0.71  0.67 0.54  0.79
## DMNR -0.17  0.85  1.31 0.86 0.80 0.74 0.93  0.91  1.05  0.98 0.85  1.19
## DILG  0.01  0.60  0.86 0.81 0.74 0.69 0.84  0.82  0.87  0.83 0.69  0.92
## CFMG  0.11  0.54  0.80 0.74 0.74 0.68 0.79  0.76  0.83  0.78 0.71  0.88
## DECI  0.07  0.50  0.74 0.69 0.68 0.64 0.73  0.72  0.77  0.73 0.66  0.82
## PREP  0.01  0.64  0.93 0.84 0.79 0.73 0.91  0.90  0.95  0.90 0.76  1.00
## FAMI -0.02  0.64  0.91 0.82 0.76 0.72 0.90  0.90  0.94  0.90 0.75  0.98
## ORAL -0.01  0.71  1.05 0.87 0.83 0.77 0.95  0.94  1.02  0.96 0.85  1.09
## WRIT -0.04  0.67  0.98 0.83 0.78 0.73 0.90  0.90  0.96  0.92 0.77  1.02
## PHYS  0.05  0.54  0.85 0.69 0.71 0.66 0.76  0.75  0.85  0.77 0.88  0.94
## RTEN -0.03  0.79  1.19 0.92 0.88 0.82 1.00  0.98  1.09  1.02 0.94  1.21
```

```r
round(sqrt(diag(var(USJudgeRatings))),2)
```

```
## CONT INTG DMNR DILG CFMG DECI PREP FAMI ORAL WRIT PHYS RTEN
## 0.94 0.77 1.14 0.90 0.86 0.80 0.95 0.95 1.01 0.96 0.94 1.10
```

```r
print('The smallest standard deviation is: ')
```

```
## [1] "The smallest standard deviation is: "
```

```r
min(round(sqrt(diag(var(USJudgeRatings))),2))
```

```
## [1] 0.77
```

```r
print('The largest standard deviation is: ')
```

```
## [1] "The largest standard deviation is: "
```

```r
max(round(sqrt(diag(var(USJudgeRatings))),2))
```

```
## [1] 1.14
```

Regarding the dispersion, we look at the interquartile range (given by the boxplots) and the empirical standard deviation. Overall, the dispersions are not very high (around 1). We find that the variables DMNR and RTEN have the largest standard deviation, while the DECI variable has the smallest.

Let's measure the correlations between the 11 first variables and the variable RTEN.

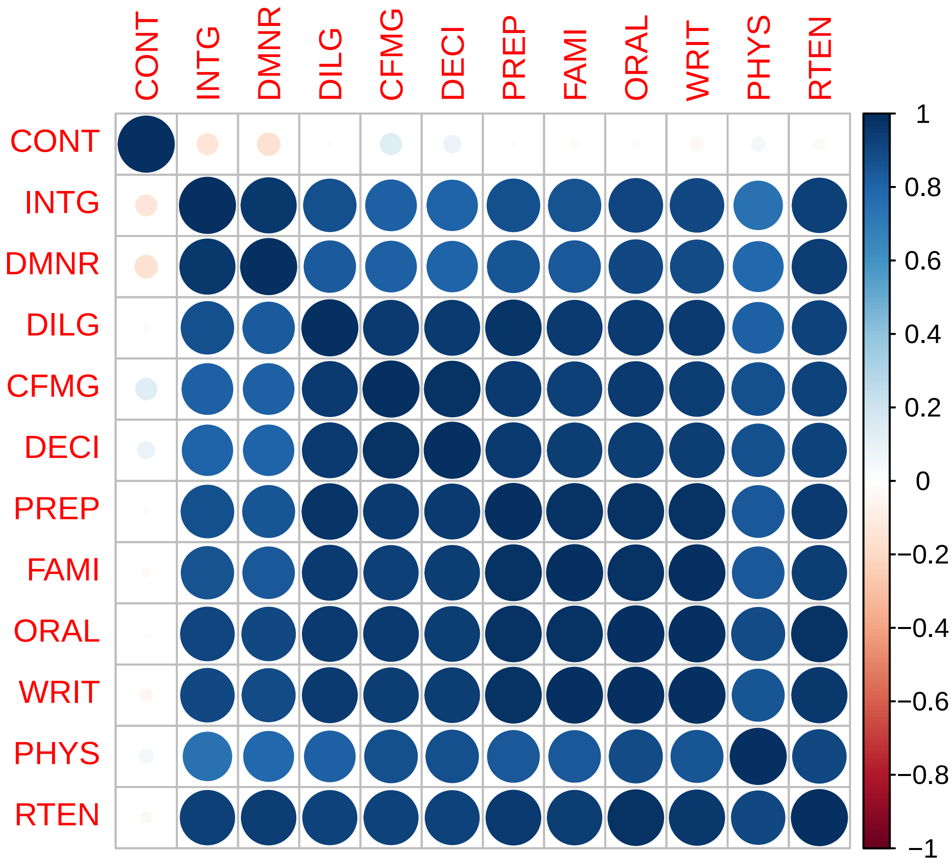
```r
round(cor(USJudgeRatings),2)
```

