MAP 531: Homework

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Problem 1: Estimating parameters of a Poisson distribution

We recall that the Poisson distribution with parameter $\theta > 0$ has a pdf given by $(p(\theta, k), k \in \mathbb{N})$ w.r.t the counting measure on $\mathbb{N}$:

$$p(\theta, k) = e^{-\theta}\frac{\theta^k}{k!}$$

Question 1

The poisson distribution is a discrete distribution since it has a countable number of possible values ($\mathbb{N}$).

In statistics, we use this distribution to compute the probability of a given number of (rare) events in a time period.

For example a poisson distribution can model:

- The number of patients arriving in an emergency room between 9 and 10am.
- The number of network failures per day.
- In quality control, the number of manufacturing defects.

Question 2

We assume that $\mathbb{X}$ follows a Poisson distribution with parameter $\theta > 0$.

We will use the fact that $e^\theta = \sum_{i=0}^{\infty}(\frac{\theta^i}{i!}), \forall\theta \in \mathbb{R}$

$$\mathbb{E}[\mathbb{X}] = \sum_{i=0}^{\infty}(i * p(\theta, i)) = \sum_{i=0}^{\infty}(i * e^{-\theta}\frac{\theta^i}{i!}) = \theta * e^{-\theta}\sum_{i=1}^{\infty}(\frac{\theta^{i-1}}{(i-1)!}) = \theta * e^{-\theta}\sum_{i=0}^{\infty}(\frac{\theta^i}{i!}) = \theta * e^{-\theta} * e^\theta = \theta$$

$$\mathbb{E}[\mathbb{X}^2] = \sum_{i=0}^{\infty}(i^2 * p(\theta, i)) = \sum_{i=0}^{\infty}(i^2 * e^{-\theta}\frac{\theta^i}{i!}) = \theta * e^{-\theta}\sum_{i=1}^{\infty}(i\frac{\theta^{i-1}}{(i-1)!}) = \theta * e^{-\theta}\sum_{i=0}^{\infty}((i+1)\frac{\theta^i}{i!})$$

$$= \theta * e^{-\theta}[\sum_{i=0}^{\infty}(i\frac{\theta^i}{i!}) + \sum_{i=0}^{\infty}(\frac{\theta^i}{i!})] = \theta * e^{-\theta}[\theta\sum_{i=0}^{\infty}(\frac{\theta^i}{i!}) + e^\theta] = \theta * e^{-\theta}[\theta * e^\theta + e^\theta] = \theta(\theta + 1)$$

$$\mathbb{V}(\mathbb{X}) = \mathbb{E}[\mathbb{X}^2] - \mathbb{E}[\mathbb{X}]^2 = \theta(\theta + 1) - \theta^2 = \theta$$

Question 3

We are provided with n independent observations of a Poisson random variable of parameter $\theta \in \Theta = \mathbb{R}_+^*$.
Our observations are $X_k \sim Pois(\theta), \forall k \in 1, ..., n$.
The corresponding statistical model is:

$$\mathcal{M}^n = (\mathbb{N}^n, \mathcal{P}(\mathbb{N}^n),\ \{\mathbb{P}_\theta^n,\ \theta \in \Theta\})$$

with $\mathbb{P}_\theta^n = \mathbb{P}_\theta \otimes ... \otimes \mathbb{P}_\theta$ (n times)
We are trying to estimate the parameter θ.

Question 4

The likelihood function is the function on θ that makes our n observations most likely.

Using the independance of the X_k:

$$l(\theta) = \prod_{k=1}^{n} e^{-\theta} \frac{\theta^{X_k}}{X_k!}$$

$$L(\theta) = log(l(\theta)) = \sum_{k=1}^{n}(-\theta + X_k log(\theta) - log(X_k!)) = -n\theta + log(\theta)\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} log(X_k!)$$

By derivating with respect to θ, we have:

$$L'(\theta) = -n + \frac{\sum_{k=1}^{n} X_k}{\theta}$$

$$L''(\theta) = -\frac{\sum_{k=1}^{n} X_k}{\theta^2} < 0$$

Since, the second derivative of the log-likelihood function is negative, the function is concave and admits a global maximum given by:

$$L'(\theta) = 0 \Leftrightarrow -n + \frac{\sum_{k=1}^{n} X_k}{\theta} = 0 \Leftrightarrow \hat{\theta}_{MLE} = \overline{X}$$

So, the maximum likelihood estimator is:

$$\hat{\theta}_{MLE} = \overline{X}$$

Question 5

Since the X_k are iid, we have that:

$$\mathbb{E}[\overline{X}] = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[X_k] = \mathbb{E}[X_1] = \theta$$

$$\mathbb{V}(\overline{X}) = \frac{1}{n^2}\sum_{k=1}^{n} \mathbb{V}(X_k) = \frac{1}{n}\mathbb{V}[X_1] = \frac{\theta}{n}$$

Applying the central limit theorem, we have that $\sqrt{n}(\hat{\theta}_{MLE} - \theta)$ converges towards a Gaussian $\mathcal{N}(0, \theta)$.

Question 6

The weak law of large numbers gives us that:

$$\hat{\theta}_{MLE} \xrightarrow{p} \theta$$

By continuous mapping, we have:

$$\sqrt{\hat{\theta}_{MLE}} \xrightarrow{p} \sqrt{\theta}$$

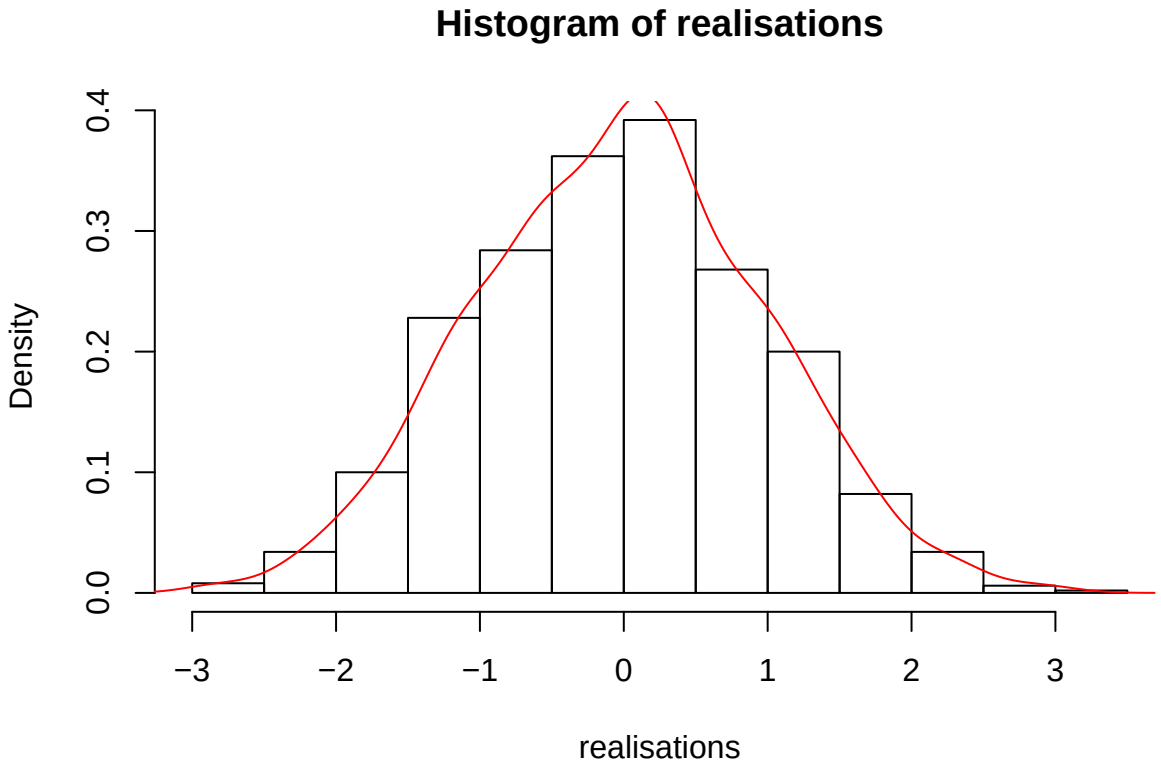
Then, by applying Slutsky's theorem, we finally have that:

$$\sqrt{n}\frac{(\hat{\theta}_{MLE} - \theta)}{\sqrt{\hat{\theta}_{MLE}}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Now, let's check this result in R by simulating 1000 times our random variable $\sqrt{n}\frac{(\hat{\theta}_{MLE}-\theta)}{\sqrt{\hat{\theta}_{MLE}}}$ with a sample size of 100:

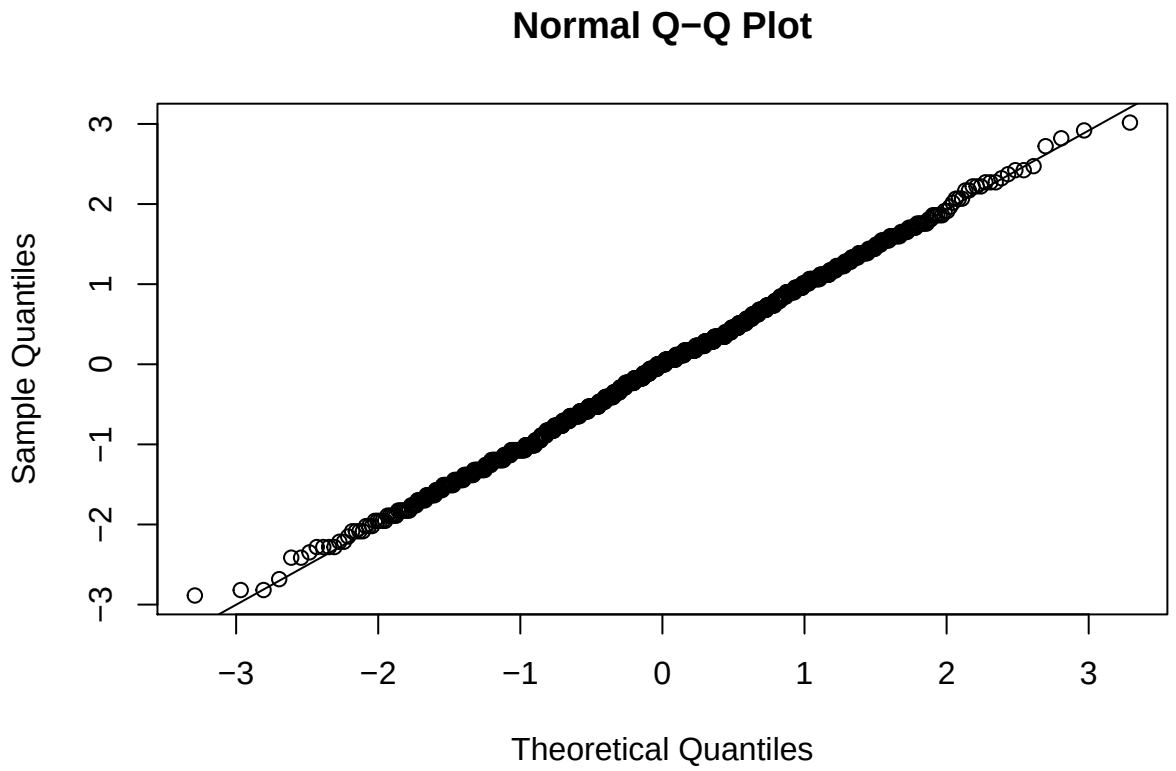
```
estim <- function(x, theta){
  n <- length(x)
  est <- sqrt(n) * (mean(x) - theta) / sqrt(mean(x))
  return(est)
  }
```

```
set.seed(23)
Nattempts = 1e3
nsample = 100
theta = 3

samples <- lapply(1:Nattempts, function(i) rpois(nsample, theta))
realisations <- sapply(samples, function(x) estim(x, theta))

hist(realisations, probability = TRUE)
d = density(realisations, kernel='gaussian')
lines(d, col = 'red')
```



The histogram confirms what we found theoretically. In fact, by plotting the density associated to the histogram we can observe a curve that represents a gaussian distribution. It is symetric around its expectation that seems to be zero. So, we can conclude that the random variable $\sqrt{n}\frac{(\hat{\theta}_{MLE}-\theta)}{\sqrt{\hat{\theta}_{MLE}}}$ follows a standard gaussian distribution.

```r
qqnorm(realisations)
qqline(realisations)
```



The Q-Q plot compares the theoritical quantiles of a standard gaussian distribution to the ones of our estimated distribution. We can observe that the points approximately lie on the line y = x, so the distributions compared are similar and this plot also confirms that the random variable $\sqrt{n}\frac{(\hat{\theta}_{MLE}-\theta)}{\sqrt{\hat{\theta}_{MLE}}}$ follows a standard gaussian distribution.

Question 7

Let $Z_n = \sqrt{n}\frac{(\hat{\theta}_{MLE}-\theta)}{\sqrt{\hat{\theta}_{MLE}}}$ be our random variable.

Denote z_α the α-quantile for the standard Normal distribution for $\alpha \in (0,\ 1)$.

$$\lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2} \leq Z_n \leq z_{1-\alpha/2}) \geq 1-\alpha \Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2}\sqrt{\frac{\hat{\theta}_{MLE}}{n}} \leq \hat{\theta}_{MLE}-\theta \leq z_{1-\alpha/2}\sqrt{\frac{\hat{\theta}_{MLE}}{n}}) \geq 1-\alpha$$

For $\alpha \in (0,1)$, an asymptotic confidence interval of level α for θ is therefore:

$$[\hat{\theta}_{MLE} - z_{1-\alpha/2}\frac{\sqrt{\hat{\theta}_{MLE}}}{\sqrt{n}};\ \hat{\theta}_{MLE} + z_{1-\alpha/2}\frac{\sqrt{\hat{\theta}_{MLE}}}{\sqrt{n}}]$$

Question 8

We apply the δ-method with $g(x) = 2\sqrt{x}$
We have: $g'(x) = \frac{1}{\sqrt{x}}$
So,

$$\sqrt{n}(g(\hat{\theta}_{MLE}) - g(\theta)) \xrightarrow{d} \mathcal{N}(0,\ g'(\theta)^2 \times \theta) \Leftrightarrow \sqrt{n}(g(\hat{\theta}_{MLE}) - g(\theta)) \xrightarrow{d} \mathcal{N}(0,1)$$

$$\Leftrightarrow \sqrt{n}(2\sqrt{\hat{\theta}_{MLE}} - 2\sqrt{\theta}) \xrightarrow{d} \mathcal{N}(0,1)$$

Question 9

Let $W_n = \sqrt{n}(2\sqrt{\hat{\theta}_{MLE}} - 2\sqrt{\theta})$ be our random variable.

We know by the last question that $W_n \xrightarrow{d} \mathcal{N}(0,1)$.

$$\lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2} \leq W_n \leq z_{1-\alpha/2}) \geq 1-\alpha \Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(-\frac{z_{1-\alpha/2}}{2\sqrt{n}} \leq \sqrt{\hat{\theta}_{MLE}} - \sqrt{\theta} \leq \frac{z_{1-\alpha/2}}{2\sqrt{n}}) \geq 1-\alpha$$

$$\Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(\sqrt{\hat{\theta}_{MLE}} - \frac{z_{1-\alpha/2}}{2\sqrt{n}} \leq \sqrt{\theta} \leq \sqrt{\hat{\theta}_{MLE}} + \frac{z_{1-\alpha/2}}{2\sqrt{n}}) \geq 1-\alpha$$

When n goes towards infinity, $\frac{z_{1-\alpha/2}}{2\sqrt{n}}$ goes to 0. Since $\sqrt{\hat{\theta}_{MLE}}$ is positive, there exists a n_0 such that $\forall n \geq n_0$, $\sqrt{\hat{\theta}_{MLE}} - \frac{z_{1-\alpha/2}}{2\sqrt{n}}$ is positive and we can take the squares in the inequality without changing the order of the inequalities:

$$\Leftrightarrow \lim_{n\to+\infty} \mathbb{P}((\sqrt{\hat{\theta}_{MLE}} - \frac{z_{1-\alpha/2}}{2\sqrt{n}})^2 \leq \theta \leq (\sqrt{\hat{\theta}_{MLE}} + \frac{z_{1-\alpha/2}}{2\sqrt{n}})^2) \geq 1-\alpha$$

For $\alpha \in (0,1)$, an asymptotic confidence interval for θ of level α is therefore:

$$[(\sqrt{\hat{\theta}_{MLE}} - \frac{z_{1-\alpha/2}}{2\sqrt{n}})^2;\ (\sqrt{\hat{\theta}_{MLE}} + \frac{z_{1-\alpha/2}}{2\sqrt{n}})^2]$$

Question 10

Based on the first moment of a poisson distribution, we easily have that:

$$\hat{\theta}_{MME} = \overline{X}$$

We can remark that $\hat{\theta}_{MME} = \hat{\theta}_{MLE}$

Based on the second moment of a poisson distribution, we have:

$$n^{-1}\sum_{k=1}^{n} X_k^2 = \hat{\theta}_2(\hat{\theta}_2 + 1) = \hat{\theta}_2^2 + \hat{\theta}_2$$

We can now construct many different estimators by replace one of the $\hat{\theta}_2$ by $\hat{\theta}_1$ or we can inverse the function $h(x) = x(x+1)$ to find an estimator of θ.

The inverse function of h on $\mathbb{R}_+^*$ is $h^{-1}(x) = \frac{1}{2}[-1 + \sqrt{4x+1}]$ and this gives us another estimator of θ:

$$\hat{\theta}_2 = \frac{1}{2}[-1 + \sqrt{(4n^{-1}\sum_{k=1}^{n} X_k^2) + 1}]$$

Question 11

$\mathbb{E}[\hat{\theta}_{MLE}] = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[X_k]$ by linearity of the expectation. So,

$$\mathbb{E}[\hat{\theta}_{MLE}] = \frac{1}{n} * n * \theta = \theta$$

Therefore, $\hat{\theta}_{MLE}$ is an unbiased estimator of θ, ie. ${b_\theta}^*(\hat{\theta}_{MLE}) = 0$

$\mathbb{V}(\hat{\theta}_{MLE}) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}(X_i)$ by independance of the X_k.

$$\mathbb{V}(\hat{\theta}_{MLE}) = \frac{1}{n^2} * n * \theta = \frac{\theta}{n}$$

The quadratic risk Q is:

$$Q = {b_\theta}^*(\hat{\theta}_{MLE})^2 + \mathbb{V}^*(\hat{\theta}_{MLE}) = 0 + \frac{\theta}{n} = \frac{\theta}{n}$$

Question 12

$\hat{\theta}_{MLE}$ is an unbiased estimator. So the Cramer-Rao bound is given by:

$$\frac{1}{I_n(\theta^*)} = \frac{1}{\mathbb{E}[-L''(\theta^*)]}$$

By derivating the log-likelihood function with respect to θ, we have:

$$L'(\theta^*) = -n + \frac{\sum_{i=1}^{n} X_k}{\theta}$$

$$-L''(\theta^*) = \frac{\sum_{i=1}^{n} X_k}{\theta^2}$$

Therefore,

$$\mathbb{E}[-L''(\theta^*)] = \frac{\sum_{i=1}^{n}\mathbb{E}[X_k]}{\theta^2} = \frac{n}{\theta}$$

and finally,

$$\frac{1}{I_n(\theta^*)} = \frac{\theta}{n} = \mathbb{V}(\hat{\theta}_{MLE})$$

Then, we can conclude that our estimator $\hat{\theta}_{MLE}$ reaches the Cramer-Rao bound and is therefore efficient.