```
##       CONT  INTG  DMNR DILG CFMG DECI PREP  FAMI  ORAL  WRIT PHYS  RTEN
## CONT  1.00 -0.13 -0.15 0.01 0.14 0.09 0.01 -0.03 -0.01 -0.04 0.05 -0.03
## INTG -0.13  1.00  0.96 0.87 0.81 0.80 0.88  0.87  0.91  0.91 0.74  0.94
## DMNR -0.15  0.96  1.00 0.84 0.81 0.80 0.86  0.84  0.91  0.89 0.79  0.94
## DILG  0.01  0.87  0.84 1.00 0.96 0.96 0.98  0.96  0.95  0.96 0.81  0.93
## CFMG  0.14  0.81  0.81 0.96 1.00 0.98 0.96  0.94  0.95  0.94 0.88  0.93
## DECI  0.09  0.80  0.80 0.96 0.98 1.00 0.96  0.94  0.95  0.95 0.87  0.92
## PREP  0.01  0.88  0.86 0.98 0.96 0.96 1.00  0.99  0.98  0.99 0.85  0.95
## FAMI -0.03  0.87  0.84 0.96 0.94 0.94 0.99  1.00  0.98  0.99 0.84  0.94
## ORAL -0.01  0.91  0.91 0.95 0.95 0.95 0.98  0.98  1.00  0.99 0.89  0.98
## WRIT -0.04  0.91  0.89 0.96 0.94 0.95 0.99  0.99  0.99  1.00 0.86  0.97
## PHYS  0.05  0.74  0.79 0.81 0.88 0.87 0.85  0.84  0.89  0.86 1.00  0.91
## RTEN -0.03  0.94  0.94 0.93 0.93 0.92 0.95  0.94  0.98  0.97 0.91  1.00
```

```r
library(corrplot)
```

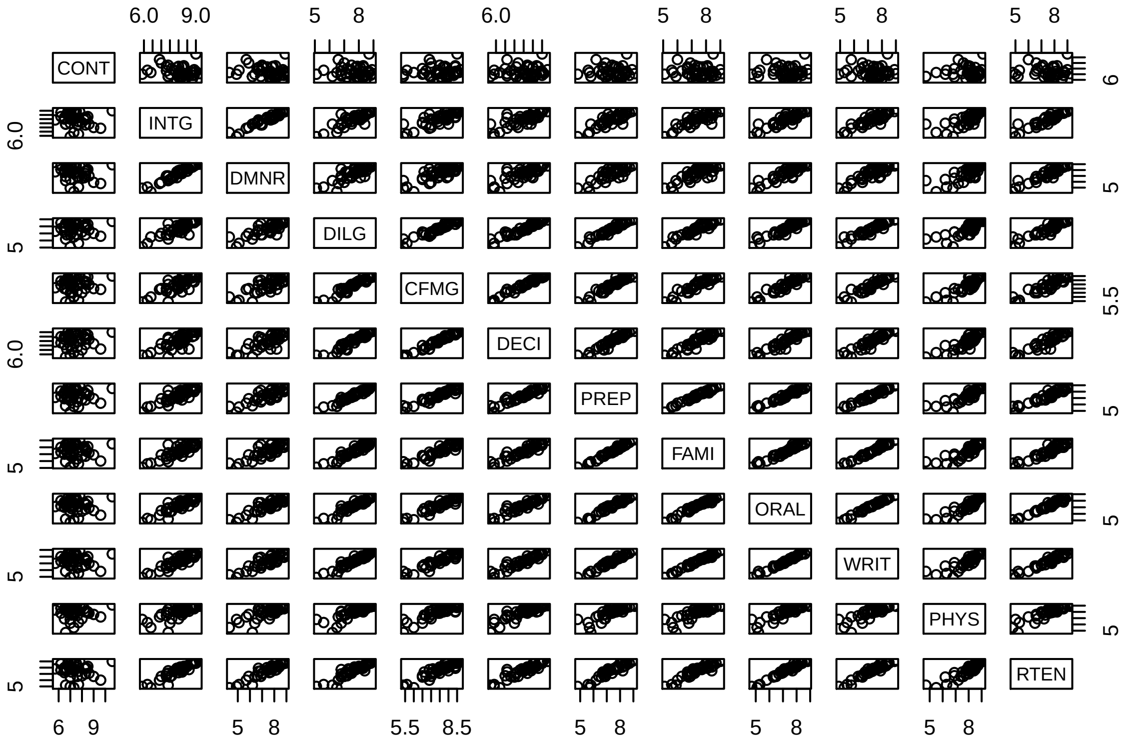
```
## corrplot 0.84 loaded
```

```r
corrplot(cor(USJudgeRatings))
```

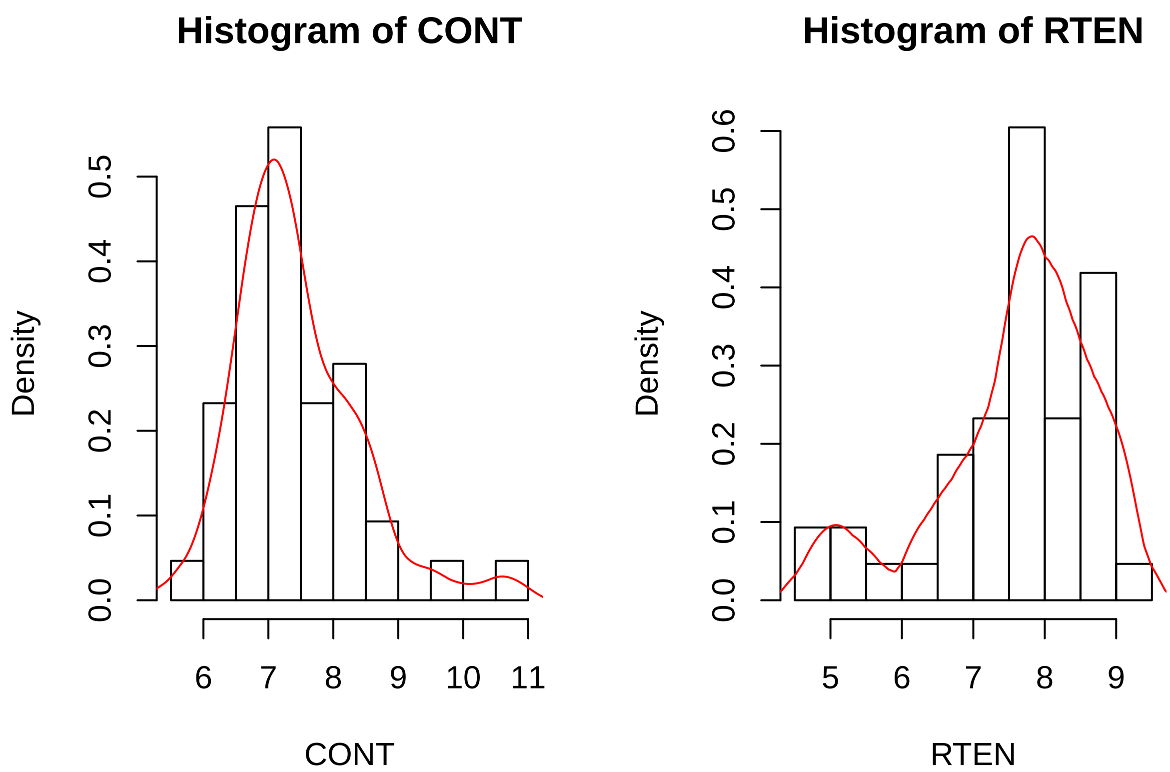


All the variables have a strong positive correlation two by two except the variable CONT which is not correlated to all the other variables. The