Question 13

$$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X_n})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \theta + \theta - \overline{X_n})^2 = \frac{1}{n}\sum_{i=1}^{n}[(X_i - \theta)^2 + (\theta - \overline{X_n})^2 + 2(X_i - \theta)(\theta - \overline{X_n})]$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i - \theta)^2 + (\theta - \overline{X_n})^2 + \frac{2}{n}(\theta - \overline{X_n})\sum_{i=1}^{n}(X_i - \theta) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \theta)^2 + (\theta - \overline{X_n})^2 + 2(\theta - \overline{X_n})(\overline{X_n} - \theta)$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i - \theta)^2 - (\theta - \overline{X_n})^2$$

Question 14

$$\mathbb{E}[(\theta - \overline{X_n})^2] = \mathbb{E}[\theta^2 - 2\theta\overline{X_n} + \overline{X_n}^2] = \theta^2 - 2\theta\mathbb{E}[\overline{X_n}] + \mathbb{E}[\overline{X_n^2}]$$
$$= -\theta^2 + \mathbb{V}(\overline{X_n}) + \mathbb{E}[\overline{X_n}]^2 = -\theta^2 + \frac{\theta}{n} + \theta^2 = \frac{\theta}{n}$$

$$\mathbb{E}[\hat{\theta}_2] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}(X_i - \theta)^2 - (\theta - \overline{X_n})^2] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[(X_i - \theta)^2] - \mathbb{E}[(\theta - \overline{X_n})^2]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{V}(X_i) - \frac{\theta}{n} = \theta(1 - \frac{1}{n})$$

Therefore the bias is:

$$b_{\hat{\theta}_2} = -\frac{\theta}{n}$$

We can get an unbiased estimator $\hat{\theta}_3$ by defining $\hat{\theta}_3 = (1 - \frac{1}{n})^{-1}\hat{\theta}_2$

Question 15

Using the previous questions, we know that:

$$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \theta)^2 - (\theta - \overline{X_n})^2$$

therefore:

$$\sqrt{n}(\hat{\theta}_2 - \theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \theta)^2 - \sqrt{n}(\theta - \overline{X_n})^2 - \sqrt{n}\theta = \sqrt{n}(\overline{Y_n} - \theta) - \sqrt{n}(\theta - \overline{X_n})^2$$

where:

$$\forall i \in [\![1, n]\!], Yi = (X_i - \theta)^2$$
$$\bar{Y_n} = \frac{1}{n}\sum_{i=1}^{n}Y_i$$

Since, we know that:

$$\mathbb{E}[Y_i] = \mathbb{V}(X_i) = \theta$$
$$\mathbb{V}(Y_i) = 2\theta^2 + \theta$$

We can apply the central limit theorem, and we find that:

$$\sqrt{n}(\overline{Y_n} - \theta) \xrightarrow{d} \mathcal{N}(0, 2\theta^2 + \theta)$$

On the other hand, we have the following equalities:

$$\sqrt{n}(\theta - \overline{X_n})^2 = \sqrt{n}(\overline{X_n} - \theta)^2 = \sqrt{n}(\overline{X_n} - \theta)(\overline{X_n} - \theta)$$

By applying the central limit theorem to the left part of the quantity, we have that:

$$\sqrt{n}(\overline{X_n} - \theta) \xrightarrow{d} \mathcal{N}(0, \theta)$$

and applying the law of large numbers to the right part gives us:

$$(\overline{X_n} - \theta) \xrightarrow{p} 0$$

Then, by applying Slutsky's theorem to these two quantities, we have that:

$$\sqrt{n}(\theta - \overline{X_n})^2 \xrightarrow{d} 0$$

Since it converges in distribution to a constant, it also converges in probability:

$$\sqrt{n}(\theta - \overline{X_n})^2 \xrightarrow{p} 0$$

So, we can apply Slutsky's theorem to $\sqrt{n}(\overline{Y_n} - \theta) - \sqrt{n}(\theta - \overline{X_n})^2$ which finally gives us that:

$$\sqrt{n}(\hat{\theta}_2 - \theta) \xrightarrow{d} \mathcal{N}(0, 2\theta^2 + \theta)$$

Now, let's compute another asymptotic confidence interval centered in $\hat{\theta}_2$.

We know by the first part of question that:

$$\frac{\sqrt{n}(\hat{\theta}_2 - \theta)}{\sqrt{2\theta^2 + \theta}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Let's use $\hat{\theta}_2$ as an estimator of θ for the denominator (by applying Slutsky's theorem).
In fact, we need to find a pivotal quantity to compute our asymptotic confidence interval.

First, we will prove that $\hat{\theta}_2 \xrightarrow{p} \theta$.

By applying the law of large numbers to $\bar{Y_n}$, we have:

$$\bar{Y_n} \xrightarrow{p} \theta$$

and we saw that:

$$(\theta - \overline{X_n})^2 \xrightarrow{p} 0$$

So, by stability of the convergence in probability, we have that $\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n} Y_i - (\theta - \overline{X_n})^2 \xrightarrow{p} \theta$ and by continuous mapping,

$$\sqrt{2\hat{\theta}_2^2 + \hat{\theta}_2} \xrightarrow{p} \sqrt{2\theta^2 + \theta}$$

Then, by applying Slutsky's theorem, we have that:

$$V_n = \sqrt{n}\frac{(\hat{\theta}_2 - \theta)}{\sqrt{2\hat{\theta}_2^2 + \hat{\theta}_2}} \xrightarrow{d} \mathcal{N}(0, 1)$$

and we can use it as a pivotal quantity to find a confidence interval for θ.

$$\lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2} \leq V_n \leq z_{1-\alpha/2}) \geq 1-\alpha \Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(-z_{1-\alpha/2}\frac{\sqrt{2\hat{\theta}_2^2 + \hat{\theta}_2}}{\sqrt{n}} \leq \hat{\theta}_2 - \theta \leq z_{1-\alpha/2}\frac{\sqrt{2\hat{\theta}_2^2 + \hat{\theta}_2}}{\sqrt{n}}) \geq 1-\alpha$$

$$\Leftrightarrow \lim_{n\to+\infty} \mathbb{P}(\hat{\theta}_2 - z_{1-\alpha/2}\frac{\sqrt{2\hat{\theta}_2^2 + \hat{\theta}_2}}{\sqrt{n}} \leq \theta \leq \hat{\theta}_2 + z_{1-\alpha/2}\frac{\sqrt{2\hat{\theta}_2^2 + \hat{\theta}_2}}{\sqrt{n}}) \geq 1-\alpha$$

For $\alpha \in (0, 1)$, an asymptotic confidence interval for θ of level α is therefore:

$$[\hat{\theta}_2 - z_{1-\alpha/2}\frac{\sqrt{2\hat{\theta}_2^2 + \hat{\theta}_2}}{\sqrt{n}};\ \hat{\theta}_2 + z_{1-\alpha/2}\frac{\sqrt{2\hat{\theta}_2^2 + \hat{\theta}_2}}{\sqrt{n}}]$$

This asymptotic confidence interval has a size equal to $2 \cdot z_{1-\alpha/2} \frac{\sqrt{2\hat{\theta}_2^2+\hat{\theta}_2}}{\sqrt{n}}$.

The asymptotic confidence interval found at the question 7 has a size equal to $2 \cdot z_{1-\alpha/2} \frac{\sqrt{\hat{\theta}_{MLE}}}{\sqrt{n}}$.

So, the range of the confidence interval found with the estimator $\hat{\theta}_2$ is larger and is therefore less precise. In fact, $2\hat{\theta}_2^2 + \hat{\theta}_2 > \hat{\theta}_{MLE}$.

Furthermore, we have that:

$$var(\hat{\theta}_2) = \frac{2\theta^2+\theta}{n} = \frac{\theta(2\theta+1)}{n} = (2\theta+1)Var(\hat{\theta}_{MLE}) > Var(\hat{\theta}_{MLE}) = \frac{1}{I_n(\theta^*)}$$

So the asymptotic variance of this estimator is larger than the variance of the MLE. It is then less efficient than the MLE.

Furthermore, the quadratic risk of this estimator is also larger than the one of the MLE since $\hat{\theta}_2$ is a biaised estimator of θ and has a larger asymptotic variance. So, $\hat{\theta}_{MLE}$ is a better estimator for θ than $\hat{\theta}_2$.