number of contacts of a lawyer with the judge doesn't seem to explain the ratings received by the judge.

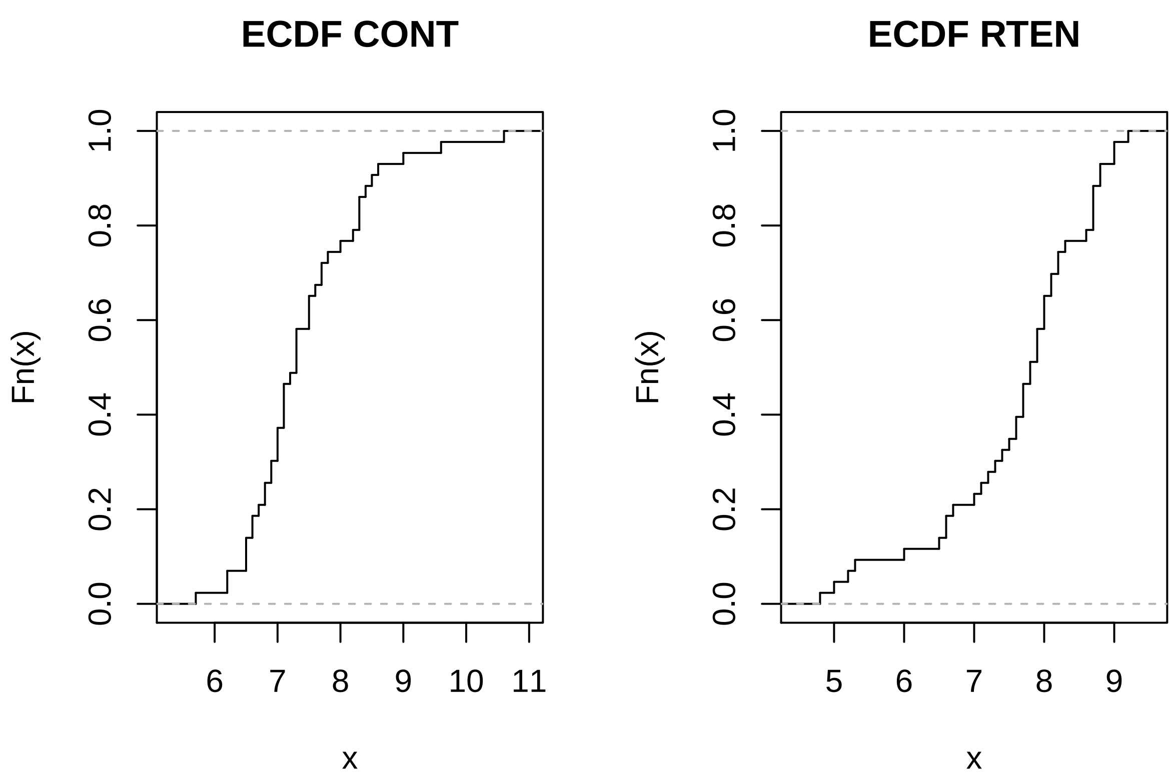
```
pairs(USJudgeRatings)
```



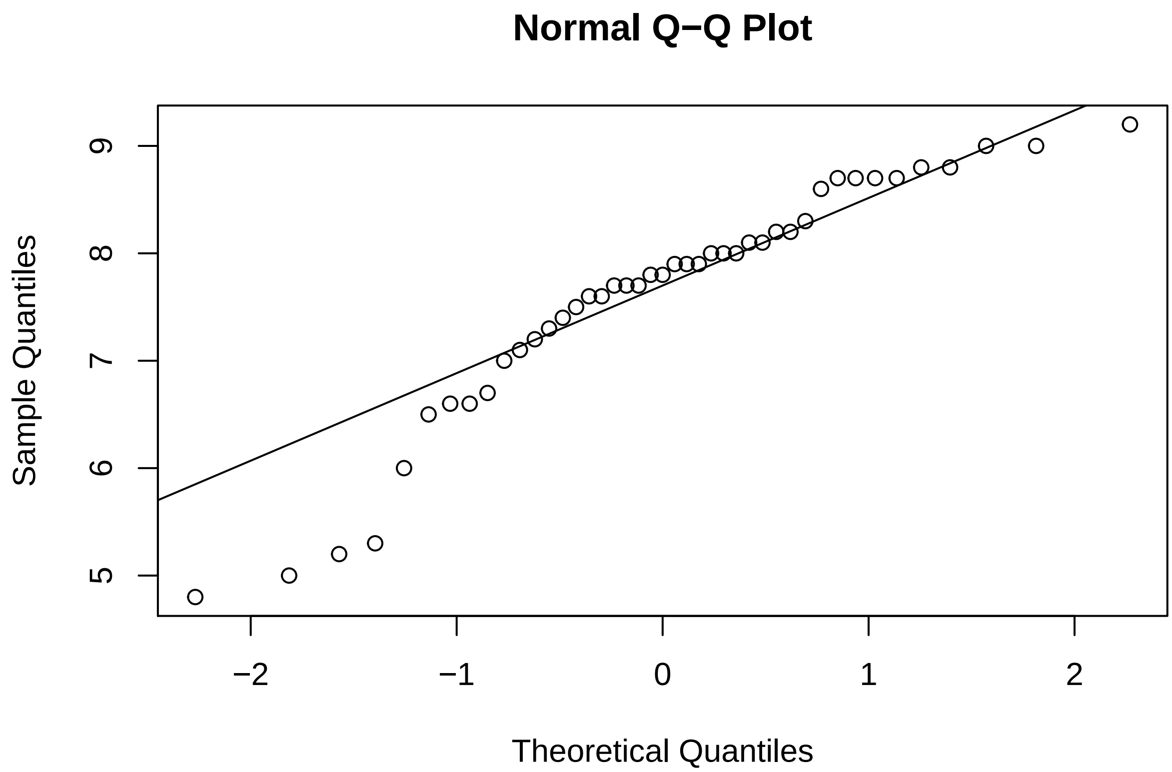