Question 16

Let $s \in \mathbb{R}$. The probability generating function of the Poisson distribution is given by:

$$G_{\mathbb{X}}(s) = \mathbb{E}[exp(s\mathbb{X})] = \sum_{k=0}^{\infty} e^{ks} e^{-\theta} \frac{\theta^k}{k!} = e^{-\theta} \sum_{k=0}^{\infty} \frac{(\theta e^s)^k}{k!} = e^{-\theta} e^{\theta e^s} = e^{\theta(e^s-1)}$$

In order to compute the first and second moment of the Poisson distribution, we can now use the moment generating function. Let's compute its first and second order derivatives.

$$G'_{\mathbb{X}}(s) = \theta e^s e^{\theta(e^s-1)}$$
$$G''_{\mathbb{X}}(s) = \theta[e^s e^{\theta(e^s-1)} + \theta e^{2s} e^{\theta(e^s-1)}] = \theta e^s [e^{\theta(e^s-1)} + \theta e^s e^{\theta(e^s-1)}]$$

Then, we have:

$$\mathbb{E}[\mathbb{X}] = G'_{\mathbb{X}}(0) = \theta$$
$$\mathbb{E}[\mathbb{X}^2] = G''_{\mathbb{X}}(0) = \theta(1+\theta)$$
$$\mathbb{V}(\mathbb{X}) = \mathbb{E}[\mathbb{X}^2] - \mathbb{E}[\mathbb{X}]^2 = \theta(1+\theta) - \theta^2 = \theta$$

We will now show that: $\mathbb{V}[(\mathbb{X}_i - \theta)^2] = 2\theta^2 + \theta$

We use the following equalities:

$$G^{(3)}_{\mathbb{X}}(s) = (1 + 3\theta e^s + \theta^2 e^{2s}) \theta e^{s+\theta(e^s-1)}$$

$$G^{(4)}_{\mathbb{X}}(s) = (1 + \theta^3 e^{3s} + 6\theta^2 e^{2s} + 7\theta e^s) \theta e^{s+\theta(e^s-1)}$$

$$\mathbb{E}[\mathbb{X}^3] = G^{(3)}_{\mathbb{X}}(0) = \theta + 3\theta^2 + \theta^3$$

$$\mathbb{E}[\mathbb{X}^4] = G^{(4)}_{\mathbb{X}}(0) = \theta^4 + 6\theta^3 + 7\theta^2 + \theta$$

Finally, we have that:

$$\mathbb{V}[(\mathbb{X}_i - \theta)^2] = \mathbb{E}[(\mathbb{X}-\theta)^4] - \mathbb{E}[(\mathbb{X}-\theta)^2]^2 = \mathbb{E}[\mathbb{X}^4] - 4\theta\mathbb{E}[\mathbb{X}^3] + 6\theta^2\mathbb{E}[\mathbb{X}^2] - 4\theta^3\mathbb{E}[\mathbb{X}] + \theta^4 - Var(\mathbb{X})^2$$
$$= \theta^4 + 6\theta^3 + 7\theta^2 + \theta - 4\theta(\theta + 3\theta^2 + \theta^3) + 6\theta^2(\theta + \theta^2) - 4\theta^4 + \theta^4 - \theta^2 = 2\theta^2 + \theta$$

Problem 2: Analysis of the USJudgeRatings dataset

This exercise is open. You are asked to use the tools we have seen together to analyze the USJudgeRatings data set. This data set is provided in the package datasets. Your analysis should be reported here and include:

- an introduction
- a general description of the data
- the use of descriptive statistics
- the use of all techniques we have seen together that might be relevant
- a conclusion

Overall, your analysis, including the graphs and the codes should not exceed 15 pages in pdf.

Introduction

We are given to analyse a dataset, named USJudgeratings, containing various ratings of state judges in the US Superior Court made by lawyers. The different variables given help us to determine if a judge is worthy staying in the US Superior Court or not. We will start by doing a general description of the data and applying descriptive statistics to better apprehend the data.

General description

We start by uploading our data.

```
data(USJudgeRatings)
```

First, let's see how the dataset is organized.

```
str(USJudgeRatings)
```

```
## 'data.frame':    43 obs. of  12 variables:
##  $ CONT: num  5.7 6.8 7.2 6.8 7.3 6.2 10.6 7 7.3 8.2 ...
##  $ INTG: num  7.9 8.9 8.1 8.8 6.4 8.8 9 5.9 8.9 7.9 ...
##  $ DMNR: num  7.7 8.8 7.8 8.5 4.3 8.7 8.9 4.9 8.9 6.7 ...
##  $ DILG: num  7.3 8.5 7.8 8.8 6.5 8.5 8.7 5.1 8.7 8.1 ...
##  $ CFMG: num  7.1 7.8 7.5 8.3 6 7.9 8.5 5.4 8.6 7.9 ...
##  $ DECI: num  7.4 8.1 7.6 8.5 6.2 8 8.5 5.9 8.5 8 ...
##  $ PREP: num  7.1 8 7.5 8.7 5.7 8.1 8.5 4.8 8.4 7.9 ...
##  $ FAMI: num  7.1 8 7.5 8.7 5.7 8 8.5 5.1 8.4 8.1 ...
##  $ ORAL: num  7.1 7.8 7.3 8.4 5.1 8 8.6 4.7 8.4 7.7 ...
##  $ WRIT: num  7 7.9 7.4 8.5 5.3 8 8.4 4.9 8.5 7.8 ...
##  $ PHYS: num  8.3 8.5 7.9 8.8 5.5 8.6 9.1 6.8 8.8 8.5 ...
##  $ RTEN: num  7.8 8.7 7.8 8.7 4.8 8.6 9 5 8.8 7.9 ...
```

The dataset is stored in a dataframe and we can observe that all the variables are numeric.

```
dim(USJudgeRatings)
```

```
## [1] 43 12
```

We are provided with n = 43 observations and p = 12 quantitative variables.

We can have a full view of the dataset by using the kable function:

```
library(knitr)
library(kableExtra)
kable(USJudgeRatings, 'latex', caption = "Ratings of US judges", booktabs = T) %>%
  kable_styling(latex_options = "striped", font_size = 6.5)
```

Table 1: Ratings of US judges

	CONT	INTG	DMNR	DILG	CFMG	DECI	PREP	FAMI	ORAL	WRIT	PHYS	RTEN
AARONSON,L.H.	5.7	7.9	7.7	7.3	7.1	7.4	7.1	7.1	7.1	7.0	8.3	7.8
ALEXANDER,J.M.	6.8	8.9	8.8	8.5	7.8	8.1	8.0	8.0	7.8	7.9	8.5	8.7
ARMENTANO,A.J.	7.2	8.1	7.8	7.8	7.5	7.6	7.5	7.5	7.3	7.4	7.9	7.8
BERDON,R.I.	6.8	8.8	8.5	8.8	8.3	8.5	8.7	8.7	8.4	8.5	8.8	8.7
BRACKEN,J.J.	7.3	6.4	4.3	6.5	6.0	6.2	5.7	5.7	5.1	5.3	5.5	4.8
BURNS,E.B.	6.2	8.8	8.7	8.5	7.9	8.0	8.1	8.0	8.0	8.0	8.6	8.6
CALLAHAN,R.J.	10.6	9.0	8.9	8.7	8.5	8.5	8.5	8.5	8.6	8.4	9.1	9.0
COHEN,S.S.	7.0	5.9	4.9	5.1	5.4	5.9	4.8	5.1	4.7	4.9	6.8	5.0
DALY,J.J.	7.3	8.9	8.9	8.7	8.6	8.5	8.4	8.4	8.4	8.5	8.8	8.8
DANNEHY,J.F.	8.2	7.9	6.7	8.1	7.9	8.0	7.9	8.1	7.7	7.8	8.5	7.9
DEAN,H.H.	7.0	8.0	7.6	7.4	7.3	7.5	7.1	7.2	7.1	7.2	8.4	7.7
DEVITA,H.J.	6.5	8.0	7.6	7.2	7.0	7.1	6.9	7.0	7.0	7.1	6.9	7.2
DRISCOLL,P.J.	6.7	8.6	8.2	6.8	6.9	6.6	7.1	7.3	7.2	7.2	8.1	7.7
GRILLO,A.E.	7.0	7.5	6.4	6.8	6.5	7.0	6.6	6.8	6.3	6.6	6.2	6.5
HADDEN,W.L.JR.	6.5	8.1	8.0	8.0	7.9	8.0	7.9	7.8	7.8	7.8	8.4	8.0
HAMILL,E.C.	7.3	8.0	7.4	7.7	7.3	7.3	7.3	7.2	7.1	7.2	8.0	7.6
HEALEY.A.H.	8.0	7.6	6.6	7.2	6.5	6.5	6.8	6.7	6.4	6.5	6.9	6.7
HULL,T.C.	7.7	7.7	6.7	7.5	7.4	7.5	7.1	7.3	7.1	7.3	8.1	7.4
LEVINE,I.	8.3	8.2	7.4	7.8	7.7	7.7	7.7	7.8	7.5	7.6	8.0	8.0
LEVISTER,R.L.	9.6	6.9	5.7	6.6	6.9	6.6	6.2	6.0	5.8	5.8	7.2	6.0
MARTIN,L.F.	7.1	8.2	7.7	7.1	6.6	6.6	6.7	6.7	6.8	6.8	7.5	7.3
MCGRATH,J.F.	7.6	7.3	6.9	6.8	6.7	6.8	6.4	6.3	6.3	6.3	7.4	6.6
MIGNONE,A.F.	6.6	7.4	6.2	6.2	5.4	5.7	5.8	5.9	5.2	5.8	4.7	5.2
MISSAL,H.M.	6.2	8.3	8.1	7.7	7.4	7.3	7.3	7.3	7.2	7.3	7.8	7.6
MULVEY,H.M.	7.5	8.7	8.5	8.6	8.5	8.4	8.5	8.5	8.4	8.4	8.7	8.7
NARUK,H.J.	7.8	8.9	8.7	8.9	8.7	8.8	8.9	9.0	8.8	8.9	9.0	9.0
O'BRIEN,F.J.	7.1	8.5	8.3	8.0	7.9	7.9	7.8	7.8	7.8	7.7	8.3	8.2
O'SULLIVAN,T.J.	7.5	9.0	8.9	8.7	8.4	8.5	8.4	8.3	8.3	8.3	8.8	8.7
PASKEY,L.	7.5	8.1	7.7	8.2	8.0	8.1	8.2	8.4	8.0	8.1	8.4	8.1
RUBINOW,J.E.	7.1	9.2	9.0	9.0	8.4	8.6	9.1	9.1	8.9	9.0	8.9	9.2
SADEN.G.A.	6.6	7.4	6.9	8.4	8.0	7.9	8.2	8.4	7.7	7.9	8.4	7.5
SATANIELLO,A.G.	8.4	8.0	7.9	7.9	7.8	7.8	7.6	7.4	7.4	7.4	8.1	7.9
SHEA,D.M.	6.9	8.5	7.8	8.5	8.1	8.2	8.4	8.5	8.1	8.3	8.7	8.3
SHEA,J.F.JR.	7.3	8.9	8.8	8.7	8.4	8.5	8.5	8.5	8.4	8.4	8.8	8.8
SIDOR,W.J.	7.7	6.2	5.1	5.6	5.6	5.9	5.6	5.6	5.3	5.5	6.3	5.3
SPEZIALE,J.A.	8.5	8.3	8.1	8.3	8.4	8.2	8.2	8.1	7.9	8.0	8.0	8.2
SPONZO,M.J.	6.9	8.3	8.0	8.1	7.9	7.9	7.9	7.7	7.6	7.7	8.1	8.0
STAPLETON,J.F.	6.5	8.2	7.7	7.8	7.6	7.7	7.7	7.7	7.5	7.6	8.5	7.7
TESTO,R.J.	8.3	7.3	7.0	6.8	7.0	7.1	6.7	6.7	6.7	6.7	8.0	7.0
TIERNEY,W.L.JR.	8.3	8.2	7.8	8.3	8.4	8.3	7.7	7.6	7.5	7.7	8.1	7.9
WALL,R.A.	9.0	7.0	5.9	7.0	7.0	7.2	6.9	6.9	6.5	6.6	7.6	6.6
WRIGHT,D.B.	7.1	8.4	8.4	7.7	7.5	7.7	7.8	8.2	8.0	8.1	8.3	8.1
ZARRILLI,K.J.	8.6	7.4	7.0	7.5	7.5	7.7	7.4	7.2	6.9	7.0	7.8	7.1

An observation in this dataset represents twelve different ratings received by a judge (given by his name) in the US Superior Court. The ratings are given by different laywers but we don't have any information on them. In order to study this dataset, we will first define properly what each variable means.

```
colnames(USJudgeRatings)
```

```
##  [1] "CONT" "INTG" "DMNR" "DILG" "CFMG" "DECI" "PREP" "FAMI" "ORAL" "WRIT"
## [11] "PHYS" "RTEN"
```

The variables are:

- *CONT* : The number of contacts of lawyer with the judge.
- *INTG* : The judicial integrity of the judge.
- *DMNR* : Demeanor of the judge.
- *DILG* : Diligence of the judge.
- *CFMG* : Case flow managed by the judge.
- *DECI* : Prompt decisions taken by the judge.
- *PREP* : How the judge is prepared for trials.
- *FAMI* : The judge's familiarity with law.
- *ORAL* : The judge's sound oral rulings.
- *WRIT* : The judge's sound written rulings.
- *PHYS* : The judge's physical ability.
- *RTEN* : Scaling if the judge is worthy to retain in the US Superior court.

Descriptive dataset analysis

Let's inspect the dataframe for missing values, outliers and errors:

```
sum(is.na(USJudgeRatings))
```

```
## [1] 0
```

There are no missing values in the dataframe.

```
summary(USJudgeRatings)
```

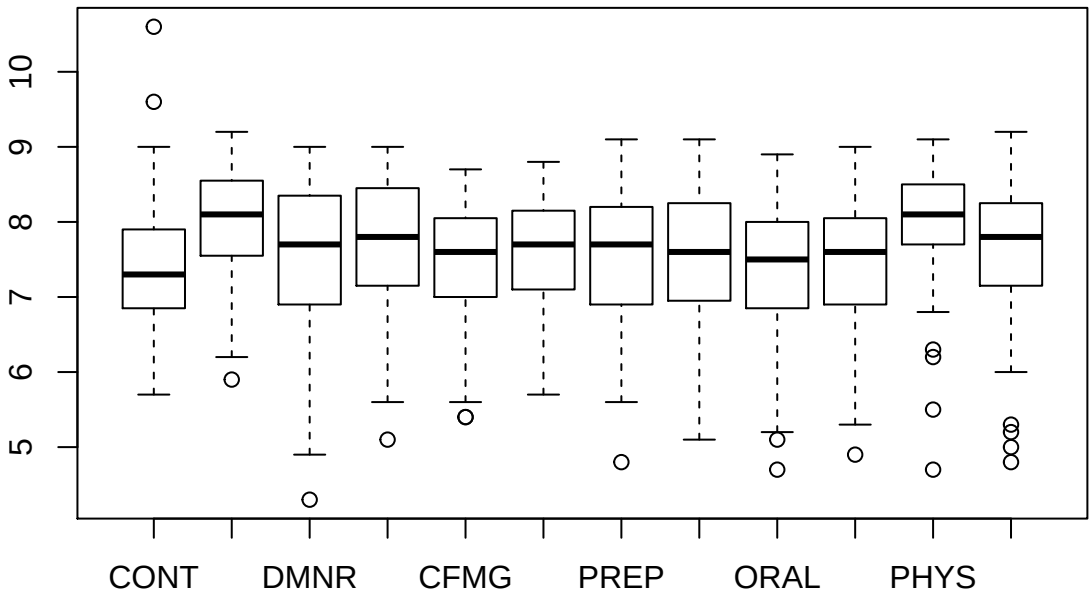
```
##       CONT             INTG            DMNR            DILG
##  Min.   : 5.700   Min.   :5.900   Min.   :4.300   Min.   :5.100
##  1st Qu.: 6.850   1st Qu.:7.550   1st Qu.:6.900   1st Qu.:7.150
##  Median : 7.300   Median :8.100   Median :7.700   Median :7.800
##  Mean   : 7.437   Mean   :8.021   Mean   :7.516   Mean   :7.693
##  3rd Qu.: 7.900   3rd Qu.:8.550   3rd Qu.:8.350   3rd Qu.:8.450
##  Max.   :10.600   Max.   :9.200   Max.   :9.000   Max.   :9.000
##       CFMG            DECI            PREP            FAMI
##  Min.   :5.400   Min.   :5.700   Min.   :4.800   Min.   :5.100
##  1st Qu.:7.000   1st Qu.:7.100   1st Qu.:6.900   1st Qu.:6.950
##  Median :7.600   Median :7.700   Median :7.700   Median :7.600
##  Mean   :7.479   Mean   :7.565   Mean   :7.467   Mean   :7.488
##  3rd Qu.:8.050   3rd Qu.:8.150   3rd Qu.:8.200   3rd Qu.:8.250
##  Max.   :8.700   Max.   :8.800   Max.   :9.100   Max.   :9.100
##       ORAL            WRIT            PHYS            RTEN
##  Min.   :4.700   Min.   :4.900   Min.   :4.700   Min.   :4.800
##  1st Qu.:6.850   1st Qu.:6.900   1st Qu.:7.700   1st Qu.:7.150
##  Median :7.500   Median :7.600   Median :8.100   Median :7.800
```

```
##  Mean   :7.293   Mean   :7.384   Mean   :7.935   Mean   :7.602
##  3rd Qu.:8.000   3rd Qu.:8.050   3rd Qu.:8.500   3rd Qu.:8.250
##  Max.   :8.900   Max.   :9.000   Max.   :9.100   Max.   :9.200
```

All the variables (except the variable CONT) are ranged between 0 and 10.
By looking at the medians and means, we know that we can have information on the distribution of the variables. In fact, means are really sensible of outliers when medians aren't. However, in our case, each variable admits a median that is close to the mean. For example, the median for the INTG variable is 8.021 and its median is equal to 8.100. So this doesn't give us any information yet on possible outliers.

In order to have more information on the distributions followed by our variables, we will now take a look at the boxplots of the variables to see if there are outliers and errors, and to compare interquartile ranges.

```
Outvals = boxplot(USJudgeRatings)
```



We observe the presence of outliers for 10 of the 12 variables (with larger values for CONT and with lower values for the other variables). Most of the variables have only one or two outliers, except the variables PHYS and RTEN, which have four outliers each. We are not provided with extra information and nothing indicates that these outliers correspond to mistakes. Thus, we will assume that they aren't mistakes and keep them in our analysis.

Note that the PHYS variable has a small interquartile range compare to the other variables. This means that all the judges seem to have a good physical ability (superior to 7) except the four outliers.
Also, the four outliers for the RTEN variable indicate that the laywers think that these four judges should not stay in the US Superior court. Therefore, we have to see if there a link between the other variables and the variable RTEN.

Let's take a closer look at some of the outliers.

```
cat('The judge with the largest number of contacts of lawyer is judge',
    rownames(USJudgeRatings)[which.max(USJudgeRatings$CONT)], '\n',
    'with a number of', max(USJudgeRatings$CONT), 'contacts.')
```

```
## The judge with the largest number of contacts of lawyer is judge CALLAHAN,R.J.
##  with a number of 10.6 contacts.
```

```
cat('The judge with the highest rating for worthiness of retention is judge',
    rownames(USJudgeRatings)[which.max(USJudgeRatings$RTEN)], '\n',
```

```
    'with a rating of', max(USJudgeRatings$RTEN))
```

```
## The judge with the highest rating for worthiness of retention is judge RUBINOW,J.E.
##  with a rating of 9.2
```

We can take a look at his ratings.

```
USJudgeRatings[which.max(USJudgeRatings$RTEN),]
```

```
##              CONT INTG DMNR DILG CFMG DECI PREP FAMI ORAL WRIT PHYS RTEN
## RUBINOW,J.E.  7.1  9.2    9    9  8.4  8.6  9.1  9.1  8.9    9  8.9  9.2
```

```
cat('The judge with the lowest rating for worthiness of retention is judge',
    rownames(USJudgeRatings)[which.min(USJudgeRatings$RTEN)], '\n',
    'with a rating of', min(USJudgeRatings$RTEN))
```

```
## The judge with the lowest rating for worthiness of retention is judge BRACKEN,J.J.
##  with a rating of 4.8
```

We can also take a look at his ratings.