```
par(mfrow=c(1,2))
hist(USJudgeRatings$CONT, probability= TRUE, main="Histogram of CONT", xlab="CONT")
d = density(USJudgeRatings$CONT, kernel = 'c', bw = 0.3)
lines(d, col="red")

hist(USJudgeRatings$RTEN, probability= TRUE, main="Histogram of RTEN" , xlab="RTEN")
d = density(USJudgeRatings$RTEN, kernel = 'o', bw = 0.3)
lines(d, col="red")
```




```r
par(mfrow=c(1,2))
plot(ecdf(USJudgeRatings$CONT), verticals = TRUE, do.points = FALSE, main = "ECDF CONT")
plot(ecdf(USJudgeRatings$RTEN), verticals = TRUE, do.points = FALSE, main = "ECDF RTEN")
```




```r
qqnorm(USJudgeRatings$RTEN)
qqline(USJudgeRatings$RTEN)
```



The QQ plots suggests that the RTEN variable is Gaussian.

Explaining the RTEN variable with a regression model

We will use RTEN as our dependent variable and try to explain it by fitting a regression model. We will try to find which of the other 11 variables explain the best our dependant variable and therefore which criterion are the most important for lawyers when evaluating if a judge is fit to stay at the Supreme Court.

use of ggplot??

```
library(e1071)
kurtosis
```

```
## function (x, na.rm = FALSE, type = 3)
## {
##     if (any(ina <- is.na(x))) {
##         if (na.rm)
##             x <- x[!ina]
##         else return(NA)
##     }
##     if (!(type %in% (1:3)))
##         stop("Invalid 'type' argument.")
##     n <- length(x)
##     x <- x - mean(x)
##     r <- n * sum(x^4)/(sum(x^2)^2)
##     y <- if (type == 1)
##         r - 3
##     else if (type == 2) {
##         if (n < 4)
##             stop("Need at least 4 complete observations.")
```

```
##         ((n + 1) * (r - 3) + 6) * (n - 1)/((n - 2) * (n - 3))
##     }
##     else r * (1 - 1/n)^2 - 3
##     y
## }
## <bytecode: 0x7fba17a24220>
## <environment: namespace:e1071>
```

```
skewness
```

```
## function (x, na.rm = FALSE, type = 3)
## {
##     if (any(ina <- is.na(x))) {
##         if (na.rm)
##             x <- x[!ina]
##         else return(NA)
##     }
##     if (!(type %in% (1:3)))
##         stop("Invalid 'type' argument.")
##     n <- length(x)
##     x <- x - mean(x)
##     y <- sqrt(n) * sum(x^3)/(sum(x^2)^(3/2))
##     if (type == 2) {
##         if (n < 3)
##             stop("Need at least 3 complete observations.")
##         y <- y * sqrt(n * (n - 1))/(n - 2)
##     }
##     else if (type == 3)
##         y <- y * ((1 - 1/n))^(3/2)
##     y
## }
## <bytecode: 0x7fba183eed40>
## <environment: namespace:e1071>
```