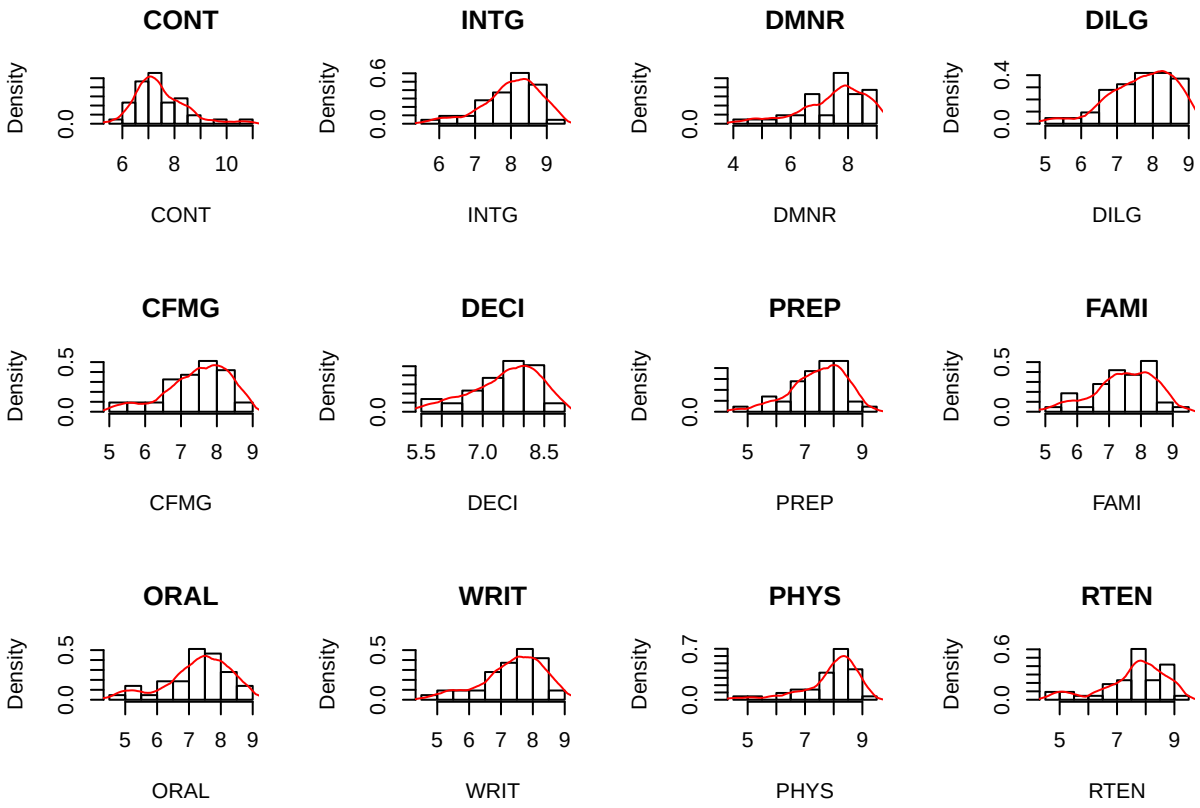
```
USJudgeRatings[which.min(USJudgeRatings$RTEN),]
```

```
##              CONT INTG DMNR DILG CFMG DECI PREP FAMI ORAL WRIT PHYS RTEN
## BRACKEN,J.J.  7.3  6.4  4.3  6.5    6  6.2  5.7  5.7  5.1  5.3  5.5  4.8
```

By comparing the ratings of these two judges, we can observe that the judge with the highest RTEN rating has better ratings in the other variables except for the variable CONT. Thus, there seems to be a link between the RTEN variable and the others (except CONT). We will look deeper into that correlation later in this report.

Let's now look at the distribution of the variables

```
par(mfrow=c(3,4))
for (var in colnames(USJudgeRatings)) {
  hist(USJudgeRatings[, var], main=var, probability=TRUE, xlab=var)
d = density(USJudgeRatings[, var], kernel = 'o', bw = 0.3)
lines(d, col="red")}
```

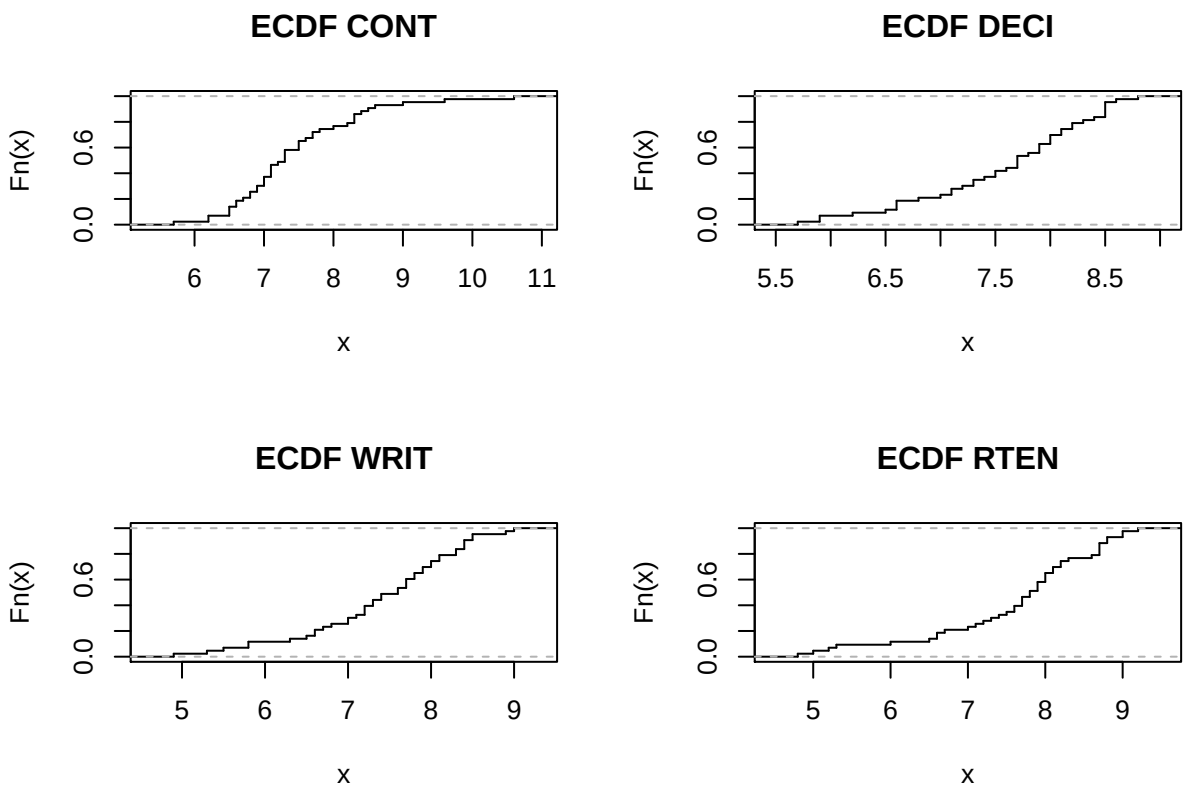


These histograms give us an idea of the distribution followed by these variables.
For example, for the CONT variable, we can see a bell curve with its highest point near the mean of the sample. So we can assume that this variable follows a gaussian distribution. However, we also see the two outliers on the histogram and they modify a bit the symmetry of the distribution.
The INTG variable also seems to follow a gaussian distribution, but it is less obvious. The bell curve is not exactly symetric. The other two variables don't seem to follow any usual distribution and it is difficult to see any real particularity.

Finally, these four variables seem to have a symmetric law if we don't look too much on the outliers. In fact, they all seem to have extreme lower values but their densities are bell curves with the highest point of probability near their means. However, these histograms aren't precise enough and we would gain in accuracy with a larger dataset.

Let's look at other statistical tools to look deeper at the distributions followed by these variables.

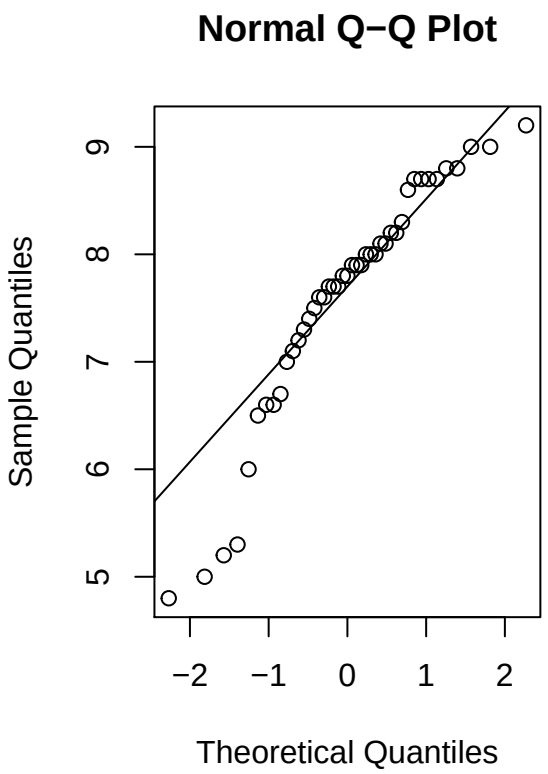
```
par(mfrow=c(2,2))
plot(ecdf(USJudgeRatings$CONT), verticals = TRUE, do.points = FALSE, main = "ECDF CONT")
plot(ecdf(USJudgeRatings$DECI), verticals = TRUE, do.points = FALSE, main = "ECDF DECI")
plot(ecdf(USJudgeRatings$WRIT), verticals = TRUE, do.points = FALSE, main = "ECDF WRIT")
plot(ecdf(USJudgeRatings$RTEN), verticals = TRUE, do.points = FALSE, main = "ECDF RTEN")
```

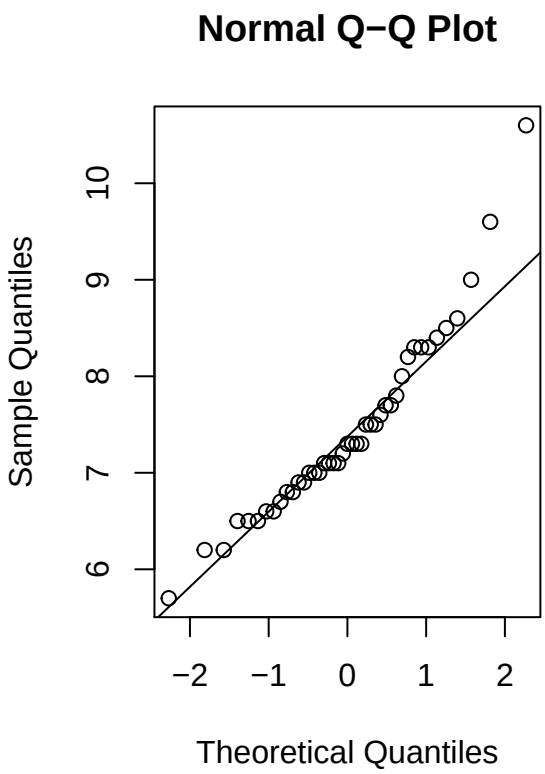


We focus on four of the variables and we look at their empirical cumulative distribution functions.
The cumulative distribution function of the variable CONT is different from the others empirical cumulative distribution functions. In fact, the range of values taken by this variable is larger than for the other variables (from 6 to 11, compared to 5 to 9). However, the shape of the functions are pretty similar. The steps are bigger for extreme values but we can associate all these ECDF to ECDF of gaussian distributions.

Finally, let's compare the quantiles of our variables to the quantiles of a gaussian distribution to see if we can really associate these variables to gaussian distributions.

```
par(mfrow=c(1,2))
qqnorm(USJudgeRatings$RTEN)
qqline(USJudgeRatings$RTEN)

qqnorm(USJudgeRatings$CONT)
qqline(USJudgeRatings$CONT)
```





The QQ-plots are really interesting to analyse for the variables RTEN and CONT. In fact, the QQ-plot for the RTEN variable shows us that until -1 the variable doesn't seem to follow a gaussian distribution but from -1 to 2, the quantiles of the variable are really close to the quantiles of a gaussian distribution of a mean approximately equal to 7.4.
The second QQ-plot suggests that the CONT variable seems to follow a Gaussian distribution until 1.5. It's the contrary of the RTEN variable.
This analysis can be explained by the outliers we found at the beginning. In fact, we found outliers with upper value for the CONT variable and outliers with lower values for the RTEN variable. This explains why we don't find a gaussian distribution for extreme values.

Let's analyze the correlation between the variables

First, we can have a look at the covariance matrix, and the standard deviation of each variable.

```
round(var(USJudgeRatings), 2)
```

```
##       CONT  INTG  DMNR DILG CFMG DECI PREP  FAMI  ORAL  WRIT PHYS  RTEN
## CONT  0.89 -0.10 -0.17 0.01 0.11 0.07 0.01 -0.02 -0.01 -0.04 0.05 -0.03
## INTG -0.10  0.59  0.85 0.60 0.54 0.50 0.64  0.64  0.71  0.67 0.54  0.79
## DMNR -0.17  0.85  1.31 0.86 0.80 0.74 0.93  0.91  1.05  0.98 0.85  1.19
## DILG  0.01  0.60  0.86 0.81 0.74 0.69 0.84  0.82  0.87  0.83 0.69  0.92
## CFMG  0.11  0.54  0.80 0.74 0.74 0.68 0.79  0.76  0.83  0.78 0.71  0.88
## DECI  0.07  0.50  0.74 0.69 0.68 0.64 0.73  0.72  0.77  0.73 0.66  0.82
## PREP  0.01  0.64  0.93 0.84 0.79 0.73 0.91  0.90  0.95  0.90 0.76  1.00
## FAMI -0.02  0.64  0.91 0.82 0.76 0.72 0.90  0.90  0.94  0.90 0.75  0.98
## ORAL -0.01  0.71  1.05 0.87 0.83 0.77 0.95  0.94  1.02  0.96 0.85  1.09
## WRIT -0.04  0.67  0.98 0.83 0.78 0.73 0.90  0.90  0.96  0.92 0.77  1.02
## PHYS  0.05  0.54  0.85 0.69 0.71 0.66 0.76  0.75  0.85  0.77 0.88  0.94
```

```
## RTEN -0.03  0.79  1.19 0.92 0.88 0.82 1.00  0.98  1.09  1.02 0.94  1.21
```

```
round(sqrt(diag(var(USJudgeRatings))),2)
```

```
## CONT INTG DMNR DILG CFMG DECI PREP FAMI ORAL WRIT PHYS RTEN
## 0.94 0.77 1.14 0.90 0.86 0.80 0.95 0.95 1.01 0.96 0.94 1.10
```

```
cat('The smallest standard deviation is: ', min(round(sqrt(diag(var(USJudgeRatings))),2)), '\n')
```

```
## The smallest standard deviation is:  0.77
```

```
cat('The largest standard deviation is: ', max(round(sqrt(diag(var(USJudgeRatings))),2)))
```

```
## The largest standard deviation is:  1.14
```

We find that the variables DMNR and RTEN have the largest standard deviations, while the DECI variable has the smallest.
In fact, we have seen at the beginning that the RTEN variable has four outliers that are far from the other values and the DMNR has an outlier that is really far from the other values.

Let's measure the correlations between the 12 variables to see if we can find is a link between them.

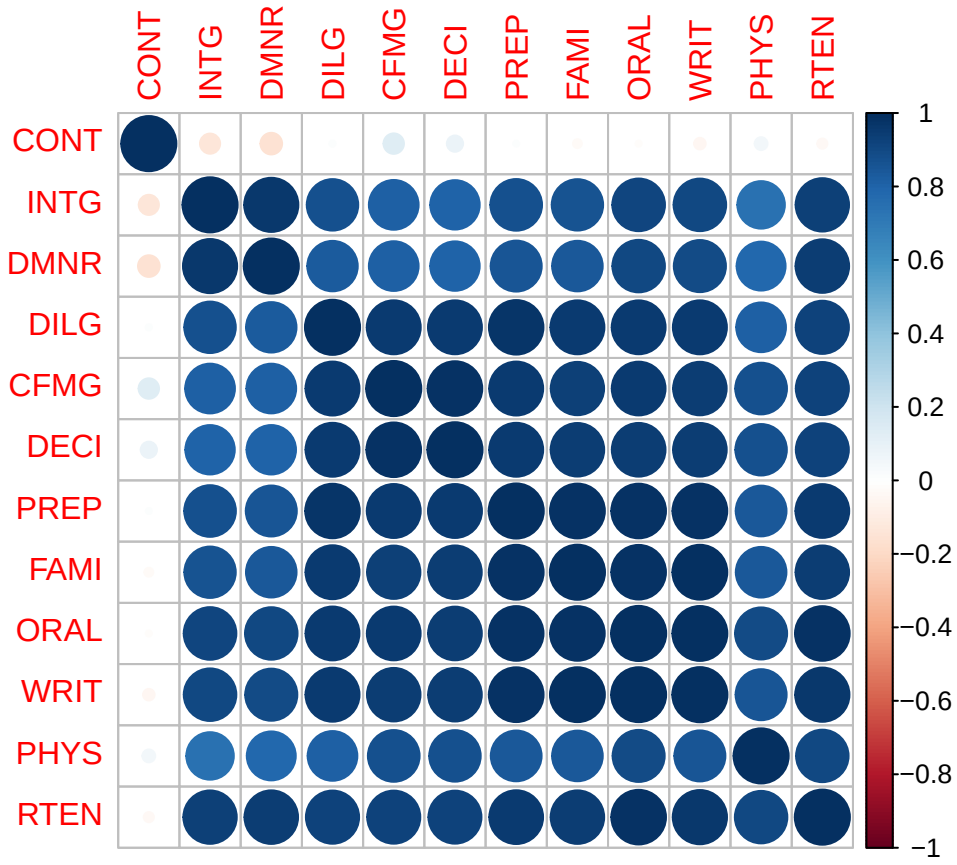
```
round(cor(USJudgeRatings),2)
```

```
##       CONT  INTG  DMNR DILG CFMG DECI PREP  FAMI  ORAL  WRIT PHYS  RTEN
## CONT  1.00 -0.13 -0.15 0.01 0.14 0.09 0.01 -0.03 -0.01 -0.04 0.05 -0.03
## INTG -0.13  1.00  0.96 0.87 0.81 0.80 0.88  0.87  0.91  0.91 0.74  0.94
## DMNR -0.15  0.96  1.00 0.84 0.81 0.80 0.86  0.84  0.91  0.89 0.79  0.94
## DILG  0.01  0.87  0.84 1.00 0.96 0.96 0.98  0.96  0.95  0.96 0.81  0.93
## CFMG  0.14  0.81  0.81 0.96 1.00 0.98 0.96  0.94  0.95  0.94 0.88  0.93
## DECI  0.09  0.80  0.80 0.96 0.98 1.00 0.96  0.94  0.95  0.95 0.87  0.92
## PREP  0.01  0.88  0.86 0.98 0.96 0.96 1.00  0.99  0.98  0.99 0.85  0.95
## FAMI -0.03  0.87  0.84 0.96 0.94 0.94 0.99  1.00  0.98  0.99 0.84  0.94
## ORAL -0.01  0.91  0.91 0.95 0.95 0.95 0.98  0.98  1.00  0.99 0.89  0.98
## WRIT -0.04  0.91  0.89 0.96 0.94 0.95 0.99  0.99  0.99  1.00 0.86  0.97
## PHYS  0.05  0.74  0.79 0.81 0.88 0.87 0.85  0.84  0.89  0.86 1.00  0.91
## RTEN -0.03  0.94  0.94 0.93 0.93 0.92 0.95  0.94  0.98  0.97 0.91  1.00
```

```
library(corrplot)
```

```
## corrplot 0.84 loaded
```

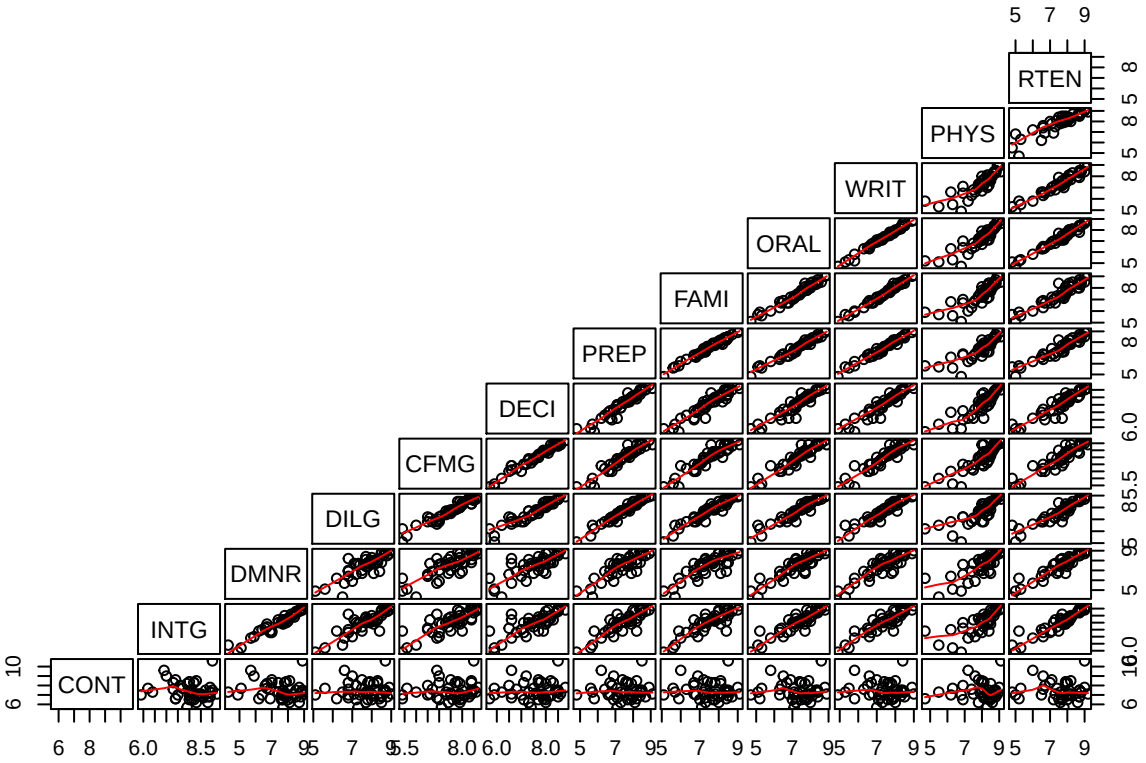
```
corrplot(cor(USJudgeRatings))
```



Obviously, we can see that there is a very strong correlation between 11 of the 12 variables. In fact, the variable CONT is the only one which isn't correlated to the other variables. The number of contacts of a judge with lawyers doesn't seem to explain the ratings received by the judge. Secondly, if we look deeper at the 11 correlated variables, we can see that the variable PHYS is the one that is the less correlated with the others.
This correlation suggests that we can create a linear model to predict a variable from the others.

To confirm this trend, we will take a look at the scatter plots between the variables.

```
pairs(USJudgeRatings, gap=1/5, lower.panel = panel.smooth, upper.panel = NULL, row1attop=FALSE)
```



These plots confirm what we suggested before. In fact, there is a linear correlation between eleven of the twelve variables (all the variables except CONT). If the subject had been to predict the values of one of these variables, we could have used a linear regression model.

Let's take a look at the skewness and kurtosis indicators

```
library(e1071)
skewness(USJudgeRatings$DILG)
```

```
## [1] -0.756197
```

```
skewness(USJudgeRatings$PREP)
```

```
## [1] -0.6572536
```

```
skewness(USJudgeRatings$FAMI)
```

```
## [1] -0.5377923
```

```
skewness(USJudgeRatings$RTEN)
```

```
## [1] -0.9373609
```

We calculate the skewness of four variables. Skewness is an indicator that measures the assymetry of a distribution. In our case, we always have a negative indicator. That means that the tail is on the left side of the distributions.

```
kurtosis(USJudgeRatings$DECI)
```

```
## [1] -0.5318582
```

```r
kurtosis(USJudgeRatings$PREP)
```

```
## [1] -0.003622203
```

```r
kurtosis(USJudgeRatings$FAMI)
```

```
## [1] -0.3653484
```

```r
kurtosis(USJudgeRatings$RTEN)
```

```
## [1] 0.2557421
```

Kurtosis is an indicator that measures the propension of a distribution to produce outliers.
However, we have seen that the variable PREP has a very lower outlier but its kurtosis is near 0. On the other side, we saw that the variable DECI doesn't have any outlier but its kurtosis is negative and near -0.5. But all these kurtosis values are ranged between -1 and 1, so their values aren't extreme and do not show any extreme particularities.

Building confidence intervals

In order to apply some statistical methods, let's construct a confidence interval for the mean of the RTEN variable. In fact, we assumed during this report that this variable follows a gaussian distribution. It can be interesting to estimate its mean without knowing its variance.

We assume that we have n = 43 observations of a gaussian distribution: $X_k \sim \mathcal{N}(\mu, \sigma^2)$, $\forall k \in 1, ..., n$ such that $\mu \in \mathbb{R},\ \sigma^2 \in \mathbb{R}_+^*$.

Our statistical model is therefore:

$$\mathcal{M}^n = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n),\ \{\mathbb{P}_\mu^n,\ (\mu, \sigma^2) \in \Theta\})$$

with $\mathbb{P}_\mu^n = \mathbb{P}_\mu \otimes ... \otimes \mathbb{P}_\mu$ (n times)

We are trying to estimate the parameter μ.

Since, we don't know the variance but we assumed that the observations are gaussians, we will use the fact that $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X_n})^2$ is an unbiaised estimator of σ^2.

We know that a confidence interval of level α for μ is (not proven):

$$[\overline{X_n} - t_{\alpha/2, df=n-1}\frac{S}{\sqrt{(n-1)}},\ \overline{X_n} - t_{\alpha/2, df=n-1}\frac{S}{\sqrt{(n-1)}}]$$

With $t_{\alpha, df=n-1}$ the quantile of level α of a Student's distribution of degree of freedom $n-1$.

Let's compute the confidence interval for μ in our case:

```r
n = 43
alpha = 0.05
moy = mean(USJudgeRatings$RTEN)
t = qt(1-alpha/2, df=n-1)
S = sqrt(sum((USJudgeRatings$RTEN - moy)^2)/(n-1))
CI = c(moy - (t * S) / (sqrt(n-1)), moy + (t * S) / (sqrt(n-1)))
print(CI)
```

```
## [1] 7.259487 7.945164
```

We find, under the assumption that the RTEN variable follows a gaussian distribution of mean μ, a confidence interval of level $\alpha = 0.05$ for μ equal to:

$$[7.259487, 7.945164]$$

Now, let's drop the gaussian distribution assumption for the RTEN variable. We can also construct a confidence interval for μ using Slutsky's theorem (which is possible because we have more than 30 observations).
In fact, we assume that our observations are independent, with a distribution with a mean equal to μ and a variance equal to σ^2.
We use the central limit theorem and we apply Slutsky's theorem, using the fact that $S \xrightarrow{p} \sigma$ (generalisation of a result found in the first exercise).

The confidence interval for μ is then given by:

$$[\overline{X_n} - z_{\alpha/2}\frac{S}{\sqrt{n}},\ \overline{X_n} - z_{\alpha/2}\frac{S}{\sqrt{n}}]$$

```
n = 43
alpha = 0.05
moy = mean(USJudgeRatings$RTEN)
z = qnorm(1-alpha/2)
S = sqrt(sum((USJudgeRatings$RTEN - moy)^2)/(n-1))
CI = c(moy - (z * S) / (sqrt(n)), moy + (z * S) / (sqrt(n)))
print(CI)
```

```
## [1] 7.273254 7.931397
```

So, if we don't assume that RTEN follows a gaussian distribution, we find a confidence interval of level $\alpha = 0.05$ for μ equal to:

$$[7.273254, 7.931397]$$

This asymptotic confidence interval has a smaller range than the previous one and is therefore more precise.

Conclusion

Overall, descriptive statistics have allowed us to better understand the dataset. The USJudgeRatings dataset gives us 12 ratings received by 43 judges. We have seen that this dataset has no missing values but contains several outliers. We found for example that 4 judges had particularly low retention ratings. This led us to look at the distributions of the different variables. By looking at histograms, empirical cumlative distribution functions and qqplots we were able to determine that some of the variables follow gaussian distributions. Then, we looked at the correlation between the variables to discover that, put aside the CONT variable, all the variables are extremely correlated. A judge either has good grades in all competencies or lower grades everywhere. Finally, we decided to build two confidence intervals to have a closer idea of the mean of the RTEN variable. This could for example allow us to build a statistical test to decide whether to keep a judge or not based on his retention rating.